A STUDY OF GENERALIZED DERIVATIONS IN RINGS

ABSTRACT

OF

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

MATHEMATICS

BY

KHALID ALI MOHAMMAD HAMDIN

UNDER THE SUPERVISION OF

PROFESSOR ASMA ALI

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)
2014
ABSTRACT

In 1991 Bresar [37] introduced notion of generalized derivation. An additive mapping \( F : R \rightarrow R \) is said to be a generalized derivation on a ring \( R \) if there exists a derivation \( d : R \rightarrow R \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \). In recent years many well known algebraists such as Beider, Bergen, Herstein, Kharchenko, Hvala, Martindale, Vukman, Bresar, Bell, Lee, Lanski etc. have made remarkable contributions in this area of study. The interest in this area was partially motivated by its useful applications in various branches of mathematics (for references see [38], [128], [129], [135], [140],[147], [149]).

The present thesis entitled "A study of generalized derivations in rings" includes a part of research work carried out by the author under the able guidance of Prof. Asma Ali, at the Department of Mathematics, Aligarh Muslim University, Aligarh. The thesis comprises five chapters and each chapter is subdivided into various sections. The definitions, examples, results, and remarks etc. have been specified with double decimal numbers. The first figure denotes the chapter, the second represents the section in the chapter and third points out the number of the definition, the example, the result or the remark as the case may be in particular chapter. For example 4.2.3 refers to the third theorem appearing in the second section of the fourth chapter.

Chapter 1 of the thesis contains some preliminary notions, basic definitions and important well known results which may be needed for the development of the subsequent text. This chapter as a matter of fact, aims at making the present thesis as self contained as possible. However, the basic knowledge of the ring theory has been presumed and no attempt is made to include the proofs of the results in this chapter.

A lot of work has been done on commutativity of prime and semiprime rings satisfying certain functional identities involving derivations or generalized deriva-
tions for example see [2], [20], [21], [32], [33], [54], [59], [60], [61], [122]. In chapter 2 we continue this type of study. Motivated by the results of Bell and Daif [32], Quadri et al [122], and Dhara [59] we prove that a nonzero left ideal $I$ of a semiprime $R$ admitting a generalized derivation $F$ with associated derivation $d$ is central if for all $x, y \in I$, one of the conditions holds: (i) $[F(x), d(y)] \pm [x, y] = 0$; (ii) $F([x, y]) \pm [d(x), F(y)] = 0$; (iii) $F([x, y]) = [F(x), y] + [d(y), x]$; (iv) $d(x)F(y) \pm xy = 0$ and (v) $F([x, y]) \pm [(F(x), y)] = 0$.

Chapter 3 is devoted to the study of rings with involution ($*$- rings) admitting a derivation. An additive mapping $x \mapsto x^*$ on a ring $R$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution on $R$. A ring equipped with an involution is called a $*$- ring or an involution ring. Bell and Kappe [35] initiated the study of derivations which act as homomorphisms or as antihomomorphisms. They proved that if $R$ is a semiprime ring and $d$ is a derivation of $R$ which acts as a homomorphism or as an antihomomorphism on $I$, a nonzero ideal of $R$, then $d = 0$. We obtain the result for a generalized $(\alpha, \beta)$- derivation of a $*$- prime ring. In fact we prove that if $R$ is a $*$- prime ring with a generalize $(\alpha, \beta)$- derivation acting as a homomorphism or an antihomomorphism on a nonzero $*$- Lie ideal $U$ of $R$, then $U \subseteq Z(R)$.

Vukman [136, Theorem 4] proved that if $R$ is a semiprime ring with a derivation $d$ and $\alpha$ is an automorphism of $R$ such that the mapping $x \mapsto d(x) + \alpha(x)$ is commuting on $R$, then $d$ and $\alpha - I$, where $I$ is an identity map on $R$, map $R$ into $Z(R)$, the centre of $R$. In chapter 4 we extend the above result as follows: Let $R$ be a semiprime ring. Suppose that $F : R \rightarrow R$ is a generalized derivation with an associated derivation $d$ and $\alpha$ is an automorphism of $R$. If the mapping $x \mapsto F(x) + \alpha(x)$ is commuting on $R$, then $d$ and $\alpha - I$ map $R$ into $Z(R)$. We also extend Theorem 10 and Theorem 11 of the mentioned paper.
Chapter 5 deals with the study of generalized biderivations of prime and semi-prime rings. Nurcan [14] defined a generalized biderivation in rings. Let $D : R \times R \rightarrow R$ be a biadditive map on a ring $R$. A biadditive mapping $\Delta : R \times R \rightarrow R$ is said to be a generalized biderivation of $R$ if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $y \mapsto D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $x \mapsto D(x, y)$ for all $x, y \in R$. The main result of the chapter extends Theorem 1 of Vukman [138] which states that if $U$ is a noncentral square closed Lie ideal of a prime ring $R$ of characteristic not two, $D : R \times R \rightarrow R$ is an additive mapping and $\Delta : R \times R \rightarrow R$ is a symmetric generalized biderivation of $R$ which is commuting on $U$, then $\Delta = 0$ on $U$.

In the end, an exhaustive bibliography of existing material related to the subject matter of thesis is included which may serve as source of material for those, interested in the domain of research.

One paper of the author related to chapter 5 has been accepted for publication in International journal of Algebra and several papers related to the material of other chapters are in process of acceptance.
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Dedicated

To

My Beloved

Family

Khalid Ali Mohammad Hamdin
Aligarh Muslim University, Aligarh-India
CERTIFICATE

This is to certify that the thesis entitled "A Study of Generalized Derivation in Rings" is based on a part of the research work of Mr. Khalid Ali Mohammad Hamdin carried out under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh. To the best of my knowledge, the work included in the thesis is original and has not been submitted to any other University or Institution for the award of the degree of Doctor of Philosophy in Mathematics.

It is further certified that Mr. Khalid Ali Mohammad Hamdin has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.

CHAIRMAN
DEPARTMENT OF MATHEMATICS
A.M.U., ALIGARH

Supervisor

Prof. Asma Ali
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Acknowledgement

All praises and thanks to Allah, the Almighty, the Lord of the worlds, the most beneficent, the most merciful, Who has taught the use of the pen, and Whose benign benediction granted me the courage, patience and strength to embark upon this work and carry it to its successful completion.

I feel deeply honoured in expressing my profound sense of gratitude to my supervisor Prof. Asma Ali, Department of Mathematics, Aligarh Muslim University, Aligarh. Without whose able supervision and inspiring guidance this work would not have taken the present shape. Despite her busy schedule, she has been kind enough for imparting me the most valuable guidance in all possible ways. I will always be grateful to her for generating my interest in Mathematics vis-a-vis Theory of Rings and Near Rings. She has been an inspiring source with unquestionable intellect and academic excellence. Her timely advice and guidance has a great impact on me. The critical comments, she rendered during the discussion have gone a long way in my understanding and presentation of the contents of this thesis. I wish I could be as perfect and determined like her in my future endeavors.

I am highly indebted to put on record my profound thanks to Prof. Murtaza A. Quadri, former Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh for his inspiration and encouragement during the completion of my work in spite of his multifaceted busy schedules.

I would like to express my gratitude to Prof. Mohd. Ashraf, Department of Mathematics, Aligarh Muslim University, Aligarh, for his invaluable support and useful suggestions for reviving and commenting on my thesis from time to time.
I extend my sincere thanks to Prof. Zafar Ahsan, Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, for providing me all the departmental facilities whenever needed.

I am extremely grateful to Dr. Nadeem-Ur-Rehman, Dr. Shakir Ali, Dr. Rekha Rani and Dr. Deepak Kumar for their useful suggestions and support.

I am immensely grateful to my colleagues Dr. Faiza Shujat, Mr. Phool Miyan, Mr. Shahoor Khan, Mrs. Ambreen Bano, Mr. Farhat Ali and Mr. Inzamul-Haque for their fruitful discussions and cooperation during the course of this work.

I would like to thank for the support and advice to all my seniors specially Dr. Radwan Al-Omary and Dr. Abdullah Kayed Noman who continuously enhance my aspiration, encouragement and guidance. Further they are grateful for bearing with me the lengthy discussions sometime stretching to odd hour.

I have no words to express my gratitude and thanks to my heartiest indebtedness parents Mr. Ali Mohammad Hamdin and Mrs. Fatima Abdullah Moheb for their limitless sacrifices to enrich my future. They were always with me in good as well as in bad times in order to keep me focused towards my goal. I would also like to express my thanks to my wife Mrs. Lateefa and dearest children Mr. Hisham and Ms. Shema for their best wishes and cooperation.

Finally, I express my indebtedness to my glorious and esteemed institution Ali-garh Muslim University, Aligarh.

Sept. 20. 2014

(Khalid Ali Mohammad Hamdin)
Preface

In 1991 Bresar [37] introduced notion of generalized derivation. An additive mapping $F : R \to R$ is said to be a generalized derivation on a ring $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In recent years many well known algebraists such as Beider, Bergen, Herstein, Kharchenko, Hvala, Martindale, Vukman, Bresar, Bell, Lee, Lanski etc. have made remarkable contributions in this area of study. The interest in this area was partially motivated by its useful applications in various branches of mathematics (for references see [38], [128], [129], [135], [140], [147], [149]).

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Chapter 3 is devoted to the study of rings with involution (\( \ast \)-rings) admitting a derivation. An additive mapping \( x \rightarrow x^\ast \) on a ring \( R \) satisfying \((xy)^\ast = y^\ast x^\ast \) and \((x^\ast)^\ast = x \) for all \( x, y \in R \) is called an involution on \( R \). A ring equipped with an involution is called a \( \ast \)-ring or an involution ring. Bell and Kappe [35] initiated the study of derivations which act as homomorphisms or as antihomomorphisms. They proved that if \( R \) is a semiprime ring and \( d \) is a derivation of \( R \) which acts as a homomorphism or as an antihomomorphism on \( I \), a nonzero ideal of \( R \), then \( d = 0 \). We obtain the result for a generalized \((\alpha, \beta)\) - derivation of a \( \ast \) - prime ring. In fact we prove that if \( R \) is a \( \ast \) - prime ring with a generalize \((\alpha, \beta)\) - derivation acting as a homomorphism or an antihomomorphism on a nonzero \( \ast \)-Lie ideal \( U \) of \( R \), then \( U \subseteq Z(R) \).

Vukman [136, Theorem 4] proved that if \( R \) is a semiprime ring with a derivation \( d \) and \( \alpha \) is an automorphism of \( R \) such that the mapping \( x \rightarrow d(x) + \alpha(x) \) is commuting on \( R \), then \( d \) and \( \alpha - I \), where \( I \) is an identity map on \( R \), map \( R \) into \( Z(R) \), the centre of \( R \). In chapter 4 we extend the above result as follows: Let \( R \) be a semiprime ring. Suppose that \( F : R \rightarrow R \) is a generalized derivation with an associated derivation \( d \) and \( \alpha \) is an automorphism of \( R \). If the mapping \( x \rightarrow F(x) + \alpha(x) \) is commuting on \( R \), then \( d \) and \( \alpha - I \) map \( R \) into \( Z(R) \). We also extend Theorem 10
and Theorem 11 of the mentioned paper.

Chapter 5 deals with the study of generalized biderivations of prime and semiprime rings. Nurcan [14] defined a generalized biderivation in rings. Let $D : R \times R \to R$ be a biadditive map on a ring $R$. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized biderivation of $R$ if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $y \mapsto D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $x \mapsto D(x, y)$ for all $x, y \in R$. The main result of the chapter extends Theorem 1 of Vukman [138] which states that if $U$ is a noncentral square closed Lie ideal of a prime ring $R$ of characteristic not two, $D : R \times R \to R$ is an additive mapping and $\Delta : R \times R \to R$ is a symmetric generalized biderivation of $R$ which is commuting on $U$, then $\Delta = 0$ on $U$.

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Chapter 1

Preliminaries

1.1 Introduction

This chapter is devoted to collect some basic notions and important terminology with a view to making our thesis as self contained as possible. Of course, the elementary knowledge of the algebraic concepts such as groups, rings, ideals, fields and homomorphisms etc. has been presumed and no attempt will be made to discuss them here. Some key results and well-known theorems are also included which we shall require for the development of the subject matter in the subsequent chapters. For their proofs, the references are mentioned for those who develop the interest in them. Most of the material included in this chapter occurs in standard literatures namely, Goodearl [63, 64], Herstein [65, 67], Jacobson [77], McCoy [105], Lam [83, 84]. Suitable examples and necessary remarks are given at proper places to make the exposition self contained as much as possible. Other examples will be given from time to time in the sequel.

1.2 Some ring theoretic concepts

In the present section we give a brief exposition of some important terminology in ring theory. Throughout the thesis, unless otherwise mentioned, \( R \) denotes an associative ring having at least two elements. For any pair of elements \( x, y \in R \),
the commutator $xy - yx$ will be denoted by $[x, y]$ and anti-commutator $xy + yx$ by $x \circ y$. The symbols $N(R)$, $C(R)$ and $Z(R)$ denote the set of nilpotent elements, the commutator and the centre of the ring $R$.

**Definition 1.2.1 (Characteristic of a Ring)** The smallest positive integer $n$ (if exists) such that $nx = 0$ for all $x \in R$ is called the characteristic of the ring $R$. If there exists no such integer, then we say that $R$ has characteristic zero. If $R$ has unity $1$, then the characteristic of $R$ is also the smallest positive integer $n$ for which $n.1 = 0$. Some time we shall denote the characteristic of $R$ by $\text{Char}R$.

**Definition 1.2.2 (Idempotent Element)** An element $e$ in a ring $R$ is called idempotent if $e^2 = e$. An idempotent $e$ is said to be central idempotent if it is in the center of $R$ i.e $ex = xe$ for all $x \in R$.

**Remark 1.2.1** In a ring $R$ with unity and no nonzero divisors of zero the only idempotents are the zero and unity.

**Definition 1.2.3 (Nilpotent Element)** An element $x$ of a ring $R$ is said to be nilpotent if there exists a positive integer $n$ such that $x^n = 0$. If such a positive integer exists, then the least such integer $n \geq 1$ is called the index of nilpotency of $x$.

**Remark 1.2.2** Every nonzero nilpotent element is necessarily a divisor of zero. Indeed, if $x \neq 0$ is nilpotent, then there exists the smallest positive integer $n > 1$ such that $x^n = 0$ so that $xx^{n-1} = 0$ with $x^{n-1} \neq 0$.

**Definition 1.2.4 (Regular element)** An element $a$ in a ring $R$ is said to be regular if it is neither a left nor a right zero divisor of $R$.

**Definition 1.2.5 (Centre of a ring)** The centre $Z(R)$ of a ring $R$ is the set of
all those elements of \( R \) which commute with each element of \( R \), i.e. \( Z(R) = \{ x \in R \mid xr = xr \text{ for all } r \in R \} \).

**Remark 1.2.3** A ring \( R \) is commutative if and only if \( Z(R) = R \).

**Definition 1.2.6 (Centralizer)** Let \( S \) be a nonempty subset of a ring \( R \). Then the centralizer of \( S \) in \( R \) denoted by \( C_R(S) = \{ x \in R \mid sx = xs \text{ for all } s \in S \} \).

**Definition 1.2.7 (Ideal)** Let \( I \) be a nonempty subset of a ring \( R \) with the property that \( I \) is an additive subgroup of \( R \). Then

(i) \( I \) is a right ideal of \( R \), if \( I \) is closed under multiplication on the right by elements of \( R \).

(ii) \( I \) is a left ideal of \( R \), if \( I \) is closed under multiplication on the left by elements of \( R \).

(iii) \( I \) is an ideal of \( R \) if it is both left as well as right ideal in \( R \), i.e. for each \( a \in I \), \( r \in R \), \( ra \) and \( ar \in R \).

**Example 1.2.1** Let \( R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z} \right\} \), \( I_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,b \in \mathbb{Z} \right\} \) is a right ideal but not a left ideal of \( R \), and \( I_2 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a,b \in \mathbb{Z} \right\} \) is a left ideal but not a right ideal of \( R \).

We usually denote such an ideal by the symbol \( < S > \). If \( S \) is a finite set, then an ideal \( I \) generated by \( S \) is said to be finitely generated. In particular, if \( I \) is generated by a single element \( a \in R \), the set \( I \) is said to be a principal ideal and is denoted by \( < a > \).
Definition 1.2.8 (Nilpotent ideal) An ideal $I$ of a ring $R$ is said to be a nilpotent ideal if there exists a positive integer $n$ such that $I^n = (0)$.

Example 1.2.2 Let $M$ be the ring of all $2 \times 2$ upper triangular matrices over integers. Then the ideal generated by \[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\] is nilpotent.

Example 1.2.3 Consider the ring $\mathbb{Z}_{p^n}$, where $p$ is a fixed prime and $n > 1$. $\mathbb{Z}_{p^n}$ has exactly one ideal for each positive divisor of $p^n$ and no other ideals; these are simply principal ideals $(p^k) = p^k\mathbb{Z}_{p^n}$ $(0 \leq k \leq n)$. For $0 < k \leq n$, we have $(p^k)^n = (p^{kn}) = (0)$. So each proper ideal of $\mathbb{Z}_{p^n}$ is nilpotent.

Definition 1.2.9 (Nil ideal) An ideal $I$ of a ring $R$ is said to be nil if every element of $R$ is nilpotent.

Remark 1.2.4 Every nilpotent ideal is nil but a nil ideal need not be nilpotent in general.

Example 1.2.4 In a commutative ring

$$R = \mathbb{Z}[x_1, x_2, \ldots]/(x_1^2, x_2^3, x_3^4, \ldots)$$

the ideal $I = (x_1, x_2, x_3, \ldots)$ generated by $x_1, x_2, x_3, \ldots$ is nil ideal but not nilpotent.

Remark 1.2.5 The sum of any finite number of nil (nilpotent) ideals of a ring is again nil (nilpotent).

Definition 1.2.10 (Commutator ideal) The commutator ideal $C(R)$ of a ring $R$ is the ideal generated by all commutators $[x, y]$ with $x, y \in R$.

Definition 1.2.11 (Maximal ideal) An ideal $M$ in a ring $R$ is said to be
a maximal ideal in \( R \) if \( M \neq R \) and there exists no ideal \( I \) in \( R \) such that \( M \subseteq I \subseteq R \).

**Remark 1.2.6** If \( M(\neq R) \) is a maximal ideal of \( R \), then for any ideal \( A \) of \( R \), \( M \subseteq A \subseteq R \) holds only when either \( A = M \) or \( A = R \).

**Definition 1.2.12 (Prime Ideal)** An ideal \( P \) in a ring \( R \) is said to be a prime ideal in \( R \) if for any two ideals \( A \) and \( B \) in \( R \), \( AB \subseteq P \) implies that \( A \subseteq P \) or \( B \subseteq P \).

**Remark 1.2.7** Equivalently, an ideal \( P \) in a ring \( R \) is prime if and only if any one of the following holds:

(i) If \( x, y \in R \) such that \( xRy \subseteq P \), then \( x \in P \) or \( y \in P \).

(ii) If \( (x) \) and \( (y) \) are principal ideals in \( R \) such that \( (x)(y) \subseteq P \), then \( x \in P \) or \( y \in P \).

(iii) If \( U \) and \( V \) are right (or left) ideals in \( R \) such that \( UV \subseteq P \), then \( U \subseteq P \) or \( V \subseteq P \).

**Definition 1.2.13 (Semiprime ideal)** An ideal \( I \) in a ring \( R \) is said to be a semiprime if for any ideal \( A \) in \( R \), whenever \( A^2 \subseteq I \) then \( A \subseteq I \).

**Definition 1.2.14 (Torsion Free Element)** An element \( x \) of a ring \( R \) is said to be \( n \)-torsion free if \( nx = 0 \) implies that \( x = 0 \). If for every \( x \in R \) \( nx = 0 \), implies \( x = 0 \), then \( R \) is called \( n \)-torsion free.

**Definition 1.2.15 (Radical ideal)** An ideal \( I \) of a ring \( R \) is said to be a radical ideal of \( R \) if for \( a \in R \), \( a^n \in I \) for some integer \( n \geq 1 \), implies that \( a \in I \).
Definition 1.2.16 (Jacobson radical) The Jacobson radical of a ring $R$, denoted by $\text{rad } R = \cap\{M \mid M$ is a maximal ideal of $R\}$.

Remark 1.2.8 If $\text{rad } R = (0)$, then $R$ is said to be a ring without Jacobson radical.

Definition 1.2.17 (Prime radical) The prime radical of a ring $R$ denoted by $\beta(R) = \cap\{P \mid P$ is a prime ideal of $R\}$.

Remark 1.2.9 If $\beta(R) = (0)$, we say that the ring $R$ is without prime radical or has zero prime radical.

Example 1.2.5 The ring $F[x]$ of formal power series over a field $F$ has zero prime radical.

Definition 1.2.18 (Annihilator) Let $R$ be a ring and $S$ be a subset of $R$. Then the left annihilator of $S$ denoted by $\text{A}_l(S) = \{x \in R \mid xs = (0), \text{ for all } s \in S\}$. Similarly the right annihilator of $S$ denoted by $\text{A}_r(S) = \{x \in R \mid sx = (0), \text{ for all } s \in S\}$.

Definition 1.2.19 (Prime ring) A ring $R$ is said to be prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$.

Remark 1.2.10 A ring $R$ is prime if and only if the zero ideal $(0)$ is a prime ideal in $R$.

Definition 1.2.20 (Semiprime ring) A ring $R$ is said to be semiprime if for $a \in R$, $aRa = (0)$ implies that $a = 0$.

Remark 1.2.11 A ring $R$ is semiprime if it has no nonzero nilpotent ideals.

Definition 1.2.21 (Simple ring) A ring $R$ is said to be simple if it has no proper nonzero ideals.
Example 1.2.6 The matrix ring $M_2(\mathbb{Z}_2)$ is a simple ring.

Definition 1.2.22 (Semisimple ring) A ring $R$ is said to be semisimple if its Jacobson radical is zero.

Definition 1.2.23 (Subdirectly irreducible ring) A ring $R$ is said to be subdirectly irreducible if the intersection of all nonzero ideals of $R$ is nonzero.

Definition 1.2.24 (Direct sum and subdirect sum of rings) Let $S_i, \ i \in U$ be a family of rings indexed by the set $U$ and $S$ denote the set of all functions defined on the set $U$ such that for each $i \in U$, the value of function at $i$ is an element of $S_i$. If addition and multiplication in $S$ are defined as: $(a+b)(i) = a(i) + b(i)$, $(ab)(i) = a(i)b(i)$ for all $a, b \in S$, then $S$ is a ring which is called the complete direct sum of rings $S_i, \ i \in U$. The set of all functions in $S$ which take the value zero at all but at most finite number of elements $i \in U$, is a subring of $S$ which is called the discrete direct sum of rings $S_i, \ i \in U$. However, if $U$ is a finite set, then the complete (discrete) direct sum of rings $S_i, \ i \in U$, is called a direct sum of rings $S_i, \ i \in U$. Let $T$ be a subring of the direct sum $S$ of rings $S_i, \ i \in U$, let $\theta_i \in U$ be a homomorphism of $S$ onto $S_i$ defined by $\theta_i(a) = a(i)$ for $a \in S$. If $T \theta_i = S_i$ for every $i \in U$, then $T$ is said to be a subdirect sum of the family of the rings $S_i, \ i \in U$.

Definition 1.2.25 (Lie and Jordan structures) Let $R$ be an associative ring, we can induce on $R$ using its operations two structures as follows:

(i) For all $x, y \in R$, the Lie product $[x, y] = xy - yx$.

(ii) For all $x, y \in R$, the Jordan product $xoy = xy + yx$.

Remark 1.2.12 For any $x, y, z \in R$, the following identities hold:
(i) \([xy,z] = x[y,z] + [x,z]y\) and \([x,yz] = [x,y]z + y[x,z]\)

(ii) \([[x,y],z] + [[y,z],x] + [[z,x],y] = 0\). This is known as Jacobi's identity.

(iii) \(x(oyz) = (xoy)z - y[x,z] = y(xoz) + [x,y]z\)

(iv) \((xoz) = x(yoz) - [x,z]y = (xoz)y + x[y,z]\).

**Definition 1.2.26 (Lie (Jordan) ring)** Let \(\mathcal{R}\) be a ring. We can induce on \(\mathcal{R}\) using its multiplicative structure, the operation \(\text{Lie}\) (resp. Jordan) defined product in this ring to be \([a,b] = ab - ba\) (resp. \(a \circ b = ab + ba\)) for all \(a, b \in \mathcal{R}\), where the product \(ab\) signifies the product of \(a\) and \(b\) in the ring \(\mathcal{R}\) itself.

**Definition 1.2.27 (Lie (Jordan) subring)** A nonempty subset \(A\) of a ring \(\mathcal{R}\) is said to be a Lie (resp. Jordan) subring of \(\mathcal{R}\) if \(A\) is an additive subgroup of \(\mathcal{R}\) and \(a, b \in A\), implies that \([a,b]\) (resp. \((a \circ b)\)) is also in \(A\).

**Definition 1.2.28 (Lie (Jordan) ideal)** An additive subgroup \(L\) of \(\mathcal{R}\) is said to be a Lie (resp. Jordan) ideal of \(\mathcal{R}\) if whenever \(u \in L\) and \(r \in \mathcal{R}\), then \([u,r] \in L\) (resp. \((u \circ r) \in L\)).

**Example 1.2.7** Let \(\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\}\). Then it is easy to check that

\(L = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}\) is a Lie ideal of \(\mathcal{R}\) and \(J = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}\) is a Jordan ideal of \(\mathcal{R}\).

**Definition 1.2.29 (Square closed Lie ideal)** A Lie ideal \(L\) is said to be a square closed Lie ideal of \(\mathcal{R}\) if \(x^2 \in L\) for all \(x \in L\).

If \(L\) is a square closed Lie ideal of \(\mathcal{R}\), then \(xy + yx = (x + y)^2 - x^2 - y^2 \in L\). Since \(xy - yx \in L\) for all \(x, y \in L\), \(2xy \in L\) for all \(x, y \in L\).
1.3 Some Well-Known Results

In this section we state some well known results which may be frequently referred in the subsequent text. For their proofs, the references are mentioned against the respective result for those who develop interest in them.

Theorem 1.3.1 ([73, Huang and Shitai]). A group can not be a union of two its proper subgroups.

Theorem 1.3.2 ([145, Wedderburn]). A finite division ring is a field.

Theorem 1.3.3 ([65, Herstein]). Let $R$ be a prime ring and $0 \neq S$ a right ideal of $R$. Suppose that $a \in S$, such that $a^n = 0$ for a fixed integer $n$. Then $R$ has a nonzero nilpotent ideals.

Theorem 1.3.4 ([65, Herstein]). Let $R$ be a simple ring of characteristic not 2 and $L$ be a Lie ideal of $R$. Then $L \subseteq Z(R)$ or $[R, R] \subseteq L$.

Theorem 1.3.5 ([134, Vincenzo]). Let $R$ be a prime ring and $L$ be a Lie ideal of $R$.

(i) If $L$ is noncommutative, then the subring generated by $L$ contains a nonzero ideal of $R$.

(ii) If $L$ is noncommutative, then there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. 
Chapter 2

Generalized derivations of semiprime rings

2.1 Introduction

In 1991 Bresar [37] introduced the notion of a generalized derivation. An additive mapping $F : R \rightarrow R$ is said to be a generalized derivation on a ring $R$ if there exists a derivation $d : R \rightarrow R$ such that for all $x, y \in R$, $F(xy) = F(x)y + x d(y)$. Generalized derivations have been preliminarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting. Hvala [75] initiated the algebraic study of generalized derivations in rings and extended some results on derivations to generalized derivations. In 1992 Bell and Daif [32] proved that if $I$ is a nonzero ideal of a semiprime $R$ and $d$ is a derivation of $R$ such that for all $x, y \in I$ $d([x, y]) = \pm [x, y]$, then $I$ is a central ideal. Further Quadri et.al [122] obtained the result for a generalized derivation of a prime ring. Very recently Dhara [59] generalized this result for a semiprime ring. Motivated by the aforementioned results we prove that a nonzero one sided ideal $I$ of a semiprime ring $R$ admitting a generalized derivation $F$ with associated nonzero derivation $d$ is central if for all $x, y \in I$ one of the conditions holds: (i) $[F(x), d(y)] \pm [x, y] = 0$; (ii) $F([x, y]) \pm [d(x), F(y)] = 0$; (iii) $F((x, y)) = [F(x), y] + [d(y), x]$; (iv) $d(x) F(y) \pm xy = 0$ and (v) $F([x, y]) \pm [F(x), y] = 0$.

Finally we investigate the following conditions: (i) $F(xy) \pm (xy) = 0$
(ii) \( F(x^2) \pm x^2 = 0 \) (iii) \( F(x)F(y) - (x \circ y) = 0 \) for all \( x, y \) in some appropriate subset of \( R \).

2.2

Definition 2.2.1 (Derivation) An additive mapping \( d : R \rightarrow R \) is said to be a derivation on a ring \( R \) if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \).

Example 2.2.1 The most natural example of a nontrivial derivation is the usual differentiation on the ring \( F[x] \) of polynomials over a field \( F \).

An additive mapping \( F_{a,b} : R \rightarrow R \) is called a generalized inner derivation if \( F_{a,b}(x) = ax + xb \) holds, for some \( a, b \in R \). It can be easily checked that if \( F_{a,b} \) is a generalized inner derivation, then for \( x, y \in R \)

\[
F_{a,b}(xy) = F_{a,b}(x)y + x[y, b]
= F_{a,b}(x)y + xI_b(y),
\]

where \( I_b(y) = yb - by \) is an inner derivation.

In view of the above observation [37] Bresar introduced the concept of a generalized derivation in rings as follows:

Definition 2.2.2 (Generalized derivation) An additive mapping \( F : R \rightarrow R \) is said to be a generalized derivation on a ring \( R \) if there exists a derivation \( d : R \rightarrow R \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \).

Example 2.2.2 Let \( R \) be a ring and \( a \) be a fixed element of \( R \). Define a mapping \( \delta : R \rightarrow R \) by \( \delta(x) = [x, a] = xa - ax \), for all \( x \in R \). Then \( \delta \) is a derivation of \( R \) which is called the inner derivation.
Example 2.2.3 Let $S$ be any ring and $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$. Define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then $F$ is a generalized derivation with associated derivation $d$ given by $d(x) = e_{11}x - xe_{11}$.

Example 2.2.4 Let $R$ be either the ring $H$ of real quaternions or the subring $K$ of $H$ consisting of all elements $a + b_1 + c_1 + d_1$, where $a, b, c, d$ are integers. Define a map $F : R \to R$ by $F(x) = ix + xi$, for all $x \in R$. Then $F$ is a generalized derivation with associated derivation $d(x) = I_i(x) = [x, i]$.

Generally, we do not mention the derivation $d$ associated with the generalized derivation $F$, rather prefer to call $F$ simply a generalized derivation. We may observe that the concept of a generalized derivation includes the concept of a derivation and a generalized inner derivation, also that of a left multiplier when $d = 0$.

Remark 2.2.1 The following example is sufficient to show that a generalized derivation need not be a derivation in general.

Example 2.2.5 Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in GF(2) \right\}$. Define a map $F : R \to R$ by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and a derivation $d : R \to R$ by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then it can be verified that $F$ is generalized derivation with associated derivation $d$ but not a derivation.

Definition 2.2.3 (Centralizing mapping) Let $S$ be a nonempty subset of a ring $R$. A mapping $f : R \to R$ is said to be centralizing (resp. commuting) on $S$ if for all $x \in S \ [f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$).
The two basic and obvious examples of a commuting map of a ring $R$ are the identity map and every map having its range in $Z(R)$, the centre of $R$.

**Example 2.2.6** Consider a ring $R = R_1 \oplus R_2$, where $R_1$ is a noncommutative ring having a nonzero derivation $d_1$ and $R_2$ is a commutative domain. Then $R$ is a noncommutative ring and $d : R \rightarrow R$ defined by $d(x_1, x_2) = (d_1(x_1), 0)$ is a nonzero commuting derivation on $R$.

**Example 2.2.7** Let $R$ be a 3-dimensional algebra over a field of characteristic 2, with basis $\{u_0, u_1, u_2\}$ and multiplication defined by

$$u_i u_j = \begin{cases} u_0 & \text{if } (i, j) = (1, 2) \\ 0 & \text{otherwise} \end{cases}.$$ 

Let $d$ be the linear transformation on $R$ defined by $d(u_0) = 0, d(u_1) = u_1, d(u_2) = u_2$. It is easily verified that $d$ is a centralizing derivation on $R$.

Over the last several years, a number of authors studied the commutativity in prime and semiprime rings admitting derivations and generalized derivations. In [32] Bell and Daif proved that if $R$ is a semiprime ring, $I$ a nonzero ideal and $d$ is a derivation of $R$ such that $d([x, y]) = \pm [x, y]$, for all $x, y \in I$, then $I$ is central ideal. In particular, if $I = R$, then $R$ is commutative. Further Quadri et. al. [122] generalized this result replacing derivation $d$ with a generalized derivation in a prime ring $R$. More precisely they proved that a prime ring $R$ admitting a generalized derivation $F$ is commutative if one of the following holds: (i) $F([x, y]) + [x, y] = 0$; (ii) $F([x, y]) - [x, y] = 0$; for all $x, y \in I$, a two sided ideal of $R$. Very recently Dhara in [59] generalized this result for a semiprime ring as follows:

**Theorem 2.2.1.** If $R$ is a semiprime ring and $F$ is a generalized derivation of $R$ with associated derivation $d$, $I$ a nonzero ideal of $R$ such that $d(I) \neq (0)$, then $I \subseteq Z(R)$ in case one the following holds:
(i) \(F([x,y]) + [x,y] = 0\) for all \(x,y \in I\)

(ii) \(F([x,y]) - [x,y] = 0\) for all \(x,y \in I\).

We extend the above mentioned results and prove that a nonzero left ideal \(I\) of a semiprime ring \(R\) is central if for all \(x,y \in I\) one of following conditions is satisfied:

(i) \([F(x),d(y)] \pm [x,y] = 0\),

(ii) \(F([x,y]) \pm [d(x),F(y)] = 0\),

(iii) \(F([x,y]) = [F(x),y] + [d(y),x]\),

(iv) \(d(x)F(y) \pm xy = 0\),

(v) \(F([x,y]) \pm [F(x),y] = 0\).

We begin our discussion with the following results.

**Lemma 2.2.1** ([31, Lemma 3.1]). Let \(R\) be a 2-torsion free semiprime ring and \(I\) be a nonzero left ideal of \(R\). If \(a,b \in R\) and \(axb + bxa = 0\) for all \(x \in I\), then \(axb = bxa = 0\) for all \(x \in I\).

**Lemma 2.2.2** ([34, Theorem 3]). Let \(R\) be a semiprime ring and \(I\) be a nonzero left ideal of \(R\). If \(R\) admits a nonzero derivation \(d\) centralizing on \(I\), then \(R\) contains a nonzero central ideal.

**Lemma 2.2.3** ([72, Corollary 2.1]). Let \(R\) be a 2-torsion free semiprime ring, \(L\) a noncentral Lie ideal of \(R\) and \(a,b \in L\).

(i) If \(aLa = (0)\), then \(a = 0\).

(ii) If \(aL = (0)\) (or \(La = (0)\)), then \(a = 0\).
(iii) If $L$ is square closed and $aLb = (0)$, then $ab = 0$ and $ba = 0$.

Lemma 2.2.4 ([84, Chapter 4]). Let $R$ be a semiprime ring.

(i) The centre of $R$ contains no nonzero nilpotent elements

(ii) $R$ does not contain any nonzero nilpotent left ideal

(iii) If $P$ is a nonzero prime ideal of $R$ and $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$.

Theorem 2.2.2. Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Suppose $R$ admits a generalized derivation $F$ with associated nonzero derivation $d$ such that $F([x, y]) = [F(x), y] + [d(y), x]$ for all $x, y \in I$. If $Id(I) \neq (0)$, then $[I, I]d(I) = 0$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

Proof. Let

$$F([x, y]) = [F(x), y] + [d(y), x] \text{ for all } x, y \in I. \tag{2.2.1}$$

Replacing $y$ by $yx$ in (2.2.1) and using (2.2.1), we find that

$$2[x, y]d(x) = y[F(x), x] + y[d(x), x] \text{ for all } x, y \in I. \tag{2.2.2}$$

Substitute $ry$ for $y$ for all $r \in R, y \in I$ in (2.2.2) and use (2.2.2) to get $2[x, r]yd(x) = 0$ for all $x, y \in I, r \in R$. Since $R$ is 2-torsion free, we have

$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.2.3}$$

Replacing $r$ by $rs$, we get

$$0 = r[x, s]yd(x) + [x, r]syd(x) = [x, r]syd(x) \text{ for all } x, y \in I, s, r \in R. \tag{2.2.4}$$

That is

$$[x, R]Ryd(x) = (0), \text{ for all } x, y \in I. \tag{2.2.5}$$
Moreover, since $R$ is semiprime, it follows that

$$[x, R]Id(x) = (0), \text{ for all } x \in I. \quad (2.2.6)$$

Take the family $\{P_\alpha\}$ of prime ideals of $R$ such that $\cap P_\alpha = (0)$. Let $x_1$ be a fixed element of $I$, then for any $x_2 \in I$, and by (2.2.6), we have

$$(0) = [x_1 + x_2, R]Id(x_1 + x_2) = [x_1, R]Id(x_2) + [x_2, R]Id(x_1). \quad (2.2.7)$$

Further, for any $P_\alpha$ (2.2.5) and Lemma 2.2.4 (iii) yield that, either $Id(x_1) \subseteq P_\alpha$, or $[x_1, R] \subseteq P_\alpha$. In case $Id(x_1) \subseteq P_\alpha$, then using it, we get $[x_2, R]Id(x_1) \subseteq P_\alpha$. On the other hand, if $[x_1, R] \subseteq P_\alpha$, then by (2.2.7), again we have $[x_2, R]Id(x_1) \subseteq P_\alpha$. Therefore, in any case, $[I, R]Id(I) \subseteq P_\alpha$, for any $\alpha$. This implies that

$$[I, R]Id(I) \subseteq \cap P_\alpha = (0). \quad (2.2.8)$$

In particular, we get $[I, I]RIId(I) = (0)$ and from this we also get $[I, I]d(I)R[I, I]d(I) = (0)$. Hence, by the semiprimeness of $R$, we obtain

$$[I, I]d(I) = (0). \quad (2.2.9)$$

Moreover, for any $r, s \in R, x, y, z \in I$ by (2.2.8), we obtain

$$0 = [r, s]yld(z) = [r, s]xtd(z). \quad (2.2.10)$$

Replacing $y$ with $ty$ in (2.2.10), for any $t \in R$, we get $[r, s]xtyd(z) = 0$, that is $[R, R]RRIId(I) = (0)$ i.e $[R, R]Id(I)R[R, R]Id(I) = (0)$. Again by the semiprimeness of $R$, it follows that $[R, R]Id(I) = (0)$ and we have $[R, R]RIId(I) = (0)$. This implies that $[Id(I), R]R[R]Id(I), R] = (0)$, i.e $Id(I) \subseteq Z(R)$. Therefore for all $x, y, z \in I$, we have $xd(z)y + xzd(y) = xzd(xy) \in Z(R)$, and since $xd(xy) \in Z(R)$, it also follows that $xd(z)y \in Z(R)$, for any $x, y, z \in I$. But $Id(I) \neq (0)$, there exist $x_0, z_0 \in I$ such that $0 \neq x_0d(z_0) = \alpha \in Z(R)$. Hence, for all $y \in I$, we get $\alpha y \in Z(R)$, that is $\alpha I \subseteq Z(R)$, as required.
Theorem 2.2.3. Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Suppose $R$ admits generalized derivations $F$ and $G$ with associated nonzero derivations $d$ and $g$ respectively such that $[F(x), y] + [x, G(y)] = 0$ for all $x, y \in I$. If $Id(I) \neq (0)$, then $[I, I]d(I) = 0$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

Proof Let

$$[F(x), y] - [x, G(y)] = 0 \text{ for all } x, y \in I. \quad (2.2.11)$$

Replacing $y$ by $yx$ in (2.2.11), we obtain

$$y[F(x), x] = [x, y]g(x) + y[x, g(x)] \text{ for all } x, y \in I. \quad (2.2.12)$$

Again replace $y$ by $ry$ in (2.2.12) and use (2.2.12), to get

$$[x, r]yg(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.2.13)$$

Substituting $ry$ for $y$, we have $[x, r]Ryg(x) = 0$ for all $x, y \in I$ and $r \in R$, which is the same identity as (2.2.5) in the proof of Theorem 2.2.2. Thus arguing as in the proof of Theorem 2.2.2, we obtain the required result.

The proof runs on the same lines in case $[F(x), y] + [x, G(y)] = 0$ for all $x, y \in I$.

Theorem 2.2.4. Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Suppose $R$ admits a generalized derivation $F$ with associated nonzero derivation $d$ such that $d(x)F(y) + xy = 0$ for all $x, y \in I$. If $Id(I) \neq (0)$, then $[I, I]d(I) = 0$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

Proof Suppose that

$$d(x)F(y) - xy = 0 \text{ for all } x, y \in I. \quad (2.2.14)$$

Replacing $y$ by $yx$ in (2.2.14) and using it, we get

$$d(x)yd(x) = 0 \text{ for all } x, y \in I. \quad (2.2.15)$$
This implies that \([y, z]d(x)R[y, z]d(x) = 0\) for all \(x, y, z \in I\). Since \(R\) is semiprime, we have \([y, z]d(x) = 0\) for all \(x, y, z \in I\). That is

\[ [I, I]d(I) = (0). \]  

(2.2.16)

Moreover, for any \(r, s \in R, x, y, z \in I\), it follows that

\[ 0 = [r, s]yd(z) = [r, s]zxyd(z). \]  

(2.2.17)

Replacing \(y\) with \(ty\) in (2.2.17), for any \(t \in R\), we get \([r, s]xtyd(z) = 0\), that is \([R, R]IRd(I) = [R, R]Id(I)R[R, R]Id(I) = (0)\). Again by the semiprimeness of \(R\), we have \([R, R]Id(I) = (0)\) and hence \([R, R]RIId(I) = (0)\). This implies that \([Id(I), R]RIId(I), R = (0)\), that is \(Id(I) \subseteq Z(R)\). Therefore for all \(x, y, z \in I\), we have \(xd(x)y + xzd(y) = xd(zy) \in Z(R)\). Since \(xd(zy) \in Z(R)\), it also follows that \(xd(z)y \in Z(R)\), for all \(x, y, z \in I\). But \(Id(I) \neq (0)\), there exist \(x_0, x_0 \in I\) such that \(0 \neq x_0d(x_0) = \alpha \in Z(R)\). Hence, for all \(y \in I\), we get \(\alpha y \in Z(R)\), that is \(\alpha I \subseteq Z(R)\), as required.

The proof is same for the case \(d(x)F(y) + xy = 0\) for all \(x, y \in I\).

**Theorem 2.2.5.** Let \(R\) be a 2-torsion free semiprime ring and \(I\) be a nonzero left ideal of \(R\). Suppose that \(R\) admits a generalized derivations \(F\) with associated nonzero derivation \(d\) such that \(F([x, y]) = [F(x), y] = 0\) for all \(x, y \in I\). If \(Id(I) \neq (0)\), then \([I, I]d(I) = 0\) and there exists \(0 \neq \alpha \in Z(R)\) such that \(\alpha I \subseteq Z(R)\).

**Proof** Let

\[ F([x, y]) - [F(x), y] = 0 \text{ for all } x, y \in I. \]  

(2.2.18)

Replacing \(y\) by \(yx\) in (2.2.18), we get

\[ [x, y]d(x) - y[F(x), x] = 0 \text{ for all } x, y \in I. \]  

(2.2.19)
Again replacing $y$ by $ry$ in (2.2.19) and using (2.2.19), we find that $[x, r]yd(x) = 0$
for all $x, y, z \in I$, which is the same identity as (2.2.3). Thus arguing as in the proof
Theorem 2.2.2, we get the required result.

Similarly we can prove the case if $F([x, y]) + [F(x), y] = 0$ for all $x, y \in I$.

2.3

**Theorem 2.3.1.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left
ideal of $R$ such that $A_r(I) = 0$. Let $F$ be a generalized derivation of $R$ with asso-
ciated nonzero derivation $d$ satisfying $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$. If
$[F(x), d(y)] \pm [x, y] = 0$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

**Proof** Let

$$[F(x), d(y)] + [x, y] = 0 \text{ for all } x, y \in I. \quad (2.3.1)$$

Substituting $xy$ for $x$ in (2.3.1), we obtain

$$F(x)[y, d(y)] + [x, d(y)]d(y) = 0 \text{ for all } x, y \in I. \quad (2.3.2)$$

Replacing $x$ by $yx$ in (2.3.2), we have $yF(x)[y, d(y)] + d(y)x[y, d(y)] + y[x, d(y)]d(y) +$
$[x, d(y)]yd(y) = 0$ for $x, y \in I$. Using (2.3.2), we find that $d(y)x[y, d(y)] + [y, d(y)]xd(y) =$
0 for all $x, y \in I$. Hence by Lemma 2.2.1, we get $d(y)x[y, d(y)] = 0$ and $[y, d(y)]x[y, d(y)] =$
0, for all $x, y \in I$. Since $I$ is a left ideal of $R$, it follows that

$$I[d(y), y]RI[d(y), y] = 0 \text{ for all } y \in I. \quad (2.3.3)$$

Semiprimeness of $R$ yields that

$$I[d(y), y] = 0 \text{ for all } y \in I. \quad (2.3.4)$$

Since $A_r(I) = 0$, we have $[d(y), y] = 0$ for all $y \in I$. By Lemma 2.2.2, we get $I \subseteq Z(R)$.

Now suppose that $[F(x), d(y)] - [x, y] = 0$ for all $x, y \in I$. Then replacing $x$ by $xy$, we
get $F(x)[y, d(y)] + [x, d(y)]d(y) = 0$ for all $x, y \in I$ which is (2.3.2). Hence proceeding as above we get $I \subseteq Z(R)$.

**Theorem 2.3.2.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$ such that $A_r (I) = (0)$. Let $F$ be a generalized derivation of $R$ with associated nonzero derivation $d$ satisfying $F(xy) = xF(y) + d(x)y$ for all $x, y \in R$. If $F([x,y]) \pm [d(x), F(y)] = 0$ for all $x, y \in I$, then $R$ contains a nonzero central ideal.

**Proof** Let

$$F([x,y]) - [d(x), F(y)] = 0 \text{ for all } x, y \in I. \quad (2.3.5)$$

Replacing $y$ by $xy$ in (2.3.5) and using (2.3.5), we find that

$$d(x)[x, y] = [d(x), x]F(y) + d(x)[d(x), y] \text{ for all } x, y \in I. \quad (2.3.6)$$

Substituting $yx$ for $y$ in (2.3.6) and using (2.3.6), we get $[d(x), x]yd(x) + d(x)y[d(x), y] = 0$ for all $x, y \in I$. By Lemma 2.2.1, we obtain that $[d(x), x]yd(x) = 0$ for all $x, y \in I$. This yields that $I[d(x), x]R[1, x] = 0$ for all $x \in I$. Since $R$ is semiprime, $I[d(x), x] = 0$ for all $x \in I$. By hypothesis $A_r (I) = 0$, we have $[d(x), x] = 0$ for all $x \in I$. Application of Lemma 2.2.2 yields that $I \subseteq Z(R)$.

The proof runs on the same lines if $F([x,y]) + [d(x), F(y)] = 0$ for all $x, y \in I$.

**Theorem 2.3.3.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero square closed Lie ideal of $R$ such that $L \nsubseteq Z(R)$. Suppose that $F$ and $G$ are generalized derivations of $R$ with associated derivations $d$ and $g$ respectively. If $F(x)x \pm xG(x) = 0$ for all $x \in L$ and $g(L) \subseteq L$, then $g$ is commuting on $L$.

**Proof** Suppose

$$F(x)x - xG(x) = 0 \text{ for all } x \in L.$$
Linearizing, we get

\[ F(x)y + F(y)x = xG(y) + yG(x) \text{ for all } x, y \in L. \] (2.3.7)

Replacing \( x \) by \( 2xy \) in (2.3.7), we obtain

\[ 2(F(x)y^2 + xd(y)y + F(y)xy - xyG(y) - yG(x)y - yxg(x)) = 0 \]
for all \( x, y \in L. \) (2.3.8)

Since \( R \) is 2-torsion free, we have

\[ F(x)y^2 + xd(y)y + F(y)xy - xyG(y) - yG(x)y - yxg(x) = 0 \]
for all \( x, y \in L. \) (2.3.9)

Right multiplying by \( y \) to the relation (2.3.7), we get

\[ F(x)y^2 + F(y)xy = xG(y)y + yG(x)y \text{ for all } x, y \in L. \] (2.3.10)

Combining (2.3.9) and (2.3.10), we obtain

\[ xd(y)y = yxg(y) + x[y, G(y)] \text{ for all } x, y \in L. \] (2.3.11)

Now, replacing \( x \) by \( 2zx \) in (2.3.11), we get

\[ 2(zxd(y)y = yzg(y) + zx[y, G(y)]) \text{ for all } x, y, z \in L. \] (2.3.12)

Left multiplying (2.3.11) by \( z \), we obtain

\[ zxd(y)y = zyg(y) + zx[y, G(y)] \text{ for all } x, y, z \in L. \] (2.3.13)

From (2.3.12) and (2.3.13), we get

\[ [y, z]xg(y) = 0 \text{ for all } x, y, z \in L. \] (2.3.14)

Replacing \( x \) by \( 2r[x, u] \) in (2.3.14), we get

\[ 2[y, z]r[x, u]g(y) = 0 \text{ for all } u, x, y, z \in L, r \in R. \] (2.3.15)

Using 2-torsion freeness of \( R \), we have

\[ [y, z]r[x, u]g(y) = 0 \text{ for all } u, x, y, z \in L, r \in R. \] (2.3.16)
This implies that
\[ [y, z]g(y)R[x, u]g(y) = 0 \text{ for all } u, x, y, z \in L. \]
(2.3.17)

Semiprimeness of \( R \) yields that
\[ [y, z]g(y) = 0 \text{ for all } y, z \in L. \]
(2.3.18)

Thus \( [x, z]g(y) = 0 \) for all \( x, y, z \in L \). Since \( g(L) \subseteq L \), replacing \( x \) by \( 2g(z)x \), we get
\[ 2[g(z), z]xg(y) = 0 \text{ for all } x, y, z \in L. \]
Substituting \( z \) for \( y \), we obtain
\[ 2[g(z), z]g(z) = 0 \text{ for all } x, y, z \in L. \]
(2.3.19)

This implies that \( 2[g(z), z][g(z), z] = 0 \) for all \( x, z \in L \). Since \( R \) is 2-torsion free, it follows that \( [g(z), z]L[g(z), z] = 0 \) for all \( z \in L \). Using Lemma 2.2.3, we get
\[ [g(z), z] = 0 \text{ for all } z \in L \] and hence \( g \) is commuting on \( L \).

Similarly we can prove the result for the case \( F(x)x + xG(x) = 0 \) for all \( x \in L \).

**Theorem 2.3.4.** Let \( R \) be a semiprime ring and \( I \) be a nonzero left ideal of \( R \) such that \( A_r(I) = (0) \). Let \( F \) be a generalized derivation of \( R \) with associated nonzero derivation \( d \). If \( [x, F(x)] \in Z(R) \) for all \( x \in I \) and \( Z(R) \neq (0) \), then \( R \) contains a nonzero central ideal.

**Proof** Suppose
\[ [x, F(x)] \in Z(R) \text{ for all } x, y \in I. \]
(2.3.20)

Replacing \( x \) by \( x + y \), in (2.3.20), we get
\[ [x, F(y)] + [y, F(x)] \in Z(R) \text{ for all } x, y \in I. \]
(2.3.21)

Substituting \( zy \) for \( y \), where \( z \in Z(R) \), we find
\[ [x, F(y)z + yd(z)] + [y, F(x)]z \in Z(R) \text{ for all } x, y \in I. \]
(2.3.22)
Commuting with $w$, we get

$$([(x, F(y))z + [y, F(x)])z, w] + [[x, y], w]d(z) = 0 \text{ for all } w, x, y, z \in I. \quad (2.3.23)$$

Using (2.3.21), we obtain

$$[[x, y], w]d(z) = 0 \text{ for all } w, x, y, z \in I. \quad (2.3.24)$$

Replacing $w$ by $rw$ in (2.3.24), we have

$$[[x, y], rw]d(z) = 0 \text{ for all } w, x, y, z \in I, \text{ and } r \in R. \quad (2.3.25)$$

Substituting $rs$ for $r$ in (2.3.25), we get

$$[[x, y], rs]w(d(z) = 0 \text{ for all } w, x, y, z \in I, \text{ and } r, s \in R. \quad (2.3.26)$$

Replacing $w$ by $w[y, x]$, we get

$$[[x, y], r] Rw[y, x]d(z) = 0 \text{ for all } w, x, y, z \in I, \text{ and } r \in R. \quad (2.3.27)$$

Right multiplying (2.3.26) by $[y, x]$, we have

$$[[x, y], r] Rw[d(z)]y, x] = 0 \text{ for all } w, x, y, z \in I, \text{ and } r \in R. \quad (2.3.28)$$

Combining (2.3.27) and (2.3.28), we obtain

$$[[x, y], r] Rw[[x, y], d(z)] = 0 \text{ for all } w, x, y, z \in I, \text{ and } r \in R. \quad (2.3.29)$$

Replacing $r$ by $d(z)$ and left multiplying (2.3.29) by $w$ semiprimeness of $R$ yields that $w[[x, y], d(z)] = 0$ for all $w, x, y, z \in I$. i.e,

$$U[[x, y], d(z)] = 0 \text{ for all } w, x, y, z \in I. \quad (2.3.30)$$

Since $A_r(I) = (0)$, it follows that

$$[[x, y], d(z)] = 0 \text{ for all } x, y, z \in I. \quad (2.3.31)$$

Substituting $yx$ for $y$ in (2.3.30), we get

$$[x, y][x, d(z)] = 0 \text{ for all } x, y, z \in I. \quad (2.3.32)$$
Now replace $y$ by $d(z)y$ in (2.3.32), to get

$$y[x, d(z)]Ry[x, d(z)] = 0 \text{ for all } x, y, z \in I. \quad (2.3.33)$$

Since $R$ is semiprime, we have

$$I[x, d(z)] = 0 \text{ for all } x, z \in I. \quad (2.3.34)$$

Again using the fact that $A_r(I) = (0)$, we have $[x, d(z)] = 0$ for all $x, z \in I$. Now replacing $z$ by $x$, we have $[x, d(x)] = 0$ for all $x \in I$ and $R$ contains a nonzero central ideal by Lemma 2.2.2.

\section{2.4}

Recently in order to generalize a result of Quadri et.al [122] Dhara [59] proved that in a semiprime ring $R$ with a generalized derivation $F$, a nonzero ideal $I \subseteq Z(R)$ if one of the conditions holds: (i) $F(x \circ y) + (x \circ y) = 0$ and (ii) $F(x \circ y) - (x \circ y) = 0$ for all $x, y \in I$. In this section we investigate some related conditions (i) $F(x \circ y) - (x \circ y) = 0$; (ii) $F(x \circ y) + (x \circ y) = 0$; (iii) $F(x^2) - x^2 = 0$; (iv) $F(x^2) + x^2 = 0$; (v) $F(x)F(y) - (x \circ y) = 0$ and prove that a nonzero one sided ideal $I$ of a semiprime ring $R$ admitting a generalized derivation $F$ is contained in the centre of $R$ if one of the conditions (i)-(v) is satisfied.

**Theorem 2.4.1.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Suppose that $R$ admits a generalized derivation $F$ with associated nonzero derivation $d$ such that $F(x \circ y) \neq (x \circ y) = 0$ for all $x, y \in I$. If $Id(I) \neq (0)$, then $[I, I]d(I) = 0$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** Suppose that $F(x \circ y) - (x \circ y) = 0$, for all $x, y \in I$. This can be rewritten as

$$F(x)y + xd(x) + F(y)x + yd(x) - x \circ y = 0 \text{ for all } x, y \in I. \quad (2.4.1)$$
Substitute $yx$ for $y$ in (2.4.1), to obtain

$$F(x)yx + xd(y)x + F(y)x^2 + yd(x)x + xyd(x) + yxd(x) - (x \circ y)x = 0. \quad (2.4.2)$$

Using (2.4.1), we get

$$(x \circ y)d(x) = 0 \text{ for all } x, y \in I. \quad (2.4.3)$$

Substitute $ry$ for $y$ in (2.4.3) and use (2.4.3), to get

$$[x, r]yd(x) = 0 \text{ for all } x, y, z \in I, r \in R. \quad (2.4.4)$$

Now arguing as in the proof of Theorem 2.2.2, we get the required result.

Similarly we can prove the case $F(x \circ y) + (x \circ y) = 0$, for all $x, y \in I$.

**Theorem 2.4.2.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a nonzero left ideal of $R$. Let $F$ be a generalized derivation of $R$ with associated derivation $d$ such that $F(x^2) \neq x^2 = 0$ for all $x \in I$, If $Id(I) \neq (0)$, then $[I, I]d(I) = 0$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

**Proof** Let

$$F(x^2) = x^2 \text{ for all } x \in I. \quad (2.4.5)$$

Substituting $x + y$ for $x$ in (2.4.5), we have

$$F(x^2 + y^2 + xy + yx) = x^2 + y^2 + xy + yx \text{ for all } x, y \in I. \quad (2.4.6)$$

Combining (2.4.5 ) and (2.4.6), we obtain $F(xy + yx) = xy + yx$ for all $x, y \in I$ i.e $F(x \circ y) - x \circ y = 0$ for all $x, y \in I$. Hence result follows by Theorem 2.4.1.

Similarly we can prove the case $F(x^2) + x^2 = 0$ for all $x \in I$. 
Theorem 2.4.3. Let $R$ be a semiprime ring and $I$ be a nonzero left ideal of $R$. Suppose $R$ admits a generalized derivation $F$ with associated nonzero derivation $d$. If $Id(I) \neq (0)$ and $F(x)F(y) - (x \circ y) = 0$ for all $x, y \in I$, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.

Proof. If $F = 0$, then we have $x \circ y = 0$ for all $x, y \in I$. Replacing $y$ by $ry$, we get $[x, r]y = 0$ for all $x, y \in I, r \in R$. Since $I$ is a left ideal, we can get $[x, r]r = [r, y] = 0$ for all $x, y \in I, r \in R$. By semiprimeness, $[x, r] = 0$ for all $x \in I, r \in R$ then $I \subseteq Z(R)$.

Suppose that $F \neq 0$, we have

$$F(x)F(y) - (x \circ y) = 0 \text{ for all } x, y \in I. \tag{2.4.7}$$

Replacing $y$ by $xy$ in (2.4.7), we obtain

$$F(x)F(y)x + F(x)yd(x) - (x \circ y)x = 0 \text{ for all } x, y \in I. \tag{2.4.8}$$

Using (2.4.7), we get

$$F(x)yd(x) = 0 \text{ for all } x, y \in I. \tag{2.4.9}$$

Replacing $y$ by $F(z)y$ in (2.4.9), we get $F(x)F(x)yd(x) = 0$ for all $x, y, z \in I$. Using (2.4.7), we get

$$(x \circ z)yd(x) = 0 \text{ for all } x, y, z \in I. \tag{2.4.10}$$

Replacing $z$ by $rz$ in (2.4.10) and using (2.4.10), we get

$$[x, r]zyd(x) = 0 \text{ for all } x, y, z \in I, r \in R. \tag{2.4.11}$$

Substitute $sy$ for $y$ in (2.4.11), to get

$$[x, r]zyd(x) = 0 \text{ for all } x, y, z \in I, r \in R. \tag{2.4.12}$$

Replacing $s$ by $d(u)s$ in (2.4.12), we obtain

$$[x, r]zd(u)syd(x) = 0 \text{ for all } x, y, z \in I, r, s \in R. \tag{2.4.13}$$
Replacing \(s\) by \(s[x,r]\) in (2.4.13), we get

\[
[x,r]zd(u)R[x,r]yd(x) = 0 \text{ for all } u,x,y,z \in I \text{ and } r \in R.
\] (2.4.14)

Semiprimeness of \(R\) yields that

\[
[x,r]yd(x) = 0 \text{ for all } x,y,z \in I, r \in R.
\] (2.4.15)

In particular, we get \([I,I]RId(I) = (0)\) and from this we also have \([I,I]d(I)R[I,I]d(I) = (0)\). Hence, by the semiprimeness of \(R\) one has

\[
[I,I]d(I) = (0) \text{ for all } x,y,z \in I, r \in R.
\] (2.4.16)

Moreover, for any \(r,s \in R, x,y,z \in I\) and by (2.4.15), it follows that

\[
0 = [rx,s]yd(z) = [r,s]xyd(z) \text{ for all } x,y,z \in I, r \in R.
\] (2.4.17)

Replacing \(y\) with \(ty\) in (2.4.17), for any \(t \in R\), we get \([r,s]xtyd(z) = 0\), that is \([R,R]IRId(I) = (0)\). Again by the semiprimeness of \(R\), we have \([R,R]IId(I) = (0)\) and an excessive use yields that \([R,R]RId(I) = (0)\). This implies that \([Id(I), R]R[Id(I), R] = (0)\), that is \(Id(I) \subseteq Z(R)\). Therefore for \(x,y,z \in I\), we have \(xd(z)y + xzd(y) = xd(zy) \in Z(R)\), and it follows that \(xd(z)y \in Z(R)\), for any \(x,y,z \in I\). Moreover, by \(Id(I) \neq (0)\), there exist \(x_0, z_0 \in I\) such that \(0 \neq z_0d(x_0) = \alpha \in Z(R)\). Hence, for all \(y \in I\), we get \(\alpha y \in Z(R)\), that is \(\alpha I \subseteq Z(R)\).
Chapter 3

Generalized \((\alpha, \beta)\)--derivations of \(*\)--prime rings

3.1 Introduction

This chapter deals with the study of rings with involution \((*\text{--rings})\) admitting a derivation. An additive mapping \(x \mapsto x^\ast\) on a ring \(R\) satisfying \((xy)^\ast = y^\ast x^\ast\) and \((x^\ast)^\ast = x\) for all \(x, y \in R\) is called an involution on \(R\). The ring equipped with an involution is called a \(*\text{--ring}\) or an involution ring. The most natural example of a \(*\text{--ring}\) is the real (or complex) matrix ring with the usual involution that is the transposition (conjugation). In 1989 Bell and Kappe [35] initiated the study of derivations which act as homomorphisms or as antihomomorphisms. In the same paper they proved that if \(R\) a semiprime ring and \(d\) is a derivation on \(R\) which is either an endomorphism or an antiendomorphism on a nonzero ideal \(I\) of \(R\), then \(d = 0\). Recently Asma et. al [3] extended the result for \((\alpha, \beta)\)--derivation \(d\) acting as a homomorphism or an antihomomorphism on a nonzero Lie ideal of \(R\). In section 3.2, we prove the similar result for a \(*\text{--prime}\) ring admitting a generalized \((\alpha, \beta)\)--derivation. In section 3.3, we study the conditions \(F([u, v]) = [F(u), v]_{\alpha, \beta} + [d(v), u]_{\alpha, \beta}\) and \(F(uov) = (F(u)ov)_{\alpha, \beta} + (d(v)ov)_{\alpha, \beta}\) and prove that a nonzero Lie ideal of a \(*\text{--prime}\) ring \(R\) admitting a generalized derivation \(F\) with associated derivation \(d\) is central if it satisfies any one of the above conditions.
3.2

Following Herstein [67] we define an involution ring\((\ast-\text{ring})\) as follows:

\textbf{Definition 3.2.1} (Involution ring or \(\ast-\text{ring}\)) An additive mapping \(x \mapsto x^\ast\) on a ring \(R\) satisfying \((xy)^\ast = y^\ast x^\ast\) and \((x^\ast)^\ast = x\) for all \(x, y \in R\) is called an involution on \(R\). The ring equipped with an involution is called a \(\ast-\text{ring}\) or an involution ring.

\textbf{Example 3.2.1} The most natural example of a \(\ast-\text{ring}\) is the real (or complex) matrix ring with the usual involution that is the transposition (conjugation).

\textbf{Definition 3.2.2} (\(\ast\)-Prime ring) A \(\ast\)-ring \(R\) is said to be a \(\ast\)-prime ring if \(aRb = aRb^\ast = 0\) or \(a^\ast Rb = aRb = 0\) implies that either \(a = 0\) or \(b = 0\).

\textbf{Remark 3.2.1} Every prime ring with an involution is a \(\ast\)-prime but converse need not be true in general.

\textbf{Example 3.2.2} Let \(R\) be a ring and \(S = R \times R^\ast\), where \(R^\ast\) is the opposite ring of \(R\). Define involution \(\ast\) on \(S\) by \((x, y)^\ast = (y, x)\). Since \((0, x)S(x, 0) = 0\), \(S\) is not a prime ring. But it can verified that if \((a, b)S(c, d) = (a, b)S(c, d)^\ast = 0\), then either \((a, b) = 0\) or \((c, d) = 0\). Hence \(S\) is \(\ast\)-prime.

\textbf{Definition 3.2.3} (\(\ast\)-Ideal) A left (resp. right, two sided) ideal \(I\) of a \(\ast\)-ring \(R\) is called a left (resp. right, two sided) \(\ast\)-ideal if \(I^\ast = I\).

\textbf{Example 3.2.3} Let \(Z\) be the ring of integers. Let \(R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}\).

Define a map \(\ast : R \rightarrow R\) by \(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^\ast = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}\). It is easy to check that
\[ I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\} \text{ is a } \ast \text{-ideal of } R. \]

**Definition 3.2.4 (\ast - Lie Ideal)** A Lie ideal \( U \) of a \( \ast \)-ring \( R \) is said to be a \( \ast \)-Lie ideal if \( U^\ast = U \).

**Example 3.2.4** In Example 3.2.3 if we take \( U = \left\{ \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} \) then \( U \) is a \( \ast \)-Lie ideal of \( R \).

**Definition 3.2.5 (\ast -Prime Ideal)** An ideal \( P \) in a \( \ast \)-ring \( R \) is said to be a \( \ast \)-prime ideal if \( (P \neq R) \) is a \( \ast \)-ideal and for \( \ast \)-ideals \( I, J \) of \( R \), \( IJ \subseteq P \) implies that \( I \subseteq P \) or \( J \subseteq P \).

**Example 3.2.5** Let \( F \) be any field and \( R = F[x] \) be the polynomial ring over \( F \). Define \( \ast : R \rightarrow R \) by \( (f(x))^\ast = f(-x) \) for all \( f(x) \in R \). Then it is easily verified that \( xR \) is a \( \ast \)-prime ideal of \( R \).

**Definition 3.2.6 (Symmetric elements)** An element \( x \) in a \( \ast \)-ring \( R \) is said to be symmetric (resp. skew-symmetric) if \( x^\ast = x \) (resp. \( x^\ast = -x \)).

The set of symmetric and skew-symmetric elements of a \( \ast \)-ring \( R \) will be denoted by \( S_\ast(R) \).

**Definition 3.2.7 (Normal element)** An element \( x \) in a \( \ast \)-ring \( R \) is said to be normal if \( xx^\ast = x^\ast x \). If all the elements of \( R \) are normal, then \( R \) is called a normal ring.

**Example 3.2.6** The ring of real quaternions is normal.
In his famous book "Structure of Rings" Jacobson [77] introduced the notion of $(s_1, s_2)$-derivation which is later more commonly known as $(\alpha, \beta)$-derivation or $(\theta, \varphi)$-derivation.

**Definition 3.2.8** ($(\alpha, \beta)$-derivation) Let $\alpha, \beta$ be endomorphisms of a ring $R$. An additive mapping $d : R \to R$ is said to be a $(\alpha, \beta)$-derivation of $R$ if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$.

A mapping $a \mapsto \alpha(a)b - b\beta(a)$, where $b$ is a fixed element of $R$ is a $(\alpha, \beta)$-derivation. Such a $(\alpha, \beta)$-derivation is said to be inner. A $(\alpha, I)$-derivation (resp. a $(I, \beta)$-derivation), where $I$ is the identity automorphism on $R$ is called simply a $\alpha$-derivation (resp. $\beta$-derivation). Of course, a $(I, I)$-derivation is an ordinary derivation on $R$.

Inspired by the definition of ($(\alpha, \beta)$-derivation) the notion of generalized $(\alpha, \beta)$-derivation has been defined by Ashraf, Asma and Shakir in [18].

**Definition 3.2.9** (Generalized $(\alpha, \beta)$-derivation) Let $(\alpha, \beta)$ be automorphisms of a ring $R$. An additive mapping $F : R \to R$ is said to be a generalized $(\alpha, \beta)$-derivation of $R$ if there exists a $(\alpha, \beta)$-derivation $d : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$, $d$ is called an associated $(\alpha, \beta)$-derivation of $F$.

We shall call a generalized $(\alpha, I)$-derivation as a generalized $\alpha$-derivation, where $I$ is the identity automorphism of $R$. Similarly a generalized $(I, \beta)$-derivation will be called as a generalized $\beta$-derivation.

**Example 3.2.7** Let $S$ be any ring and $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in S \right\}$. Let $\alpha, \beta$ be automorphisms of $R$ such that $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$, and $\beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. 
Define maps $F, d : R \to R$ by $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ and $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$.

It is straightforward to check that $F$ is a generalized $(\alpha, \beta)$-derivation with associated $(\alpha, \beta)$-derivation $d$.

We shall use without explicit mention the following basic identities:

(i) $[xy, z] = x[y, z] + [x, z]y$

(ii) $[x, yz] = y[x, z] + [x, y]z$

(iii) $[[x, y], z] + [y, z, x] + [[x, x], y] = 0$ (this identity is generally known as Jacobi identity).

(iv) $x \circ (yz) = (x \circ y)z - y[x, x] = y(x \circ z) + [x, y]z$

(v) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ y)z + x[y, z]$

(vi) $[xy, z]_{\alpha, \beta} = x[y, z]_{\alpha, \beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha, \beta}y$

(vii) $[x, yz]_{\alpha, \beta} = \beta(y)[x, z]_{\alpha, \beta} + [x, y]_{\alpha, \beta} \alpha(z)$

(viii) $[xy \circ z]_{\alpha, \beta} = x(y \circ z)_{\alpha, \beta} - [x, \beta(z)]y = x[y, \alpha(z)] + (x \circ z)_{\alpha, \beta}y$

(ix) $[x \circ yz]_{\alpha, \beta} = \beta(y)(x \circ z)_{\alpha, \beta} + [x, y]_{\alpha, \beta} \alpha(z) = (x \circ y)_{\alpha, \beta} \alpha(z) - \beta(y)[x, z]_{\alpha, \beta}$

**Lemma 3.2.1 ([61, Lemma 3.1]).** Let $R$ be a semiprime ring and $I$ be a nonzero ideal of $R$. Suppose $\theta, \varphi$ are epimorphisms of $R$ and $d$ is a $(\theta, \varphi)$-derivation of $R$ such that $\varphi(I)d(I) \neq 0$. If for all $x \in I$, $[R, \varphi(x)]\varphi(I)d(x) = 0$, then $R$ contains a nonzero central ideal.

**Lemma 3.2.2 ([113, Lemma 2.3]).** Let $U$ be a nonzero $*$-Lie ideal of a 2-torsion free $*$-prime ring $R$. If $[U, U] = 0$, then $U \subseteq Z(R)$. 
Lemma 3.2.3 ([114, Lemma 4]). If $U \not\subseteq Z(R)$ is a \*-Lie ideal of a 2-torsion free \*-prime ring $R$ and $a, b \in R$ such that $aUb = 0 = a^*Ub$ then either $a = 0$ or $b = 0$.

Lemma 3.2.4 ([124, Lemma 6]). Let $R$ be a 2-torsion free \*-prime ring, $d$ a nonzero $(\alpha, \beta)$-derivation of $R$ which commutes with $\ast$ and $U$ be a nonzero \*-Lie ideal of $R$. If $d(U) = 0$, then $U \subseteq Z(R)$.

In [35] Bell and Kappe initiated the study of derivations which act as homomorphisms or as antihomomorphisms on a ring. They proved that if $R$ is a semiprime ring and $d$ is a derivation on $R$ which is either an endomorphism or an antiendomorphism on $I$, a nonzero ideal of $R$, then $d = 0$. Of course, derivations which are not endomorphisms or antiendomorphisms on $R$ may behave as such on certain subset of $R$; for example any derivation $d$ behaves as the zero endomorphism on the subring $C$ consisting of all constants (i.e. the elements for which $d(x) = 0$). In fact in a semiprime ring $R$, $d$ may behave as an endomorphism on a proper ideal of $R$. However as noted in [35], the behaviour of $d$ is somewhat restricted in the case of a prime ring. Recently Asma et. al [3] considered $(\alpha, \beta)$-derivation $d$ acting as a homomorphism or an antihomomorphism on a nonzero Lie ideal of a prime ring and concluded that $d = 0$. In this section we establish similar result for a \*-prime ring admitting a generalized $(\alpha, \beta)$-derivation.

Theorem 3.2.1. Let $R$ be a 2-torsion free \*-prime ring and $U$ be a nonzero square closed \*-Lie ideal of $R$. Suppose $F : R \rightarrow R$ is a generalized $(\alpha, \alpha)$-derivation with associated nonzero $(\alpha, \alpha)$-derivation $d$ such that $\ast$ commutes with $F, d$ and $\alpha$.

(i) If $F$ acts as a homomorphism on $U$, then $U \subseteq Z(R)$.

(ii) If $F$ acts as an antihomomorphism on $U$, then $U \subseteq Z(R)$.

Proof (i) Let $F$ act as a homomorphism on $U$. Then we have

$$F(uv) = F(u)\alpha(v) + \alpha(u)d(v) = F(u)F(v), \text{ for all } u, v \in U.$$ (3.2.1)
Replacing $v$ by $2vw$ in (3.2.1) and using the fact that $\text{char} \mathcal{R} \neq 2$, we get

\[ F(u)\alpha(v)\alpha(w) + \alpha(u)d(v)\alpha(w) + \alpha(u)\alpha(v)d(w) \]

\[ = F(u)F(v)\alpha(w) + F(u)\alpha(v)d(w), \text{ for all } u, v, w \in U. \]  \hspace{1cm} (3.2.2)

Multiplying (3.2.1) by $\alpha(w)$ from right, we obtain

\[ F(u)\alpha(v)\alpha(w) + \alpha(u)d(v)\alpha(w) = F(u)F(v)\alpha(w), \text{ for all } u, v, w \in U. \]  \hspace{1cm} (3.2.3)

Comparing (3.2.3) and (3.2.2), we have

\[ F(u)\alpha(v)d(w) - \alpha(u)\alpha(v)d(w) = 0, \text{ for all } u, v, w \in U. \]

\[ (F(u) - \alpha(u))\alpha(v)d(w) = 0 \]

for all $u, v, w \in U$. This yields that

\[ \alpha^{-1}(F(u) - \alpha(u))U \alpha^{-1}(d(u)) = 0, \text{ for all } u \in U. \]  \hspace{1cm} (3.2.4)

If $u \in S_\ast(R) \cap U$, then

\[ \alpha^{-1}(F(u) - \alpha(u))U \alpha^{-1}(d(u)) = (\alpha^{-1}(F(u) - \alpha(u)))^*U \alpha^{-1}(d(u)). \]

Thus for some $u \in S_\ast(R) \cap U$ either $\alpha^{-1}(F(u) - \alpha(u)) = 0$ or $\alpha^{-1}(d(u)) = 0$, by Lemma 3.2.3. But for any $u \in U$, $u - u^*, u + u^* \in S_\ast(R) \cap U$. Therefore for some $u \in U$ either $\alpha^{-1}(F(u - u^*) - \alpha(u - u^*)) = 0$ or $\alpha^{-1}(d(u - u^*)) = 0$ i.e. $\alpha^{-1}(F(u) - \alpha(u)) = (\alpha^{-1}(F(u) - \alpha(u))^*)$ or $\alpha^{-1}(d(u)) = (\alpha^{-1}(d(u)))^*$. If $\alpha^{-1}(F(u) - \alpha(u)) = (\alpha^{-1}(F(u) - \alpha(u))^*)$, then by (3.2.4) $\alpha^{-1}(F(u) - \alpha(u))U \alpha^{-1}(d(u)) = (\alpha^{-1}(F(u) - \alpha(u)))^*U \alpha^{-1}(d(u))$.

Thus by Lemma 3.2.3 either $F(u) = \alpha(u)$ or $\alpha^{-1}(d(u)) = 0$. On the other hand if $\alpha^{-1}(d(u)) = (\alpha^{-1}(d(u)))^*$, then again by (3.2.4), Lemma 3.2.3 gives either $F(u) = \alpha(u)$ or $\alpha^{-1}(d(u)) = 0$. Let $A = \{u \in U \mid F(u) = \alpha(u)\}$ and $B = \{u \in U \mid \alpha^{-1}(d(u)) = 0\}$, then $A$ and $B$ are two additive subgroups of $U$ whose union is $U$. Using brauer's trick we have either $A = U$ or $B = U$. If $A = U$, then $F(u) = \alpha(u)$ for
all $u \in U$ and by (3.2.1) $\alpha(u)d(v) = 0$ for all $u, v \in U$. Replacing $u$ by $2uv$ and using the fact that $\text{char} R \neq 2$, we get $\alpha(u)\alpha(w)d(v) = 0$ i.e.,

$$uU\alpha^{-1}(d(v)) = 0, \text{ for all } u, v \in U.$$

(3.2.5)

If $u \in S_*(R) \cap U$, then $uU\alpha^{-1}(d(v)) = u^*U\alpha^{-1}(d(v))$ and arguing as above we obtain either $d(u) = 0$, since by hypothesis $U$ is a nonzero Lie ideal of $R$. Now applying Lemma 3.2.4 we get $U \subseteq Z(R)$.

(ii) Let $F$ act as an antihomomorphism on $U$. Then

$$F(u)\alpha(v) + \alpha(u)d(v) = F(v)F(u), \text{ for all } u, v \in U.$$

(3.2.6)

Replacing $u$ by $2uv$ in (3.2.6) and using the fact that $\text{char} R \neq 2$, we get

$$F(u)\alpha(v)\alpha(v) + \alpha(u)d(v)\alpha(v) + \alpha(u)\alpha(v)d(v) = F(v)F(u)\alpha(v)$$

$$+F(v)\alpha(u)d(v), \text{ for all } u, v \in U.$$

(3.2.7)

Multiplying (3.2.6) by $\alpha(v)$ from right, we get

$$F(u)\alpha(v)\alpha(v) + \alpha(u)d(v)\alpha(v) = F(v)F(u)\alpha(v), \text{ for all } u, v \in U.$$

(3.2.8)

Comparing (3.2.7) and (3.2.8), we have

$$\alpha(u)\alpha(v)d(v) = F(v)\alpha(u)d(v), \text{ for all } u, v \in U.$$

(3.2.9)

Again replacing $u$ by $2uw$ and using the fact that $\text{char} R \neq 2$, we obtain

$$\alpha(w)\alpha(u)\alpha(v)d(v) = F(v)\alpha(w)\alpha(u)d(v), \text{ for all } u, v, w \in U.$$

(3.2.10)

Now using (3.2.9), we have

$$[F(v), \alpha(w)]\alpha(u)d(v) = 0 \text{ for all } u, v, w \in U.$$
This implies that
\[ \alpha^{-1}([F(v), \alpha(w)]) U \alpha^{-1}(d(v)) = 0 \text{ for all } u, v, w \in U. \quad (3.2.11) \]

If \( v \in U \cap S_*(R) \), then we have
\[ \alpha^{-1}([F(v), \alpha(w)]) U \alpha^{-1}(d(v)) = (\alpha^{-1}([F(v), \alpha(w)]) U \alpha^{-1}(d(v)). \]
This implies that either \( \alpha^{-1}([F(v), \alpha(w)]) = 0 \) or \( \alpha^{-1}(d(v)) = 0 \) for each fixed \( v \in U \cap S_*(R) \). For any \( v \in U, v - v^*, v + v^* \in U \cap S_*(R) \). Thus for some fixed \( v \in U \) either \( \alpha^{-1}([F(v - v^*), \alpha(w)]) = 0 \) or \( \alpha^{-1}(d(v - v^*)) = 0 \) i.e. \( \alpha^{-1}([F(v), \alpha(w)]) = (\alpha^{-1}([F(v), \alpha(w)]) \alpha^{-1}(d(v))^* \]
or \( \alpha^{-1}d(v) = (\alpha^{-1}(d(v)))^* \). If \( \alpha^{-1}([F(v), \alpha(w)]) = (\alpha^{-1}([F(v), \alpha(w)]) \alpha^{-1}(d(v))^* \]
then by \( (3.2.11) \) we have \( \alpha^{-1}([F(v), \alpha(w)]) U \alpha^{-1}(d(v)) = (\alpha^{-1}([F(v), \alpha(w)]) \alpha^{-1}(d(v))^* \]
By Lemma 3.2.3 either \( \alpha^{-1}([F(v), \alpha(w)]) = 0 \) or \( \alpha^{-1}(d(v)) = 0 \). On the other hand if \( \alpha^{-1}(d(v)) = (\alpha^{-1}(d(v)))^* \), then again \( (3.2.11) \), together with Lemma 3.2.3 yields that either \( [F(v), \alpha(w)] = 0 \) or \( d(v) = 0 \). Let \( L = \{ v \in U : d(v) = 0 \} \) and \( M = \{ v \in U : [F(v), \alpha(w)] = 0 \} \). Then it can be seen that \( L \) and \( M \) are additive subgroups of \( U \) whose union is \( U \). By Brauer's trick either \( L = U \) or \( M = U \). If \( L = U \), then \( d(U) = 0 \) and by Lemma 3.2.4 we get \( U \subseteq Z(R) \). On the other hand if \( [F(v), \alpha(w)] = 0 \) for all \( v, w \in U \), then replacing \( v \) by \( 2w \) and using the fact that \( \text{char} R \neq 2 \), we get
\[ \alpha(v)[d(w), \alpha(w)] + [\alpha(v), \alpha(w)] d(w) = 0, \text{ for all } v, w \in U. \quad (3.2.12) \]
Now replacing \( v \) by \( 2v \) in \( (3.2.12) \), using \( (3.2.12) \) and the fact that \( \text{char} R \neq 2 \), we have \( [\alpha(s), \alpha(w)] \alpha(v) d(w) = 0 \), i.e.,
\[ [s, w] U \alpha^{-1}(d(w)) = 0, \text{ for all } s, w \in U. \quad (3.2.13) \]
Let \( w \in U \cap S_*(R) \), then \( [s, w] U \alpha^{-1}(d(w)) = [s, w]^* U \alpha^{-1}(d(w)) = 0 \). Thus by Lemma 3.2.3 either \( [s, w] = 0 \) or \( \alpha^{-1}(d(w)) = 0 \) for fixed \( w \in U \). For any \( w \in U \), we have \( w - w^*, w + w^* \in U \cap S_*(R) \). Therefore, for some \( w \in U \) either \( [s, w - w^*] = 0 \) or \( \alpha^{-1}(d(w - w^*)) = 0 \) i.e. \( [s, w] = [s, w]^* \), or \( \alpha^{-1}(d(w)) = (\alpha^{-1}(d(w)))^* \) for some fixed \( w \in U \). If \( [s, v] = [s, w]^* \), then by \( (3.2.13) \) \( [s, w] U \alpha^{-1}(d(w)) = [s, w]^* U \alpha^{-1}(d(w)) \). By Lemma 3.2.3 either \( [s, w] = 0 \) or \( \alpha^{-1}(d(w)) = 0 \). On the other hand if \( \alpha^{-1}(d(w)) = (\alpha^{-1}(d(w)))^* \), then again \( (3.2.13) \) together with Lemma 3.2.3 yield that either \( [s, w] = \)
0 or $\alpha^{-1}(d(w)) = 0$. Let $G = \{v \in U \mid [s, w] = 0\}$ and $H = \{v \in U \mid d(v) = 0\}$. Arguing in the similar manner as above we have either $[U, U] = 0$ or $d(U) = 0$. If $[U, U] = 0$, then by Lemma 3.2.2 $U \subseteq Z(R)$. Later yields that $U \subseteq Z(R)$, by Lemma 3.2.4.

**Theorem 3.2.2.** Let $R$ be a semiprime ring and $I$ be a nonzero ideal of $R$. Suppose $F : R \rightarrow R$ is a generalized $(\alpha, \alpha)$-derivation with associated $(\alpha, \alpha)$-derivation $d$ such that $\alpha(I) d(I) \neq 0$. If $F$ acts as a homomorphism on $R$, then $I \subseteq Z(R)$.

**Proof** By hypothesis

\[ F(xy) = F(x)\alpha(y) + \alpha(x)d(y) = F(x)F(y), \text{ for all } x, y \in I. \tag{3.2.14} \]

Replacing $y$ by $yz$, we get

\[ F(x)\alpha(y)\alpha(z) + \alpha(x)d(y)\alpha(z) + \alpha(x)\alpha(y)d(z) = F(x)F(y)\alpha(z) \]
\[ + F(x)\alpha(y)d(z), \text{ for all } x, y, z \in I. \tag{3.2.15} \]

Multiplying (3.2.14) on the right by $\alpha(z)$, we obtain

\[ F(x)\alpha(y)\alpha(z) + \alpha(x)d(y)\alpha(z) = F(x)F(y)\alpha(z), \text{ for all } x, y, z \in I. \tag{3.2.16} \]

Now comparing (3.2.15) and (3.2.16), we have

\[ \alpha(x)\alpha(y)d(z) = F(x)\alpha(y)d(z), \text{ for all } x, y, z \in I. \tag{3.2.17} \]

Substituting $xz$ for $x$ in (3.2.17), we have

\[ \alpha(x)\alpha(z)\alpha(y)d(z) = F(x)\alpha(z)\alpha(y)d(z) \]
\[ + \alpha(x)d(z)\alpha(y)d(z), \text{ for all } x, y, z \in I. \tag{3.2.18} \]

Replacing $y$ by $yz$ in (3.2.17), we have

\[ \alpha(x)\alpha(z)\alpha(y)d(z) = F(x)\alpha(z)\alpha(y)d(z), \text{ for all } x, y, z \in I. \tag{3.2.19} \]

Comparing (3.2.19) and (3.2.18), we find that $\alpha(x)d(z)\alpha(y)d(z) = 0$, for all $x, y, z \in I$. Substituting $rx$ for $y$ we obtain $\alpha(x)d(z)\alpha(r)\alpha(x)d(z) = 0$, for all $x, z \in I, r \in R$. 
that is \(\alpha(x)d(z)R\alpha(x)d(z) = 0\), for all \(x, y, z \in I\). Since \(R\) is semiprime, \(\alpha(x)d(z) = 0\), for all \(x, z \in I\). Thus, we have \(d(z)\alpha(x)d(z) = 0\), for all \(x, z \in I\). Thus

\[
\alpha^{-1}(d(z))I\alpha^{-1}(d(z)) = 0, \text{ for all } z \in I.
\]  
(3.2.20)

Multiplying (3.2.20) by \(u\) from right, we obtain

\[
\alpha^{-1}(d(z))IR\alpha^{-1}(d(z))U = 0, \text{ for all } z \in I.
\]  
(3.2.21)

Using semiprimeness of \(R\), we have

\[
\alpha^{-1}(d(z))I = 0, \text{ for all } z \in I.
\]  
(3.2.22)

By (3.2.20)

\[
I\alpha^{-1}(d(z))R\alpha^{-1}(d(z)) = 0, \text{ for all } z \in I.
\]  
(3.2.23)

Since \(R\) is semiprimeness, we get

\[
I\alpha^{-1}(d(z)) = 0, \text{ for all } z \in I.
\]  
(3.2.24)

Subtracting (3.2.24) and (3.2.22), we have

\[
[\alpha^{-1}(d(z)), I] = 0, \text{ for all } z \in I.
\]  
(3.2.25)

Operating \(\alpha\) both sides in (3.2.25), we get

\[
[d(z), \alpha(I)] = 0, \text{ for all } z, u \in I.
\]  
(3.2.26)

Replace \(z\) by \(zu\) in (3.2.26), to have

\[
\alpha(z)[d(u), \alpha(I)] + [\alpha(z), \alpha(u)]d(u) = 0, \text{ for all } z, u \in I.
\]  
(3.2.27)

Again replacing \(z\) by \(rz\) in (3.2.27), we get

\[
\alpha(r)\alpha(z)[d(u), \alpha(I)] + \alpha(r)[\alpha(z), \alpha(u)]d(u) + [\alpha(r), \alpha(u)]\alpha(z)d(u) = 0, \text{ for all } z, u \in I, r \in R.
\]  
(3.2.28)

Multiplying (3.2.27) by \(\alpha(r)\) from left, we get

\[
\alpha(r)\alpha(z)[d(u), \alpha(I)] + \alpha(r)[\alpha(z), \alpha(u)]d(u) = 0, \text{ for all } z, u \in I. 
\]  
(3.2.29)

Now comparing (3.2.28) and (3.2.29), we have

\[
[\alpha(r), \alpha(I)]\alpha(z)d(u) = 0, \text{ for all } z, u \in I, r \in R.
\]  
(3.2.30)

Application of lemma 3.2.1 yields that \(R\) contains a nonzero central ideal.
3.3

Theorem 3.3.1. Let $R$ be a 2-torsion free $*$-prime ring, and $U$ be a nonzero square closed $*$- Lie ideal of $R$. Suppose that $\alpha$ and $\beta$ are automorphisms of $R$ and $F : R \to R$ is a generalized $(\alpha, \beta)$-derivation with associated nonzero $(\alpha, \beta)$-derivation $d$ such that $*$ commutes with $F, d, \alpha$ and $\beta$. If for all $u, v \in U$ $F([u, v]) = [F(u), v]_{\alpha, \beta} + [d(v), u]_{\alpha, \beta}$, then $U \subseteq Z(R)$.

Proof By hypothesis, we have

$$F([u, v]) = [F(u), v]_{\alpha, \beta} + [d(v), u]_{\alpha, \beta} \text{ for all } u, v \in U. \quad (3.3.1)$$

Substituting $[u, ru]$ for $u$ in (3.3.1), we get $F([u, ru], v) = [F([u, r]u), v]_{\alpha, \beta} + [d(v), [u, r]u]_{\alpha, \beta}$ for all $u, v \in U, r \in R$ i.e.,

$$F([u, r][u, v]) + F([u, r], v)u = [F([u, r]u), v]_{\alpha, \beta} + [d(v), [u, r]u]_{\alpha, \beta} \text{ for all } u, v \in U, r \in R \text{ i.e.,}$$

$$F[u, r]\alpha[u, v] + \beta[u, r]d[u, v] + F[u, r], v]_{\alpha, \beta} + \beta[u, r][d(v), u]_{\alpha, \beta} + [d(v), [u, r]_{\alpha, \beta}]$$

By hypothesis, we have

$$F[u, r]\alpha[u, v] + \beta[u, r]d[u, v] + F[u, r], v]_{\alpha, \beta} + [d(v), [u, r]_{\alpha, \beta}] = [F[u, r]\alpha(u), v]_{\alpha, \beta} + [\beta[u, r]d(u), v]_{\alpha, \beta} + \beta[u, r][d(v), u]_{\alpha, \beta} + [d(v), [u, r]_{\alpha, \beta}]$$

Hence

$$F[u, r]\alpha[u, v] + \beta[u, r]d([u, v]) + F[u, r], v]_{\alpha, \beta} + [d(v), [u, r]_{\alpha, \beta}] + \beta[u, r][d(u), v]_{\alpha, \beta} + [\beta[u, r]d(u), v]_{\alpha, \beta} + [d(v), [u, r]_{\alpha, \beta}]$$

for all $u, v \in U, r \in R$. 

After simplification, we get

\[
F[u, r] \alpha(u) \alpha(v) - F[u, r] \alpha(v) \alpha(u) + \beta[u, r] d(u) \alpha(v) + \beta[u, r] \beta(v) d(v) - \beta[u, r] d(v) \alpha(u) - \beta[u, r] \beta(v) d(u) + F[u, r] \alpha(u) \alpha(v) - \beta[u, r] \beta(v) d(u) + \beta[u, r] \beta(v) d(u) - \beta(v) F[u, r] \alpha(u) + \beta[u, r] d(u) \alpha(v) - \beta[u, r] \beta(v) d(u) + \beta[u, r] \beta(v) d(u) - \beta(v) \beta[u, r] d(u) + \beta[u, r] d(v) \alpha(u) - \beta[u, r] \beta(u) d(u) + d(u) \alpha[u, r] \alpha(u) - \beta[u, r] d(v) \alpha(u).
\]

We find that, \( \beta[u, r] \beta(u) d(v) = \beta[u, r] d(v) \alpha(u) \) for all \( u, v \in U, r \in R \). This gives us

\[
\beta[u, r][d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U, r \in R. \tag{3.3.2}
\]

Replacing \( r \) by \( sr \) in (3.3.2), we get

\[
\beta[u, sr][d(v), u]_{\alpha, \beta} = 0, \text{ for all } u, v \in U \text{ and } s, r \in R.
\]

This implies that

\[
\beta[u, s] R[d(v), u]_{\alpha, \beta} = 0, \text{ for all } u, v \in U \text{ and for all } s \in R. \tag{3.3.3}
\]

If \( u \in S_*(R) \cap U \), then (3.3.3), yields that

\[
\beta[u, s] R[d(v), u]_{\alpha, \beta} = (\beta[u, s])^* R[d(v), u]_{\alpha, \beta} = 0. \tag{3.3.4}
\]

Thus, for some \( u \in S_*(R) \cap U \) either \( \beta[u, s] = 0 \) or \( [d(v), u]_{\alpha, \beta} = 0 \). But for any \( u \in U, u - u^* \), \( u + u^* \in S_*(R) \cap U \). Therefore, for some \( u \in U \) either \( \beta[u - u^*] = 0 \) or \( [d(v), u - u^*]_{\alpha, \beta} = 0 \). If \( \beta[u - u^*, s] = 0 \), then \( \beta[u, s] = (\beta[u, s])^* \) and from equation (3.3.3), we obtain \( \beta[u, s] R[d(v), u]_{\alpha, \beta} = (\beta[u, s])^* R[d(v), u]_{\alpha, \beta} \). On the other hand if \( [d(v), u - u^*] = 0 \), then \( [d(v), u] = [d(v), u]^* \) and again by (3.3.3), we have \( \beta[u, s] R[d(v), u]_{\alpha, \beta} = \beta[u, s] R[d(v), u]_{\alpha, \beta} \). Hence by Lemma 3.2.3 either \( \beta[u, s] = 0 \) or \( [d(v), u]_{\alpha, \beta} = 0 \). Let \( L = \{ u \in U \mid \beta[u, s] = 0 \} \) and \( K = \{ u \in U \mid [d(v), u]_{\alpha, \beta} = 0 \} \). Then \( L \) and \( K \) are additive subgroups of \( U \) whose union is \( U \). Using Brauer’s trick, we get either \( L = U \) or \( K = U \). If \( L = U \), then \( [u, s] = 0 \) for all \( u \in U \) \( s \in R \) that is \( U \subseteq Z(R) \). On the other hand if \( K = U \), then

\[
[d(v), u]_{\alpha, \beta} = 0, \text{ for all } v, u \in U. \tag{3.3.5}
\]
Replacing \( v \) by \( 2uv \) in (3.3.5), we get

\[
\beta(v)[d(u), u]_{a,b} + [\beta(v), \beta(u)]d(u) = 0. \tag{3.3.6}
\]

Replacing \( v \) by \( u \) in (3.3.5), we get

\[
[d(u), u]_{a,b} = 0, \text{ for all } u \in U. \tag{3.3.7}
\]

Thus (3.3.6), yields that

\[
[\beta(v), \beta(u)]d(u) = 0, \text{ for all } u, v \in U. \tag{3.3.8}
\]

Now substituting \( 2uv \) for \( v \), we obtain \([\beta(z)\beta(v), \beta(u)]d(u) = 0, \text{ for all } u, v, z \in U. \]

That is

\[
\beta(z)[\beta(v), \beta(u)]d(u) + [\beta(z), \beta(u)]\beta(v)d(u) = 0, \text{ for all } u, v, z \in U.
\]

Using (3.3.8), we have

\[
[\beta(z), \beta(u)]\beta(v)d(u) = 0, \text{ for all } u, v, z \in U. \tag{3.3.9}
\]

Since \( \beta \) is an automorphism of \( R \), it follows that

\[
[z, u]U\beta^{-1}(d(u)) = 0, \text{ for all } u, v, z \in U. \tag{3.3.10}
\]

If \( u \in S_*(R) \cap U \), then by (3.3.10), we have \([z, u]U\beta^{-1}(d(u)) = [z, u]^*U\beta^{-1}(d(u))\).

Thus for some \( u \in S_*(R) \cap U \) either \( \beta^{-1}(d(u)) = 0 \) or \([z, u] = 0 \) by Lemma 3.2.3.

But for any \( u \in U \) \( u - u^* \), \( u + u^* \in S_*(R) \cap U \). Therefore, for some \( u \in U \) either \( \beta^{-1}(d(u - u^*)) = 0 \) or \([z, u - u^*] = 0 \) i.e. \([z, u] = [z, u]^* \) or \( \beta^{-1}(d(u)) = (\beta^{-1}(d(u)))^* \). If \([z, u] = [z, u]^* \), then by (3.3.10), we have \([z, u]U\beta^{-1}(d(u)) = [z, u]^*U\beta^{-1}(d(u)) \) and by Lemma 3.2.3 either \([z, u] = 0 \) or \( \beta^{-1}(d(u)) = 0 \). Otherwise \( \beta^{-1}(d(u)) = (\beta^{-1}(d(u)))^* \),

gives that \([z, u]U\beta^{-1}(d(u)) = [z, u]U(\beta^{-1}(d(u)))^* \) and again by Lemma 3.2.3 we have either \([z, u] = 0 \) or \( \beta^{-1}(d(u)) = 0 \). Now let \( A = \{ u \in U \mid [z, u] = 0 \} \) and \( B = \{ u \in U \mid d(u) = 0 \} \). Then \( A \) and \( B \) are additive subgroups of \( U \) whose union is \( U \). Using Brauer's trick, we have either \( A = U \) or \( B = U \). If \( A = U \), then \([U, U] = 0 \) and by Lemma 3.2.2, we get \( U \subseteq Z(R) \). If \( B = U \), then \( d(U) = (0) \) and Lemma 3.2.4 completes the proof.
Theorem 3.3.2. Let \( R \) be a 2-torsion free *-prime ring and \( U \) be a nonzero square closed *- Lie ideal of \( R \). Suppose that \( \alpha \) and \( \beta \) are automorphisms of \( R \) and \( F : R \to R \) is a generalized \((\alpha, \beta)\)-derivation with associated nonzero \((\alpha, \beta)\)-derivation \( d \) such that * commutes with \( F, d, \alpha \) and \( \beta \). If for all \( u, v \in U \) \( F(u \circ v) = (F(u) \circ v)_{\alpha, \beta} + (d(v) \circ u)_{\alpha, \beta} \), then \( U \subseteq Z(R) \).

Proof By hypothesis \( F(u \circ v) = (F(u) \circ v)_{\alpha, \beta} + (d(v) \circ u)_{\alpha, \beta} \) for all \( u, v \in U \). Replacing \( u \) by \([u, ru]\), we obtain

\[
F([u, r]\circ v) = (F([u, r]\circ v)_{\alpha, \beta} + (d(v) \circ [u, r]u)_{\alpha, \beta}
\]

for all \( u, v \in U, r \in R \). (3.3.11)

This gives,

\[
F(((u, r] \circ v)u + [u, r][u, v]) = ((F([u, r]\alpha(u) + \beta[u, r]d(u)) \circ v)_{\alpha, \beta}
\]

+ (d(v) \circ [u, r]u)_{\alpha, \beta} \text{ for all } u, v \in U, r \in R. \tag{3.3.12}

That is

\[
F(((u, r] \circ v)u) + F([u, r][u, v]) = (F([u, r]\alpha(u) \circ v)_{\alpha, \beta}
\]

+ (\beta[u, r]d(u)) \circ v)_{\alpha, \beta} + (d(v) \circ [u, r]u)_{\alpha, \beta} \text{ for all } u, v \in U, r \in R. \tag{3.3.13}

Thus,

\[
F([u, r] \circ v)\alpha(u) + \beta([u, r] \circ v)d(u) + F([u, r][u, v] + \beta[u, r]d[u, v]
\]

= (F([u, r] \circ v)_{\alpha, \beta}\alpha(u) + F([u, r]\alpha[u, v] + (\beta[u, r] \circ v)_{\alpha, \beta}d(u)
\]

+ \beta[u, r][d(u), \alpha(v)] + (d(v) \circ [u, r])_{\alpha, \beta}\alpha(u) + \beta[u, r][d(v), u]_{\alpha, \beta}
\]

for all \( u, v \in U, r \in R \). (3.3.14)

Using hypothesis, we have

\[
\beta([u, r][\beta(v)d(u) + \beta(v)[u, r]d(u) + \beta[u, r]d[u, v] =
\]

\[
(\beta[u, r] \circ v)_{\alpha, \beta}\alpha(u) + \beta[u, r][d(u), \alpha(v)] + \beta[u, r][d(v), u]_{\alpha, \beta}
\]

for all \( u, v \in U, r \in R \). (3.3.15)

This implies that

\[
\beta[u, r][\beta(v)d(u) + \beta[u, r]d[u, v] = \beta[u, r][\alpha(v)d(u) +
\]

\[
\beta[u, r][d(u), \alpha(v)] + \beta[u, r][d(v), u]_{\alpha, \beta} \text{ for all } u, v \in U, r \in R. \tag{3.3.16}
\]
After simplification, we get
\[ \beta[u, r](\alpha(v)d(u) + d(u)\alpha(v) - \alpha(v)d(u) + [d(v), u]_{\alpha, \beta} - \beta(v)d(u) - d(u)\alpha(v) - \beta(u)d(v) + d(v)\alpha(u) + \beta(v)d(u)) = 0 \text{ for all } u, v \in U, r \in R \text{ i.e.,} \]
\[ \beta[u, r]([d(v), u]_{\alpha, \beta} + [d(v), u]_{\alpha, \beta}) = 0. \]

This implies that
\[ 2\beta[u, r][d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U \text{ and } r \in R. \] Since \( R \) is 2-torsion free, we have
\[ \beta[u, r][d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U, r \in R. \]
Replacing \( r \) by \( rs \), we get \( \beta[u, rs][d(v), u]_{\alpha, \beta} = 0. \)

Thus,
\[ \beta(r[u, s])[d(v), u]_{\alpha, \beta} + \beta([u, r]s)[d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U \text{ and } r, s \in R. \]
Since \( \beta \) is an automorphism, we get \( \beta(r)\beta([u, s])[d(v), u]_{\alpha, \beta} + \beta([u, r])\beta(s)[d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U \text{ and } r, s \in R \) which gives \( \beta[u, r][d(v), u]_{\alpha, \beta} = 0 \text{ for all } u, v \in U \text{ and } r \in R. \) Now arguing in the similar manner as in the proof of Theorem 3.3.1, we get the required result.
Chapter 4

Identities with generalized derivations on semiprime rings

4.1 Introduction

Posner [121, Theorem 2] proved that existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Later on Mayne [106, Theorem 1] proved that in case there exists a nontrivial centralizing automorphism on a prime ring, then the ring is commutative. Vukman [136] investigated identities with derivations and automorphisms on semiprime rings and obtained that if there exist a derivation \( d : R \rightarrow R \) and an automorphism \( \alpha : R \rightarrow R \) where \( R \) is a 2-torsion free semiprime ring, such that \( [d(x)x + x\alpha(x), x] = 0 \) for all \( x \in R \), then \( d \) and \( \alpha - I \), \( I \) an identity mapping, map \( R \) into its centre. In section 4.2 we prove the following theorem which extend the above mentioned result of Vukman: Let \( R \) be a semiprime ring. Suppose that \( F : R \rightarrow R \) is a generalized derivation with an associated derivation \( d \) and \( \alpha \) is an automorphism of \( R \). If the mapping \( x \rightarrow F(x) + \alpha(x) \) is commuting on \( R \), then \( d \) and \( \alpha - I \) map \( R \) into \( Z(R) \). We also extend Theorem 10 and Theorem 11 of Vukman [136]. In section 4.3 we prove that if \( I \) is a nonzero ideal of a semiprime ring \( R \) admitting a generalized \((\alpha, \beta)\)-derivation \( F \) with an associated \((\alpha, \beta)\)-derivation \( d \) satisfying either of the conditions \( [F(x) + d(x), x]_{\alpha, \beta} = 0 \) or \((F(x) \circ x)_{\alpha, \beta} = 0 \) for all \( x \in I \) such that \( \beta(I)d(I) \neq 0 \), then \( R \) has a nonzero central ideal.
4.2

Following lemmas are required to prove our main results.

**Lemma 4.2.1 ([53, Lemma 1])**. Let $R$ be a $n$-torsion free ring. Suppose that $t_1, t_2, t_3, \ldots, t_n \in R$ satisfy $kt_1 + k^2t_2 + \ldots + k^nt_n = 0$ for $k = 1, 2, \ldots, n$. Then $t_i = 0$ for all $i$.

**Lemma 4.2.2 ([131, Proposition 2.3])**. Let $R$ be a semiprime ring and let $d : R \to R$ be a commuting $\alpha$-derivation on $R$. In this case, $d$ maps $R$ into its center.

**Lemma 4.2.3 ([136, Theorem 6])**. Let $R$ be 2-torsion free semiprime ring and let $f : R \to R$ be an additive centralizing mapping on $R$. Then $f$ is commuting on $R$.

**Lemma 4.2.4 ([136, Lemma 3])**. Let $R$ be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.

**Lemma 4.2.5 ([136, Theorem 11])**. Let $R$ be a 2-torsion free semiprime ring. Suppose that there exist a derivation $D : R \to R$ and automorphism $\alpha : R \to R$, such that the mapping $x \mapsto D(x)x + x\alpha(x)$ is commuting on $R$. In this case, $D$ and $\alpha - I$ map $R$ into $Z(R)$.

**Lemma 4.2.6 ([137, Theorem 4])**. Let $R$ be a 2-torsion free semiprime ring. Suppose that an additive mapping $F : R \to R$ satisfies $[[F(x), x], x] = 0$ for all $x \in R$. Then, $[F(x), x] = 0$ holds for all $x \in R$.

**Lemma 4.2.7 ([141, Lemma 1])**. Let $R$ be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case,
\((a + c)xb = 0\) is satisfied for all \(x \in R\).

**Lemma 4.2.8 ([151, Lemma 1.3])**. Let \(R\) be a semiprime ring. Suppose that there exists \(a \in R\) such that \(a[x,y] = 0\) holds for all \(x,y \in R\). Then \(a \in Z(R)\).

In [136, Theorem 4] Vukman proved that if \(R\) is a semiprime ring, \(d : R \rightarrow R\) is a derivation of \(R\) and \(\alpha\) is an automorphism of \(R\) such that the mapping \(x \rightarrow d(x) + \alpha(x)\) is commuting on \(R\), then \(d\) and \(\alpha - I\) map \(R\) into \(Z(R)\), the centre of \(R\). We extend the result replacing \(d\) by a generalized derivation \(F\) of \(R\) as follows:

**Theorem 4.2.1.** Let \(R\) be a semiprime ring. Suppose that \(F : R \rightarrow R\) is a generalized derivation with an associated derivation \(d : R \rightarrow R\) and \(\alpha\) is an automorphism of \(R\). If the mapping \(x \rightarrow F(x) + \alpha(x)\) is commuting on \(R\), then \(d\) and \(\alpha - I\), where \(I\) is an identity map on \(R\), map \(R\) into \(Z(R)\).

**Proof** By hypothesis

\[
[F(x) + \alpha(x), x] = 0 \quad \text{for all} \quad x \in R, \quad (4.2.1)
\]

Linearizing (4.2.1), we get

\[
[F(x) + \alpha(x), y] + [F(y) + \alpha(y), x] = 0 \quad \text{for all} \quad x,y \in R, \quad (4.2.2)
\]

Substituting \(yx\) for \(y\) in (4.2.2) and using (4.2.1), we obtain

\[
[F(x) + \alpha(x), y]x + [F(y), x]x + y[d(x), x] + [y, x]d(x) + \alpha(y)[\alpha(x) + \alpha(y)][\alpha(x), x] = 0 \quad \text{for all} \quad x,y \in R. \quad (4.2.3)
\]

Using (4.2.2), we have

\[
[\alpha(y), x]G(x) + y[d(x), x] + [y, x]d(x) + \alpha(y)[\alpha(x), x] = 0 \quad \text{for all} \quad x,y \in R, \quad (4.2.4)
\]
where \( G(x) \) denotes \( \alpha(x) - x \). Replacing \( xy \) for \( y \) in (4.2.4), we get

\[
[\alpha(x), x] \alpha(y) G(x) + \alpha(x)[\alpha(y), x] G(x) + xy[d(x), x] + x[y, x] d(x) + \alpha(x) \alpha(y)[\alpha(x), x] = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.5)

Replacing \( \alpha(y) \) for \( y \) in the above relation, we obtain

\[
[\alpha(x), x] y G(x) + \alpha(x)[y, x] G(x) + xy[d(x), x] + x[y, x] d(x) + \alpha(x) y[\alpha(x), x] = 0
\]

for all \( x, y \in R \).

(4.2.6)

Left multiplying (4.2.4) by \( x \) replacing \( \alpha(y) \) by \( y \) and then subtracting from (4.2.6), we get

\[
[G(x), x] y G(x) + G(x)[y, x] G(x) + G(x) y[G(x), x] = 0 \quad \text{for all} \quad x, y \in R,
\]  

(4.2.7)

where \( [\alpha(x), x] = [G(x), x] \), which reduces to

\[
x G(x) y G(x) + G(x) y(-G(x) x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.8)

Applying Lemma 4.2.7, the above relation gives

\[
[G(x), x] y G(x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.9)

Substituting in the above relation \( y x \) for \( y \), we obtain

\[
[G(x), x] y x G(x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.10)

Right multiplying (4.2.9) by \( x \) and then subtracting from (4.2.10), we get

\[
[G(x), x] y[G(x), x] = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.11)

Semiprimeness of \( R \) yields that

\[
[G(x), x] = 0 \quad \text{for all} \quad x \in R.
\]  

(4.2.12)

We have therefore, \([\alpha(x), x] = 0\), for all \( x \in R \), which together with the relation (4.2.1) yields that
\[ [F(x), x] = 0 \text{ for all } x \in R. \] \hspace{1cm} (4.2.13)

Linearization of the above relation gives

\[ [F(x), y] + [F(y), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.14)

Replacing \(yx\) for \(y\) in (4.2.14) and using (4.2.13), we obtain

\[ [F(x), y]x + [F(y), x]x + [yd(x), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.15)

Right multiplying (4.2.14) by \(x\) and then subtracting from (4.2.15), we get

\[ [yd(x), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.16)

Substituting \(d(x)y\) for \(y\) in (4.2.16) and using (4.2.16), we obtain

\[ [d(x), x]yd(x) = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.17)

Replacing \(y\) by \(yx\) in (4.2.17), we get

\[ [d(x), x]yx d(x) = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.18)

Right multiplying (4.2.17) by \(x\) and then subtracting from (4.2.18), we obtain

\[ [d(x), x]yd(x), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.19)

Semiprimeness of \(R\) yields that

\[ [d(x), x] = 0 \text{ for all } x \in R. \] \hspace{1cm} (4.2.20)

Linearizing (4.2.20), we get

\[ [d(x), y] + [d(y), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.21)

Replacing \(y\) by \(yx\) in (4.2.21), we obtain

\[ x[d(x), y] + d(x)[y, x] + x[d(y), x] = 0 \text{ for all } x, y \in R. \] \hspace{1cm} (4.2.22)
Multiplying (4.2.21) by \(x\) from left, we get

\[xd(x, y) + xd(y, x) = 0 \text{ for all } x, y \in R. \quad (4.2.23)\]

Subtracting (4.2.22) and (4.2.23), we have

\[d(x)[y, x] = 0 \text{ for all } x, y \in R. \quad (4.2.24)\]

Substituting \(x + z\) for \(x\) in the above relation, we obtain \(d(x)[y, z] + d(x)[y, x] = 0\) for all \(x, y, z \in R\). Replacing \(y\) by \(d(x)\) in the previous relation, we get

\[d(x)[d(x), z] = 0 \text{ for all } x, z \in R. \quad (4.2.25)\]

Substituting \(yz\) for \(z\) in (4.2.25) and using it, we get

\[d(x)y[d(x), z] = 0 \text{ for all } x, z \in R. \quad (4.2.26)\]

Substituting \(zy\) for \(y\) in (4.2.26) and left multiplying from \(z\) and then subtracting from (4.2.26), we get \([d(x), z]y[d(x), z] = 0\) for all \(x, y, z \in R\). Semiprimeness of \(R\) yields that \([d(x), z] = 0\) for all \(x, z \in R\), that is \(d(x) \in Z(R)\) for all \(x \in R\) which completes the proof.

**Corollary 4.2.1.** Let \(R\) be a 2-torsion free semiprime ring. Suppose that \(F : R \to R\) is a generalized derivation with an associated derivation \(d : R \to R\) and \(\alpha\) is an automorphism of \(R\) such that the mapping \(x \to F(x) + \alpha(x)\) is centralizing on \(R\). In this case, \(d\) and \(\alpha - I\) map \(R\) into \(Z(R)\).

**Proof** The proof is an immediate consequence of Lemma 4.2.3 and Theorem 4.2.1

**Corollary 4.2.2.** Let \(R\) be a noncommutative 2-torsion free prime ring. Suppose that \(F : R \to R\) is a generalized derivation with an associated derivation \(d : R \to R\) and \(\alpha\) is an automorphism of \(R\) such that the mapping \(x \to F(x) + \alpha(x)\) is centralizing on \(R\). In this case, \(d = 0\) and \(\alpha = I\).
Proof By Lemma 4.2.3, we get $[F(x) + \alpha(x), x] = 0$ for all $x \in R$ and hence Theorem 4.2.1 completes the proof.

In [136, Theorem 10] Vukman proved that if $R$ is a semiprime ring, $d : R \rightarrow R$ is a derivation and $\alpha$ is an automorphism of $R$ such that $d(x)x + x(\alpha(x) - x) = 0$ for all $x \in R$, then $d = 0$ and $\alpha = I$. We obtain this result in case of a generalized derivation as follows:

**Theorem 4.2.2.** Let $R$ be a semiprime ring. Suppose that $F : R \rightarrow R$ is a generalized derivation with an associated derivation $d : R \rightarrow R$ and $\alpha$ is an automorphism of $R$ such that $F(x)x + x(\alpha(x) - x) = 0$ for all $x \in R$. Then $d = 0$ and $\alpha = I$.

**Proof** By hypothesis, we have

$$F(x)x + xG(x) = 0 \text{ for all } x \in R,$$

where $G(x)$ stands for $\alpha(x) - x$. The linearization of the above relation yields that

$$F(x)y + F(y)x + xG(y) + yG(x) = 0 \text{ for all } x, y \in R. \tag{4.2.28}$$

Substituting $yx$ for $y$ in (4.2.28), we obtain

$$F(x)yx + F(y)x^2 + yd(x)x + xG(y)\alpha(x) + xyG(x) + yxG(x) = 0 \text{ for all } x, y \in R. \tag{4.2.29}$$

Right multiplying (4.2.28) by $x$ and then subtracting from (4.2.29), we get

$$yd(x)x + xG(y)G(x) + xyG(x) + yxG(x) - yG(x)x = 0 \text{ for all } x, y \in R. \tag{4.2.30}$$

Replacing $y$ by $xy$ in (4.2.30), we get

$$xyd(x)x + xG(x)\alpha(y)G(x) + x^2G(y)G(x) + x^2yG(x) + xyG(x) - xyG(x)x = 0 \text{ for all } x, y \in R. \tag{4.2.31}$$
Left multiplying (4.2.30) by \( x \), we obtain

\[
xyd(x)x + x^2 G(y)G(x) + x^2 yG(x) + xyxG(x) - xyG(x)x = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.32)

Comparing (4.2.31) and (4.2.32), we get

\[
xG(x)\alpha(y)G(x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.33)

Replacing \( \alpha(y) \) by \( y \) in (4.2.33), we obtain

\[
xG(x)yG(x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.34)

Replacing \( y \) by \( xy \) in (4.2.34), we have

\[
xG(x)yxG(x) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.35)

Semiprimeness of \( R \) yields that

\[
xG(x) = 0 \quad \text{for all} \quad x \in R.
\]  

(4.2.36)

Applying Lemma 4.2.4, we get \( G = 0 \). Using this relation in (4.2.27) we obtain \( F(x)x = 0 \) for all \( x \in R \). Again by Lemma 4.2.4, we get \( F = 0 \). This implies that

\[
F(x) = 0 \quad \text{for all} \quad x \in R.
\]  

(4.2.37)

Replacing \( x \) by \( xy \) in (4.2.37) and using (4.2.37), we find

\[
xd(y) = 0 \quad \text{for all} \quad x, y \in R.
\]  

(4.2.38)

In particular \( xd(x) = 0 \) for all \( x \in R \). Using Lemma 4.2.4, we conclude that \( d = 0 \).


The following theorem is an extension of Theorem 11 of Vukman [136].

**Theorem 4.2.3.** Let \( R \) be a 2-torsion free semiprime ring. Suppose that \( F : R \to R \) is a generalized derivation with an associated derivation \( d : R \to R \) and \( \alpha \) is an automorphism of \( R \) such that the mapping \( x \to F(x)x + x\alpha(x) \) is commuting on \( R \). In
this case, $d$ and $\alpha - I$ map $R$ into $Z(R)$.

Proof. By hypothesis

$$[F(x)x + x\alpha(x), x] = 0 \text{ for all } x \in R.$$  \hspace{1cm} (4.2.39)

Linearizing (4.2.39) and using Lemma 4.2.1, we obtain

$$[A(x), y] + [F(x)y + F(y)x + x\alpha(y) + y\alpha(x), x] = 0 \text{ for all } x \in R,$$  \hspace{1cm} (4.2.40)

where $A(x)$ stands for $F(x)x + x\alpha(x)$. Replacing $yx$ for $y$ in (4.2.40), we get

$$[A(x), yx] + [F(x)y + F(y)x, x] + y[d(x)x, x] + x[a(y)\alpha(x), x] + [yx\alpha(x), x] = 0 \text{ for all } x, y \in R.$$  \hspace{1cm} (4.2.41)

Using (4.2.40), we get

$$x[\alpha(y), x]G(x) - y[\alpha(x), x]x + [y, x][\alpha(x), x] + y[d(x), x]x + [y, x]d(x)x +$$

$$x\alpha(y)[\alpha(x), x] + yx[\alpha(x), x] = 0$$  \hspace{1cm} (4.2.42)

for all $x, y \in R$,

where $G(x)$ denotes $\alpha(x) - x$. Substituting $xy$ for $y$ and replacing $\alpha(y)$ by $y$ in (4.2.42), we get

$$x[\alpha(x), x]yG(x) + x\alpha(x[y, x]G(x) - xy[\alpha(x), x]x + x[y, x][\alpha(x), x] +$$

$$xy[d(x), x]x + x[y, x]d(x)x + x\alpha(x)y[\alpha(x), x] = 0$$  \hspace{1cm} (4.2.43)

for all $x, y \in R$.

Left multiplying (4.2.42) by $x$, we get

$$x^2[\alpha(y), x]G(x) - xy[\alpha(x), x]x + x[y, x][\alpha(x), x] + xy[d(x), x]x + x[y, x]d(x)x +$$

$$x^2\alpha(y)[\alpha(x), x] + xyx[\alpha(x), x] = 0$$  \hspace{1cm} (4.2.44)

for all $x, y \in R$. 
Substituting $\alpha(y)$ for $y$ in (4.2.44), we have
\[
x^2[y,x][G(x) - xy[\alpha(x), x][x + x[y, x][\alpha(x), x]] + xy[d(x), x]x + x[y, x][d(x)]x + \\
x^2y[\alpha(x), x] + xy[x[\alpha(x), x] = 0
\]
for all $x, y \in R$. \hfill (4.2.45)

Subtracting (4.2.43) from (4.2.45), we obtain
\[
x[G(x), x][G(x) + xG(x)[yG(x), x] + xG(x)[y, x]G(x) = 0 \text{ for all } x, y \in R, \quad (4.2.46)
\]
where $[G(x), x] = [\alpha(x), x]$. Collecting terms, the above relation can be written as
\[
-x^2G(x)yG(x) + xG(x)yG(x)x = 0 \text{ for all } x, y \in R. \quad (4.2.47)
\]
Substituting $yx$ for $y$ in the above relation gives
\[
-x^2G(x)yxG(x) + xG(x)yxG(x)x = 0 \text{ for all } x, y \in R. \quad (4.2.48)
\]
Applying Lemma 4.2.7, in the above relation gives
\[
x[G(x), x][yxG(x) = 0 \text{ for all } x, y \in R. \quad (4.2.49)
\]
Substituting $yx$ for $y$ in the above relation, then multiplying (4.2.49) from the right side by $x$, and then subtracting the relations so obtained, we arrive at
\[
x[G(x), x][yx[G(x), x] = 0 \text{ for all } x, y \in R. \quad \text{That is}
\]
\[
x[G(x), x]Rx[G(x), x] = 0
\]
for all $x, y \in R$. Since $R$ is semiprime, it follows that
\[
x[\alpha(x), x] = 0 \text{ for all } x \in R. \quad (4.2.50)
\]
Applying (4.2.50) in (4.2.39), we get
\[
[F(x), x]x = 0 \text{ for all } x \in R. \quad (4.2.51)
\]
Linearizing (4.2.50) and using Lemma 4.2.1, we get

\[ x[α(x), y] + x[α(y), x] + y[α(x), x] = 0 \quad \text{for all } x, y ∈ R. \quad (4.2.52) \]

Substituting \( xy \) for \( y \) in (4.2.52), we get

\[ x^2[α(x), y] + xα(x)[α(y), x] + xy[α(x), x] = 0 \quad \text{for all } x, y ∈ R. \quad (4.2.53) \]

Left multiplying (4.2.52) by \( x \) and then subtracting from (4.2.53), we obtain

\[ xG(x)[α(y), x] = 0 \quad \text{for all } x, y ∈ R. \quad (4.2.54) \]

Replacing \( α(y) \) for \( y \) in (4.2.54), we get

\[ xG(x)[y, x] = 0 \quad \text{for all } x, y ∈ R. \quad (4.2.55) \]

Putting in the above relation \( yz \) for \( y \), we arrive at

\[ xG(x)y[z, x] = 0 \quad \text{for all } x, y, z ∈ R. \quad (4.2.56) \]

Linearization of \( x \) and \( w \) in (4.2.56) and application of Lemma 4.2.1, give

\[ xG(x)y[z, w] + xG(w)y[z, x] + wG(x)y[z, x] = 0 \quad \text{for all } x, y, z, w ∈ R. \quad (4.2.57) \]

Putting in the above relation \( [z, w]yG(x) \) for \( y \) and applying the relation (4.2.55), we obtain \((xG(x)[z, w]y(xG(x)[z, w])) = 0 \) for all \( x, y, z, w ∈ R \). Semiprimeness of \( R \) gives

\[ xG(x)[z, w] = 0 \quad \text{for all } x, y, z, w ∈ R. \quad (4.2.58) \]

Applying Lemma 4.2.4, we obtain

\[ G(x)[z, w] = 0 \quad \text{for all } x, z, w ∈ R. \quad (4.2.59) \]

By Lemma 4.2.8, we conclude that \( G(x) ∈ Z(R) \) for all \( x ∈ R \). That is, \( α - I \) maps \( R \) into \( Z(R) \). Linearization of (4.2.51) and application of Lemma 4.2.1, yields that
\[ [F(x), y]x + [F(y), x]x + [F(x), x]y = 0 \quad \text{for all} \quad x, y \in R. \quad (4.2.60) \]

Replacing \( y \) by \( yx \) in (4.2.60), we get

\[ [F(x), y]x^2 + [F(y), x]x^2 + [yd(x), x]x + [F(x), x]yx = 0 \quad \text{for all} \quad x, y \in R. \quad (4.2.61) \]

Right multiplying (4.2.60) by \( x \) and then subtracting from (4.2.61), we obtain

\[ [yd(x), x]x = 0 \quad \text{for all} \quad x, y \in R. \quad (4.2.62) \]

Replacing \( y \) by \( d(x)y \) in (4.2.62) and using (4.2.62), we get

\[ [d(x), x]yd(x)x = 0 \quad \text{for all} \quad x, y \in R. \quad (4.2.63) \]

Replacing \( y \) by \( xy \) in (4.2.63), we obtain

\[ [d(x), x]xyd(x)x = 0 \quad \text{for all} \quad x, y \in R. \quad (4.2.64) \]

Putting first in the above relation \( yx \) for \( y \), then multiplying the relation (4.2.64) from the right side by \( x \), and subtracting the relations so obtained one from another, we arrive at \( [d(x), x]xy[d(x), x]x = 0 \) \text{ for all } \( x, y \in R \). Semiprimeness of \( R \) gives

\[ [d(x), x]x = 0 \quad \text{for all} \quad x \in R. \quad (4.2.65) \]

Thus by Lemma 4.2.5, we obtain \( d \) maps \( R \) into \( Z(R) \).

**Theorem 4.2.4.** Let \( R \) be a 2-torsion free semiprime ring. Suppose that \( F : R \to R \) is a generalized derivation with an associated derivation \( d : R \to R \) and \( \alpha \) is an automorphism of \( R \) such that \([F(x), x] \pm \alpha(x), x] = 0 \) for all \( x \in R \). In this case, \( d \) and \( \alpha - I \) map \( R \) into \( Z(R) \).

**Proof** By hypothesis, we have

\[ [[F(x), x], x] + [\alpha(x), x] = 0 \quad \text{for all} \quad x \in R. \quad (4.2.66) \]
Linearization of (4.2.66) and application of Lemma 4.2.1, yields that

\[[F(y), x], x] + [[F(x), y], x] + [[F(x), x], y] + [\alpha(y), x] + [\alpha(x), y] = 0 \text{ for all } x, y \in R. \tag{4.2.67}\]

Replacing \( y \) by \( x \) in (4.2.67) and using (4.2.66), we get

\[[F(x), x], x] = 0 \text{ for all } x \in R. \tag{4.2.68}\]

By Lemma 4.2.6, we obtain \([F(x), x] = 0 \text{ for all } x \in R\), which is (4.2.13). Now using similar techniques as we have used in the proof of Theorem 4.2.1, we obtain \( d \) maps \( R \) into \( Z(R) \). Now (4.2.68) and (4.2.66), give \([\alpha(x), x] = 0 \text{ for all } x \in R\). Hence by Lemma 4.2.2, completes the proof.

Similarly we can get the result for the case \([F(x), x] = \alpha(x), x] = 0 \text{ for all } x \in R\).

**Theorem 4.2.5.** Let \( R \) be a \( 2 \)-torsion free semiprime ring. Suppose that \( F : R \to R \) is a generalized derivation with an associated derivation \( d : R \to R \) and \( \alpha \) is an automorphism of \( R \) such that \([F(x) \pm [\alpha(x), x], x] = 0 \text{ for all } x \in R\). In this case, \( R \) is commutative and \( d \) maps \( R \) into \( Z(R) \).

**Proof** By hypothesis, we have

\[[F(x), x] + [[\alpha(x), x], x] = 0 \text{ for all } x \in R. \tag{4.2.69}\]

Linearization of (4.2.69) and application of Lemma 4.2.1, give

\[[F(y), x] + [F(x), y] + [[\alpha(y), x], x] + [[\alpha(x), y], x] + [[\alpha(x), x], y] = 0 \text{ for all } x, y \in R, \tag{4.2.70}\]

Substituting \( x \) for \( y \) in (4.2.70) and using (4.2.69), we get

\[[[\alpha(x), x], x] = 0 \text{ for all } x \in R. \tag{4.2.71}\]

This implies that

\([x, [x, \alpha(x)]] = 0 \text{ for all } x \in R. \tag{4.2.72}\)
Replace $\alpha(x)$ by $y$ in (4.2.72), to get

$$[x, [x, y]] = 0 \text{ for all } x, y \in R.$$  \hspace{1cm} (4.2.73)

This implies that $[x, y] \in Z(R)$ for all $x, y \in R$. Therefore, we can write

$$[[x, y], r] = 0 \text{ for all } x, y, r \in R.$$  \hspace{1cm} (4.2.74)

Substituting $yx$ for $y$ in (4.2.74) and using (4.2.74), we get

$$[x, y][x, r] = 0 \text{ for all } x, y, r \in R.$$  \hspace{1cm} (4.2.75)

Replacing $r$ by $ry$ in the above relation and using it, we obtain

$$[x, y[r, x, y] = 0 \text{ for all } x, y, r \in R.$$  \hspace{1cm} (4.2.76)

Semiprimeness of $R$ yields that

$$[x, y] = 0 \text{ for all } x, y \in R.$$  \hspace{1cm} (4.2.77)

Hence $R$ is commutative. Using (4.2.71) and (4.2.69), we get $[F(x), x] = 0$ for all $x \in R$, which is (4.2.13). Arguing in the similar manner as in the proof of Theorem 4.2.1, we obtain $d$ maps $R$ into $Z(R)$.

Similarly we can prove the case $[F(x) - [\alpha(x), x], x] = 0$ for all $x \in R$.

Theorem 4.2.6. Let $R$ be a 2-torsion free semiprime ring. Suppose that $F : R \to R$ is a generalized derivation with an associated derivation $d : R \to R$ and $\alpha$ is an automorphism of $R$ such that $[[F(x) \pm \alpha(x), x], x] = 0$ for all $x \in R$. In this case, $d$ maps $R$ into $Z(R)$ and $R$ is commutative.

Proof By hypothesis, we have

$$[[F(x), x], x] + [[\alpha(x), x], x] = 0 \text{ for all } x \in R.$$  \hspace{1cm} (4.2.78)
Linearization of (4.2.78) and application of Lemma 4.2.1, yield that

\[ [[F(y), x], x] + [[F(x), y], x] + [[F(x), x], y] + [[\alpha(y), x], x] + [[\alpha(x), y], x] + [[\alpha(x), x], y] = 0 \text{ for all } x, y \in R. \tag{4.2.79} \]

Substituting \( y \) for \( x \) and replacing \( \alpha(y) \) by \( y \) in (4.2.79), using (4.2.78), we get \( [[F(x), x], x] = 0 \) for all \( x \in R \), which is (4.2.68). Arguing as in proof of Theorem 4.2.4, we get \( d \) maps \( R \) into \( Z(R) \). Since \( [[F(x), x], x] = 0 \) for all \( x \in R \), (4.2.78), yields that \( [[\alpha(x), x], x] = 0 \) for all \( x \in R \), and Theorem 4.2.5 completes the proof.

Similarly we can prove the case \( [[F(x) - \alpha(x), x], x] = 0 \) for all \( x \in R \).

The following example illustrates that the above Theorems do not hold for arbitrary rings and torsion condition in the hypothesis is not superfluous.

**Example 4.2.1.** Let \( R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\} \). Then \( R \) is not 2-torsion free semiprime ring. Define maps \( F, d, \alpha : R \to R \) by

\[
F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \( F \) is a generalized derivation with an associated derivation \( d \) and \( \alpha - I \) is an \( \alpha \)-derivation of \( R \) satisfying the hypothesis of Theorem 4.2.1 to Theorem 4.2.6. But \( R \) is not commutative.

4.3

The following Theorems are in spirit of Lemma 4.2.7 and Lemma 4.2.4 proved by Vukman in [141] and [136] respectively.
Theorem 4.3.1. Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ for all $x \in I$, then $f$ is commuting on $I$.

Proof. Let

$$f(x)x = 0 \text{ for all } x \in I. \quad (4.3.1)$$

Linearizing (4.3.1), we get

$$f(x)y + f(y)x = 0 \text{ for all } x, y \in I. \quad (4.3.2)$$

Replacing $y$ by $y^2$ in (4.3.2), we obtain

$$f(x)y^2 + f(y^2)x = 0 \text{ for all } x, y \in I. \quad (4.3.3)$$

Right multiplying (4.3.2) by $y$ and subtracting from (4.3.3), we get

$$f(y^2)x - f(y)xy = 0 \text{ for all } x, y \in I. \quad (4.3.4)$$

Replacing $x$ by $xf(y)$ in (4.3.4) and using (4.3.1), we get

$$f(y^2)xf(y) = 0 \text{ for all } x, y \in I. \quad (4.3.5)$$

Right multiplying (4.3.4) by $f(y)$ and using (4.3.5), we get

$$f(y)xyf(y) = 0 \text{ for all } x, y \in I. \quad (4.3.6)$$

Replacing $x$ by $rx$ in (4.3.6), we obtain

$$f(y)rxzf(y) = 0 \text{ for all } x, y \in I. \quad (4.3.7)$$

Left multiplying (4.3.7) by $xy$, we have

$$xyf(y)rxzf(y) = 0 \text{ for all } x, y \in I. \quad (4.3.8)$$

Semiprimeness of $R$ gives that

$$xyf(y) = 0 \text{ for all } x, y \in I. \quad (4.3.9)$$
Replacing \( x \) by \( xr \) in (4.3.9), we get
\[
xryf(y) = 0 \text{ for all } x, y \in I. \tag{4.3.10}
\]
Replacing \( r \) by \( f(y)r \) in (4.3.10), we obtain
\[
xf(y)ryf(y) = 0 \text{ for all } x, y \in I. \tag{4.3.11}
\]
Substituting \( x \) for \( y \), we get \( xf(x)Rx_{s}f(x) = 0 \) for all \( x \in R \). By semiprimeness, we have
\[
xf(x) = 0 \text{ for all } x \in I. \tag{4.3.12}
\]
Subtracting (4.3.1) from (4.3.12), we have \( f(x)x - xf(x) = 0 \) for all \( x \in I \). This implies that \( [f(x), x] = 0 \) for all \( x \in I \) and hence \( f \) is commuting on \( I \).
Similarly we can prove the Theorem in case \( xf(x) = 0 \) for all \( x \in I \).

**Theorem 4.3.2.** Let \( R \) be a semiprime ring, \( I \) be a nonzero ideal of \( R \). Suppose that the relation \( axb + bxc = 0 \) holds for all \( x \in I \) and some \( a, b, c \in R \). In this case \( (a + c)xh = 0 \) is satisfied for all \( x \in I \).

**Proof** We have
\[
axb + bxc = 0 \text{ for all } x \in I. \tag{4.3.13}
\]
Replacing \( x \) by \( xy \) in (4.3.13), we get
\[
axyb + bxyc = 0 \text{ for all } x, y \in I. \tag{4.3.14}
\]
Substituting \( by \) for \( y \) in (4.3.14), we obtain
\[
axbyb + bxbyc = 0 \text{ for all } x, y \in I, b \in R. \tag{4.3.15}
\]
Right multiplying (4.3.13) by \( y \), to get
\[
axyb + bxcy = 0 \text{ for all } x, y \in I, b \in R. \tag{4.3.16}
\]
Replacing \( y \) by \( yb \) in (4.3.16) and subtracting from (4.3.15), we get
\[
bx(byc - cyb) = 0 \quad \text{for all} \quad x, y \in I, b \in R. \tag{4.3.17}
\]
Substituting \( z \) for \( x \) in (4.3.17), to find
\[
bzx(byc - cyb) = 0 \quad \text{for all} \quad x, y, z \in I, b \in R. \tag{4.3.18}
\]
Replacing \( x \) by \( cz \) in (4.3.18), we get
\[
bzx(byc - cyb) = 0 \quad \text{for all} \quad x, y, z \in I, b, c \in R. \tag{4.3.19}
\]
Left multiplying (4.3.17) by \( z \), we obtain
\[
zbx(byc - cyb) = 0 \quad \text{for all} \quad x, y, z \in I, b, c \in R. \tag{4.3.20}
\]
Replacing \( z \) by \( cz \) in (4.3.20) and subtracting from (4.3.19), we obtain
\[
(bcz - czb)x(byc - cyb) = 0 \quad \text{for all} \quad x, y, z \in I, b, c \in R. \tag{4.3.21}
\]
In particular
\[
(byc - cyb)x(byc - cyb) = 0 \quad \text{for all} \quad x, y, z \in I, b, c \in R. \tag{4.3.22}
\]
This implies that \((byc - cyb)I^2 = \{0\}\) for all \( y \in I \). Since \( R \) is semiprime, we have \((byc - cyb)I = 0\), for all \( y \in I \). It means that \((byc - cyb) \in \text{ann}(I) \cap I = \{0\}\), That is \((byc - cyb) = 0\) for all \( y \in I \). Using last relation in (4.3.13), we obtain \((a + c)xb = 0\) for all \( x \in I \).

**Theorem 4.3.3.** Let \( R \) be 2-torsion free semiprime ring, \( I \) a nonzero ideal of \( R \) and \( \alpha, \beta \) be two epimorphisms of \( R \). Suppose that \( F \) is a generalized \((\alpha, \beta)\)-derivation with an associated \((\alpha, \beta)\)-derivation \( d \) of \( R \) such that \( \beta(I)d(I) \neq 0 \). If \([F(x) + d(x), x]_{\alpha, \beta} = 0\) for all \( x \in I \), then \( R \) has a nonzero central ideal.
Proof By hypothesis

\[ [F(x) + d(x), x]_{\alpha, \beta} = 0 \quad \text{for all} \quad x \in I. \quad (4.3.23) \]

Linearizing, we get

\[ [F(x) + d(x), y]_{\alpha, \beta} + [F(y) + d(y), x]_{\alpha, \beta} = 0 \quad \text{for all} \quad x, y \in I. \quad (4.3.24) \]

Substituting \( yx \) for \( y \) in (4.3.24), we obtain

\[ [F(x) + d(x), y]_{\alpha, \beta} + [F(y) + d(y), x]_{\alpha, \beta} + [F(y) + d(y)[\alpha(x), \alpha(x)] + [\beta(y), \alpha(x)]d(x) + \beta(y)[d(x), x]_{\alpha, \beta} + [\beta(y), \alpha(x)]d(x) = 0 \quad \text{for all} \quad x, y \in I. \quad (4.3.25) \]

Using (4.3.23) and (4.3.24), (4.3.25) gives

\[ 2[\beta(y), \alpha(x)]d(x) = 0 \quad \text{for all} \quad x, y \in I. \quad (4.3.26) \]

Since \( R \) is 2-torsion free, we have

\[ \beta(y)[d(x), x]_{\alpha, \beta} + [\beta(y), \alpha(x)]d(x) = 0 \quad \text{for all} \quad x, y \in I. \quad (4.3.27) \]

Replacing \( ry \) for \( y \) in (4.3.27), we get

\[ \beta(r)\beta(y)[d(x), x]_{\alpha, \beta} + \beta(r)[\beta(y), \alpha(x)]d(x) + [\beta(r), \alpha(x)]\beta(y)d(x) = 0 \quad \text{for all} \quad x, y \in I \text{ and } r \in R. \quad (4.3.28) \]

Using (4.3.27), it reduces to

\[ [\beta(r), \beta(x)]\beta(y)d(x) = 0 \quad \text{for all} \quad x, y \in I \text{ and } r \in R. \quad (4.3.29) \]

This implies that \( [R, \beta(x)]R\beta(y)d(x) = 0 \quad \text{for all} \quad x, y \in I. \) Since \( R \) is semiprime, it must contain a family \( \Omega = \{P_\alpha \mid \alpha \in \Lambda\} \) of prime ideals such that \( \cap P_\alpha = 0. \) If \( P \) is a typical member of \( \Omega \) and \( x \in I, \) it follows that

\[ [R, \beta(x)] \subseteq P \quad \text{or} \quad \beta(I)d(x) \subseteq P \]

Let \( T_1 = \{x \in I \mid [R, \beta(x)] \subseteq P\} \) and \( T_2 = \{x \in I \mid \beta(I)d(x) \subseteq P\} \) be two additive subgroups of \( I \) such that \( T_1 \cup T_2 = I. \) Since a group cannot be a union of its proper
subgroups, either \( T_1 = I \) or \( T_2 = I \). i.e. \( [\beta(I), R] \subseteq P \) or \( \beta(I)d(I) \subseteq P \). Thus both the cases together yield that \( [\beta(I), R]\beta(I)d(I) \subseteq P \) for any \( P \in \Omega \). Therefore \( [\beta(I), R]\beta(I)d(I) \subseteq \cap P = 0 \). Thus

\[
0 = [R, \beta(RIR)]\beta(RI)d(I) = [R, R\beta(I)R]R\beta(I)d(I)
\]

and so \( 0 = [R, R\beta(I)d(I)R]R\beta(I)d(I)R \). Thus \( 0 = [R, J]RJ \), where \( J = R\beta(I)d(I)R \) is a nonzero ideal of \( R \), for \( \beta(I)d(I) \neq 0 \). i.e, \( 0 = [R, J]R[R, J] \). Since \( R \) is semiprime, it follows that \( J \subseteq Z(R) \).

**Corollary 4.3.1.** Let \( R \) be a 2-torsion free prime ring, \( I \) a nonzero ideal of \( R \) and \( \alpha, \beta \) be epimorphisms of \( R \) such that \( \beta(I) \neq 0 \). Suppose that \( F \) is a generalized \((\alpha, \beta)\)-derivation with associated nonzero \((\alpha, \beta)\)-derivation \( d \) of \( R \). If \( [F(x) + d(x), x]_{\alpha, \beta} = 0 \) for all \( x \in I \), then \( R \) is commutative.

**Proof** By Theorem 4.3.3 we see that if \( d(I) \neq 0 \), then \( R \) is commutative. Now if \( d(I) = 0 \), then \( d(R) = 0 \), a contradiction. Hence again \( R \) is commutative.

**Theorem 4.3.4.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \) and \( \alpha, \beta \) be epimorphisms of \( R \). Suppose that \( F \) is a generalized \((\alpha, \beta)\)-derivation with an associated \((\alpha, \beta)\)-derivation \( d \) of \( R \) such that \( \beta(I)d(I) \neq 0 \). If \( (F(x) \circ x)_{\alpha, \beta} = 0 \) for all \( x \in I \), then \( R \) has a nonzero central ideal.

**Proof** By hypothesis

\[
(F(x) \circ x)_{\alpha, \beta} = 0 \quad \text{for all} \quad x \in I, \tag{4.3.30}
\]

Linearizing (4.3.30), we get

\[
(F(x) \circ y)_{\alpha, \beta} + (F(y) \circ x)_{\alpha, \beta} = 0 \quad \text{for all} \quad x, y \in I. \tag{4.3.31}
\]
Substituting $yx$ for $y$ in (4.3.31), we obtain

\begin{equation}
(F(x) \circ y)_{\alpha, \beta} \alpha(x) + (F(y) \circ x)_{\alpha, \beta} \alpha(x) + F(y)[\alpha(x), \alpha(x)] + \\
\beta(y)(d(x) \circ x)_{\alpha, \beta} - [\beta(y), \beta(x)]d(x) = 0 \text{ for all } x, y \in I.
\end{equation}

(4.3.32)

Using (4.3.31), we get

\begin{equation}
\beta(y)(d(x) \circ x)_{\alpha, \beta} - [\beta(y), \beta(x)]d(x) = 0 \text{ for all } x, y \in I.
\end{equation}

(4.3.33)

Substituting $ry$ for $y$ in the above relation and using (4.3.33), we obtain

\begin{equation}
[\beta(r), \beta(x)]\beta(y)d(x) = 0 \text{ for all } x, y \in I \text{ and } r \in R
\end{equation}

(4.3.34)

This is same as (4.3.29). Arguing in the similar as in Theorem 4.3.3, we get the result.

**Theorem 4.3.5.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha, \beta$ be epimorphisms of $R$. Suppose that $F$ is a generalized $(\alpha, \beta)$-derivation with an associated $(\alpha, \beta)$-derivation $d$ of $R$ such that $\beta(I)d(I) \neq 0$. If $F(x)\alpha(x) + \beta(x)d(x) = 0$ for all $x \in I$, then $R$ has a nonzero central ideal.

**Proof** By the hypothesis, we have

\begin{equation}
F(x)\alpha(x) + \beta(x)d(x) = 0 \text{ for all } x \in I.
\end{equation}

(4.3.35)

Linearizing (4.3.35) and using it, we get

\begin{equation}
F(x)\alpha(y) + F(y)\alpha(x) + \beta(x)d(y) + \beta(y)d(x) = 0 \text{ for all } x, y \in I.
\end{equation}

(4.3.36)

Substituting $yx$ for $y$ in (4.3.36), we have

\begin{equation}
F(x)\alpha(y)\alpha(x) + F(y)\alpha(x)\alpha(x) + \beta(y)d(x)\alpha(x) + \beta(x)d(y)\alpha(x) \\
+ \beta(x)\beta(y)d(x) + \beta(y)\beta(x)d(x) = 0 \text{ for all } x, y \in I.
\end{equation}

(4.3.37)

Right multiplying (4.3.36) by $\alpha(x)$ and then subtracting from (4.3.37), we get

\begin{equation}
\beta(x)\beta(y)d(x) + \beta(y)\beta(x)d(x) = 0 \text{ for all } x, y \in I.
\end{equation}

(4.3.38)
Replacing $y$ by $ry$ in (4.3.38), we obtain

$$\beta(x)\beta(r)\beta(y)d(x) + \beta(r)\beta(y)\beta(x)d(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (4.3.39)$$

Using (4.3.38), we get

$$\beta(x)\beta(r)\beta(y)d(x) - \beta(r)\beta(x)\beta(y)d(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (4.3.40)$$

This implies that $\{\beta(r), \beta(x)\} \beta(y)d(x) = 0$ for all $x, y \in I$ and $r \in R$. This is same as (4.3.29). Arguing in the similar manner as in Theorem 4.3.3, we get the required result.
Chapter 5

Generalized biderivations of prime and semiprime rings

5.1 Introduction

In 1980, Maksa [101] introduced the concept of a biderivation. A biadditive mapping $D : R \times R \to R$ is said to be a biderivation if for all $x, y \in R$, the mappings $y \mapsto D(x, y)$ and $x \mapsto D(x, y)$ are derivations of $R$. It was shown in [102] that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mapping is closely connected with the notion of a biderivation. Every commuting additive mapping $f : R \to R$ gives rise to a biderivation on $R$. Linearizing $[f(x), x] = 0$ for all $x \in R$, we get $[f(x), y] = [x, f(y)]$ for all $x, y \in R$ and hence we note that the mapping $(x, y) \mapsto [f(x), y]$ is a biderivation on $R$ (moreover all derivations appearing are inner).

Section 5.2 deals with the study of generalized biderivations of a prime ring. The notion of a generalized biderivation was introduced by Nurcan in [14]. Let $R$ be a ring and $D : R \times R \to R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $y \mapsto D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with function $x \mapsto D(x, y)$ for all $x, y \in R$. The main result of this section
which extends a Theorem of Vukman [138, Theorem 1] is the following: Let $R$ be a prime ring of characteristic not two and $U$ be a noncentral square closed Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be an additive map. If $\Delta$ is a symmetric generalized D-biderivation such that $\Delta$ is commuting on $U$, then $\Delta = 0$ on $U$.

In section 5.3, we study symmetric biderivations of prime rings. In [32] Bell and Daif proved that if a semiprime ring $R$ admits a derivation $d$ such that $xy - d(xy) = yx - d(yx)$ for all $x, y \in R$, then $R$ is commutative. We obtain the result for a biderivation $D$ in case of a one sided ideal of a semiprime ring $R$.

Finally we investigate the commutativity of a semiprime ring $R$ satisfying various identities involving the trace of a symmetric biadditive mapping $D$ of $R$.

5.2

Definition 5.2.1 (Symmetric mapping) Let $R$ be a ring. A mapping $D : R \times R \rightarrow R$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in R$.

Definition 5.2.2 (Biadditive mapping) A mapping $D : R \times R \rightarrow R$ is called biadditive if it is additive in both arguments.

Definition 5.2.3 (Trace) A mapping $f : R \rightarrow R$ defined by $f(x) = D(x, x)$, where $D : R \times R \rightarrow R$ is a symmetric mapping is called the trace of $D$.

Remark 5.2.1

(i) The trace $f$ of $D$ satisfies the relation $f(x+y) = f(x) + f(y) + D(x, y) + D(y, x)$ for all $x, y \in R$.

(ii) If $D$ is symmetric, then the trace $f$ of $D$ satisfies the relation $f(x + y) = f(x) + f(y) + 2D(x, y)$ for all $x, y \in R$. 
Following Makse [101] we define a biderivation as follows:

**Definition 5.2.4 (Biderivation)** A biadditive mapping $D : R \times R \to R$ is said to be a biderivation on a ring $R$ if $D(xy, z) = D(x, z)y + xD(y, z)$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$.

In [14] Nurcan defined generalized biderivation in rings as follows:

**Definition 5.2.5 (Generalized biderivation)** Let $R$ be a ring and $D : R \times R \to R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \to R$ is said to be a generalized $D$-biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with the function $y \mapsto D(x, y)$ for all $x, y \in R$ as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of $R$ associated with the function $x \mapsto D(x, y)$ for all $x, y \in R$ i.e. $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$.

**Example 5.2.1** Let $R$ be a ring and $\lambda \in Z(R)$, the centre of $R$. Then the mapping $(x, y) \mapsto \lambda[x, y]$ is a biderivation on $R$.

**Example 5.2.2** Let $R$ be a ring. If $D$ is any biderivation of $R$ and $\alpha : R \times R \to R$ is a biadditive function such that $\alpha(x, yz) = \alpha(x, y)z$ and $\alpha(xy, z) = \alpha(x, z)y$ for all $x, y, z \in R$, then $D + \alpha$ is a generalized $D$-biderivation of $R$.

In [138] Vukman proved the following result: Let $R$ be a noncommutative prime ring of characteristic not two and $D : R \times R \to R$ be a symmetric biderivation with trace $f$. If $f$ is commuting, then $f = 0$. Further Bresar extended the result and proved that if $D : R \times R$ is a symmetric biderivation of a prime ring $R$ of characteristic not two and $f$ is centralizing on $R$, then $f = 0$. Motivated by the above results we prove the following theorem in the setting of a noncentral square closed Lie ideal of
a prime ring $R$ admitting a symmetric generalized $D$-biderivation.

**Theorem 5.2.1.** Let $R$ be a prime ring of characteristic not two and $U$ be a noncentral square closed Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be an additive map. If $\Delta$ is a symmetric generalized $D$-biderivation on $R$ such that $\Delta$ is commuting on $U$, then $\Delta = 0$ on $U$.

For developing the proof of the above theorem following lemmas are essential.

**Lemma 5.2.1 ([14, Lemma 4.4]).** Let $R$ be a ring and $D : R \times R \rightarrow R$ be an additive map. If $\Delta : R \times R \rightarrow R$ is a generalized $D$-biderivation, then $\Delta(x, y)z[u, v] = [x, y]zD(u, v)$ for all $x, y, z, u, v \in R$.

**Lemma 5.2.2 ([36, Lemma 4]).** Let $R$ be a 2-torsion free prime ring. If $U \not\subseteq Z(R)$ is a Lie ideal of $R$ and $aUb = (0)$, then $a = 0$ or $b = 0$.

**Lemma 5.2.3 ([66, Lemma 1]).** Let $R$ be a semiprime ring and $U$ be a nonzero Lie ideal of $R$. If $[U, U] \subseteq Z(R)$, then $U \subseteq Z(R)$.

**Lemma 5.2.4 ([67, Chapter 1]).** If $R$ is a semiprime ring, then centre of a nonzero one sided ideal is contained in the centre of $R$. As an immediate consequence, any commutative one sided ideal is contained in the centre of $R$.

**Proof of Theorem 5.2.1** By hypothesis

$$[\Delta(x, x), x] = 0 \text{ for all } x \in U. \quad (5.2.1)$$

Linearization of (5.2.1) yields that

$$[\Delta(x, x), x] + [\Delta(x, x), y] + [\Delta(x, y), x] + [\Delta(x, y), y] + [\Delta(y, x), x]$$

$$+ [\Delta(y, x), y] + [\Delta(y, y), x] + [\Delta(y, y), y] = 0 \text{ for all } x, y \in U. \quad (5.2.2)$$
Since $\Delta$ is symmetric, using (5.2.1), we obtain

$$[\Delta(x,x),y] + 2[\Delta(x,y),x] + 2[\Delta(x,y),y] + [\Delta(y,y),x] = 0 \text{ for all } x,y \in U. \ (5.2.3)$$

Substituting $-y$ for $y$ in (5.2.3), we have

$$-[\Delta(x,x),y] - 2[\Delta(x,y),x] + 2[\Delta(x,y),y] + [\Delta(y,y),x] = 0 \text{ for all } x,y \in U. \ (5.2.4)$$

Adding (5.2.3) and (5.2.4) and using char $R \neq 2$, we find

$$2[\Delta(x,y),y] + [\Delta(y,y),x] = 0 \text{ for all } x,y \in U. \ (5.2.5)$$

Replace $x$ by $2zx$ in (5.2.5), we have

$$2\Delta(x,y)[z,y] + 2[\Delta(x,y),y]z + 2x[D(z,y),y] + 2[x,y]D(z,y) + [\Delta(y,y),x]z + x[\Delta(y,y),x] = 0 \text{ for all } x,y,z \in U. \ (5.2.6)$$

In view of (5.2.5), (5.2.6) gives that

$$2\Delta(x,y)[z,y] + 2x[D(z,y),y] + 2[x,y]D(z,y) + x[\Delta(y,y),z] = 0 \text{ for all } x,y,z \in U. \ (5.2.7)$$

Substitute $y$ for $z$, to get

$$2x[D(y,y),y] + 2[x,y]D(y,y) = 0 \text{ for all } x,y \in U. \ (5.2.8)$$

Since char $R \neq 2$, we have

$$x[D(y,y),y] + [x,y]D(y,y) = 0 \text{ for all } x,y \in U. \ (5.2.9)$$

Substituting $2zx$ for $x$ in (5.2.9), using (5.2.9) and char $R \neq 2$, we obtain

$$[z,y]xD(y,y) = 0 \text{ for all } x,y,z \in U. \ (5.2.10)$$

This implies that

$$[z,y]UD(y,y) = 0 \text{ for all } y,z \in U. \ (5.2.11)$$

Lemma 5.2.2 yields that either $[z,y] = 0$ or $D(y,y) = 0$ for all $x,y \in U$. If $[z,y] = 0$ for all $z,y \in U$, then $U \subseteq Z(R)$ by Lemma 5.2.3 which is a contradiction to the
hypothesis. On the other hand, we have $D(y, y) = 0$ for all $x, y \in U$. Linearization of $y$ in the above relation yields that $D(y, z) = 0$ for all $y, z \in L$. Using Lemma 5.2.1 we have $\Delta(x, y) z[u, v] = 0$ for all $x, y, z, u, v \in U$. Again by Lemma 5.2.2 $[u, v] = 0$ for all $u, v \in U$ a contradiction, hence $\Delta(x, y) = 0$ for all $x, y \in U$ i.e. $\Delta = 0$ on $U$.

**Theorem 5.2.2.** Let $R$ be a prime ring of characteristic not two and $U$ be a noncentral square closed Lie ideal of $R$. Let $D : R \times R \rightarrow R$ be an additive map. If $\Delta$ is a symmetric generalized $D$-biderivation such that $\Delta(x, y) + [x, y] \in Z(R)$ for all $x, y \in U$, then $\Delta = 0$ on $U$.

**Proof** Let

$$\Delta(x, y) - [x, y] \in Z(R) \text{ for all } x, y \in U. \tag{5.2.12}$$

Replace $y$ by $2yw$, we get

$$2(\Delta(x, y)w + yD(x, w) - [x, y]w - y[x, w]) \in Z(R) \text{ for all } x, y, w \in U. \tag{5.2.13}$$

Since $\text{char } R \neq 2$, we have

$$\Delta(x, y)w + yD(x, w) - [x, y]w - y[x, w] \in Z(R) \text{ for all } x, y, w \in U. \tag{5.2.14}$$

Commuting (5.2.14) with $w$, to get

$$[yD(x, w), w] - [y[x, w], w] = 0 \text{ for all } x, y, w \in U. \tag{5.2.15}$$

Substitute $w$ for $x$, we obtain

$$[yD(w, w), w] = 0 \text{ for all } y, w \in U. \tag{5.2.16}$$

This implies that

$$y[D(w, w), w] - [y, w]D(w, w) = 0 \text{ for all } y, w \in U. \tag{5.2.17}$$
Substitute $2xy$ for $y$ in (5.2.17) and use (5.2.17), we get

$$[y, w] z D(w, w) = 0 \text{ for all } y, z, w \in U. \quad (5.2.18)$$

Arguing in the similar manner as in the proof of Theorem 5.2.1, we get the result.

Similarly we can proof the case $\Delta(x, y) + [x, y] \in Z(R)$ for all $x, y \in U$.

In 1992, Bell and Daif [32] proved that if $R$ is a semiprime ring admitting a derivation $d$ such that $xy - d(xy) = yx - d(yx)$ for all $x, y \in R$, then $R$ is commutative. We obtain the result for a biderivation of a semiprime ring $R$ in case of a one sided ideal of $R$.

**Theorem 5.2.3.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Suppose that there exists a symmetric biderivation $D(., .) : R \times R \rightarrow R$ such that $[x, y] - f(xy) + f(yx) = 0$ for all $x, y \in L$, where $f$ is the trace of $D$. Then $L \subseteq Z(R)$.

**Proof**  By hypothesis

$$[x, y] - f(xy) + f(yx) = 0 \text{ for all } x, y \in L. \text{ This can be rewritten as}$$

$$[x, y] = [x^2, f(y)] + [f(x), y^2] + 2xD(x, y)y - 2yD(x, y)x \text{ for all } x, y \in L. \quad (5.2.19)$$

Replace $x$ by $x + y$ in (5.2.19), to get

$$[x, y] = [x^2, f(y)] + [xy, f(y)] + [yx, f(y)] + [f(x), y^2] + 2[D(x, y), y^2]$$

$$+ 2xD(x, y)y + 2xf(y)y - 2yD(x, y)x - 2yf(y)x \text{ for all } x, y \in L. \quad (5.2.20)$$

Using (5.2.19) and (5.2.20), we obtain

$$0 = [xy, f(y)] + [yx, f(y)] + 2[D(x, y), y^2] + 2xf(y)y - 2yf(y)x \text{ for all } x, y \in L. \quad (5.2.21)$$

Replacing $y$ by $x + y$ in (5.2.21) and using (5.2.21), we have

$$2([x^2, f(y)] + [f(x), y^2] + 2xD(x, y)y - 2yD(x, y)x) = 0 \text{ for all } x, y \in L. \quad (5.2.22)$$
Since $R$ is 2-torsion free it implies that

$$[x^2, f(y)] + [f(x), y^2] + 2x D(x, y)y - 2y D(x, y)x = 0 \text{ for all } x, y \in L. \quad (5.2.23)$$

Comparing (5.2.23) and (5.2.19), we obtain $[x, y] = 0$ for all $x, y \in L$. Hence $L \subseteq Z(R)$ by Lemma 5.2.4

Using similar techniques as we have used in the proof of Theorem 5.2.3, we can prove the following:

**Theorem 5.2.4.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Suppose that there exists a symmetric biderivation $D(., .) : R \times R \rightarrow R$ such that $[x, y] = f(xy) - f(x) - f(y)$ for all $x, y \in L$, where $f$ is the trace of $D$. Then $L \subseteq Z(R)$.

**Theorem 5.2.5.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Suppose that there exists a symmetric biderivation $D(., .) : R \times R \rightarrow R$ such that $xy - D(x, x) = yx - D(y, y)$ (or $xy + D(x, x) = yx + D(y, y)$) for all $x, y \in L$. Then $L \subseteq Z(R)$.

**Proof.** Let

$$xy - D(x, x) = yx - D(y, y) \text{ for all } x, y \in L.$$

This can be rewritten as $[x, y] = f(x) - f(y)$, where $f$ is the trace of $D$. Replacing $x$ by $x + y$, we get

$$[x, y] = f(x) + 2D(x, y) = 0 \text{ for all } x, y \in L. \quad (5.2.24)$$

Now substituting $yz$ for $y$ in (5.2.24), we obtain

$$[x, y]z + y[x, z] = f(x) + 2D(x, y)z + 2yD(x, z) = 0 \text{ for all } x, y, z \in L. \quad (5.2.25)$$
Replacing \( x \) by \(-x\) in (5.2.25), we find

\[-[x,y]z - y[x,z] = f(x) - 2D(x,y)z - 2yD(x,z) = 0 \text{ for all } x, y, z \in L. \tag{5.2.26}\]

Comparing (5.2.25) and (5.2.26), we get \( 2f(x) = 0 \) for all \( x \in L \). Since \( R \) is 2-torsion free, we have \( f(x) = 0 \) for all \( x \in L \). Replacing \( x \) by \( x + y \) and using 2-torsion freeness of \( R \), we get \( D(x,y) = 0 \) for all \( x, y \in L \). Using (5.2.24), we obtain \([x,y] = 0 \) for all \( x, y \in L \). Hence \( L \subseteq Z(R) \) by Lemma 5.2.4

Similarly we can prove the case if \( xy + D(x,x) = yx + D(y,y) \) for all \( x, y \in L \).

**Theorem 5.2.6.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Suppose that there exists a symmetric biderivation \( D(\cdot, \cdot) : R \times R \to R \) such that \( yx - D(x,x) = xy - D(y,y) \) (or \( yx + D(x,x) = xy + D(y,y) \)) for all \( x, y \in L \). Then \( L \subseteq Z(R) \).

**Proof** The proof runs on the same parallel lines as of Theorem 5.2.5.

### 5.3

Bell and Martindale [34] proved that if a semiprime ring \( R \) admits a derivation \( d \) which is nonzero on a nonzero left ideal \( I \) of \( R \) and centralizing on \( I \), then \( R \) must contain a nonzero central ideal. Recently Ashraf, Asma and Shakir [23] explored the commutativity of a prime ring \( R \) admitting a generalized derivation \( F \) satisfying one of the following properties: (i) \( F(xy) \neq xy \in Z(R) \), (ii) \( F(xy) \neq xy \in Z(R) \); and (iii) \( F(x)F(y) \neq xy \in Z(R) \) for all \( x, y \in R \). Now we prove the following:

**Theorem 5.3.1.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(\cdot, \cdot) : R \times R \to R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(xy) \neq [x,y] \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).
Proof. Let

\[ f(xy) - [x, y] \in Z(R) \text{ for all } x, y, \in L. \]  \hspace{1cm} (5.3.1)

Replacing \( y \) by \( y + z \) in (5.3.1), we get

\[ f(xy) + f(xz) + 2D(xy, xz) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \]  \hspace{1cm} (5.3.2)

Since \( R \) is 2-torsion free, (5.3.2) yields that

\[ D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \]  \hspace{1cm} (5.3.3)

Substituting \( y \) for \( z \) in (5.3.3), we get

\[ f(xy) = D(xy, xy) \in Z(R) \text{ for all } x, y \in L. \]  \hspace{1cm} (5.3.4)

In view of (5.3.1), (5.3.4) yields that

\[ [x, y] \in Z(R) \text{ for all } x, y \in L. \]  \hspace{1cm} (5.3.5)

Then

\[ [[x, y], r] = 0 \text{ for all } x, y, \in L, r \in R. \]  \hspace{1cm} (5.3.6)

Replace \( x \) by \( xy \) in (5.3.6), to get

\[ [[x, y], r] = 0 \text{ for all } x, y, r \in L. \]  \hspace{1cm} (5.3.7)

This implies that

\[ [x, y][y, r] = 0 \text{ for all } x, y, r \in L. \]  \hspace{1cm} (5.3.8)

Replacing \( r \) by \( rx \) in (5.3.8), we get

\[ [x, y][y, x] = 0 \text{ for all } x, y, r \in L. \]  \hspace{1cm} (5.3.9)

This implies that

\[ [x, y][x, y] = 0 \text{ for all } x, y \in L. \]  \hspace{1cm} (5.3.10)

Since \( R \) is semiprime, we get \( [x, y] = 0 \) for all \( x, y, L \), Hence \( L \subseteq Z(R) \) by Lemma 5.2.4. The proof is same for the case \( f(xy) + [x, y] \in Z(R) \) for all \( x, y \in L \).
Theorem 5.3.2. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) + [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof The proof runs on the same parallel lines as of Theorem 5.3.1.

Theorem 5.3.3. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof Let

$$f(xy) - xy \in Z(R) \text{ for all } x, y \in L. \quad (5.3.11)$$

Replacing $y$ by $y + z$, we get

$$f(xy) + f(xz) + 2D(xy, xz) - xy - xz \in Z(R) \text{ for all } x, y, z \in L. \quad (5.3.12)$$

Comparing (5.3.11) and (5.3.12) we obtain

$$2D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.3.13)$$

Since $R$ is 2-torsion free, we have

$$D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.3.14)$$

Substituting $y$ for $z$ in (5.3.14), we get

$$f(xy) = D(xy, xy) \in Z(R) \text{ for all } x, y \in L. \quad (5.3.15)$$

Using (5.3.11), we have $xy \in Z(R)$ for all $x, y \in L$. This implies that $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 5.3.1, we get the result.

The proof is same for the case $f(xy) + xy \in Z(R)$ for all $x, y \in L$. 

Theorem 5.3.4. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(.,.) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) = yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. The proof runs on the same parallel lines as of Theorem 5.3.3.

Theorem 5.3.5. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(.,.) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x,y]) = [x,y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. Let

$$f([x,y]) - [x,y] \in Z(R) \text{ for all } x, y \in L. \quad (5.3.16)$$

Replacing $y$ by $y+z$, we have

$$f([x,y] + [x,z]) - [x,y] - [x,z] \in Z(R) \text{ i.e. } f([x,y]) + f([x,z]) + 2D([x,y],[x,z]) - [x,y] - [x,z] \in Z(R) \text{ for all } x, y, z \in L. \quad (5.3.17)$$

Using (5.3.16), we get

$$2D([x,y],[x,z]) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.3.17)$$

Substituting $y$ for $z$ in (5.3.17) and using the fact that $R$ is 2-torsion free, we find

$$f([x,y]) \in Z(R) \text{ for all } x, y \in L. \quad (5.3.18)$$

Using (5.3.16) and (5.3.18), we obtain $[x,y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in Theorem 5.3.1, we get the result.

Similarly one can prove the result if $f([x,y]) + [x,y] \in Z(R)$ for all $x, y \in L$.

Theorem 5.3.6. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(.,.) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x,y]) \neq [y,x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$. 


Proof  The proof runs on the same parallel lines as of Theorem 5.3.6.

Theorem 5.3.7. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) = xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof  Let

$$f([x, y]) - xy \in Z(R) \quad \text{for all } x, y \in L. \tag{5.3.19}$$

Replacing $y$ by $y+z$ in (5.3.19), we have $f([x, y]+[x, z]) - xy - xz \in Z(R)$ for all $x, y, z \in L$. This implies that

$$f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - xy - xz \in Z(R) \quad \text{for all } x, y, z \in L \tag{5.3.20}$$

Using (5.3.19), we obtain

$$2D([x, y], [x, z]) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{5.3.21}$$

Since $R$ is 2-torsion free, (5.3.21) yields that

$$D([x, y], [x, z]) \in Z(R) \quad \text{for all } x, y, z \in L. \tag{5.3.22}$$

In particular, if we substitute $y$ for $z$ in (5.3.22), then we have $f([x, y]) \in Z(R)$ for all $x, y \in L$. Again using (5.3.19), we get $xy \in Z(R)$ for all $x, y \in L$. This implies that $[x, y] \in Z(R)$. Arguing in the similar manner as in Theorem 5.3.1, we get the result.

Similarly we can prove the result if $f([x, y]) + xy \in Z(R)$ for all $x, y \in L$.

Theorem 5.3.8. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) = yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$. 
Proof The proof runs on the same parallel lines as of Theorem 5.3.7.

Theorem 5.3.9. Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(\cdot, \cdot) : R \times R \to R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(xy) \equiv f(x) \equiv [x, y] \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

Proof Let

\[
f(xy) - f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{5.3.23}
\]

Replacing \( y \) by \( y + z \), we get \( f(xy) + f(xz) + 2D(xy, xz) - f(x) - [x, y] - [x, z] \in Z(R) \) for all \( x, y, z \in L \). Using (5.3.23), we obtain

\[
f(xz) + 2D(xy, xz) - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{5.3.24}
\]

Substituting \(-z\) for \( z \) in (5.3.24), we get

\[
f(xz) - 2D(xy, xz) + [x, z] \in Z(R) \text{ for all } x, y, z \in L. \tag{5.3.25}
\]

Adding (5.3.24) and (5.3.25), we obtain

\[
2f(xz) \in Z(R) \text{ for all } x, y, z \in L. \tag{5.3.26}
\]

Since \( R \) is 2-torsion free, we have \( f(xy) \in Z(R) \) for all \( x, y \in L \). Using (5.3.23), we get

\[
f(x) - [x, y] \in Z(R) \text{ for all } x, y \in L. \tag{5.3.27}
\]

Replacing \( x \) by \( x + z \), in (5.3.27), we have

\[
f(x) + f(x) + 2D(x, z) - [x, y] - [z, y] \in Z(R) \text{ for all } x, z \in L. \tag{5.3.28}
\]

Again by (5.3.27) and using 2-torsion freeness of \( R \), we find \( D(x, z) \in Z(R) \). In particular \( f(x) = D(x, x) \in Z(R) \) for all \( x \in L \). Since \( f(xz) \in Z(R) \) and \( f(x) \in Z(R) \), we have \( f(xy) - f(x) \in Z(R) \) for all \( x, y \in L \). Using (5.3.23), we get \([x, y] \in Z(R) \) for all \( x, y \in L \). Arguing in the similar manner as in Theorem 5.3.1, we get the result. Similarly we can prove the result if \( f(xy) + f(z) + [x, y] \in Z(R) \) for all \( x, y \in L \).
Theorem 5.3.10. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(.,.) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(xy) + f(y) = [x,y] \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.

Proof Let

$$f(xy) - f(y) - [x,y] \in Z(R) \text{ for all } x,y \in L. \quad (5.3.29)$$

Replacing $y$ by $y + z$, we have $f(xy) + f(xz) + 2D(xy,xz) - f(y) - f(x) - 2D(y,z) - [x,y] - [x,z] \in Z(R)$ for all $x,y,z \in L$. Using (5.3.29), we get

$$2(D(xy,xz) - D(y,z)) \in Z(R) \text{ for all } x,y,z \in L. \quad (5.3.30)$$

Substituting $y$ for $z$ in (5.3.30) and using the fact that $R$ is 2-torsion free, we find

$$f(xy) - f(y) \in Z(R) \text{ for all } x,y \in L. \quad (5.3.31)$$

This implies that $[x,y] \in Z(R)$ for all $x,y \in L$. Arguing in the similar manner as in the proof Theorem 5.3.1, we get the result.

Similarly we can prove the result if $f(xy) + f(y) + [x,y] \in Z(R)$ for all $x,y \in L$.

5.4

Theorem 5.4.1. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(.,.) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x,y]) + f(x) = [x,y] \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.

Proof Suppose

$$f([x,y]) - f(x) - [x,y] \in Z(R) \text{ for all } x,y \in L. \quad (5.4.1)$$

Replacing $x$ by $x + z$ in (5.4.1), we obtain

$$f([x,y]) + f([x,y]) + 2D([x,y],[z,y]) - f(x) - f(z)$$

$$-2D(x,z) - [x,y] - [z,y] \in Z(R) \text{ for all } x,y,z \in L. \quad (5.4.2)$$
Using (5.4.1), we have

\[ 2(D([x, y], [z, y]) - D(x, z)) \in Z(R) \text{ for all } x, y, z \in L. \]  

(5.4.3)

Substituting \( x \) for \( z \) in (5.4.3) and using the fact that \( R \) is 2-torsion free, we obtain

\[ f([x, y]) - f(x) \in Z(R) \text{ for all } x, y \in L. \]  

(5.4.4)

Again using (5.4.1) and (5.4.5), we have \([x, y] \in Z(R)\) for all \(x, y \in L\). Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

Similarly we can prove the result if \( f([x, y]) + f(x) + [x, y] \in Z(R) \) for all \(x, y \in L\).

**Theorem 5.4.2.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(., .) : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f([x, y]) = f(y) + [x, y] \in Z(R) \) for all \(x, y \in L\), then \( L \subseteq Z(R) \).

**Proof** Suppose

\[ f([x, y]) - f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L. \]  

(5.4.5)

Replacing \( y \) by \( y + z \), we get

\[ f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) - f(y) - f(z) - 2D(y, z) \]
\[ - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \]  

(5.4.6)

In view of (5.4.5) and (5.4.6) yields that

\[ 2(D([x, y], [x, z]) - D(y, z)) \in Z(R) \text{ for all } x, y, z \in L. \]  

(5.4.7)

Substituting \( y \) for \( z \) in (5.4.7) and using the fact that \( R \) is 2-torsion free, we obtain

\[ f([x, y]) - f(y) = D([x, y], [x, y]) - D(y, y)) \in Z(R) \text{ for all } x, y \in L. \]  

(5.4.8)

Using (5.4.5) and (5.4.8), we have \([x, y] \in Z(R)\) for all \(x, y \in L\). Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

Similarly we can prove the Theorem if \( f([x, y]) + f(y) + [x, y] \in Z(R) \) for all \(x, y \in L\).
Using the similar techniques as we have used in the proof of Theorem 5.4.1 and 5.4.2, we can prove the following:

**Theorem 5.4.3.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) + f(x) + f(y) = [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Theorem 5.4.4.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) + f(y) + f(x) = [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Theorem 5.4.5.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) + f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof.** Let

$$f([x, y]) - f(xy) - [x, y] \in Z(R) \text{ for all } x, y \in L. \quad (5.4.9)$$

Replacing $y$ by $y + z$ in (5.4.9), we get

$$f([x, y]) + f([x, z]) + 2D([x, y], [x, z]) = f(xy) - f(xz) - 2D(xy, xz) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.10)$$

Using (5.4.9) and (5.4.10), we obtain

$$2D([x, y], [x, z]) - D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.11)$$

Since $R$ is 2-torsion free, we have

$$D([x, y], [x, z]) - D(xy, xz) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.12)$$
Substituting $y$ for $z$ in (5.4.12), we get

$$f([x, y]) - f(xy) \in Z(R) \text{ for all } x, y \in L.$$  \hspace{1cm} (5.4.13)

Using (5.4.9), we have $[x, y] \in Z(R)$ for all $x, y \in L$. Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

The proof is same for the case $f([x, y]) + f(xy) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 5.4.6.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f([x, y]) \equiv f(xy) \equiv [y, x] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** The proof runs on the same parallel lines as that of Theorem 5.4.5.

**Theorem 5.4.7.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x)f(y) \equiv [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose

$$f(x)f(y) - [x, y] \in Z(R) \text{ for all } x, y \in L.$$  \hspace{1cm} (5.4.14)

Substituting $y + z$ for $y$ in (5.4.14), we have

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - [x, y] - [x, z] \in Z(R) \text{ for all } x, y, z \in L.$$  \hspace{1cm} (5.4.15)

Using (5.4.14), we find

$$2f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L.$$  \hspace{1cm} (5.4.16)

Since $R$ is 2-torsion free, we have

$$f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L.$$  \hspace{1cm} (5.4.17)
In particular if we replace \( z \) by \( y \) in (5.4.17), then

\[
f(x)f(y) \in Z(R) \text{ for all } x, y \in L. \tag{5.4.18}
\]

Hence using (5.4.18) and (5.4.14), we obtain \([x, y] \in Z(R)\) for all \( x, y \in L \). Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

Similarly we can prove the case if \( f(x)f(y) + [x, y] \in Z(R) \) for all \( x, y \in L \).

**Theorem 5.4.8.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(\cdot, \cdot) : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x)f(y) + [y, x] \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** The proof runs on the same parallel lines as that of Theorem 5.4.7.

**Theorem 5.4.9.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(\cdot, \cdot) : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x)f(y) + xy \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** Let

\[
f(x)f(y) - xy \in Z(R) \text{ for all } x, y \in L. \tag{5.4.19}
\]

Substituting \( y + z \) for \( y \) in (5.4.19), we have

\[
f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - xy - xz \in Z(R) \tag{5.4.20}
\]

for all \( x, y, z \in L \).

Applying (5.4.19), we find

\[2f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L. \tag{5.4.21}\]

Since \( R \) is 2-torsion free, we have

\[
f(x)D(y, z) \in Z(R) \text{ for all } x, y, z \in L \tag{5.4.22}\]
In particular replacing \( z \) by \( y \) in (5.4.22) and using (5.4.19), we find

\[
f(x)f(y) \in Z(R) \text{ for all } x, y \in L. \tag{5.4.23}
\]

This implies that \( xy \in Z(R) \) for all \( x, y \in L \). Arguing in the similar manner as we have done in the proof of Theorem 5.3.1, we get the result.

Similarly we can prove the case if \( f(x)f(y) + xy \in Z(R) \) for all \( x, y \in L \).

**Theorem 5.4.10.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(.,.) : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x)f(y) \oplus fy \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** The proof runs on the same parallel lines as that of Theorem 5.4.9.

**Theorem 5.4.11.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(.,.) : R \times R \rightarrow R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x) \circ f(y) \oplus [x,y] \in Z(R) \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** Suppose

\[
f(x) \circ f(y) - [x,y] \in Z(R) \text{ for all } x, y \in L. \tag{5.4.24}
\]

Replacing \( y \) by \( y + z \) in (5.4.24), we get

\[
f(x) \circ f(y) + f(x) \circ f(z) + 2(f(x) \circ D(y,z)) - [x,y] - [x,z] \in Z(R) \text{ for all } x, y \in L. \tag{5.4.25}
\]

Comparing (5.4.24) and (5.4.25), we have

\[2(f(x) \circ D(y,z)) \in Z(R) \text{ for all } x, y, z \in L.
\]

Since \( R \) is 2-torsion free, we find

\[f(x) \circ D(y,z) \in Z(R) \text{ for all } x, y, z \in L. \tag{5.4.26}
\]
Replacing $y$ by $z$ in (5.4.26), we get

$$f(x) \circ f(y) \in Z(R) \text{ for all } x, y, z \in L.$$  \hspace{1cm} (5.4.27)

From (5.4.24) and (5.4.27), we have

$$[x, y] \in Z(R) \text{ for all } x, y \in L.$$

Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

Similarly we can prove the case if $f(x) \circ f(y) + [x, y] \in Z(R)$ for all $x, y \in L$.

**Theorem 5.4.12.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \to R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x) \circ f(y) = xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** Suppose

$$f(x) \circ f(y) - xy \in Z(R) \text{ for all } x, y \in L.$$  \hspace{1cm} (5.4.28)

Replacing $y$ by $y + z$, in (5.4.28), we have

$$f(x) \circ f(y) + f(x) \circ f(z) + 2(f(x) \circ D(y, z)) - xy - xz \in Z(R)$$

for all $x, y, z \in L$. \hspace{1cm} (5.4.29)

Comparing (5.4.28) and (5.4.29), we have

$$2(f(x) \circ D(y, z)) \in Z(R) \text{ for all } x, y, z \in L.$$  \hspace{1cm} (5.4.30)

Substitute $y$ for $z$ in (5.4.30) and using 2-torsion freeness of $R$, we get

$$f(x) \circ f(y) \in Z(R) \text{ for all } x, y \in L.$$  \hspace{1cm} (5.4.31)

Using (5.4.28) and (5.4.31), we obtain

$$xy \in Z(R) \text{ for all } x, y \in L.$$  \hspace{1cm} (5.4.32)
Interchanging the role of $x$ and $y$ in (5.4.32) and subtracting from (5.4.32), we find

$$[x, y] \in Z(R) \text{ for all } x, y \in L. \quad (5.4.33)$$

Arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.

The prove is same for the case $f(x) \circ f(y) + xy \in Z(R)$ for all $x, y \in L$.

**Theorem 5.4.13.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x) \circ f(y) + xy \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**Proof** The proof runs on the same parallel lines as of Theorem 5.4.12.

**Theorem 5.4.14.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $f(x)f(y) + x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

**proof** Suppose

$$f(x)f(y) - x \circ y = 0 \text{ for all } x, y \in L. \quad (5.4.34)$$

Replacing $y$ by $y + z$ in (5.4.34), we get

$$f(x)f(y) + f(x)f(z) + 2f(x)D(y, z) - x \circ y - x \circ z = 0 \text{ for all } x, y, z \in L. \quad (5.4.35)$$

From (5.4.34) and (5.4.35), we have

$$2f(x)D(y, z) = 0 \text{ for all } x, y, z \in L. \quad (5.4.36)$$

Using 2-torsion freeness of $R$ and replacing $y$ by $z$ in (5.4.36), we get

$$f(x)f(y) = 0 \text{ for all } x, y \in L. \quad (5.4.37)$$
Using (5.4.37) and (5.4.34), we have

\[ xy + yx = 0 \text{ for all } x, y \in L. \]  \hspace{1cm} (5.4.38)

Replace \( y \) by \( ry \) in (5.4.38) and using (5.4.38), we get

\[ [x, r]y = 0 \text{ for all } x, y \in L, r \in R. \]  \hspace{1cm} (5.4.39)

A simple calculation yields that \([x, r]R[x, r] = 0\) for all \( x, y \in L, r \in R \). Since \( R \) is semiprime, we have \([x, r] = 0\) for all \( x \in L, r \in R \). Hence \( L \subseteq Z(R) \).

Similarly we can prove the case if \( f(x)f(y) + x \circ y \in Z(R) \) for all \( x, y \in L \).

**Theorem 5.4.15.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a nonzero left ideal of \( R \). Let \( D(\cdot, \cdot) : R \times R \to R \) be a symmetric biadditive mapping and \( f \) be the trace of \( D \). If \( f(x) \circ f(y) + x \circ y = 0 \) for all \( x, y \in L \), then \( L \subseteq Z(R) \).

**Proof** Suppose

\[ f(x) \circ f(y) - x \circ y = 0 \text{ for all } x, y \in L. \]  \hspace{1cm} (5.4.40)

Replace \( y \) by \( y + z \) in (5.4.40), we have

\[ f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ D(y, z) - x \circ y - x \circ z = 0 \]

for all \( x, y, z \in L \).

Comparing (5.4.40) and (5.4.41), we get

\[ 2f(x) \circ D(y, z) = 0 \text{ for all } x, y, z \in L. \]  \hspace{1cm} (5.4.42)

Using 2-torsion freeness of \( R \) and replacing \( y \) by \( z \) in (5.4.42), we obtain

\[ f(x) \circ f(y) = 0 \text{ for all } x, y \in L. \]  \hspace{1cm} (5.4.43)

Using (5.4.43) and (5.4.40), we have

\[ x \circ y = 0 \text{ for all } x, y \in L. \]  \hspace{1cm} (5.4.44)

Arguing in the similar manner as in the proof of Theorem 5.4.14, we get the result.
Theorem 5.4.16. Let $R$ be a 2-torsion free semiprime ring and $L$ be a nonzero left ideal of $R$. Let $D(\cdot, \cdot) : R \times R \rightarrow R$ be a symmetric biadditive mapping and $f$ be the trace of $D$. If $[x, y] - f(xy) + f(yx) \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. By hypothesis

$$[x, y] - f(xy) + f(yx) \in Z(R) \text{ for all } x, y \in L. \quad (5.4.45)$$

Replacing $y$ by $y + z$ in (5.4.45), we get

$$[x, y] + [x, z] - f(xy) - f(yz) - 2D(xy, xz) + f(yx) + f(zz) + 2D(yx, zx) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.46)$$

This implies that

$$-2D(xy, xz) + 2D(yx, zx) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.47)$$

Since $R$ is 2-torsion free, we have

$$-D(xy, xz) + D(yx, zx) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.4.48)$$

Replacing $z$ by $y$ in (5.4.48), we get

$$-f(xy) + f(yx) \in Z(R) \text{ for all } x, y \in L. \quad (5.4.49)$$

Comparing (5.4.45) and (5.4.49), we get $[x, y] \in Z(R)$ and arguing in the similar manner as in the proof of Theorem 5.3.1, we get the result.
Bibliography


[70] Herstein, I. N., Center-like elements in prime rings, J. Algebra 60 (1979), 567-574.


