SOME TOPICS ON ORIENTED GRAPHS

ABSTRACT

THESIS

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy in Applied Mathematics

By Madhukar Sharma

Under the Supervision of Dr. Merajuddin

DEPARTMENT OF APPLIED MATHEMATICS
Z.H. COLLEGE OF ENGINEERING & TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2006
Tournaments are special classes of oriented graphs. Enormous amount of work has been done on this topic. Perhaps the first theorem on tournaments is the fact that every tournament contains a Hamiltonian path, and second theorem is the fact that a tournament is strongly connected if and only if every vertex is contained in cycles of all possible lengths. Another result, just as basic as these above two, gives a necessary and sufficient condition for a sequence of integers to be the score sequence of some tournament. This result is known as Landau's Theorem. Landau's Theorem attracted many researchers as nearly a dozen different proofs appeared in literature. Many of these existing proofs are discussed in the survey by Reid [4] in 1996. Brualdi and Shen [1] proved that the score sequence of a tournament satisfies a set of inequalities
which are stronger than the well known set of inequalities of Landau [3].

Degree imbalance sequences of simple digraphs have been studied by Dhruv et al. [2]. They gave the concept of imbalance of a vertex and found a necessary and sufficient condition for a sequence of integers to be an imbalance sequence.

Hypertournaments are generalization of tournaments. Hypertournaments have been studied by a number of authors [4]. These authors raised the problem of extending the most important results on tournaments to hypertournaments. Instead of scores of vertices in tournament, Zhou et al. [5] considered scores and losing scores of vertices in a $k$-hypertournament. Zhou et al. [5] also found a necessary and sufficient condition for a sequence of nonnegative integers to be score and losing score sequence of a $k$-
hypertournament, which is a generalization of Landau’s Theorem [3] on tournaments to $k$-hypertournaments.

Chapter 1 provides the necessary ground to understand the contents presented in the subsequent sections.

In chapter 2 we study strong score sequences and self-converse strong score sequences of tournaments with repetitions. We mention some known results on the score sequences and a brief review of self-converse, regular, near-regular score sequences. We define score sequences with repetitions. The strong score sequences with a single repetition of length four are characterized and the total number of strong score sequences with a single repetition is obtained. We characterize the self-converse strong score sequences with a single repetition of length three and five and report the number of self-converse strong score sequences with a single repetition.
In chapter 3 we study some properties of imbalance sequences in oriented graphs. We characterize irreducible imbalance sequences of oriented graphs and bounds for imbalance $b_i$ of a vertex $v_i$ of an oriented graph are found. In the last we report a result of an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

In chapter 4 we obtain some stronger inequalities for degree imbalances in oriented graphs. We give a different proof of Theorem 3.8. In the last we investigate occurrence of equalities in inequalities (4.1) given in Theorem 4.1.

In the last chapter 5 we study properties of score, losing score and total score sequences of $k$-hypertournaments. We find some stronger inequalities for score and losing score sequences of $k$-hypertournaments. We report bounds for scores and
losing scores. In the last, we discuss total score sequences of \( k \)-hypertournaments and characterize total score sequences of strong \( k \)-hypertournaments.

References


SOME TOPICS ON ORIENTED GRAPHS

THESIS
SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy in Applied Mathematics

By
Madhukar Sharma

Under the Supervision of
Dr. Merajuddin

DEPARTMENT OF APPLIED MATHEMATICS
Z.H. COLLEGE OF ENGINEERING & TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)
2006
Dedicated to my
Father
Late Swami Kartikaya Sharma
Certificate

Certified that the thesis entitled "Some topics on oriented graphs" being submitted by Mr. Madhukar Sharma, in partial fulfilment of the requirement for the award of the degree of Doctor of Philosophy, is a record of his own work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted for the award of any other degree or diploma.

Chairman

supervisor
ACKNOWLEDGEMENTS

First and foremost, I render infinite thanks to God, Who sustains me and gave me the strength, determination and discipline to complete the thesis.

Dr. Merajuddin has been more than a thesis supervisor to me. I express my deep sense of gratitude to him for his constant encouragement and excellent supervision. His valuable suggestions and active patience throughout the thesis work have helped in improving the presentation of the thesis. I am very thankful to him for building in me a right kind of attitude towards research. His warmth, friendliness, and fatherly affection have undoubtedly made my association with him a cherished phase in my life.

The present dissertation is nothing but an encapsulation of my parents blessing who have been a source of constant inspiration to me at every stage during my research work, without their love and blessing this work could not be completed.

I have had a good fortune to work with my colleagues and friends Parvez Ali, Syed Ajaz Kareem Kirmani, Naseem Ahamed Khan, Dr. Dinesh Singh, Dr. Dharavendra Singh and Saket Dwivedi.
It was a pleasure to work with them and I thank them all for having shared my experience and thoughts throughout the last several years.

Indeed, merely mentioning names cannot pay thanks but I am grateful to the entire members of Applied Mathematics Department, AMU, Aligarh.

I am also thankful to Science Library staff, J.N.U. Delhi and I.I.T. Kanpur Library staff for their co-operation in complementing this task.

Last but certainly not least, I would like to thank the Aligarh Muslim University, Aligarh for providing me all the facilities.

MADHUKAR SHARMA
(MADHUKAR SHARMA)
CONTENTS

Chapter 1 Introduction .............................................. 1-11

1.1 General introduction ............................................. 1
1.2 Synopsis of the thesis ............................................. 5
1.3 Basic definitions .................................................. 7
Notations .................................................................. 11

Chapter 2 Strong score sequences with single repetitions ........ 12-52

2.1 Introduction .......................................................... 12
2.2 Strong score sequences with single repetition ................. 15
2.3 Self-converse strong score sequences with single repetition ......................................................... 29
2.4 Self-converse tournament score sequences having two consecutive repetitions 49

Chapter 3 On imbalance sequences of oriented graphs ............. 53-77

3.1 Introduction .......................................................... 53
3.2 Necessary and sufficient condition ................................ 55
3.3 Construction of an oriented graph with a given imbalance sequence ........................................ 57
3.4 Irreducible imbalance sequences of oriented graphs... 59
3.5 The bounds of imbalances................................. 66
3.6 Lexicographic enumeration of imbalance sequences... 68
3.7 Self-converse Imbalance Sequences....................... 73

Chapter 4 Some Inequalities for imbalances in oriented graphs 78-92
4.1 Some stronger inequalities for imbalances................. 78
4.2 Equality in stronger inequalities.......................... 86

Chapter 5 On score sequences of k-hypertournaments 93-114
5.1 Introduction.................................................... 94
5.2 Score and losing score sequences of k-hypertournaments.......................... 95
5.3 Existence of score and losing score sequences of k-hypertournaments.......................... 96
5.4 Some stronger inequalities for scores and losing scores in k-hypertournaments.......................... 99
5.5 Total score sequences of k-hypertournaments.......... 110

References 115-121
CHAPTER ONE
CHAPTER 1

INTRODUCTION

1.1 General Introduction

There are many interesting areas of research in directed graphs. One such in the field of directed graphs is the tournament. The tournament theory is one of the richest theories in oriented graphs.

Upto 1965, much research work was carried out on tournaments but the results were all scattered. Harary et al. [29] in 1965 for the first time made a systematic study of the tournaments and presented the known results in their book [29]. After a year only Harary et al. [28] published a paper containing some more new results along with a bibliography. The pioneering work on tournaments has been reported by Moon [39] in his book "Topics on Tournaments", which was published in 1968. A very useful and extensive survey as well as bibliography (containing 95 references) of the work on tournaments has been published by Reid and Beineke [45], in the book "Selected Topics in Graph Theory". A short,
specialized survey appeared in 1981 by Beineke [7]. Many results on the scores of tournaments along with a bibliography can be found in Merajuddin [38].

The recent general survey as well as bibliography by the Reid [43] (containing 170 references) has been published in 1996, which covers the development of the past 43 years from 1953 to 1996. Finally, the most recent list of publication connected with tournaments are given by Antel [2], which gives a year-wise list of publications containing 218 references. For the survey and bibliographies we refer to Reid [43] and Antel [2].

Perhaps the first theorem on tournaments is the fact that every tournament contains a Hamiltonian path, and second theorem is the fact that a tournament is strongly connected if and only if every vertex is contained in cycles of all possible lengths. Another result, just as basic as these above two, gives a necessary and sufficient condition for a sequence of integers to be the score sequence of some tournament. This result is known as Landau's Theorem. Landau's Theorem attracted many researchers as nearly a dozen different proofs appeared
in literature. Many of these existing proofs are discussed in the survey by Reid [43] in 1996. For other proofs we refer to Griggs and Reid [23], and Brualdi and Shen [11].

The important research areas of tournaments are the spanning path, scores, extremal problems, the automorphism of tournaments, regularities in tournaments, frequency sets, score sets, scores in multipartite tournaments, kings in tournaments and multipartite tournaments, tournament colouring problems, and isomorphism problems. For many open problems, we refer to Reid [43].

One of the important aspects of tournaments is the score sequences, which are studied in detail in chapters 2. The literature on the scores of the tournaments can be found in [2,4,7,11,17,20,21,22,23,28,29,32,36,37,38,39,43,45].

Landau’s Theorem [36] is the tournament analog of Erdös-Gallai Theorem [18] for graphical sequences. First time Chen et al. [14] gave the concept of degree sequences with single repetitions in 1995. For the literature related to degree sequences with single repetitions, we refer to [13,14,31].
Degree imbalance sequences of simple digraphs (without repeated arcs) have been studied by Dhruv et al. [15]. They gave the concept of imbalance of a vertex and found a necessary and sufficient condition for a sequence of integers to be an imbalance sequence.

Hypertournaments has been studied by a number of authors (Cf. Assous[3], Barbut and Bialostocki[6], Frankl[19], and Gutin and Yeo [25]). These authors raise the problem of extending the most important results on tournaments to hypertournaments. Instead of scores of vertices in tournament, Zhou et al. [47] considered scores and losing scores of vertices in a $k$-hypertournament. Zhou et al. [47] also found a necessary and sufficient condition for a sequence of nonnegative integers to be score and losing score sequence of a $k$-hypertournament, which is a generalization of Landau's Theorem [36] on tournaments to $k$-hypertournaments. After two years Koh and Ree [35] defined total score of a vertex $v_i$ and gave a condition for a sequence of integers to be a total score sequence of a $k$-hypertournament. They also found some conditions for the existence of $k$-hypertournament matrices.
with constant score sequence called regular $k$-hypertournament matrices. Many open problems on the same topic may be found in Reid [43] (section 8) and for related references we refer to [3,6,19,25,34,35,43,47].

1.2 Synopsis of the thesis

Here we mention the contributions contained in the thesis. Chapter 1 provides the necessary ground to understand the contents presented in the subsequent sections.

In chapter 2 we study strong score sequences and self-converse strong score sequences of tournaments with repetitions. We mention some known results on the score sequences and a brief review of self-converse, regular, near-regular score sequences. We define score sequences with repetitions. The strong score sequences with a single repetition of length four are characterized and the total number of strong score sequences with a single repetition is obtained. We characterize the self-converse strong score sequences with a single repetition of length three and five and report the number of self-converse strong score sequences with a single repetition.
In chapter 3 we study some properties of imbalance sequences in oriented graphs. We characterize irreducible imbalance sequences of oriented graphs and bounds for imbalance $b_v$ of a vertex $v$, of an oriented graph are found. In the last we report a result of an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

In chapter 4 we obtain some stronger inequalities for degree imbalances in oriented graphs. We give a different proof of Theorem 3.8. In the last we investigate occurrence of equalities in inequalities (4.1) given in Theorem 4.1.

In the last chapter 5 we study properties of score, losing score and total score sequences of $k$-hypertournaments. We find some stronger inequalities for score and losing score sequences of $k$-hypertournaments. We report bounds for scores and losing scores. In the last, we discuss total score sequences of $k$-hypertournaments and characterize total score sequences of strong $k$-hypertournaments.
1.3 Basic Definitions

For the basic definitions and concepts we refer to [12,13,16,21,23,30,31,33,38,39,40, 41]. Few basic definitions are given below.

**Definition 1.1.** A graph $G = (V,E)$ is an ordered pair, where $V$ is a finite, nonempty set whose elements are termed as vertices, and $E$ is a set of unordered pair of distinct vertices of $V$. Each element $e = (a,b) \in E$ (where $a,b \in V$), is called an edge and is said to join the vertices $a$ and $b$.

**Definition 1.2.** A sub-graph of a graph $G$, is a graph having all of its vertices and edges in $G$.

**Definition 1.3.** If the edge $e = (a,b) \in E$, then $a$ and $b$ are both said to be incident with $e$ and adjacent to each other.

**Definition 1.4.** A digraph $D = (V,A)$ is defined to be an ordered pair of sets $(V,A)$, where $V$ is finite and nonempty set and $A$ is a set of ordered pairs of distinct elements of $V$. The elements of $A$ are called arcs. If $e = (a,b)$ is an arc of digraph, then $a$ is adjacent to $b$ and $b$ is adjacent from $a$. 

**Definition 1.5.** An oriented graph is a digraph having no symmetric pair of directed arcs.

**Definition 1.6.** The out-degree \( d^+(v) \) of a vertex \( v \) is the number of vertices adjacent from it, and the in-degree \( d^-(v) \) is the number of vertices adjacent to it.

**Definition 1.7.** A directed walk in a digraph \( D \), is an alternating finite sequence.

\[ W = v_0 \ e_1 \ v_1 \ ... \ e_k \ v_k, \]

of vertices and arcs such that for \( i = 1,2, \ldots , k , \ e_i = (v_{i-1}, v_i) \).

The number \( k \) of arcs in \( W \) is called length of \( W \). A closed walk has the same first and last vertices and a spanning walk contains all the vertices.

**Definition 1.8.** A path is a walk in which all the vertices are distinct.

**Definition 1.9.** A cycle is a closed path.

**Definition 1.10.** If there is a path from a vertex \( u \) to the vertex \( v \), then \( v \) is said to be reachable from \( u \).

**Definition 1.11.** A digraph \( D \) is said to be strong or strongly connected if every two vertices are mutually reachable.
Definition 1.12. The converse digraph $D'$ of a digraph $D$ has the same vertex set as $D$ and $(u,v) \in E(D')$ if and only if $(v,u) \in E(D)$.

Definition 1.13. A complete graph $K_n$ is a simple graph in which each pair of distinct vertices is joined by an edge.

Definition 1.14. A strong component $S$ of a digraph $D$ is a maximal strong sub-graph.

Definition 1.15. Two Digraphs $D_1 = (V_1,E_1)$ and $D_2 = (V_2,E_2)$ are said to be isomorphic, denoted by $D_1 \cong D_2$, if there is a one-to-one, onto mapping $f: V_1 \rightarrow V_2$, such that $(u,v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$.

Definition 1.16. A digraph is said to be self-converse if $D \cong D'$.

Definition 1.17. A tournament $T = (V,E)$ is a complete oriented graph with vertex set $V$ and arc set $E$, i.e., for every pair of vertices $u$ and $v$ either $(u,v)$ is an arc or $(v,u)$ is an arc. but not both.

Definition 1.18. In any tournament a vertex $u$ dominates $v$ if $(u,v)$ is an arc.
Definition 1.19. A tournament is called transitive if, whenever vertex \( u \) dominates \( v \), and \( v \) dominates \( w \), then \( u \) dominates \( w \).

Definition 1.20. In a tournament, the score of a vertex \( v \), denoted \( s(v) \) or \( s_v \), is the number of vertices dominated by \( v \). Thus \( s_v \) is the out degree of \( v \).

Definition 1.21. Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) and \( C = (c_1, c_2, \ldots, c_n) \) with \( c_1 \leq c_2 \leq \ldots \leq c_n \) be sequences of integers of order \( n \). Then \( B \) precedes \( C \) if there exist a positive integer \( k \leq n \) such that \( b_i = c_i \) for each \( 1 \leq i \leq k - 1 \) and \( b_k < c_k \). \( B = C \) if \( b_i = c_i \) for \( 1 \leq i \leq n \).

We write \( B \preceq C \) if \( B \) precedes \( C \), and we say that \( C \) is a successor of \( B \). If \( B \preceq C \) and \( C \preceq D \), then \( B \preceq D \), where \( D = (d_1, d_2, \ldots, d_n) \) with \( d_1 \leq d_2 \leq \ldots \leq d_n \).

Definition 1.22. We say that \( C \) is an immediate successor of \( B \) if there is no \( D \) such that \( B \preceq D \preceq C \).

Definition 1.23. An enumeration of all sequences of a given order with the property that the immediate successor of any sequence follows it in the list is called a lexicographic enumeration.

Definition 1.24. A tournament \( T \) is said to be self-converse if \( T = T' \).
Notations

\[ \square \quad \text{The end of a proof.} \]

\[ S \setminus \{x\} \quad \text{The sequence obtained from } S \text{ by removing the entry } x, \text{ where } x \in S. \]

\[ [n] \quad \{1,2,\ldots,n\}. \]

\[ A \setminus B \quad \text{The set difference } A - B. \]

\[ B \preceq C \quad B \text{ precedes } C. \]

**Note.** All the tables are generated by programs in FORTRAN language and these programs are available with the author.
CHAPTER TWO
CHAPTER 2

STRONG SCORE SEQUENCES WITH SINGLE REPETITION

In this chapter we study strong score sequences and self-converse strong score sequences of tournaments with repetitions. We mention some known results on the score sequences and a brief review of self-converse, regular, near-regular score sequences. We define score sequences with repetitions. The strong score sequences with a single repetition of length four are characterized and the total number of strong score sequences with a single repetition is obtained. We characterize the self-converse strong score sequences with a single repetition of length three and five and report the number of self-converse strong score sequences with a single repetition.

2.1 Introduction

A tournament $T = (V, E)$ of order $n$ is a complete oriented graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and arc set $E$, i.e..
each pair of distinct vertices $v_i$ and $v_j$ is joined by exactly one of the arcs $(v_i, v_j)$ or $(v_j, v_i)$ but not both. If $(v_i, v_j) \in E$, we say that $v_i$ dominates $v_j$. The score $s(v_i)$ or simply $s_i$, is the number of vertices dominated by $v_i$.

A sequence $S = (s_1, s_2, \ldots, s_n)$ with $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq n-1$ is called a score sequence, if there exists a tournament $T$ with scores $s_i$, $i = 1, 2, \ldots, n$.

**Definition 2.1.** A score sequence $S = (s_1, s_2, \ldots, s_n)$ is said to be realizable by a tournament $T$ if there exists a tournament $T$ with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ such that $s_i = s(v_i)$ for $i = 1, 2, \ldots, n$. Such a $T$ is known as realization of $S$.

A score sequence $S = (s_1, s_2, \ldots, s_n)$ is regular if $s_1 = s_2 = \ldots = s_n$ and near-regular if the maximum difference between its scores is one.

Below we report a necessary and sufficient condition for determining when a sequence is realizable by a tournament.

**Theorem 2.1.** [36] Let $S = (s_1, s_2, \ldots, s_n)$ be a non-decreasing sequence of $n$ non-negative integers not exceeding $n - 1$. Let $S_1$ be obtained from $S$ by deleting one entry $s_i$ and reducing
(n - 1 - s_i) largest entries of S by 1. Then S is a score sequence if and only if S_1 is.

And more directly we have the following.

**Theorem 2.2.** [36] A non-decreasing sequence \( S = (s_1, s_2, ..., s_n) \) of non-negative integers is a score sequence of a tournament if and only if, for \( 1 \leq k \leq n \)

\[
\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}
\]

with equality for \( k = n \).

Above Theorem 2.2 is known as *Landau Theorem*.

**Definition 2.2.** A score sequence \( S \) is said to be strong if all the tournaments with score sequence \( S \) are strong.

Theorem 2.2 induces a result to find which score sequences are strong.

**Corollary 2.3.** [28] A non-decreasing sequence \( S = (s_1, s_2, ..., s_n) \) of non-negative integers is a strong score sequence of a tournament if and only if, for \( 1 \leq k \leq n - 1 \),

\[
\sum_{i=1}^{k} s_i > \frac{k(k-1)}{2}
\]

(2.2)
and \[ \sum_{i=1}^{n} s_i = \frac{n(n-1)}{2} \]

Beineke and Eggleton (unpublished) noted in 1970's that in applying Theorem 2.2 and Corollary 2.3, one needs only to check the inequalities (2.1) and (2.2) for those values of \( k \) for which \( s_k < s_{k-1} \), as reported in Reid [43]. Gervacio [21] gave another characterization for the realization of strong score sequences. Gervacio [21,22] also investigated the construction of tournaments with a given score sequence.

2.2 Strong score sequences with single repetition

Let \( S = (s_1, s_2, \ldots, s_n) \) be a score sequence of a tournament of order \( n \), then \( 0 \leq s_i \leq n - 1 \) for \( 1 \leq i \leq n \). Now we can allocate \( n \) distinct integer values lying between 0 and \( n - 1 \), on \( n \) places in non-decreasing order only in one way and this score sequence is \((0,1,\ldots,n-1)\), which has all entries distinct and is transitive.

Let \( S = (s_1, s_2, \ldots, s_n) \) be a strong score sequence of order \( n \), then \( 1 \leq s_i \leq n - 2 \) for \( 1 \leq i \leq n \). In this case we have to allocate maximum \( n-2 \) distinct integers in nondecreasing
order, lying between 1 and \( n - 2 \), on \( n \) places, so either one entry must be repeated at least three times, or, two or more entries must be repeated at least two times in the sequence. Thus there is no strong score sequence which has all entries distinct and also there is no strong score sequence which has a single repetition of one entry two times and all other entries distinct.

In this section we study the strong score sequences, which has only a single repetition of one entry, and all other entries distinct. We define,

**Definition 2.3.** A score sequence \( S = (s_1, s_2, \ldots, s_n) \) is said to contain a single repetition \( p \) of length \( q \) if there are exactly \( q \) equal entries \( s_i = s_{i+1} = \ldots = s_{i+q-1} = p \), and all other entries of the sequence are distinct.

We know that there is only one strong score sequence of odd order \( n \) (\( n \geq 3 \)), which has all entries same. This is given by

\[
S = \left( \frac{n-1}{2} \frac{n-1}{2} \ldots \frac{n-1}{2} \right) \quad \text{or, } S = \left( n \times \frac{n-1}{2} \right)
\]

This is a regular score sequence.
We also observe that there is only one strong score sequence of even order \( n \) \( (n \geq 4) \), which has two repetitions \( \frac{n-2}{2} \) and \( \frac{n}{2} \) each of length \( \frac{n}{2} \), i.e.,

\[
S = \left( \frac{n-2}{2}, \frac{n-2}{2}, \ldots, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2} \right)
\]

or \( S = \left( \frac{n}{2} \times \frac{n-2}{2}, \frac{n}{2} \times \frac{n-2}{2}, \ldots, \frac{n}{2} \times \frac{n}{2} \right) \), which is a near-regular score sequence.

**Example 2.1.** This example represents all the 6 strong score sequences having a single repetition of order 8, arranged in lexicographic order.

<table>
<thead>
<tr>
<th>Sequence Number</th>
<th>Score Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(1,2,3,4,4,4,4,6)</td>
</tr>
<tr>
<td>2.</td>
<td>(1,2,4,4,4,4,4,5)</td>
</tr>
<tr>
<td>3.</td>
<td>(1,3,3,3,3,4,5,6)</td>
</tr>
<tr>
<td>4.</td>
<td>(1,3,4,4,4,4,4,4)</td>
</tr>
<tr>
<td>5.</td>
<td>(2,3,3,3,3,3,5,6)</td>
</tr>
<tr>
<td>6.</td>
<td>(3,3,3,3,3,3,4,6)</td>
</tr>
</tbody>
</table>

*Table (2.1)*
Let $sr(n)$ denotes the number of strong score sequences with a single repetition of order $n$. A strong score sequence with a single repetition have order at least three. No result is known which gives the values of $sr(n)$ for all $n$ ($n \geq 3$). Table (2.2) lists the values of $sr(n)$ for some values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sr(n)$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>14</td>
<td>22</td>
<td>42</td>
<td>78</td>
<td>144</td>
<td>270</td>
<td>510</td>
</tr>
</tbody>
</table>

*Table (2.2)*

Let $S$ be a strong score sequence of order $n$, with a single repetition (say $p$) of length five. As $p$ is repeated five times exactly two of the $n-2$ possible entries from 1,2,...,$n-2$ are missing. Let the missing entries be $x$ and $y$, we write

$$S = (1, 2, \ldots, p-1, 5 \times p, p+1, \ldots, n-2) \setminus (x, y).$$

Let $S = (s_1, s_2, \ldots, s_n)$ be a score sequence of order $n$ and $S' = (n - 1 - s_1, n - 1 - s_2, \ldots, n - 1 - s_n)$ be the converse score sequence of $S$.

The following result shows the similarity between $S'$ and $S$. 

Theorem 2.4. \( S = (s_1, s_2, \ldots, s_n) \) is a strong score sequence with a single repetition \( p \) of length \( q \), if and only if the converse score sequence \( S' = (s_1', s_2', \ldots, s_n') \) is a strong score sequence with a single repetition \( n - 1 - p \) of the same length \( q \), where
\[
s_i' = n - 1 - s_i, \quad i = 1, 2, \ldots, n.
\]

Proof. We know that \( S \) is strong if and only if \( S' \) is strong. As \( S \) has a single repetition \( p \) of length \( q \), then \( S' \) will have a single repetition \( n - 1 - p \) of the same length \( q \).

Now we report results, which characterize strong score sequences with single repetition of length four.

Theorem 2.5. For \( n \) even and \( 6j \leq n < 6(j+1), \ j = 1, 2, \ldots \) there are \( 2j \) strong score sequences \( S_k \) and \( S_k' \) of order \( n \), having a single repetition of length 4, where
\[
S_k = (1, 2, \ldots, \frac{n + 2(k - 2)}{2}, 4 \times \frac{n + 2(k - 1)}{2}, \frac{n + 2k}{2}, \ldots, n - 2) \backslash \left( \frac{n + 6k - 4}{2} \right)
\]

for \( k = 1, 2, \ldots, j \).

Proof. Since \( n \) is even, let \( n = 2m \), then \( 3j \leq m < 3(j + 1), \ j = 1, 2, \ldots, \) and
Chapter 2: Strong score sequences with single repetition

\[ S_k = (1,2,\ldots,m+k-2,4\times(m+k-1),m+k,\ldots,2m-2) \setminus (m+3k-2) \]

\[ k = 1,2,\ldots,j. \quad (2.4) \]

Now we show that \( S_k \) is a strong score sequence.

Consider,

\[ \sum_{i=1}^{2m} s_i = \{1 + 2 + \ldots + (2m-2)\} + 3(m+k-1) - (m+3k-2) \]

\[ = \frac{2m(2m-1)}{2}, \text{ thus the equality holds for } n = 2m. \]

Now for strongness we have to show that

\[ \sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \quad 1 \leq l < 2m. \quad (2.5) \]

**Case 1.** For \( 1 \leq l \leq m + k - 1 \)

\[ \sum_{i=1}^{l} s_i = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}, \text{ thus the result holds for } 1 \leq l \leq m+k-1. \]

**Case 2.** For \( m + k \leq l \leq m + k + 2 \). As \( s_{m-k} = s_{m+k+1} = s_{m+k+2} \), we only consider \( l = m + k + 2 \)

\[ \sum_{i=1}^{m+k+2} s_i = \sum_{i=1}^{m+k-1} s_i + 3(m+k+1) \]

\[ = \frac{(m+k-1)(m+k)}{2} + 3(m+k+1) \]
Chapter 2: Strong score sequences with single repetition

\[
\frac{(m+k-1)(m+k+6)}{2} \quad (2.6)
\]

\[
\frac{(m+k+2)(m+k+1)}{2} + (m+k-4), \ m \geq 3, \ k \geq 1. \quad (2.7)
\]

**Subcase 2.1.** When \( m = 3, \ k = 1 \), then \( m + k - 4 = 0 \), then we get \( S_1 = (1,2,3,3,3,3) \) which is strong.

**Subcase 2.2.** In the remaining two cases either \( m \geq 3 \) and \( k > 1 \) or \( m > 3 \) and \( k \geq 1 \), thus \( m + k - 4 > 0 \), hence, from equation (2.7), we get

\[
\sum_{i=1}^{m+k+2} s_i > \frac{(m+k+2)(m+k+1)}{2},
\]

so the result holds for \( l = m + k + 2 \).

The missing entry is \( m + 3k - 2 \), so consider the case:

**Case 3.** For \( m + k + 3 \leq l \leq m + 3k \).

\[
\sum_{i=1}^{l} s_i = \sum_{i=1}^{m+k+2} s_i + \sum_{i=m+k+3}^{l} s_i
\]

\[
= \frac{(m+k-1)(m+k+6)}{2} + (s_{m+k+3} + s_{m+k+4} + \ldots + s_l)
\]

(Using condition (2.6))

\[
= \frac{(m+k-1)(m+k+6)}{2} + \left\{ (m+k) + (m+k+1) + \ldots + (l-3) \right\}
\]
Subcase 3.1. When \( m = 3k, 4k + 3 \leq l \leq 6k \) and equation (2.8) becomes,

\[
\sum_{i=1}^{l} s_i = \frac{l(l-1)}{2} + 2(6k-l) \tag{2.9}
\]

taking \( l < 6k \), the condition (2.9) becomes

\[
\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \text{ for } m = 3k.
\]

Subcase 3.2. When \( m > 3k, m+k+3 \leq l \leq m+3k \) from condition (2.8),

\[
\sum_{i=1}^{l} s_i \geq \frac{l(l-1)}{2} + 3m + 3k - 2(m+3k), \text{ (as } l \leq m+3k) \\
\]

\[
= \frac{l(l-1)}{2} + (m-3k)
\]

so, \( \sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \) \( \text{ (as } m > 3k) \), so inequalities hold in this subcase.

Case 4. For \( m+3k+1 \leq l < 2m \)

\[
\sum_{i=1}^{l} s_i = \{1 + 2 + \ldots + (l-2)\} + 3(m+k-1) - (m+3k-2)
\]
\[ = \frac{l(l - 1)}{2} + (2m - 1) \]

\[ > \frac{l(l - 1)}{2}. \quad (as \ l < 2m). \]

Hence by condition (2.2) \( S_k \) for \( k = 1, 2, \ldots, j \) are strong score sequences of order \( n \) with a single repetition \( \frac{n + 2(k - 1)}{2} \) of length four. Also by Theorem 2.3 \( S_k' \) for \( k = 1, 2, \ldots, j \) are the strong score sequences of same order as \( S_k \) with a single repetition \( \frac{n - 2k}{2} \) of length four. This proves the Theorem. \( \square \)

**Remark 2.1.** In score sequences (2.4), if we set \( k = 1 \), then we get for \( 3j \leq m < 3(j + 1) \), \( j = 1, 2, \ldots, \)

\[ S_1 = (1, 2, \ldots, m - 1, 4 \times m, m + 1, \ldots, 2m - 2) \backslash (m + 1) \]

i.e., \( S_1 = (1, 2, \ldots, m - 1, 4 \times m, m + 2, \ldots, 2m - 2) \) \( (2.10) \)

and converse of sequence \( S_1 \),

\[ S_1' = (1, 2, \ldots, m - 3, 4 \times (m - 1), m, \ldots, 2m - 2) \] \( (2.11) \)

Now from \( S_1 \), if we delete one repeated score, i.e., \( m \) and reducing \( 2m - 1 \cdot m \) (= \( m - 1 \)) largest entries by one, we get a score sequence
\( S_1 = (1, 2, \ldots, m-2, 3 \times (m-1), m, \ldots, 2m-3) \) \hspace{1cm} (2.12)

Now if we apply the same operation on \( S_1' \), i.e., we delete the repeated entry \( m-1 \), then we get again \( S_1 \). Which is a strong score sequence, having a single repetition \( m-1 \) of length three.

Hence for \( k = 1 \), the strong score sequences (2.4), after deleting the one repeated entry \( m \) and reducing \( m-1 \) largest entries by 1, always gives the strong score sequence with a single repetition \( m-1 \) of length three of order \( 2m-1 \). While the converse of score sequence (2.4), after the same operation also gives the same score sequence.

Now we give a result which characterizes the strong score sequences with a single repetition \( p \) of length four for order \( n \geq 9 \) and \( n \) odd.

**Theorem 2.6.** For \( n \) odd and \( 6j + 3 \leq n < 6j + 9, j = 1, 2, \ldots \) there are \( 2j \) strong score sequences \( S_k \) and \( S_k' \) of order \( n \), having a single repetition \( \frac{n+2k-1}{2} \) of length four, where
Chapter 2: Strong score sequences with single repetition

\[
S_k = (1, 2, \ldots, \frac{n+2k-3}{2}, 4 \times \frac{n+2k-1}{2}, \frac{n+2k+1}{2}, \ldots, n-2) \setminus \frac{n+6k-1}{2}, \text{ and } k = 1, 2, \ldots, j
\]

(2.13)

**Proof.** Since \( n \) is odd, let \( n = 2m + 1 \), so \( 3j + 1 \leq m < 3j + 4 \), \( j = 1, 2, \ldots, \) and

\[
S_k = (1, 2, \ldots, m + k - 1, 4 \times (m + k), m + k + 1, \ldots, 2m - 1) \setminus (m + 3k)
\]

(2.14)

Here we note that the missing entry \( m + 3k \), will appear after the repeated entry \( m + k \), because for \( k \geq 1 \), \( m + 3k > m + k \). Now we show that \( S_k \) is a strong score sequence. Consider

\[
\sum_{i=1}^{2m+1} s_i = \{1+2+ \ldots + (2m-1)\} + 3(m+k) - (m+3k)
\]

\[
= \frac{(2m + 1)(2m)}{2}, \text{ thus the equality holds for } l = 2m + 1.
\]

For strongness, now we have to show that

\[
\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \text{ for } 1 \leq l < 2m + 1.
\]

(2.15)

**Case 1.** For \( 1 \leq l \leq (m+k) \)

\[
\sum_{i=1}^{l} s_i = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}, \text{ so the inequalities hold in this case.}
\]
Case 2. For \( l = m + k + 3 \)

\[
\sum_{i=1}^{m+k+3} S_i = \sum_{i=1}^{m+k+1} S_i + 3(m+k) = \frac{(m+k)(m+k+1)}{2} + 3(m+k) = \frac{(m+k)(m+k+7)}{2} \\
> \frac{(m+k+3)(m+k+2)}{2}, \text{ (as } m \geq 4, k \geq 1), \text{ inequalities (2.15) hold for } l = m + k + 3.
\]

Since missing entry is \( m + 3k \). So consider the following cases.

Case 3. For \( m + k + 4 \leq l \leq m + 3k + 2 \),

\[
\sum_{i=1}^{l} S_i = \sum_{i=1}^{m+k+3} S_i + (S_{m+k+4} + \ldots + S_l) \\
= \frac{(m+k)(m+k+7)}{2} + \{(m+k+1) + \ldots + (l-3)\} \\
= \frac{l(l-1)}{2} + 3m + 3k - 2l + 3 \quad (2.16)
\]

Subcase 3.1. When \( m = 3k+1, 4k+5 \leq l \leq 6k+3 \), then missing entry becomes 6\(k+1\), so we leave \( l = 6k+3 \), because we have
already proved the equality. From equation (2.16), for \( m = 3k+1 \),

\[
\sum_{i=1}^{l} s_i = \frac{l(l-1)}{2} + 12k - 2l + 6 \tag{2.17}
\]

since \( 6k + 3 > l \) so \( 12k - 2l + 6 > 0 \) hence from condition (2.17),
we get \( \sum_{i=1}^{l} s_i > \frac{l(l-1)}{2} \), so inequalities hold in this case.

**Subcase 3.2.** For \( m > 3k+1 \) and \( m+k+4 \leq l \leq m+3k+2 \), from equation (2.16), we have

\[
\sum_{i=1}^{l} s_i = \frac{l(l-1)}{2} + 3m + 3k + 3 - 2l
\]

\[\Rightarrow \sum_{i=1}^{l} s_i \geq \frac{l(l-1)}{2} + 3m + 3k + 3 - 2(m + 3k + 2) \text{ (as } l \leq m + 3k + 2)\]

\[\Rightarrow \sum_{i=1}^{l} s_i = \frac{l(l-1)}{2} + m - 3k - 1\]

\[\Rightarrow \sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \text{ (as } m > 3k+1 \text{ so } m - 3k - 1 > 0).\]

Inequalities hold for \( m+k+4 \leq l < m+3k+2 \).

**Case 4.** For \( m + 3k + 3 \leq l \leq 2m + 1 \), we get

\[
\sum_{i=1}^{l} s_i = \{1 + 2 + \ldots + (l-2)\} + 3(m+k) - (m+3k)
\]
\[
= \frac{(l-2)(l-1)}{2} + 2m
\]
\[
= \frac{l(l-1)}{2} - l + 2m+1
\]
\[
> \frac{l(l-1)}{2}, \text{ (as } l < 2m+1), \text{ so inequalities hold in this case.}
\]

Hence by condition (2.2) \( S_k \) for \( k = 1, 2, \ldots, j \) shall be strong score sequences, each of order \( n \), with a single repetition \( \frac{n+2k-1}{2} \) of length four. Also by Theorem 2.3 \( S_k' \) for \( k = 1, 2, \ldots, j \) are the strong score sequences of same order as \( S_k \) with a single repetition \( \frac{n-2k-1}{2} \) of length four. This completes the proof. \( \Box \)

Let \( sr4(n) \) denoted the number of strong score sequences with a single repetition \( p \) of length 4. Table 2.3 lists the values of \( sr4(n), n \geq 6 \), for some values of \( n \), the general result for \( sr4(n) \) is not known.

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( sr4(n) )</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Table (2.3)
Table 2.4 and Table 2.5 list the values of $sr5(n)$ and $sr6(n)$, the number of strong score sequences with a single repetition of length 5 and 6 respectively, for some values of $n$. The general results for $sr5(n)$ and $sr6(n)$ are not known.

<table>
<thead>
<tr>
<th>$n$</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sr5(n)$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

Table (2.4)

<table>
<thead>
<tr>
<th>$n$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sr6(n)$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>16</td>
<td>20</td>
<td>28</td>
<td>40</td>
</tr>
</tbody>
</table>

Table (2.5)

2.3 Self-converse strong score sequences with single repetition

**Definition 2.4.** A score sequence $S = (s_1, s_2, \ldots, s_n)$ is said to be self-converse if all the tournaments $T$, having the score sequence $S$ are self-converse, i.e., $T \cong T'$. If $S = (s_1, s_2, \ldots, s_n)$ is a score sequence of a tournament $T$, then $S'$ score sequence of $T'$, is given by
$S' = (n-1-s_1, \ldots, n-1-s_n)$.

All the score sequences of order fewer than four are self-converse.

In 1979, Eplett [17] characterized the self-converse score sequences.

**Theorem 2.7.** [17] A score sequence $S = (s_1, s_2, \ldots, s_n)$ is self-converse if and only if

$$s_i + s_{n+1-i} = n - 1, \text{ for } 1 \leq i \leq n.$$

(2.18)

Theorem 2.7 is equivalent to saying that $S$ is self-converse if and only if $S' = S$. One needs to verify the condition (2.18) only for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$. All regular and near regular score sequences are self-converse.

Let $T$ be a tournament with score sequence $S$. The strong components of $S$ are the score sequences of the strong components of $T$. The following result gives all the strong components of $T$.

**Theorem 2.8.** [4] Let $T$ be a tournament and $S = (s_1, s_2, \ldots, s_n)$ be the score sequence of $T$. Suppose
Chapter 2: Strong score sequences with single repetition

\[ \sum_{i=1}^{p} s_i = \frac{p(p - 1)}{2} \]  \hspace{1cm} (2.19)

\[ \sum_{i=1}^{q} s_i = \frac{q(q - 1)}{2} \]  \hspace{1cm} (2.20)

and \[ \sum_{i=1}^{q} s_i > \frac{k(k - 1)}{2}, \text{ for } p + 1 \leq k \leq q - 1 \]  \hspace{1cm} (2.21)

where \( 0 \leq p < q \leq n \).

Then the sub-tournament induced by the vertices \( \{ v_{p+1}, \ldots, v_q \} \) is a strong component of \( T \) with score sequence \( (s_{p+1} - p, \ldots, s_q - p) \).

The above theorem shows that the strong components of \( S \) are determined by the successive values of \( k \) for which

\[ \sum_{i=1}^{k} s_i = \frac{k(k - 1)}{2}, \ 1 \leq k \leq n. \]  \hspace{1cm} (2.22)

Now we discuss the self-converse strong score sequences which have some repeated entries.

**Example 2.2.** The following example represents all 8 self-converse strong score sequences with a single repetition, of order 9 arranged in lexicographic order.
Table (2.6)

<table>
<thead>
<tr>
<th>Sequence number</th>
<th>Score sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(1,2,3,4,4,4,5,6,7)</td>
</tr>
<tr>
<td>2.</td>
<td>(1,2,4,4,4,4,4,6,7)</td>
</tr>
<tr>
<td>3.</td>
<td>(1,3,4,4,4,4,4,5,7)</td>
</tr>
<tr>
<td>4.</td>
<td>(1,4,4,4,4,4,4,4,7)</td>
</tr>
<tr>
<td>5.</td>
<td>(2,3,4,4,4,4,4,5,6)</td>
</tr>
<tr>
<td>6.</td>
<td>(2,4,4,4,4,4,4,4,6)</td>
</tr>
<tr>
<td>7.</td>
<td>(3,4,4,4,4,4,4,4,5)</td>
</tr>
<tr>
<td>8.</td>
<td>(4,4,4,4,4,4,4,4,4)</td>
</tr>
</tbody>
</table>

**Theorem 2.9.** If $S$ is a self-converse score sequence of order $n$ $(n \geq 3$ and odd) having a single repetition, then the repeated entry will be $(n-1)/2$.

**Proof.** Let $n = 2m+1$ and $S = (s_1, s_2, ..., s_{2m+1})$ be a self-converse score sequence.

Thus $s_i + s_{2m+2-i} = 2m$

At $i = m+1$ $2s_{m+1} = 2m$

i.e., $s_{m+1} = m$.

It may be noted that the single repeated entry is always $m$. 

Remark 2.2. There is no self-converse score sequence $S$ of even order $n$ having a single repetition of one entry.

Theorem 2.10. If $S = (s_1, \ldots, px \times \frac{n-1}{2}, \ldots, n-1-s_1)$ is a self-converse strong score sequence of odd order $n$ ($n \geq 3$) having a single repetition of length $p$. Then for every $n$ there exist self-converse strong score sequences $S'$ and $S''$ of order $n+2$, having a single repetition of length $p$ and $p+2$ respectively, given by

(i) $S' = (1, s_1+1, \ldots, px \times \frac{n+1}{2}, \ldots, n-s_1, n)$

(ii) $S'' = (s_1+1, \ldots, (p+2)\times \frac{n+1}{2}, \ldots, n-s_1)$

Proof. (i) Suppose that $T(S)$ is a tournament with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, which realizes score sequence $S$. For a realization of score sequence $S'$, adding two vertices $u$ and $w$ to $T(S)$ in such a way that all the vertices of $T(S)$ dominate vertex $u$, the vertex $w$ dominates all vertices of $T(S)$ and $u$ dominates $w$. Hence tournament $T(S')$ is as shown in figure 2.1, which realizes $S'$. 
Clearly $T(S')$ is strong and self-converse also contain a single repetition of $\frac{n+1}{2}$ of the same length $p$.

(ii) Suppose $T(S)$ is a realization of score sequence $S$. For a realization of score sequence $S^{**}$ adding two vertices $u$ and $w$ to $T(S)$ in such a way that vertex $u$ dominates vertices $v_1, v_2, \ldots, v_{(n+1)/2}$, vertices $v_{(n+3)/2}, \ldots, v_n$ dominate vertex $u$, vertices $v_1, v_2, \ldots, v_{(n+1)/2}$ dominate vertex $w$, vertex $w$ dominates
vertices \( v_{(n+3)/2}, \ldots, v_n \) and vertex \( w \) dominates vertex \( u \). A realization of tournament \( T(S^{**}) \) shown in following figure 2.2.

Clearly \( T(S^{**}) \) is strong and self-converse also contain a single repetition of \( \frac{n+1}{2} \) of length \( p+2 \).
Let sc(n) represents the number of self-converse strong score sequences of order n with single repetition. Following Table (2.7) lists the values of sc(n) for some values of n.

<table>
<thead>
<tr>
<th>n</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>sc(n)</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
</tr>
</tbody>
</table>

**Table (2.7)**

We state the following conjecture:

**Conjecture 2.1.** If n is odd then

\[ sc(n) = 2^{(n-3)/2}, \text{ for } n \geq 3. \]  

(2.23)

Below we characterize self-converse strong score sequences with single repetition of length three.

**Theorem 2.11.** If n \(\geq 3\) and is odd, then

\[ S = (1, 2, \ldots, \frac{n-3}{2}, 3 \times \frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2) \]  

(2.24)

is a unique self-converse strong score sequence of order n with a single repetition \(\frac{n-1}{2}\) of length three.
**Proof.** As \( n \) is odd, let \( n = 2m + 1, \ m \geq 1 \), sequence (2.24) becomes.

\[
S = (1, 2, ..., m - 1, 3m, m + 1, ..., 2m - 1).
\]

(2.25)

Here we have,

\[
s_i = \begin{cases} 
  i, & \text{for } 1 \leq i \leq m \\
  m, & \text{for } m + 1 \leq i \leq m + 2 \\
  i - 2, & \text{for } m + 3 \leq i \leq 2m + 1.
\end{cases}
\]

(2.26)

First we show that sequence \( S \) is a strong score sequence.

Consider

\[
\sum_{i=1}^{2m+1} s_i = \sum_{i=1}^{m} s_i + \sum_{i=m+1}^{m+2} s_i + \sum_{i=m+3}^{2m+1} s_i
\]

\[
= (1 + 2 + ... + m) + 2m + \{ (m+1) + (m+2) + ... + (2m-1) \}
\]

\[
= \frac{(2m+1)(2m)}{2}, \text{ hence equality holds.}
\]

It remains to prove that

\[
\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \text{ for } 1 \leq l \leq 2m.
\]

**Case 1.** For \( 1 \leq l \leq m \), we have
\[ \sum_{i=1}^{l} s_i = (1+2+ \ldots + l) \]

\[ = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}, \text{ so inequalities hold in this case.} \]

**Case 2.** For \( m+1 \leq l \leq m+2 \), as \( s_{m+1} = s_{m+2} \), we only consider \( l = m + 2 \)

\[ \sum_{i=1}^{m+2} s_i = \sum_{i=1}^{m} s_i + s_{m+1} + s_{m+2} \]

\[ = \frac{m(m+5)}{2}. \tag{2.27} \]

**Subcase 2.1.** Let \( m = 1 \), now \( S = (1,1,1) \), which is strong.

**Subcase 2.2.** Let \( m > 1 \), we have

\[ \sum_{i=1}^{m+2} s_i = \frac{m(m+5)}{2}, \quad \{ \text{from equation (2.27)} \} \]

\[ > \frac{(m+2)(m+1)}{2}, \]

for \( m > 1 \) so inequalities hold for \( l = m + 2 \).

**Case 3.** Let \( m + 3 \leq l \leq 2m \), we have

\[ \sum_{i=1}^{l} s_i = \sum_{i=1}^{m} s_i + 2m + \sum_{i=m+1}^{l} s_i \]
Chapter 2: Strong score sequences with single repetition

\[ m(m+1) \div 2 + 2m + \{(m+1) \div (m+2) + \ldots + (l-2)\} \]

\[ = \frac{m(m+5)}{2} \div \frac{(l-m-2)(l+m-1)}{2} \]

\[ = \frac{(l^2-3l+4m+2)}{2} \]

\[ \geq \frac{(l^2-3l+2l+2)}{2}, \text{ (as } l \leq 2m) \]

\[ = \frac{l(l-1)}{2} + 1 > \frac{l(l-1)}{2}, \]

so inequalities hold for \( m + 3 \leq l \leq 2m \).

Thus \( S \) is a strong score sequence with a single repetition \( \frac{(n-1)}{2} \) of length three.

Now we show score sequence \( S \) is self-converse

(i) For \( 1 \leq i \leq m - 1 \), we have

\[ s_i + s_{(2m+1)+1-i} = s_i + s_{2m+2-i} \]

\[ = i + (2m + 2 - i) - 2 \]

(from condition (2.26) when \( 1 \leq i \leq m - 1 \) than \( m + 3 \leq 2m + 2 - i \leq 2m + 1 \))

so, \( s_i \text{ and } s_{(2m+1)+1-i} \leq 2m \)
(ii) For $i = m$, we have

$$s_i + s_{(2m + 1) - 1} = s_i + s_{2m + 2 - i} = 2m$$

(from equation (2.26) when $i = m$ than $2m + 2 - i = m + 2$)

Hence from Theorem 2.7, the score sequence $S$ is self-converse.

Lastly, for uniqueness, consider a self-converse strong score sequence of order $2m + 1$, with a single repetition of length three, let

$$S = (s_1, s_2, \ldots, s_m, s_{m+1}, s_{m+2}, \ldots, s_{2m+1})$$

Where $s_i = i + 1 - 1, \quad 1 \leq i \leq m$

$$s_m + 2 = s_{m+1} = s_m$$

$$s_i = 2m - s_{2m+2-i}, \quad m + 3 \leq i \leq 2m + 1$$

Since $2s_{m+1} = 2m$, which gives $t = 1$, hence

$$S = (1,2,\ldots,m,m,m,\ldots,2m),$$

which is strong. This proves the uniqueness of the Theorem.
Another Proof. For uniqueness consider a self-converse strong score sequence of order \(2m + 1\) with a single repetition of length three. Let

\[S = (s_1, s_2, \ldots, s_{m}, s_{m+1}, s_{m+2}, \ldots, s_{2m+1})\]

Where \(s_m = s_{m+1} = s_{m+2} = p\) (say)

As \(S\) is self-converse,

\[2s_{m+1} = 2m\]

so,

\[p = m\]

Also \(S\) is strong so, \(1 \leq s_i \leq m - 1\) for \(i = 1, 2, \ldots, m - 1\). Thus \(s_i = i\) for \(i = 1, 2, \ldots, m - 1\). Hence the result. \(\Box\)

The next result deals with the strong score sequences of even order, which has a single repetition \(p\) of length three.

**Theorem 2.12.** There is no strong score sequence of even order \(n\) \((n \geq 4)\), which has a single repetition of length three.

**Proof.** As \(S\) is strong so \(1 \leq s_i \leq n - 2\). If one entry, say \(p\), repeats three times then \(p\) occupies three positions in the sequence and now we are to allocate remaining \(n - 3\) entries at \(n - 3\) positions. Thus \(s_1 = 1\) and \(s_n = n - 2\). For \(n\) even, let
Chapter 2: Strong score sequences with single repetition

42

(2.28)

be a strong score sequence of even order \( n \) with single repetition \( p \) of length three.

By Landau's Theorem,

\[
\sum_{s=1}^{n} s_s = \frac{n(n-1)}{2}
\]

\[
\Rightarrow \sum_{s=1}^{p} s_s + \sum_{s=p+1}^{p+2} s_s + \sum_{s=p+3}^{n} s_s = \frac{n(n-1)}{2}
\]

\[
\Rightarrow \frac{p(p+1)}{2} + 2p + \frac{(n-p-2)(p+n-1)}{2} = \frac{n(n-1)}{2}
\]

\[
\Rightarrow p = \frac{n-1}{2}.
\]

Hence \( p \) can not be a positive integer as \( n \) is even, a contradiction. Thus we proved the result.

Next result characterizes self-converse strong score sequences having a single repetition of length five.

Theorem 2.13. For \( n \geq 5 \) and \( n \) odd, there are \( \frac{n-3}{2} \) self-converse strong score sequences \( S_k \) of order \( n \), having a single repetition \( \frac{n-1}{2} \) of length five, where
Chapter 2: Strong score sequences with single repetition

\[ S_k = (1, 2, \ldots, \frac{n-3}{2}, 5 \times \frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2) \setminus (\frac{n-1}{2} - k, \frac{n-1}{2} + k) \]

for \( k = 1, 2, \ldots, \frac{n-3}{2} \).

(2.29)

**Proof.** As \( n \) is odd and \( n \geq 5 \), on putting \( n = 2m + 3 \), \( m \geq 1 \). \( S_k \) becomes.

\[ S_k = (1, 2, \ldots, m, 5 \times (m + 1), m + 2, \ldots, 2m + 1) \setminus (m - k + 1, m + k + 1) \]

(2.30)

When \( m \) is equal to 1, then \( S_k \) becomes \((2, 2, 2, 2, 2)\), which is strong score sequence as well as self-converse. Hence the result holds for \( m = 1 \).

Now we show that result is true for \( m \geq 2 \). As missing entries are \( m - k + 1 \) and \( m + k + 1 \). Thus from (3.14) we get

\[
\begin{align*}
    s_i &= i & \text{for } 1 \leq i \leq m - k \\
    s_i &= i + 1 & \text{for } m - k + 1 \leq i \leq m \\
    s_i &= m + 1 & \text{for } m + 1 \leq i \leq m + 4 \\
    s_i &= i - 3 & \text{for } m + 5 \leq i \leq m + k + 3 \\
    s_i &= i - 2 & \text{for } m + k + 4 \leq i \leq 2m + 3.
\end{align*}
\]

(2.31)
\[
\sum_{i=1}^{m+2} s_i = (1 + 2 + \ldots + 2m+1) + 4(m+1) (m+k+1) - (m+k+1) \\
\cdot (2m+1)(m+1) + 2(m+1) \\
\to (2m+3)(2m+2) \quad \text{thus equality holds for } n = 2m + 3.
\]

Now for strictness we have to show that,
\[
\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \text{ for } 1 \leq l \leq 2m + 2. \tag{2.32}
\]

**Case 1.** For \(1 \leq l \leq m - k\), we have
\[
\sum_{i=1}^{l} s_i = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}, \quad \text{for } 1 \leq l \leq m - k, \text{ so inequalities hold.}
\]

**Case 2.** For \(m - k + 1 \leq l \leq m\), we have
\[
\sum_{i=1}^{l} s_i = \sum_{i=1}^{m} s_i + \sum_{i=m}^{l} s_i \\
= \frac{l(l-1)}{2} + (2l - m + k) \\
\geq \frac{l(l-1)}{2} + \{2(m - k + 1) - m + k\}, \quad \text{(as } l \geq m - k + 1) \\
= \frac{l(l-1)}{2} + (m - k + 2)
\]
so \( \sum_{i=1}^{n} s_i \geq \frac{l(l-1)}{2} \), (as \( k \leq m \)), inequalities hold in this case.

**Case 3.** Let \( m + 1 \leq l \leq m + 4 \), as \( s_{m+1} = s_{m+2} = s_{m+3} = s_{m+4} \), we consider only \( l = m + 4 \),

\[
\sum_{i=1}^{n} s_i = \{1+2+\ldots+(m+1)\} + 4(m+1) - (m-k+1)
\]

\[
= \frac{(m+4)(m+3)}{2} + (m+k-2)
\]

\[
> \frac{(m+4)(m+3)}{2}, \text{(as } m \geq 2, k \geq 1\text{), inequality holds for } l = m + 4.
\]

**Case 4.** For \( m + 5 \leq l \leq 2m + 2 \), we divide the case 4 into two subcases, as \( m = k \) and \( m > k \).

**Subcase 4.1.** For \( m = k \), \( k + 5 \leq l \leq 2k + 2 \) the missing entries become 1 and \( 2k + 1 \). Thus \( S_k \) becomes,

\[
S_k = (2,3,\ldots,k,5x(k+1),k+2,\ldots,2k)
\]

(2.33)

and

\[
\sum_{i=1}^{n} s_i = (2+3+\ldots+k)+5(k+1)+\{(k+2)+\ldots+(l-3)\}
\]

\[
= \frac{(k-1)(k+2)}{2} + 5(k+1) + \frac{(l-k-4)(l+k-1)}{2}
\]
Chapter 2: Strong score sequences with single repetition

\[
\begin{align*}
&= \frac{l(l-1)}{2} + (4k - 2l + 6) \\
&\geq \frac{l(l-1)}{2} + \{4k - 2(2k + 2) + 6\}, \text{ (as } l \leq 2k+2) \\
&= \frac{l(l-1)}{2} + 2 \\
&> \frac{l(l-1)}{2}, \text{ hence inequalities hold in this subcase.}
\end{align*}
\]

**Subcase 4.2.** For \( m > k, m + 5 \leq l \leq 2m + 2, \) consider the following two subcases.

**Subcase 4.2.1.** For \( m + 5 \leq l \leq m + k + 3, m > k, \) we have

\[
\begin{align*}
\sum_{s_i} &= \{1+2 + \ldots + (m+1)\} + 4(m+1) - (m-k+1) \\
&\quad + \{s_{m+5} + \ldots + s_l\} \\
&= \frac{(m+1)(m+2)}{2} + 3m+k+3 + \{(m+2)+(m+3) + \ldots + (l-3)\} \\
&= \frac{l(l-1)}{2} + (3m + k - 2l + 6) \\
&\geq \frac{l(l-1)}{2} + \{3m + k - 2(m + k + 3) + 6\}
\end{align*}
\]
= \frac{l(l-1)}{2} + (m - k)

> \frac{l(l-1)}{2}, \text{ (as } m > k), \text{ inequalities hold in this subcase.}

Subcase 4.2.2. For \( m + k + 4 \leq l \leq 2m + 2, m > k, \text{ we have} \\
\sum_{i=1}^{l} s_i = \{1+2+ ... + (l-2)\} - (m-k+1) - (m+k+1) + 4(m+1)

= \frac{l(l-1)}{2} + (2m - l + 3)

\geq \frac{l(l-1)}{2} + \{2m - (2m + 2) + 3\}, \quad \text{ (as } l \leq 2m + 2) \\
= \frac{l(l-1)}{2} + 1

> \frac{l(l-1)}{2}, \text{ so inequalities hold in this subcase.}

Hence by condition (2.2), the sequence \( S_k \) is a strong score sequence of order \( n \) contains a single repetition \( \frac{n-1}{2} \) of length five.

It remains to prove that the score sequences \( S_k \), for \( k = 1,2, ..., \frac{n-3}{2} \), are self-converse.
(i) For $1 \leq i \leq m - k$, we have

\[ s_i + s_{(2m+3)+1-i} = s_i + s_{2m+4-i} \]

\[ = i + (2m+4-i) - 2, \text{ (from equation 2.33)} \]

\[ = 2m+2. \]

(ii) For $m - k + 1 \leq i \leq m$, we have

\[ s_i + s_{(2m+3)+1-i} = s_i + s_{2m+4-i} \]

\[ = (i + 1) + (2m + 4 - i) - 3 \]

\[ = 2m + 2. \text{ (from equation (2.33))} \]

(iii) For $i = m + 1$, we have

\[ s_i + s_{(2m+3)+1-i} = s_{m+1} + s_{m+3} \]

\[ = (m+1) + (m+1), \text{ (from equation (2.33))} \]

\[ = 2m + 2. \]

Hence the equation (2.18) is satisfied for $i = 1, 2, \ldots, \left\lfloor \frac{2m+3}{2} \right\rfloor$, so the score sequences $S_k$, for $k = 1, 2, \ldots, \frac{n-3}{2}$, are self-converse. □
2.4 Self-converse tournament score sequences having two consecutive repetitions

In this section we discuss some properties of self-converse score sequences having two consecutive repetitions.

**Definition 2.5.** A score sequence of the form

\[ S = (s_1, \ldots, p \times t, p \times (t+1), \ldots, s_n) \]

is said to be a score sequence with two consecutive repetitions if two consecutive entries, say \(t\) and \(t+1\), repeat \(p\) times and remaining other are distinct.

**Theorem 2.14.** If \(S\) is a self-converse score sequence of even order \(n\) with two consecutive repetitions, then the repeated entries are \(n-2\) and \(n\).

**Proof.** Let \(n = 2m\) and \(S = (s_1, s_2, \ldots, s_n)\) be a self-converse score sequence having two consecutive repetitions.

Thus \(s_i + s_{2m-i+1} = 2m-1\)

At \(i = m\), \(s_m + s_{m+1} = 2m-1\)

\[ \Rightarrow \quad 2s_m + 1 = 2m-1 \]

\[ \Rightarrow \quad s_m = m - 1. \]
Remark 2.3. There is no self-converse score sequence of odd order $n$ having two consecutive repetitions.

Next result deals with strong self-converse score sequences having two consecutive repetitions.

**Theorem 2.15.** If $S = (s_1, \ldots, p \times \frac{n-2}{2}, p \times \frac{n}{2}, \ldots, n-1-s_1)$ be a self-converse score sequence of even order $n$ ($n \geq 4$) with two consecutive repetitions. Then

\[(i) \quad S' = (1, s_1+1, \ldots, p \times \frac{n}{2}, p \times \frac{n+2}{2}, \ldots, n-s_1, n)\]

\[(ii) \quad S'' = (s_1+1, \ldots, (p+1) \times \frac{n}{2}, (p+1) \times \frac{n+2}{2}, \ldots, n-s_1)\]

are strong self-converse score sequences of order $n+2$ having two consecutive repetitions.

**Proof.** The proof is similar to the proof of Theorem 2.10. Hence we omit the details. The following figures 2.3 and 2.4 represent tournaments $T(S')$ and $T(S'')$ realizing score sequences $S'$ and $S''$. 
Tournament $T(S^*)$

Figure 2.3
Let $ts(n)$ represents the number of self-converse strong score sequences of order $n$ having two consecutive repetitions.

We state the following conjecture for the value of $ts(n)$:

**Conjecture 2.2.** If $n$ is even then

$$ts(n) = 2^{(n-2)/2} \text{ for } n \geq 4.$$
CHAPTER THREE
CHAPTER 3

ON IMBALANCE SEQUENCES OF ORIENTED GRAPHS

Dhruv et al. [15] gave the concept of imbalance of a vertex $v$ and reported a necessary and sufficient condition for a sequence of integers to be an imbalance sequence of a simple directed graph. In this chapter we study some properties of imbalance sequences in oriented graphs. We characterize irreducible imbalance sequences of oriented graphs and bounds for imbalance $b_i$ of a vertex $v_i$ of an oriented graph are obtained. In the last we report a result on an imbalance sequence for a self-converse tournament and conjecture that it is true for oriented graphs.

3.1 Introduction

A sequence of integers is graphic if it is a degree sequence of a simple undirected graph. For characterization of graphic sequences we refer to [18,26,27,30]. Dhruv et al. [15] defined the imbalance $b(v)$ of a vertex $v$. 
**Definition 3.1.** The imbalance $b(v)$ of a vertex $v$ in a digraph is defined as $d^+(v) - d^-(v)$, where $d^+(v)$ is the out-degree of $v$ and $d^-(v)$ is the in-degree of $v$.

Below we report some definitions as given in [15].

**Definition 3.2.** A sequence of integers $A = (a_1, a_2, ..., a_n)$ with $a_1 \geq a_2 \geq ... \geq a_n$ is feasible if it has sum zero and satisfies

$$\sum_{i=1}^{k} a_i \leq k(n - k) \text{ for } 1 \leq k < n$$

(3.1)

**Definition 3.3.** $a_i$ is said to be smaller than $a_j$ if $a_i < a_j$ or if $a_i = a_j$ and $i < j$.

From a feasible sequence $A = (a_1, a_2, ..., a_n)$, we form another feasible sequence $\hat{A} = (\hat{a}_2, ..., \hat{a}_n)$ by deleting $a_1$ and adding 1 to the $a_i$ smallest elements of $A$.

The complication in defining $\hat{A}$ arise when $a_n - a_1 > 1$; in this case there is a gap consisting of elements to which we add zero. The following example with $n = 9$ and $a_1 = 5$ produces such a gap, since $a_5 = a_4 = 2$.

$$A = \begin{pmatrix} 5, 3, 2, 2, 2, -5, -5, -6 \end{pmatrix}$$

$$- \begin{array}{cccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{array}$$
Chapter 3: On imbalance sequences of oriented graphs

\[ \hat{A} : (-, 3, 3, 3, 2, 2, -4, -4, -5) \]

**Theorem 3.1.** [15] If \( A \) is feasible, then \( \hat{A} \) is feasible.

### 3.2 Necessary and sufficient condition

The next Theorem 3.2 provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple directed graph (without repeated arcs).

**Theorem 3.2.** [15] A sequence is realizable as an imbalance sequence if and only if it is feasible.

The above result is equivalent to saying that a sequence of integers \( B = (b_1, b_2, ..., b_n) \) with \( b_1 \geq b_2 \geq ... \geq b_n \) is an imbalance sequence of a simple directed graph (without repeated arcs) if and only if

\[
\sum_{i=1}^{k} b_i \leq k(n - k) \text{ for } 1 \leq k < n \tag{3.2}
\]

with equality when \( k = n \).

On arranging the imbalance sequence in nondecreasing order, we obtain the following Corollary 3.3.
**Corollary 3.3.** A sequence of integers \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \cdots \leq b_n \) is an imbalance sequence of a simple directed graph (without repeated arcs) if and only if

\[
\sum_{i=1}^{k} b_i \geq k(k - n), \text{ for } 1 \leq k < n
\]

(3.3)

with equality when \( k = n \).

**Proof.** Let \( \bar{b}_i = b_{n-i+1} \). Then the sequence \( \bar{B} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n) \) satisfies condition (3.2).

We have

\[
\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} \bar{b}_{n-i+1}
\]

\[
= \sum_{i=1}^{n-k} \bar{b}_{n-i+1} - \sum_{i=k+1}^{n} \bar{b}_{n-i+1}
\]

\[
= 0 - (\bar{b}_{n-k} + \bar{b}_{n-k+1} + \cdots + \bar{b}_1)
\]

\[
= - \sum_{i=1}^{n-k} \bar{b}_i
\]

\[
\geq - (n - k)(n - (n - k)) \text{ (from Condition 3.2)}
\]

\[
= k(k - n),
\]

where \( 1 \leq k \leq n - 1 \) and equality holds when \( k = n \). \( \square \)
3.3 Construction of an oriented graph with a given imbalance sequence

Klietman and Wang [33] observed that Havel and Hakimi [26,30] argument works with the deletion of the any element $d_k$ of the degree sequence $(d_1,d_2,...,d_n)$ with $d_1 \leq d_2 \leq ... \leq d_n$, subtracting 1 from the $d_k$ largest other elements.

The analogous statement about imbalance sequence is false. Dhruv et al. [15] considered the imbalance sequence $(3,1,-1,-3)$ of a transitive tournament. Deleting the element $1$ and adding 1 to the smallest imbalance gives $(3,-1,-2)$, which has no realization by a simple digraph.

Theorem 3.2 provides us an algorithm to construct an oriented graph from a given imbalance sequence. At each stage we form $\hat{B} = (\hat{b}_2,...,\hat{b}_n)$ from $B = (b_1,b_2,...,b_n)$ by deleting the largest imbalance $b_1$ and adding 1 to $b_1$ smallest elements of $B$. Arcs of an oriented graph are defined by $\mathcal{U}_i \rightarrow v$ if and only if $\hat{b}_i \neq b_v$. If this procedure applied recursively, then

(i) it test whether $B$ is an imbalance sequence and if $B$ is
an imbalance sequence, then

(ii) an oriented graph $D_B$ with imbalance sequence $B$ is constructed.

**Example 3.1.**

Suppose $n = 5$ and $B = (2,0,0,0,-2)$

<table>
<thead>
<tr>
<th>Stage</th>
<th>$B$</th>
<th>Arcs of $D_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(2,0,0,0,-2)$</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>$(-1,0,0,-1)$</td>
<td>$1 \rightarrow 2, 5$</td>
</tr>
<tr>
<td>3.</td>
<td>$(-,-,0,0,0)$</td>
<td>$2 \rightarrow 5$</td>
</tr>
</tbody>
</table>

**Table 3.1**

![Figure 3.1](image.png)
Theorem 3.4. [14] Among realizations of $B = (b_1, b_2, \ldots, b_n)$, the oriented graph $D_B$ has fewest arcs.

3.4 Irreducible imbalance sequences of oriented graphs

An oriented graph $D$ is reducible if it is possible to partition its vertices into two nonempty sets $V_1$ and $V_2$ in such a way that every vertex of $V_2$ is adjacent to all vertices of $V_1$. Let $D_1$ and $D_2$ be induced digraphs having vertex sets $V_1$ and $V_2$ respectively. Then $D$ consists of all the arcs of $D_1, D_2$ and every vertex of $D_2$ is adjacent to all vertices of $D_1$. We write $D = [D_1, D_2]$. If this is not possible, then the oriented graph $D$ is irreducible. Let $D_1, D_2, \ldots, D_k$ be irreducible oriented graphs with disjoint vertex sets. $D = [D_1, D_2, \ldots, D_k]$ denotes the oriented graph having all arcs of $D_m$, $1 \leq m \leq k$, and every vertex of $D_j$ is adjacent to all vertices of $D_i$ with $1 \leq i < j \leq k$. $D_1, D_2, \ldots, D_k$ are called irreducible components of $D$. Such decomposition is known as irreducible component decomposition of $D$ and is unique.

Definition 3.4. An imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is said to be irreducible if all the oriented graphs with the imbalance sequence $B$ are irreducible.
In case of tournaments, the score sequence $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \ldots \leq s_n$ used to decide whether a tournament $T$ having the score sequence $S$ is strong or not [28]. This is not true in case of oriented graphs. For example oriented graphs $D_1$ and $D_2$ in Figure (3.2) both have imbalance sequence $(0,0,0)$, but $D_1$ is strong and $D_2$ is not.

![Figure (3.2)](image)

The following Theorem characterizes irreducible imbalance sequences.

**Theorem 3.5.** Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be an imbalance sequence of oriented graph. Then $B$ is irreducible if and only if

$$\sum_{i=1}^{k} b_i > k(k-n) \text{ for } 1 \leq k \leq n-1 \quad (3.4)$$
and $\sum_{i=1}^{n} b_i = 0$. \hspace{1cm} (3.5)

**Proof.** Suppose $D$ is an oriented graph with vertex set $V$, having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$. Equality condition (3.5) is obvious. To prove inequalities (3.4), let $U$ be the set of $k$ vertices with the smallest imbalances, the arcs within $U$ contribute nothing to $\sum_{i=1}^{k} b_i$, and the ordered pairs $(V \setminus U) \times U$ contributes at most $-1$ to each $v \in U$, so

$$\sum_{i=1}^{k} b_i \geq -k(n-k)$$

$$= k(k-n), \text{ for } 1 \leq k \leq n-1 \hspace{1cm} (3.6)$$

Since $D$ is irreducible, there must exist at least one arc from a vertex of $U$ to a vertex of $V \setminus U$.

So condition (3.6) becomes,

$$\sum_{i=1}^{k} b_i \geq k(k-n) + 2$$

$$> k(k-n), \text{ for } 1 \leq k \leq n-1.$$ 

For the converse, suppose that conditions (3.4) and (3.5) hold. Hence from Corollary 3.3 there exist an oriented graph $D$
having imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \).

Suppose that such an oriented graph is reducible. Then there exist a vertex set \( W \) with \( k \) vertices (\( k < n \)), such that every vertex of \( V \setminus W \) is adjacent to all the vertices of \( W \). Hence

\[
\sum_{i=1}^{k} b_i = k (k - n),
\]

a contradiction, proving the converse part. \( \square \)

Rearranging the imbalance sequence \( B = (b_1, b_2, \ldots, b_n) \) in order \( \tilde{b}_1 \geq \tilde{b}_2 \geq \ldots \geq \tilde{b}_n \), we obtain following Corollary 3.6.

**Corollary 3.6.** Let \( D \) be an oriented graph having imbalance sequence \( B = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) \) with \( \tilde{b}_1 \geq \tilde{b}_2 \geq \ldots \geq \tilde{b}_n \). Then \( D \) is irreducible if and only if

\[
\sum_{i=1}^{k} \tilde{b}_i < k(n - k) \text{ for } 1 \leq k \leq n
\]

and \( \sum_{i=1}^{n} \tilde{b}_i = 0 \).

**Note:** The proof is similar to the proof of Corollary 3.3.

The next result is an extension of Theorem 3.5.
Theorem 3.7. Let $D$ be an oriented graph having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$. Suppose that

$$\sum_{i=1}^{p} b_i = p(p - n),$$

$$\sum_{i=1}^{q} b_i = q(q - n)$$

and

$$\sum_{i=1}^{k} b_i > k(k - n), \text{ for } p + 1 \leq k \leq q - 1,$$

where $0 \leq p < q \leq n$.

Then subdigraph induced by the vertices $\{v_{p+1}, v_{p+2}, \ldots, v_q\}$ is an irreducible component of $D$ with imbalance sequence

$$(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q).$$

Proof. Suppose imbalance of vertex $v_i$ in oriented graph $D$ is $b_i$, $1 \leq i \leq n$. Since $\sum_{i=1}^{q} b_i = q(q - n)$, so clearly each vertex of $W = \{v_{q+1}, v_{q+2}, \ldots, v_n\}$ dominates all vertices of $\{v_1, v_2, \ldots, v_q\}$. Thus the vertices within $W$ contributes $-(n - q)$ to imbalance of every vertex of $\{v_1, v_2, \ldots, v_q\}$.

Also $\sum_{i=1}^{p} b_i = p(p - n)$, so each vertex of $V = \{v_{p+1}, v_{p+2}, \ldots, v_q\}$ dominates all vertices of $U = \{v_1, v_2, \ldots, v_p\}$. So
vertices within $U$ contribute $p$ to imbalance of every vertex of $V$.

Hence the imbalance sequence of subdigraph induced by vertices $\{v_{p+1}, v_{p+2}, \ldots, v_q\}$ is

$$(b_{p+1} + n - q - p, b_{p+2} + n - q - p, \ldots, b_q + n - q - p)$$

i.e., $$(b_{p+1} + n - p - q, b_{p+2} + n - p - q, \ldots, b_q + n - p - q).$$

Now we have to show that above imbalance sequence is irreducible. We have

$$\sum_{i=1}^p b_i > k(k - n)$$

$$\Rightarrow \sum_{i=1}^p b_i + \sum_{i=p+1}^q b_i > k(k - n)$$

$$\Rightarrow p(p - n) + \sum_{i=p+1}^q b_i + (k - p)(n - p - q) > k(k - n) + (k - p)(n - p - q)$$

$$\Rightarrow \sum_{i=p+1}^q (b_i + n - p - q) > k(k - n) + (k - p)(n - p - q) - p(p - n)$$

$$= k^2 - kp - kq + pq$$

$$= (k-p)k - (k-p)q$$

$$= (k-p)(k-q).$$
Thus \[ \sum_{i=p+1}^{k} (b_i + n - p - q) > (k - p)((k - p) - (q - p)), \]

and \[ \sum_{i=p+1}^{q} (b_i + n - p - q) = \frac{q}{n} b_i + (q - p)(n - p - q) \]

\[ = \frac{q}{n} b_i - \sum_{i=p+1}^{k} b_i + (q - p)(n - p - q) \]

\[ = q(q-n) - p(p-n) + (q - p)(n - p - q) \]

\[ = 0. \]

Hence by Theorem 3.5 the imbalance sequence is irreducible.

\[ \square \]

Theorem 3.7 shows that the irreducible components of \( B \) are determined by the successive values of \( k \) for which

\[ \sum_{i=1}^{k} b_i = k(k - n) \text{ for } 1 \leq k \leq n \] (3.7)

We explain it with the help of following example.

**Example 3.2.** Let \( B = (-6, -5, -4, 1, 1, 1, 6, 6) \). Equation (3.7) is satisfied for \( k = 3, 6 \) and 8. So the irreducible components of \( B \) are \((-1, 0, 1), (0, 0, 0) \) and \((0, 0)\).
3.5 The bounds of imbalances

The converse of an oriented graph $D$ is an oriented graph $D'$, obtained by reversing orientation of all arcs of $D$. Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be imbalance sequence of an oriented graph $D$.

Then $B' = (-b_n, -b_{n-1}, \ldots, -b_1)$.

Next result gives lower and upper bounds for the imbalance $b_i$ of a vertex $v_i$ of an oriented graph $D$.

**Theorem 3.8.** If $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of an oriented graph $D$, then for each $i$, $i - n \leq b_i \leq i - 1$.

**Proof.** First, we prove that $b_i \geq i - n$.

Suppose that $b_i < i - n$ then, for every $k < i$ $b_k \leq b_i < i - n$.

So that,
\begin{align*}
\sum_{k=1}^{i} b_k &< \sum_{k=1}^{i} (i-n) \\
\Rightarrow \sum_{k=1}^{i} b_k &< i(i-n).
\end{align*}

As \( B = (b_1, b_2, \ldots, b_n) \) is an imbalance sequence so, by Corollary 3.3,
\begin{align*}
\sum_{k=1}^{i} b_k &\geq i(i-n).
\end{align*}

Which is a contradiction. Hence
\begin{equation}
(i-n) \leq b_i. \tag{3.8}
\end{equation}

The second inequality is dual to the first. In the converse oriented graph \( D' \) with imbalance sequence \( B' = (b'_1, b'_2, \ldots, b'_n) \).

We have
\begin{align*}
b'_{n-i+1} &\geq (n-i+1)-n \text{ (using condition 3.8)} \\
&= 1 - i
\end{align*}

but \( b_i = -b'_{n-i+1} \)

So, \( b_i \leq - (1-i) = i - 1 \).

Proving the result. \( \square \)
3.6 Lexicographic enumeration of imbalance sequences

If we know the first sequence in a lexicographic enumeration, then we can generate all the imbalance sequences of a given order provided we know how to get the immediate successor of any given imbalance sequence. To generate the immediate successor of any given imbalance sequence we use the idea of Gervacio [20].

We note two important facts, first, the imbalance sequence \((- (n-1), - (n-3), \ldots, n-3, n-1)\) is not successor of any imbalance sequence of order \(n\), hence it is always the first in the lexicographic enumeration. Second, \((b_1, b_2, \ldots, b_n)\) has no successor if and only if \(b_n - b_1 \leq 0\).

The immediate successor \((b_1', b_2', \ldots, b_n')\) with \(b_1' \leq b_2' \leq \ldots \leq b_n'\) of imbalance sequence \((b_1, b_2, \ldots, b_n)\) with \(b_1 \leq b_2 \leq \ldots \leq b_n\) can be find as follows.
begin

1. Determine the maximum $k$ such that $b_n - b_k \geq 2$.

2. for $i \leftarrow 1$ until $k - 1$ do

   $b'_{i} = b_{i};$

3. $b'_{k} = b_{k} + 1;$

4. $j = k + 1$

5. $b'_{i} = b_{k} + 1;$

6. while $\sum_{i=1}^{k} b'_{i} \geq j(j-1)$ do

   begin

   7. $j = j + 1;$

   8. goto 5;

   end

9. $t \leftarrow j;$

10. $b'_{i} = t(t - n) - \sum_{i=1}^{t} b'_{i};$

11. for $i \leftarrow t + 1$ until $n$ do

    $b'_{i} = 2i - n - 1$

end
Chapter 3: On imbalance sequences of oriented graphs

The following example lists all imbalance sequences of order 3 arranged in lexicographic order.

**Example 3.3**

<table>
<thead>
<tr>
<th>Sequence No.</th>
<th>Imbalance sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(-2, 0, 2 )</td>
</tr>
<tr>
<td>2.</td>
<td>(-2, 1, 1 )</td>
</tr>
<tr>
<td>3.</td>
<td>(-1, -1, 2 )</td>
</tr>
<tr>
<td>4.</td>
<td>(-1, 0, 1 )</td>
</tr>
<tr>
<td>5.</td>
<td>( 0, 0, 0 )</td>
</tr>
</tbody>
</table>

**Table 3.2**

Below we list the number of imbalance sequences of order $n$, denoted by $b(n)$. However a general formula for $b(n)$ is not known.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(n)$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>59</td>
<td>247</td>
<td>1111</td>
<td>5302</td>
<td>26376</td>
<td>125670</td>
</tr>
</tbody>
</table>

**Table 3.3**
**Definition 3.5.** Let $B = (b_1, b_2, \ldots, b_m)$ with $b_1 < b_2 < \ldots < b_m$ and $C = (c_1, c_2, \ldots, c_n)$ with $c_1 < c_2 < \ldots < c_n$ are two imbalance sequences of order $m$ and $n$ respectively. Then we define

$$B + C = (b_1 - n, b_2 - n, \ldots, b_m - n, c_1 + m, c_2 + m, \ldots, c_n + m)$$

The plus operation defined above is not commutative but it is associative.

Now we establish some results dealing with imbalance sequences that are tournament analogue to Merajuddin [38].

**Theorem 3.9.** Let $B_1 = (b_1, b_2, \ldots, b_n)$ with $b_1 < b_2 < \ldots < b_n$ and $B_2 = (-n, b_1 + 1, b_2 + 1, \ldots, b_n + 1)$. Then $B_1$ is $m^{th}$ imbalance sequence of order $n$ if and only if $B_2$ is the $m^{th}$ imbalance sequence of order $(n + 1)$.

**Proof.** Suppose $D_1$ be a realization of $B_1$. Then $D_2 = [K, D_1]$, where $K$ is an oriented graph of order 1, is a realization of $B_2$. This shows that $B_2$ is an imbalance sequence when $B_1$ is an imbalance sequence. For converse, suppose $D$ be a realization of $B_2$. We can write $D = [U, W]$, where $U$ is an oriented graph of order 1. Clearly $W$ is a realization of $B_1$. This shows that $B_1$ is an imbalance sequence when $B_2$ is an imbalance sequence.
The unique correspondence shows that both are occupying the same position. □

Let $b_k(n)$ denotes the number of imbalance sequences of order $n$, in nondecreasing order, having imbalance $k$ at least once, for $1 - n \leq k \leq n - 1$. Then we have the following results.

**Theorem 3.10.**

(i) $b_k(n) = b_{-k}(n)$

(ii) $b_{1-n}(n) = b(n-1)$

(iii) $b_{n-1} = b(n-1)$

**Proof.** (i) This is equivalent to proving that whenever $B = (b_1, b_2, ..., b_n)$ with $b_1 \leq b_2 \leq ... \leq b_n$ is an imbalance sequence, then $B' = (-b_n, -b_{n-1}, ..., b_1)$ is also an imbalance sequence. This always happens, since $B$ is an imbalance sequence of an oriented graph $D$ if and only if $B'$ is an imbalance sequence of oriented graph $D'$, the converse of $D$.

(ii) Let $B_1 = (b_1, b_2, ..., b_{n-1})$ be the last imbalance, i.e., $b(n-1)^{th}$ imbalance sequence of order $n-1$. By Theorem 3.9 $B_2 = (- (n-1), b_1+1, b_2+1, ..., b_{n-1}+1)$ is the $b(n-1)^{th}$ imbalance
sequence of order \( n \). Now we show that there does not exist any imbalance sequence \( B_3 = (t_1, t_2, \ldots, t_n) \), \( B_3 \neq B_2 \) such that \( t_1 = -(n-1) \) and \( B_2 \preceq B_3 \).

Suppose that there exists one such \( B_3 \). Then by Theorem 3.9, \( B_4 = (t_2-1, \ldots, t_{n-1}) \) is an imbalance sequence of order \( n - 1 \) and \( B_1 \preceq B_4 \), a contradiction as \( B_1 \) is the last imbalance sequence of order \( (n-1) \). Thus \( B_2 \) is the last imbalance sequence of order \( n \) in which the first entry is \( -(n-1) \).

Hence \( b_{1-n}(n) = b(n-1) \).

(iii) Putting \( k = n - 1 \) in Theorem 3.10(i), we get

\[
b_{n-1} = b_{1-n}
\]

and from Theorem 3.10(ii),

\[
b_{1-n} = b(n-1)
\]

Hence \( b_{n-1} = b(n-1) \).

3.8 Self-converse imbalance sequences

Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) be an imbalance sequence of an oriented graph \( D \). Then the imbalance sequence \( B' \) of the oriented graph \( D' \), the converse
of $D$, is given by $(−b_n, −b_{n−1},..., −b_1)$. An oriented graph $D$ is said to be self-converse if $D \cong D'$.

**Definiton 3.6.** An imbalance sequence $B = (b_1, b_2, ..., b_n)$ with $b_1 \leq b_2 \leq ... \leq b_n$ is self-converse if all the oriented graph having imbalance sequence $B$ are self-converse.

Next result characterizes self-converse imbalance sequences of tournaments.

**Theorem 3.11** A sequence of integers $B = (b_1, b_2, ..., b_n)$ with $b_1 \leq b_2 \leq ... \leq b_n$ is self-converse if and only if

$$b_i + b_{n−i+1} = 0, \text{ for } 1 \leq i \leq n.$$

**Proof.** Consider a tournament $T$ having $B = (b_1, b_2, ..., b_n)$ with $b_1 \leq b_2 \leq ... \leq b_n$ and $S = (s_1, s_2, ..., s_n)$ with $s_1 \leq s_2 \leq ... \leq s_n$ as its imbalance and score sequences. As tournament $T$ is self-converse so by Theorem 2.7 [17], we have

$$s_i + s_{n−i+1} = n−1$$

$$\Rightarrow d_i^r + d_{n−i+1} = n−1$$

$$\Rightarrow 2d_i^r + 2d_{n−i+1} = 2(n−1)$$

$$\Rightarrow d_i^r − (n−1−d_i^r) + d_{n−i+1} − (n−1−d_{n−i+1}) = 0$$
Chapter 3: On imbalance sequences of oriented graphs

\[ (d'_v - d'_i) + (d'_{i+1} - d'_{v-1}) = 0 \]
\[ \Rightarrow b_i + b_{n-i+1} = 0. \]

This proves the necessity of Theorem. Converse also follows from Theorem 2.7.

Now we state a conjecture:

**Conjecture 3.1.** The above result is also true for oriented graphs.

Below we obtain following results on self-converse imbalance sequences of tournaments.

**Theorem 3.12.** If \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence of a tournament, then \( B + B' \) is a self-converse imbalance sequence.

**Proof.** Here \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) is an imbalance sequence. By the definition of converse of imbalance sequence, \( B' = (-b_n, -b_{n-1}, \ldots, -b_1) \).

So, by the definition 3.5, we have

\[ B + B' = (b_1 - n, b_2 - n, \ldots, b_n - n, -b_n + n, \ldots, -b_1 + n) \]
\[ = (t_1, t_2, \ldots, t_{2n}), \text{ say} \]
where

\[ t_i = \begin{cases} 
  b_i - n, & \text{for } 1 \leq i \leq n \\
  -b_{2n-i+1} + n, & \text{for } n + 1 \leq i \leq 2n.
\end{cases} \]

Clearly \( t_i + t_{2n-i+1} = 0 \), for \( 1 \leq i \leq 2n \).

Hence \( B + B' \) is self-converse. \( \square \)

**Theorem 3.13.** Let \( B = (b_1, b_2, \ldots, b_n) \) with \( b_1 \leq b_2 \leq \ldots \leq b_n \) be a self-converse imbalance sequence and \( C \) be any other imbalance sequence in nondecreasing order. Then \( C + B + C' \) is a self-converse imbalance sequence.

**Proof.** Suppose that \( C = (c_1, c_2, \ldots, c_m) \) with \( c_1 \leq c_2 \leq \ldots \leq c_m \) is an imbalance sequence of order \( m \). Then by definition of converse

\[ C' = (-c_m, -c_{m-1}, \ldots, -c_1) \]

and by definition 3.5

\[ C + B + C' = (c_1-m-n, c_2-m-n, \ldots, c_m-m-n, b_1, b_2, \ldots, b_n, -c_m + m + n, -c_{m-1} + m + n, \ldots, -c_1 + m + n) \]

\[ = (r_1, r_2, \ldots, r_{2m+n}), \text{ say} \]

Where \( r_i = \begin{cases} 
  c_i - m - n, & \text{for } 1 \leq i \leq m \\
  b_i - m, & \text{for } m + 1 \leq i \leq m + n \\
  -c_{2m+n-i+1} + m + n, & \text{for } m+n+1 \leq i \leq 2m+n.
\end{cases} \)
Case (i). For $1 \leq j \leq m$,

$$r_j + r_{2m+n-j+1} = c_j - m - n - c_j + m + n$$

{ when $1 \leq j \leq m$ then $m + n + 1 \leq 2m + n - j + 1 \leq 2m + 1$ }

$$\Rightarrow r_j + r_{2m+n-j+1} = 0.$$ 

Case (ii). For $m + 1 \leq j \leq m + n$,

$$r_j + r_{2m+n-j+1} = b_{j+m} + b_{m+n-j+1}$$

$$= b_k + b_{n-k+1} \text{ for } k = j - m \text{ and } 1 \leq k \leq n$$

$$= 0.$$

As $B = (b_1, b_2, \ldots, b_n)$ is a self-converse imbalance sequence, so

$$b_i + b_{n-i+1} = 0, \text{ for } 1 \leq i \leq n.$$

From above we have,

$$r_j + r_{2m+n-j+1} = 0, \text{ for } 1 \leq j \leq 2m + n.$$

Hence $C + B + C'$ is a self-converse imbalance sequence. $\square$
CHAPTER FOUR
CHAPTER 4

SOME INEQUALITIES FOR IMBALANCES IN ORIENTED GRAPHS

Some stronger inequalities for scores in tournaments are given by Brualdi and Shen [11]. In this chapter, analogues to Brualdi and Shen [11] we obtain some stronger inequalities for degree imbalances in oriented graphs and thus we give a different proof of Theorem 3.8. In the last we investigate occurrence of equality in inequalities (4.1) given in Theorem 4.1.

4.1 Some stronger inequalities for imbalances

The imbalance sequences of oriented graphs have been studied in chapter 3. Theorem 3.2 gives a necessary and sufficient condition for a sequence of nonnegative integers to be the imbalance sequence of some oriented graph.

An extension of Theorem (3.2) is,
**Theorem 4.1.** A sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of an oriented graph if and only if for every subset $I \subseteq [n],$

\[ \sum_{i \in I} b_i \geq |I|(|I| - n), \quad (4.1) \]

with equality when $|I| = n.$

**Proof.** First consider the necessity of Theorem. Consider an oriented graph $D$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n.$ Suppose $U$ is a set of vertices such that $U = \{v_i : i \in I\}.$ Clearly $|U| = |I|.$ Then vertices within set $U$ contributes nothing to $\sum_{i \in U} b_i$ and the vertices of $V \setminus U$ contribute atleast $-|I|(n-|I|),$ i.e., $|I|(|I| - n).$ Thus

\[ \sum_{i \in I} b_i \geq |I|(|I| - n). \]

For the sufficiency, suppose $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is a sequence of integers satisfying condition 4.1. When $I = [k]$ and $1 \leq k \leq n,$ then condition 4.1 become

\[ \sum_{i \in I} b_i \geq k(k - n). \quad (4.2) \]
So by Corollary 3.3 there exist an oriented graph having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$.

When $I \neq [k]$ and $1 \leq k \leq n$, we have

$$\sum_{i \in I} b_i \geq \sum_{i=1}^{|I|} b_i \geq |I|(|I| - n) \text{ (using condition 4.2)}$$

Hence again from Corollary 3.3 there exist an oriented graph having imbalance sequence $B = (b_1, b_2, \ldots, b_n)$. 

Now we show that the imbalance sequence of an oriented graph satisfies the inequalities (4.3) below, which are individually stronger than the inequalities (4.1).

**Theorem 4.2.** A sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of an oriented graph if and only if for every subset $I \subseteq [n]$,

$$\sum_{i \in I} b_i \geq \sum_{i \in I} (i - 1) + \frac{1}{2} |I|(|I| - 2n + 1) \quad (4.3)$$

with equality when $|I| = n$.

**Proof.** The sufficient part of the Theorem follows from Theorem 4.1, since

$$\sum_{i \in I} (i - 1) \geq \sum_{i=1}^{|I|} (i - 1)$$
Chapter 4: Some inequalities for imbalances in oriented graphs

Hence, 

\[ \frac{1}{2} |I|(|I|-1) \]

Thus inequalities (4.3) imply inequalities (4.1).

Now we prove that imbalance sequence of an oriented graph satisfies (4.3).

Suppose \( B = (b_1, b_2, \ldots, b_n) \) is an imbalance sequence of an oriented graph such that \( b_1 \leq b_2 \leq \ldots \leq b_n \). For any subset \( I \subseteq [n] \), define

\[ f(I) = \sum_{i \in I} b_i - \sum_{i \in I} (i-1) - \frac{1}{2} |I|(|I|-2n+1) \]

First we choose \( I \) to have \( f(I) \) minimum and secondly to have \( |I| \) minimum.

Claim. \( I = \{ i : 1 \leq i \leq |I| \} \).

Otherwise there exist \( i \notin I \) and \( j \in I \) such that \( j = i + 1 \).

Then \( b_i \leq b_j \), since
Chapter 4: Some inequalities for imbalances in oriented graphs

\[ f(I) - f(I \setminus \{j\}) = \sum_{i \in I} b_i - \sum_{i \in I} (i - 1) - \frac{1}{2} |I|(|I| - 2n + 1) \]

\[ - \left[ \sum_{i \in I} b_i - b_i - \sum_{i \in I} (i - 1) + (j - 1) - \frac{1}{2} (|I| - 1)(|I| - 2n) \right] \]

\[ = b_j - (j - 1) + \frac{1}{2} (2n - 2|I|) \]

\[ = b_j - (j - n + |I| - 1) \]

So, \( f(I) - f(I \setminus \{j\}) = b_j - (j - n + |I| - 1) < 0 \)

\[ \Rightarrow \quad b_j < j - n + |I| - 1 \quad (4.4) \]

Also,

\[ f(I \cup \{i\}) - f(I) \]

\[ = \sum_{m \in I} b_m + b_i - \left[ \sum_{m \in I} (m - 1) + (i - 1) \right] - \frac{1}{2} (|I| + 1)(|I| - 2n + 2) \]

\[ - \sum_{m \in I} b_m + \sum_{m \in I} (m - 1) + \frac{1}{2} |I|(|I| - 2n + 1) \]

\[ = b_i - (i - 1) + \frac{1}{2} (2n - 2|I| - 2) \]

\[ = b_i - (i - n + |I|) \]

Thus \( f(I \cup \{i\}) - f(I) = b_i - (i - n + |I|) \geq 0 \)

\[ \Rightarrow \quad b_i - (i - n + |I|) \geq 0 \]
\[
\Rightarrow \quad b_i \geq (i - n + |I|) \tag{4.5}
\]

From conditions (4.4) and (4.5), we have

\[
i - n + |I| \leq b_i \leq b_j \leq j - n + |I| - 1
\]

\[
\Rightarrow \quad i - n + |I| \leq b_i \leq b_j < i - n + |I|, \quad (\text{as } j = i + 1)
\]

a contradiction. This proves the claim.

Therefore,

\[
f(I) = \sum_{i=1}^{|I|} b_i - \sum_{i=1}^{|I|} (i-1) - \frac{1}{2} |I|(|I| - 2n + 1)
\]

\[
= \sum_{i=1}^{|I|} b_i - \frac{1}{2} |I|(|I| - 1) - \frac{1}{2} |I|(|I| - 2n + 1)
\]

\[
= \sum_{i=1}^{|I|} b_i - |I|(|I| - n)
\]

\[
\geq 0
\]

Where the inequality follows from Theorem 4.1. By the choice of subset \( I \), Theorem 4.2 follows. \( \Box \)

Clearly Theorem 4.2 gives a lower bound for \( \sum_{i=1}^{|I|} b_i \). It is natural to ask for an upper bound for \( \sum_{i=1}^{|I|} b_i \). The following Corollary 4.3 gives an upper bound for \( \sum_{i=1}^{|I|} b_i \).
Corollary 4.3. A sequence of integers $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of an oriented graph if and only if for every subset $I \subseteq [n],$

$$\sum_{i \in I} b_i \leq \sum_{i \in I} (i-1) - \binom{|I|}{2}$$

with equality when $|I| = n.$

Proof. Let $J = [n] \setminus I,$ then

$$\sum_{i \in J} b_i \leq \sum_{i \in J} (i-1) - \binom{|J|}{2},$$

with equality when $|J| = n,$ if and only if

$$\sum_{i \in I} b_i \geq \sum_{i \in J} (i-1) + \frac{1}{2} |J|(|J| - 2n + 1),$$

with equality when $|J| = n.$ Which imply,

$$\sum_{i \in [n]} b_i - \sum_{i \in I} b_i \geq \sum_{i \in [n]} (i-1) - \sum_{i \in I} (i-1) + \frac{1}{2} (n-|I|)(1-|I|-n)$$

$$\Rightarrow 0 - \sum_{i \in I} b_i \geq \frac{1}{2} n(n-1) - \sum_{i \in I} (i-1) + \frac{1}{2} (n - n^2 - |I| + |I|^2)$$

$$\Rightarrow \sum_{i \in I} b_i \leq \sum_{i \in [n]} (i-1) - \frac{1}{2} |I|(|I| - 1)$$
Theorem 4.4. Suppose that a sequence of integers $B = (b_1, b_2, ..., b_n)$ with $b_1 \leq b_2 \leq ... \leq b_n$ is an imbalance sequence of an oriented graph. Then for each $i$,

$$i - n \leq b_i \leq i - 1.$$ 

**Proof.** Putting $I = \{i\}$ in Theorem 4.2 we get

$$\sum_{r \in I} b_r \geq \sum_{r \in I} (i - 1) + \frac{1}{2} (1 - 2n + 1)$$

$$\Rightarrow \quad b_i \geq (i - 1) + (1 - n)$$

$$\Rightarrow \quad b_i \geq (i - n). \quad (4.7)$$

Again putting $I = \{i\}$ in Corollary 4.3 we have

$$b_i \leq (i - 1) - \binom{1}{2}$$
Combining conditions (4.7) and (4.8), we get the desired result. □

4.2 Equality in stronger inequalities

In this section, we investigate existence of equality in inequalities (4.3). For any integers $r$ and $s$ with $r \leq s$, let $[r,s]$ denotes the set of all integers between $r$ and $s$.

We have following results.

**Theorem 4.5.** Let $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ be an imbalance sequence of an oriented graph. If

\[
\sum_{i \in I} b_i = \sum_{i \in I} (i - 1) + \frac{1}{2} |I|(|I| - 2n + 1) \quad (4.9)
\]

for some $I \subseteq [n]$, then one of the following holds

(i) $I = [1, |I|]$ and

\[
\sum_{i \in I} b_i = |I|(|I| - n)
\]

(ii) $I = [t, t + |I| - 1]$ for some $t$, $2 \leq t \leq n - |I| + 1$,

\[
\sum_{i \in I} b_i = (t + |I| - 1)(t + |I| - 1 - n)
\]
Chapter 4: Some inequalities for imbalances in oriented graphs

and \( b_i = t + |I| - 1 - n \), for all \( i \leq t + |I| - 1 \).

(iii) \( I = [1, r] \cup [r + t, t + |I| - 1] \) for some \( r \) and \( t \) such that

\[
1 \leq r \leq |I| - 1 \quad \text{and} \quad 2 \leq t \leq n - |I| + 1,
\]

\[
\sum_{i=1}^{r} b_i = r(r - n), \quad \sum_{i=1}^{t+|I|-1} b_i = (t + |I| - 1)(t + |I| - 1 - n) \quad \text{and}
\]

\[
b_i = (r + t + |I| - n - 1) \quad \text{for all} \quad i, \; r + 1 \leq i \leq t + |I| - 1.
\]

**Proof.** For any subset \( J \subseteq [n] \), we define \( f(J) \) as in Theorem 4.2, i.e.,

\[
f(J) = \sum_{i \in J} b_i - \sum_{i \not\in J} (i - 1) - \frac{1}{2} |J|(|J| - 2n + 1)
\]

So,

\[
f(I) = \sum_{i \in I} b_i - \sum_{i \not\in I} (i - 1) - \frac{1}{2} |I|(|I| - 2n + 1).
\]

By condition (4.9), we have

\[
f(I) = 0 \quad \text{and} \quad f(J) \geq 0 \quad \text{for all} \quad J \subseteq [n].
\]

Now we prove that \( I \) is one of the three types as given in the statement of the Theorem 4.5. Contrary suppose that this is not true. Then there exist indices \( i, j, k, \) and \( I \) with \( j = i + 1 \) and \( l = k + 1 \) such that \( \{i, k\} \cap I = \emptyset \) and \( \{j, l\} \subseteq I \). Thus \( b_i \leq b_j \) and \( b_k \leq b_l \).
We have

\[ f(I) - f(I \setminus \{j, l\}) = \sum_{i \in I} b_i - \sum_{i \in I} (i-1) - \frac{1}{2} |I|(|I| - 2n + 1) - \sum_{i \in I} b_i + b_j + b_i + \left\{ \sum_{i \in I} (i-1) - (j-1) - (l-1) \right\} + \frac{1}{2} (|I| - 2)(|I| - 2n - 1) \]

\[ = b_j + b_i - (j + l + 2|I| - 2n - 3) \tag{4.10} \]

And

\[ f(I \cup \{i, k\}) - f(I) = \sum_{m \in I} b_m + b_i + b_k - \left\{ \sum_{m \in I} (m-1) + (i-1) + (k-1) \right\} - \frac{1}{2} (|I| + 2)(|I| - 2n + 3) - \sum_{m \in I} b_m + \sum_{m \in I} (m-1) + \frac{1}{2} |I|(|I| - 2n + 1) \]

\[ = b_i + b_k - (i + k - 2n + 2|I| + 1). \tag{4.11} \]

Since \( f(I) - f(I \setminus \{j, l\}) \leq 0 \) and \( f(I \cup \{i, k\}) - f(I) \geq 0 \), therefore from equations (4.10) and (4.11), we get

\[ b_j + b_i - (j + l + 2|I| - 2n - 3) \leq 0 \]

and \[ b_i + b_k - (i + k - 2n + 2|I| + 1) \geq 0 \]

\[ \Rightarrow b_j + b_i \leq (j + l + 2|I| - 2n - 3) \]
and \( b_i + b_k \geq (i + k - 2n + 2|l| + 1) \)

\[ \Rightarrow \quad i + k - 2n + 2|l| + 1 \leq b_i + b_k \leq b_j + b_l \leq j + l + 2|l| - 2n - 3, \text{ this gives,} \]

\[ i + k - 2n + 2|l| + 1 \leq i + k - 2n + 2|l| - 1 \quad \text{(as } j = i + 1 \text{ and } l = k + 1) \]

Which imply \( 1 \leq -1 \), a contradiction. This proves that \( l \) satisfies one of the conditions (i), (ii) or (iii).

**Case (i).** \( l = [1, |l|] \), then

\[
f(l) = \sum_{i=1}^{\lfloor l \rfloor} b_i - \sum_{i=1}^{\lfloor l \rfloor} (i-1) - \frac{1}{2}|l|(|l| - 2n + 1)
\]

\[
= \sum_{i=1}^{\lfloor l \rfloor} b_i - \frac{1}{2}|l|(|l| - 1) - \frac{1}{2}|l|(|l| - 2n + 1)
\]

\[
= \sum_{i=1}^{\lfloor l \rfloor} b_i - |l|(|l| - n),
\]

and so, \( \sum_{i=1}^{\lfloor l \rfloor} b_i = |l|(|l| - n) \), as \( f(l) = 0 \).

For other two cases we require the following claim.

**Claim 1.** If there exit indices \( i \) and \( j \) with \( j = i + 1 \) such that \( i \notin I \) and \( j \in I \), then
Chapter 4: Some inequalities for imbalances in oriented graphs

\[ f(I \cup \{i\}) = f(I \setminus \{j\}) = 0. \]

Since, \( f(I) - f(I \setminus \{j\}) = b_j - (j - n + |I| - 1) \leq 0, \)

and \( f(I \cup \{i\}) - f(I) = b_i - (i - n + |I|) \geq 0 \)

\[ \Rightarrow \quad b_j \leq j - n + |I| - 1 \quad \text{and} \quad b_i \geq i - n + |I| \]

\[ \Rightarrow \quad i - n + |I| \leq b_i \leq b_j \leq j - n + |I| - 1 \]

\[ \Rightarrow \quad i - n + |I| \leq b_i \leq b_j \leq i - n + |I|. \]

This implies that equalities hold throughout all the above inequalities. Thus

\[ f(I \cup \{i\}) = f(I) = f(I \setminus \{j\}) = 0. \]

**Case (ii).** \( I = [t, t + |I| - 1] \) for some \( t, 2 \leq t \leq n - |I| + 1. \)

By applying Claim 1 recursively, we have

\[ f(t + |I| - 1) = 0 \]

i.e., \( b_{t, |I| - 1} = (t + |I| - 2) - \frac{1}{2}(2 - 2n) = 0 \)

\[ \Rightarrow \quad b_{t, |I| - 1} = (t + |I| - 1 - n). \]

Since, \( \sum_{i=1}^{r} b_i \geq (t + |I| - 1) (t + |I| - 1 - n) \)
and \( b_1 \leq b_2 \leq \ldots \leq b_{|I|-1} \), hence equality holds throughout all the above inequalities.

**Case (iii).** \( I = [1, r] \cup [r + t, t + |I| - 1] \) for some \( r \) and \( t \) such that \( 1 \leq r \leq |I| - 1 \) and \( 2 \leq t \leq n - |I| + 1 \). Again applying claim 1 recursively, we have

\[
\begin{align*}
f([1, r]) &= f([1, t + |I| - 1]) = f([1, r] \cup \{t + |I| - 1\}) = 0
\end{align*}
\]

Therefore,

\[
\begin{align*}
f([1, r]) &= \sum_{i=1}^{r} b_i - r(r - n) = 0 \quad (4.12)
\end{align*}
\]

\[
\begin{align*}
f([1, t + |I| - 1]) &= \sum_{i=1}^{r+|I|-1} b_i - (t + |I| - 1)(t + |I| - 1 - n) = 0
\end{align*}
\]

(4.13)

and \( f([1, r] \cup \{t + |I| - 1\}) - f([1, r]) \)

\[
\begin{align*}
&= \sum_{i=1}^{r} b_i + b_{|I|-1} - \left\{ \sum_{i=1}^{r} (i-1) + (t + |I| - 2) \right\} \\
&\quad - \frac{1}{2} (r+1)(r-2n+2) - \sum_{i=1}^{r} b_i + \sum_{i=1}^{r} (i-1) - \frac{1}{2} r(r - 2n + 1) \\
&= b_{|I|-1} - (|I| + r + t - n)
\end{align*}
\]

As \( f([1, r] \cup \{t + |I| - 1\}) - f([1, r]) = 0 \), we have
Chapter 4: Some inequalities for imbalances in oriented graphs

\[ b_{r+1} = |T| + r + t - n. \]

Since,

\[ \sum_{i=r+1}^{s} b_i = \sum_{i=1}^{s} b_i - \sum_{i=1}^{r} b_i, \]

\[ = (t + |T| - 1) (t + |T| - 1 - n) - r(r - n) \]

( using equations (4.12) and (4.13))

\[ = |T|^2 + 2t|T| - n|T| - 2|T| + t^2 - r^2 - nt + rn - 2t \]

\[ + n + 1 \]

\[ = (|T| + r + t - n - 1) (|T| - r + t - 1), \]

and \( b_{r+1} \leq b_{r+2} \leq \ldots \leq b_{s+|T|-1}, \) we have

\[ b_i = r + t + |T| - n - 1 \text{ for all } i, \ r + 1 \leq i \leq t + |T| - 1. \]

This completes the proof of the Theorem 4.5. \( \square \)
CHAPTER FIVE
CHAPTER 5

ON SCORE SEQUENCES OF

k-HYPERTOURNAMENTS

Hypertournaments have been studied by a number of authors (Cf. Assous[3], Barbut and Bialostocki[6], Frankl[18], Gutin and Yeo[25]). Reid [43], in section 8, described several results on hypertournaments obtained by the authors and posed some interesting problems on the topic. In particular, he raised the problem to extend the most important results on tournaments to hypertournaments. In this chapter we study properties of score, losing score and total score sequences of k-hypertournaments. We find some stronger inequalities for score and losing score sequences of k-hypertournaments. We report bounds for scores and losing scores. In the last, we discuss total score sequences of a k-hypertournament and characterize total score sequence of a strong k-hypertournament.
Chapter 5: On score sequences of k-hypertournaments

5.1 Introduction

Hypergraphs are generalization of graphs. While edges of a graph are pairs of the vertices of graphs, edges of a hypergraph are subsets of the vertex set, consisting of at least two vertices. An edge consists of k-vertices termed as k-edge.

**Definition 5.1.** A k-hypergraph is a hypergraph all of whose edges are k-edges.

**Definition 5.2.** Given two nonnegative integers \( n \) and \( k \) with \( n \geq k > 1 \), a \( k \)-hypertournament on \( n \) vertices is a pair \((V,E)\), where \( V \) is a set of vertices with \( |V| = n \) and \( E \) is a set of \( k \)-tuples of vertices, called \( k \)-arcs, such that for any \( k \)-subset \( S \) of \( V \), \( E \) contains exactly one of \( k! \), \( k \)-tuples whose entries belongs to \( S \). Note that if \( n < k \), then \( E = \phi \); this kind of hypertournament is known as a null-hypertournament. Clearly a 2-hypertournament is a tournament.

For example, let \( V = \{1,2,3,4\} \) be the set of vertices, and \( E = \{ e_1 = (1,2,3), e_2 = (1,2,4), e_3 = (1,3,4), e_4 = (2,3,4)\} \) be the set of all 3-arcs. Then \( H = (V,E) \) is a 3-hypertournament.

**Definition 5.3.** Let \( H = (V,E) \) denote a \( k \)-hypertournament on \( n \) vertices. An \((x,y)\)-path in \( H \) is a sequence \((x =) v_1 e_1 v_2 e_2 v_3 ... v_{r-1}\)
$e_{t-1}v_t \ (= y)$ of distinct vertices $v_1, v_2, \ldots, v_t$, $t \geq 1$, and distinct arcs $e_1, e_2, \ldots, e_{t-1}$ such that $v_i$ precedes $v_{i+1}$ in $e_i$, $1 \leq i \leq t - 1$.

Thus $1e_12e_24$ is a path in the 3-hypertournament given above.

**Definition 5.4.** A $k$-hypertournament $H$ is strong if for any two vertices $x \in V$ and $y \in V$, $H$ contains both an $(x,y)$-path and a $(y,x)$-path.

### 5.2 Score and losing score sequences of $k$-hypertournaments

Zhou et al. [47] defined score and losing scores of vertices in a $k$-hypertournament $H$.

**Definition 5.5.** For every vertex $v_i$ of a $k$-hypertournament $H$, the score of vertex $v_i$ is denoted by $s(v_i)$ (or simply $s_i$), and defined by the number of arcs containing $v_i$ in which $v_i$ is not the last element. Similarly, the losing score $r(v_i)$ (or simply $r_i$) of vertex $v_i$ is the number of arcs containing $v_i$ in which $v_i$ is the last element.

**Example 5.1.** The score and losing scores of 3-hypertournament $H = (V,E)$, where $V = \{1,2,3,4\}$ and $E = \{(1,2,3),(3,4,1),(1,2,4),(3,2,4)\}$ are

$s_1 = 2$, $s_2 = 3$, $s_3 = 2$, $s_4 = 1$ and
Definition 5.6. The losing score sequence of a $k$-hypertournament is a sequence of nonnegative integers $(r_1, r_2, \ldots, r_n)$ with $r_1 \leq r_2 \leq \ldots \leq r_n$, where $r_i$ is a losing score of some vertex in a $k$-hypertournament $H$.

Example 5.2. Consider a 4-hypertournament $H$ of order 5 with vertex set $V = \{1, 2, 3, 4, 5\}$ and set of 4-arcs $E = \{(2, 3, 1, 4), (2, 5, 3, 1), (1, 4, 5, 2), (1, 3, 4, 5), (2, 3, 4, 5)\}$. The losing score sequence of $H$ is $(0, 1, 1, 1, 2)$

5.3 Existence of score and losing score sequences of $k$-hypertournaments

Zhou et al. [47] characterized score and losing score sequences of a $k$-hypertournament analogous to Landau's Theorem[36].

Theorem 5.1. [47] Given two nonnegative integers $n$ and $k$ with $n \geq k > 1$, a sequence of integers $R = (r_1, r_2, \ldots, r_n)$ with $r_1 \leq r_2 \leq \ldots \leq r_n$ is a losing score sequence of a $k$-hypertournament $H$ if and only if

$$\sum_{i=1}^{l} r_i \geq \binom{l}{k}, \text{ for } l = 1, 2, \ldots, n$$

(5.1)
with equality when \( l = n \).

Koh and Ree [35] gave a short proof of the above
Theorem using Hall’s Theorem.

An extension of (5.1) is,

\[
\sum_{i \in I} r_i \geq \binom{|I|}{k}, \text{ for each subset } I \subseteq [n] = \{1,2,\ldots,n\} \quad (5.2)
\]

with equality when \(|I| = n\).

Since \( s_i + r_i = \binom{n-1}{k-1} \), for each \( i = 1,2,\ldots,n \), then from
inequalities (5.1), the condition for a given sequence \( (s_1, s_2,\ldots, s_n) \) to be a score sequence of a \( k \)-hypertournament is given
below.

**Corollary 5.2.** [35] Given two nonnegative integers \( n \) and \( k \)
with \( n \geq k > 1 \), a sequence of integers \( S = (s_1, s_2,\ldots, s_n) \) with \( s_1 \geq s_2 \geq \ldots \geq s_n \) is a score sequence of a \( k \)-hypertournament \( H \) if
and only if

\[
\sum_{i=1}^l s_i \leq l \binom{n-1}{k-1} \binom{l}{k}, \text{ for } l = 1,2,\ldots,n \quad (5.3)
\]

with equality when \( l = n \).
Rearranging the score sequence in nondecreasing order, \( \tilde{s}_1 \leq \tilde{s}_2 \leq \ldots \leq \tilde{s}_n \), Koh and Ree [34,35] obtained the same inequalities for \( \tilde{s}_i \), as in Zhou et al. [47].

\[
\sum_{i=1}^{l} \tilde{s}_i \geq l \left( \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k} \right), \quad (l = 1, 2, \ldots, n)
\]

(5.4)

with equality when \( l = n \).

From inequalities (5.1) and \( s_i + r_i = \binom{n-1}{k-1} \), we get an extension of (5.3) is,

\[
\sum_{i \in I} s_i \leq |I| \binom{n-1}{k-1} - \binom{|I|}{k}, \quad \text{for each subset } I \subseteq [n],
\]

(5.5)

with equality when \(|I| = n\).

**Corollary 5.3.** [35] Sequences \( s_1 \leq s_2 \leq \ldots \leq s_n \) and \( r_1 \geq r_2 \geq \ldots \geq r_n \) are the score and losing score sequences of a \( k \)-hypertournament if and only if they satisfy

\[
s_i + r_i = \binom{n-1}{k-1}, \quad \text{for all } i = 1, 2, \ldots, n
\]

and

\[
\sum_{i=1}^{l} s_i \geq l \left( \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k} \right),
\]
and \( \sum_{i=1}^{l} r_i \leq \binom{n}{k} - \binom{n-l}{k} \), for \( l = 1,2,\ldots,n \) and equalities hold when \( l = n \).

**Lemma 5.4.** [47] Let \( H \) be a \( k \)-hypertournament of order \( n \) and \( S = (s_1,s_2,\ldots,s_n) \) with \( s_1 \leq s_2 \leq \ldots \leq s_n \) as its score sequence. Then

\[
\sum_{i=1}^{n} s_i = (k-1) \binom{n}{k}.
\]

### 5.4 Some stronger inequalities for scores and losing scores in \( k \)-hypertournaments

Analogous to Brualdi and Shen [11], we prove that the losing score sequence of a \( k \)-hypertournament \( H \) of order \( n \) satisfies the inequalities (5.6) below, that are individually stronger than the inequalities (5.2).

**Theorem 5.5.** A sequence of integers \( R = (r_1,r_2,\ldots,r_n) \) with \( r_1 \leq r_2 \leq \ldots \leq r_n \) is a losing score sequence of a \( k \)-hypertournament \( H \), if and only if for each subset \( I \subseteq [n] \),

\[
\sum_{i \in I} r_i \geq \frac{1}{2} \sum_{i \in I} \frac{(i-1)}{k-1} + \frac{1}{2} \binom{|I|}{k}
\]

(5.6)

with equality when \( |I| = n \).
Proof. The sufficiency part of the Theorem follows from inequalities (5.2), since

\[ \frac{1}{2} \sum_{i \in I} \left( \frac{i-1}{k-1} \right) \geq \frac{1}{2} \sum_{i \in I} \left( \frac{i-1}{k-1} \right) \]

\[ = \frac{1}{2} \left[ \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{|I|-1}{k-1} \right] \]

\[ = \frac{1}{2} \binom{|I|}{k} \]

From inequalities (5.2), we have

\[ \sum_{i \in I} r_i \geq \binom{|I|}{k}, \text{ with equality when } |I| = n. \]

Thus inequalities (5.6) imply inequalities (5.2).

Now we prove that the losing score sequence of a $k$-hypertournament satisfy (5.6). Suppose that $R = (r_1, r_2, \ldots, r_n)$ with $r_1 \leq r_2 \leq \ldots \leq r_n$ is a losing score sequence of a $k$-hypertournament $H$. For any subset $I \subseteq [n]$, we define

\[ f(I) = \sum_{i \in I} r_i - \frac{1}{2} \sum_{i \in I} \left( \frac{i-1}{k-1} \right) - \frac{1}{2} \binom{|I|}{k}. \]

First we choose $I$ to have $f(I)$ minimum and secondly to have $|I|$ minimum.

Claim. $I = \{ i : 1 \leq i \leq |I| \}$. 

Otherwise, there exist \( i \notin I \) and \( j \in I \) such that \( j = i + 1 \).

Then \( r_i \leq r_j \), since

\[
f(I) - f(I \setminus \{j\}) =
\]

\[
\sum_{m \in I} r_m - \frac{1}{2} \sum_{m \in I} \frac{m-1}{k-1} - \frac{1}{2} \binom{|I|}{k} - \sum_{m \in I} r_m - r_j - \frac{1}{2} \left( \sum_{m \in I} \frac{m-1}{k-1} - \binom{j-1}{k} - \frac{1}{2} \binom{|I|-1}{k} \right)
\]

\[
= r_j - \frac{1}{2} \left( \binom{j-1}{k-1} + \binom{|I|-1}{k} \right)
\]

and,

\[
f(I \cup \{i\}) - f(I) =
\]

\[
\sum_{m \in I} r_m + r_i - \frac{1}{2} \sum_{m \in I} \frac{m-1}{k-1} + \binom{i-1}{k-1} - \frac{1}{2} \binom{|I|+1}{k} - \sum_{m \in I} r_m + \frac{1}{2} \sum_{m \in I} \frac{m-1}{k-1} + \frac{1}{2} \binom{|I|}{k}
\]

\[
= r_i - \frac{1}{2} \left( \binom{i-1}{k-1} + \binom{|I|+1}{k} - \binom{|I|}{k} \right)
\]

\[
= r_i - \frac{1}{2} \left( \binom{i-1}{k-1} + \binom{|I|}{k-1} \right)
\]

Therefore

\[
r_j - \frac{1}{2} \left( \binom{j-1}{k-1} + \binom{|I|-1}{k-1} \right) = f(I) - f(I \setminus \{j\}) < 0
\]
and \( r_i - \frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{|I|}{k-1} \right] = f(I \cup \{i\}) - f(I) \geq 0 \)

This gives

\[
\frac{1}{2} \left[ \binom{j-1}{k-1} + \binom{|I|}{k-1} \right] < r_j < \frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{|I|}{k-1} \right] < r_i < \frac{1}{2} \left[ \binom{j-1}{k-1} + \binom{|I|}{k-1} \right]
\]

Thus we have

\[
\frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{|I|}{k-1} \right] < r_i \leq r_j < \frac{1}{2} \left[ \binom{j-1}{k-1} + \binom{|I|}{k-1} \right]
\]

That is,

\[
\frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{|I|}{k-1} \right] < \frac{1}{2} \left[ \binom{j-1}{k-1} + \binom{|I|}{k-1} \right]
\]

\[
\Rightarrow \quad \left[ \left( \binom{|I|}{k-1} - \binom{|I|-1}{k-1} \right) \right] < \left[ \left( \binom{i}{k-1} - \binom{i-1}{k-1} \right) \right] < 0
\]

\[
\Rightarrow \quad \binom{|I|-1}{k-2} < \binom{i-1}{k-2}
\]

A contradiction as \( i \leq |I| \). This proves the claim.

Therefore

\[
f(I) = \sum_{i=1}^{x} \left[ \binom{i-1}{k-1} + \frac{1}{2} \sum_{j=1}^{x} \left( \binom{i-1}{k-1} - \frac{1}{2} \binom{|I|}{k} \right) \right]
\]
Chapter 5: On score sequences of k-hypertournaments

\[ \sum_{i=1}^{k} r_i - \frac{1}{2} \left( \frac{n - |I|}{k - 1} + \frac{n}{k} - \frac{1}{2} \binom{n}{k} \right) \]

where the inequality follows from inequalities (5.2). By the choice of the subset \( I \), Theorem 5.5 follows. \( \square \)

Obviously, Theorem 5.5 gives a lower bound for \( \sum_{i=1}^{k} r_i \).

It is natural to ask for an upper bound for \( \sum_{i=1}^{k} r_i \). The following Corollary 5.6 gives the upper bound for \( \sum_{i=1}^{k} r_i \).

**Corollary 5.6.** A sequence of integers \( R = (r_1, r_2, \ldots, r_n) \) with \( r_1 \leq r_2 \leq \ldots \leq r_n \) is a losing score sequence of a \( k \)-hypertournament \( H \), if and only if for each subset \( I \subseteq [n] \),

\[ \sum_{i \in I} r_i \leq \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} + \frac{1}{2} \binom{n}{k} - \frac{1}{2} \binom{n - |I|}{k} \]  \hspace{1cm} (5.7)

with equality when \( |I| = n \).

**Proof.** Let \( J = [n] \setminus I \). Then

\[ \sum_{i \in I} r_i \leq \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} + \frac{1}{2} \binom{n}{k} - \frac{1}{2} \binom{n - |I|}{k}, \]
with equality when $|I| = n$, if and only if

$$\sum_{i \in I} r_i \geq \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} + \frac{1}{2} \binom{|I|}{k},$$

with equality when $|I| = n$. Therefore

$$\sum_{i \in [n]} r_i - \sum_{i \in I} r_i \geq \frac{1}{2} \left[ \sum_{i \in [n]} \binom{i-1}{k-1} - \sum_{i \in I} \binom{i-1}{k-1} \right] + \frac{1}{2} \binom{n-|I|}{k}$$

$$\Rightarrow \binom{n}{k} - \sum_{i \in I} r_i \geq \frac{1}{2} \left[ \binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n-1}{k-1} \right] - \sum_{i \in I} \binom{i-1}{k-1}$$

$$= \frac{1}{2} \binom{n}{k} - \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} + \frac{1}{2} \binom{n-|I|}{k}$$

$$\Rightarrow \sum_{i \in I} r_i \leq \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} + \frac{1}{2} \binom{n}{k} - \frac{1}{2} \binom{n-|I|}{k},$$

with equality when $|I| = n$.

Therefore Corollary 5.6 follows from Theorem 5.5.

Landau [36] gave the bounds for $s_i$, score of vertex $v_i$ in a tournament $T$ with score sequence $S$.

**Lemma 5.7.** [36] Suppose that a sequence of integers $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \ldots \leq s_n$ is a score sequence of a tournament $T$. Then for each $i$,
\[ \frac{i-1}{2} \leq s_i \leq \frac{n+i-2}{2} \]

The next result is an analogue to Lemma 5.7 on losing scores in a $k$-hypertournament.

**Theorem 5.8.** Suppose that a sequence of integers $R = (r_1, r_2, \ldots, r_n)$ with $r_1 \leq r_2 \leq \ldots \leq r_n$ is a losing score sequence of a $k$-hypertournament $H$ of order $n$. Then for each $i$,

\[
\frac{1}{2} \binom{i-1}{k-1} \leq r_i \leq \frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{n-1}{k-1} \right]
\]

**Proof.** Choosing $l = \{i\}$ Theorem 5.5 gives,

\[
\sum_{a \in l} r_a \geq \frac{1}{2} \sum_{a \in l} \binom{i-1}{k-1} + \frac{1}{2} \binom{|l|}{k}
\]

\[
\Rightarrow \quad r_i \geq \frac{1}{2} \binom{i-1}{k-1} + \frac{1}{2} \binom{1}{k}
\]

\[
\Rightarrow \quad r_i \geq \frac{1}{2} \binom{i-1}{k-1}, \text{ as } \binom{1}{k} = 0 \text{ for } k > 1
\]

Again from Corollary 5.6 putting $l = \{i\}$, we get

\[
r_i \leq \frac{1}{2} \binom{i-1}{k-1} + \frac{1}{2} \binom{n}{k} - \frac{1}{2} \binom{n-1}{k} \]

\[
= \frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{n-1}{k-1} \right]
\]

Hence the result. \(\square\)
The next result is a consequence of Theorem 5.5, which shows that the score sequence of a $k$-hypertournament $H$ of order $n$ satisfies inequalities (5.8) below that are individually stronger than the inequalities (5.5).

**Corollary 5.9.** A sequence of integers $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \geq s_2 \geq \ldots \geq s_n$ is a score sequence of a $k$-hypertournament $H$ if and only if for every subset $I \subseteq [n],$

$$\sum_{i \in I} s_i \leq |I| \binom{n-1}{k-1} - \sum_{i \in I} \binom{i-1}{k-1} - \frac{1}{2} \binom{|I|}{k}$$

(5.8)

with equality when $|I| = n.$

**Proof.** From Theorem 5.5, we have

$$\sum_{i \in I} r_i \geq \frac{1}{2} \left[ \sum_{i \in I} \binom{i-1}{k-1} + \binom{|I|}{k} \right], \text{ with equality when } |I| = n.$$

But we have $s_i + r_i = \binom{n-1}{k-1},$ for $i = 1, 2, \ldots, n.$

So,

$$\sum_{i \in I} \left( \binom{n-1}{k-1} - s_i \right) \geq \frac{1}{2} \left[ \sum_{i \in I} \binom{i-1}{k-1} + \binom{|I|}{k} \right]$$

$$\Rightarrow \sum_{i \in I} s_i \leq |I| \binom{n-1}{k-1} - \sum_{i \in I} \binom{i-1}{k-1} - \frac{1}{2} \binom{|I|}{k},$$

with equality when $|I| = n.$
As \( r_i = \binom{n-1}{k-1} \), for each \( i = 1, 2, \ldots, n \), from Corollary 5.6, we easily get the following result.

**Corollary 5.10.** A sequence of integers \( S = (s_1, s_2, \ldots, s_n) \) with \( s_1 \geq s_2 \geq \ldots \geq s_n \) is a score sequence of a \( k \)-hypertournament \( H \) if and only if for every subset \( I \subseteq [n] \),

\[
\sum_{i \in I} s_i \geq \left| I \right| \binom{n-1}{k-1} - \frac{1}{2} \sum_{i \in I} \left( \binom{i-1}{k-1} - \binom{n-|I|}{k} \right)
\]

with equality holds when \( |I| = n \).

**Proof.** From Corollary 5.6, we have

\[
\sum_{i \in I} r_i \leq \frac{1}{2} \left[ \sum_{i \in I} \left( \binom{i-1}{k-1} + \binom{n}{k} \right) - \binom{n-|I|}{k} \right]
\]

\[
\Rightarrow \sum_{i \in I} \left( \binom{n-1}{k-1} - s_i \right) \leq \frac{1}{2} \left[ \sum_{i \in I} \left( \binom{i-1}{k-1} + \binom{n}{k} \right) - \binom{n-|I|}{k} \right]
\]

\[
\Rightarrow \left| I \right| \binom{n-1}{k-1} - \sum_{i \in I} s_i \leq \frac{1}{2} \left[ \sum_{i \in I} \left( \binom{i-1}{k-1} + \binom{n}{k} \right) - \binom{n-|I|}{k} \right]
\]

\[
\Rightarrow \sum_{i \in [n]} s_i \geq \left| I \right| \binom{n-1}{k-1} - \frac{1}{2} \sum_{i \in I} \left( \binom{i-1}{k-1} - \binom{n-|I|}{k} \right)
\]

with equality holds when \( |I| = n \). \( \square \)

The next result is a generalization of Lemma 5.7 which follows from the above Corollaries 5.9 and 5.10.
**Corollary 5.11.** Suppose that \( S = (s_1, s_2, \ldots, s_n) \) with \( s_1 \geq s_2 \geq \ldots \geq s_n \) is a score sequence of a \( k \)-hypertournament \( H \) of order \( n \).

Then for each \( i \),

\[
\frac{1}{2} \left[ \binom{n-1}{k-1} - \binom{i-1}{k-1} \right] \leq s_i \leq \binom{n-1}{k-1} - \frac{1}{2} \binom{i-1}{k-1}
\]

**Proof 1.** Suppose \( R = (r_1, r_2, \ldots, r_n) \) with \( r_1 \leq r_2 \leq \ldots \leq r_n \) is a losing score sequence of some \( k \)-hypertournament of order \( n \).

Then there exist a score sequence \( S = (s_1, s_2, \ldots, s_n) \) with \( s_1 \geq s_2 \geq \ldots \geq s_n \) with \( r_i + s_i = \binom{n-1}{k-1}, 1 \leq i \leq n \). Then

\[
s_i = \binom{n-1}{k-1} - r_i = \binom{n-1}{k-1} - \frac{1}{2} \left[ \binom{i-1}{k-1} + \binom{n-1}{k-1} \right] \quad (5.9)
\]

(Using Theorem 5.8)

\[
= \frac{1}{2} \left[ \binom{n-1}{k-1} - \binom{i-1}{k-1} \right]
\]

Again from Theorem 5.8 and equation (5.9), we get

\[
s_i \leq \binom{n-1}{k-1} - \frac{1}{2} \binom{i-1}{k-1}.
\]

This completes the proof.

**Proof 2.** Putting \( I = \{i\} \) in Corollary 5.9, we get
\[
\sum_{i \in \mathcal{I}} s_i \leq \binom{n-1}{k-1} - \frac{1}{2} \sum_{i \in \mathcal{I}} \binom{i-1}{k-1} - \frac{1}{2} \binom{|\mathcal{I}|}{k}
\]

\Rightarrow s_i \leq \binom{n-1}{k-1} - \frac{1}{2} \binom{i-1}{k-1} \quad \text{as} \quad k > 1 \quad \text{so} \quad \binom{|\mathcal{I}|}{k} = \binom{1}{k} = 0

Also putting \( I = \{i\} \) in Corollary 5.10, we get

\[
s_i \geq \binom{n-1}{k-1} - \frac{1}{2} \binom{i-1}{k-1} - \frac{1}{2} \binom{n-1}{k-1}
\]

\[
= \binom{n-1}{k-1} - \frac{1}{2} \binom{i-1}{k-1} - \frac{1}{2} \binom{n-1}{k-1}
\]

\[
= \frac{1}{2} \left[ \binom{n-1}{k-1} - \binom{i-1}{k-1} \right]
\]

Hence the result. \( \square \)

The following results due to Zhou et al. [47] characterize the score and losing score sequences of a strong \( k \)-hypertournament.

**Lemma 5.12.** [47] A sequence of integers \( S = (s_1, s_2, \ldots, s_n) \) with \( s_1 \leq s_2 \leq \ldots \leq s_n \) is a score sequence of a strong \( k \)-hypertournament \( H \) with \( n > k \) if and only if

\[
\sum_{i=1}^{l} s_i > l \binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k}, \quad (l = 1, 2, \ldots, n-1), \quad (5.10)
\]

and

\[
\sum_{i=1}^{n} s_i = (k-1) \binom{n}{k}.
\]
Rearranging the score sequence in nonincreasing order, \( \bar{s}_1 \geq \bar{s}_2 \geq \ldots \geq \bar{s}_n \), we obtain

\[
\sum_{i=1}^{l} \bar{s}_i < \binom{n-1}{k-1} (\binom{l}{k}), \quad (l = 1, 2, \ldots, n-1)
\]

and

\[
\sum_{i=1}^{n} \bar{s}_i = (k-1) \binom{n}{k}. \quad (5.11)
\]

**Lemma 5.13.** [7] A sequence of integers \( R = (r_1, r_2, \ldots, r_n) \) with \( r_1 \leq r_2 \leq \ldots \leq r_n \) is a losing score sequence of a strong \( k \)-hypertournament \( H \) with \( n > k \) if and only if

\[
\sum_{i=1}^{l} r_i > \binom{l}{k}, \quad (l = 1, 2, \ldots, n-1)
\]

and

\[
\sum_{i=1}^{n} r_i = \binom{n}{k}. \quad (5.12)
\]

### 5.5 Total score sequences of \( k \)-hypertournaments

Koh and Ree [35] defined the total score \( t_i \) of a vertex \( v_i \) as \( s_i - r_i \), where \( s_i \) and \( r_i \) are score and losing score of vertex \( v_i \). They found a necessary and sufficient condition for a sequence of integers to be a total score sequence of a \( k \)-hypertournament.

**Lemma 5.14.** [35] A nonincreasing sequence of integers \( t_1 \geq t_2 \geq \ldots \geq t_n \) is a total score sequence of a \( k \)-hypertournament \( H \) of
order \( n \) if and only if \( t_i \) has the same parity as that of \( \binom{n-1}{k-1} \)
for each \( i = 1, 2, \ldots, n \),

\[
\sum_{i=1}^{l} t_i \leq l \binom{n-1}{k-1} - 2 \binom{l}{k}, \quad (l = 1, 2, \ldots, n)
\]

with equality when \( l = n \).

Now we give a condition for a sequence of integers to be total score sequence of a strong \( k \)-hypertournament.

**Theorem 5.15.** A sequence of integers \((t_1, t_2, \ldots, t_n)\) with \( t_1 \geq t_2 \geq \ldots \geq t_n \) is a total score sequence of a strong \( k \)-hypertournament \( H \) of order \( n \) with \( n > k \) if and only if \( t_i \) has the same parity as that of \( \binom{n-1}{k-1} \) for each \( i = 1, 2, \ldots, n \),

\[
\sum_{i=1}^{l} t_i \leq l \binom{n-1}{k-1} - 2 \binom{l}{k}, \quad (l = 1, 2, \ldots, n-1) \tag{5.13}
\]

and

\[
\sum_{i=1}^{n} t_i = (k-2) \binom{n}{k}. \tag{5.14}
\]

**Proof.** Let a nonincreasing sequence of integers \( t_1 \geq t_2 \geq \ldots \geq t_n \) be a total score sequence of a strong \( k \)-hypertournament \( H \) of order \( n \). Then there exist strong score and losing score sequences \( S = (s_1, s_2, \ldots, s_n) \), \( s_1 \geq s_2 \geq \ldots \geq s_n \) and \( R = \)
(r_1, r_2, \ldots, r_n), r_1 \leq r_2 \leq \ldots \leq r_n \text{ with } t_i = s_i - r_i. \text{ Then Theorem 5.15 follows from conditions (5.11) and (5.12).}

Since \( t_i = s_i - r_i \) and \( \binom{n-1}{k-1} = s_i + r_i \),

then \( t_i = \binom{n-1}{k-1} - 2r_i \)

and hence \( t_i \) has same parity with \( \binom{n-1}{k-1} \) for \( i = 1, 2, \ldots, n \).

For the converse, suppose that a nonincreasing sequence of integers \( (t_1, t_2, \ldots, t_n) \) with \( t_1 \geq t_2 \geq \ldots \geq t_n \), which satisfy the conditions (5.13) and (5.14), given in the statement of the Theorem 5.15.

For each \( i = 1, 2, \ldots, n \), let

\[
s_i = \frac{1}{2} \left( \binom{n-1}{k-1} + t_i \right) \text{ and } r_i = \frac{1}{2} \left( \binom{n-1}{k-1} - t_i \right)
\]

Then \( t_1 < \binom{n-1}{k-1} \), (as \( s_i < \binom{n-1}{k-1} \))

and \( t_n = \sum_{i=1}^{n} t_i - \sum_{i=1}^{n-1} t_i \)

\[
> (k-2)\binom{n}{k} - \left\{ \binom{n-1}{k-1} - 2\binom{n-2}{k} \right\}.
\]
Chapter 5: On score sequences of k-hypertournaments

So, \(-\binom{n-1}{k-1}\) < \(t_i\) < \(\binom{n-1}{k-1}\) and hence \(s_i > 0\) and \(r_i < 0\). The sequences \(S = (s_1, s_2, \ldots, s_n)\) and \(R = (r_1, r_2, \ldots, r_n)\) respectively nonincreasing and nondecreasing. So, for \(1 \leq l \leq n - 1\)

\[
\sum_{i=1}^{l} s_i = \sum_{i=1}^{l} \frac{1}{2} \left[ \binom{n-1}{k-1} + t_i \right]
\]

\[
= \frac{1}{2} \left[ \binom{n-1}{k-1} + \sum_{i=1}^{l} t_i \right]
\]

\[
< \frac{1}{2} \left[ \binom{n-1}{k-1} + \binom{n-1}{k-1} - 2 \binom{l}{k} \right]
\]

\[
= \binom{n-1}{k-1} - \binom{l}{k},
\]

and, similarly

\[
\sum_{i=1}^{l} r_i = \frac{1}{2} \left[ \binom{n-1}{k-1} - \sum_{i=1}^{l} t_i \right] > \binom{l}{k}, \text{ with}
\]

\[
\sum_{i=1}^{n} s_i = (k-1) \binom{n}{k} \quad \text{and} \quad \sum_{i=1}^{n} r_i = \binom{n}{k}.
\]

So by conditions (5.11) and (5.12), there exist a strong \(k\)-hypertournament \(H\) with \(S = (s_1, s_2, \ldots, s_n)\) and \(R = (r_1, r_2, \ldots, r_n)\)
as its score and losing score sequences, and hence \( t_i = s_i - r_i \), for \( i = 1, 2, \ldots, n \) as its total score sequence. 

REFERENCES
REFERENCES

[1]. Alspach, Brian and Reid, K.B., *Degree frequencies in digraphs and tournaments*, J. Graph Theory 2 (1978), 241-249.


References


[18]. Erdös, P. and Gallai, T., Graphs with prescribed degrees of vertices (In Hungarian), Math. Lapok. 11 (1960), 264-274.


[20]. Gervacio, S.V., Score sequences, Lexicographic enumeration and tournament construction, Proc. of the First Japan Conference on Graph Theory and


[43]. Reid, K.B., *Tournaments, Scores, Kings, Generalizations and Special Topics*, In, *Surveys on Graph Theory* (edited...


