DISCRETE CODED WAVEFORMS FOR SIGNAL PROCESSING IN RADAR

SYNOPSIS

THESIS
SUBMITTED FOR THE AWARD OF THE DEGREE OF
Doctor of Philosophy
IN
ELECTRONICS ENGINEERING

BY
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UNDER THE SUPERVISION OF
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ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)
1997
**SYNOPSIS**

The work presented in this thesis is concerned with the design of discrete coded waveforms for improving range resolution and clutter performance of radar systems. This approach to signal design offers many advantages in terms of waveform shaping and digital implementation of processors. Assuming a matched filter receiver, the bulk of the work is concentrated on studying the autocorrelation function properties of these waveforms, which are directly related to the range resolution. Pulse trains of this type are also useful in synchronising digital communication systems. Sequences of this type can also be used as an orthogonal code in a simple coding system.

There is a broad class of discrete coding waveforms or sequences, that can be found with good autocorrelation function. The class of binary sequences is important. These are easy to generate and store using standard logic circuits, cross-correlation is simpler and transmitters operate continuously at peak power. The binary sequences having values ±1 such as Barker sequences with good autocorrelation function, have the most uniform distribution of energy for the given length. However, Barker sequences with a length greater than thirteen do not exist. Furthermore, the autocorrelation function of binary sequences contains significant sidelobes as compared to the non-binary sequences. The non-uniform or multilevel
sequences such as Huffman sequence have got autocorrelation function much similar to a pulse. But the cost for this achievement is paid in the form of circuit complexity which is due to the need of amplitude and phase modulation in its implementation. However, the use of digital microcircuits becoming more attractive in signal processing applications due to their flexibility, cheapness and compactness, the generation of multilevel sequences are no longer difficult. Few disadvantages are also associated with multilevel sequences. The distribution of energy is not uniform throughout the sequence length, the energy efficiency of the multilevel sequences are poor as compared to the binary sequences.

An attempt is made to solve the signal design problem using numerical optimisation methods that incorporate the fixed amplitude constraint. In particular, a constraint optimisation technique is developed for synthesising binary sequences with good autocorrelation function properties. It is well known that by choosing a large number of elements randomly, sequences where r.m.s. sidelobe levels are of the order $\sqrt{N}$ can be found. However, it can be expected that in the statistical synthesis a large number of sidelobes will exceed $\sqrt{N}$. It will be shown that by proper choice of the sequence both average and peak sidelobe can be held at a lower value and this yield a better range resolution and clutter rejection.
Moreover, the problem of designing pairs and sets of phase coded pulse trains with low autocorrelation sidelobes and small mutual cross-correlation is considered. Such sequences have many practical application. For example they may be used as address codes in a time division multiple access (TDMA) system, where information from several data sources is to be transmitted over a channel.

In the case of non-binary sequences such as Huffman codes (impulse equivalent sequences) a synthesis method based on the weighting sequences of the optimum inverse (least square energy) filters is presented which yields sequences with good energy efficiency and near pulse like autocorrelation function. It is shown that the tap gains or weighting coefficients of the optimum inverse filter has better autocorrelation function than the input sequence itself. This fact is being utilized to generate the sequence of extremely small autocorrelation function sidelobes by using the process of inversion a number of times.

Although, the range sidelobes can be reduced quite effectively by proper signal design methods, for some applications they might still be too large. Thus, sidelobe reduction methods, which minimise the detection loss subject to a set of sidelobe constraints, are presented by mismatching the receiver filter.

An artificial neural network (ANN) based approach is also developed which can completely reduce the autocorrelation sidelobes down
to zero value at the output of the matched filter. Various possibilities are explored and their behaviour in the presence of noise is considered. ANN has found wide applications in pattern recognition. The pattern recognition is also applicable in radar signal detection or classification processes. In the present case radar signals can be treated as data patterns. The basic theories of statistical hypothesis testing, decision theory etc. apply in this situation. Indeed, many pattern recognition techniques employ likelihood ratios, when the statistics are known apriori, and pattern recognition or classification is then merely an applications of multiple hypothesis testing. Nevertheless, some interesting techniques presented under the name of pattern recognition, learning machines, artificial intelligence etc. are applicable to radar system and should be available for the radar designers consideration.

Finally, in the case where significant target velocity is encountered it becomes necessary to consider not only the range but also the velocity resolution (Doppler shift) properties of the transmitted waveform. Combined range and velocity resolution depends on the complete waveform structure in time and frequency. Hence signals with good resolution in one parameter may perform very poorly when combined resolution in both parameters is considered. For combined resolution in range and velocity the waveform must be investigated in terms of the complete MF response in delay and Doppler. This generalised response for various types of pulse trains is considered by using the standard range Doppler ambiguity description.
(ABF) of Woodward. The ABF plays a central part in the analysis of combined resolution. This is so because the width of the main response peak of the ABF serves as a measure for close-target visibility in range-Doppler, while the low level response and subsidiary spikes give an indication of the self-clutter and target masking problem by mutual interference.
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ABSTRACT

The work presented in this thesis is concerned with the design of discrete coded waveforms for improving range resolution and clutter performance of radar systems. This approach to signal design offers many advantages in terms of waveform shaping and digital implementation of processors. Assuming a matched filter receiver the bulk of the work is concentrated on studying the autocorrelation function properties of these waveforms, which are directly related to the range resolution. Pulse trains of this type are also useful in synchronizing digital communication systems. Sequences of this type can also be used as an orthogonal code in a simple coding system.

There is a broad class of discrete coding waveform or sequences, that can be found with good autocorrelation function. The class of binary sequences is important. These are easy to generate and store using standard logic circuits, cross-correlation is simpler and transmitters operate continuously at peak power due to uniform distribution of energy. However, the auto-correlation function of binary sequences contains significant sidelobes as compared to the non-binary sequences. The non-binary or multilevel sequences such as Huffman sequence have got auto-correlation function much similar to a pulse, but the distribution of energy is not uniform throughout the sequence length, therefore, the energy efficiency of the multilevel sequences are poor as compared to the binary sequences.

An attempt is made to solve the signal design problem using numerical optimization methods. It is well known that by choosing a large number of elements randomly, sequences whose r.m.s. sidelobe levels are of the order $\sqrt{N}$ can be found. However, it can be expected that in the statistical synthesis a large number of sidelobes will exceed $\sqrt{N}$. It will be shown that by proper choice of the sequence both average and peak sidelobes can be held at a lower value and this yield a better range resolution and clutter rejection.

Moreover, the problem of designing pairs and sets of phase coded pulse trains with low auto-correlation side lobes and small mutual cross correlation is
considered. Such sequences have many practical applications. For example they may be used as address codes in a time division multiple access (TDMA) systems, where information from several data sources is to be transmitted over a channel.

In the case of non-binary sequences such as Huffman codes a synthesis method based on the weighting sequences of the optimum inverse filters is presented which yields sequences with good energy efficiency and near pulse like autocorrelation function.

Although, the range sidelobes can be reduced quite effectively by proper signal design methods, for some applications they might still be too large. Thus sidelobe reduction methods, which minimize the detection loss are presented by mismatching the receiver filter.

An artificial neural network (ANN) based approach is also developed which can completely reduce the autocorrelation sidelobes down to zero value at the output of the matched filter. Various possibilities are explored and their behaviour in the presence of noise is considered.

Finally, in the case where significant target velocity is encountered it becomes necessary to consider not only the range but also the velocity resolution (Doppler shift) properties of the transmitted waveform. For combined resolution in range and velocity the waveform must be investigated in terms of the complete MF response in delay and doppler. This is done by using the standard range doppler ambiguity description (ABF) of Woodward.
ACKNOWLEDGEMENT

The author would like to express his gratitude to his supervisor Prof. F. Ghani whose genuine interest, valuable suggestions and encouragement made this work possible. Thanks are also due to the Chairman, Department of Electronics Engg. for the facilities provided. Thanks are also due to Mr. Akmal for typing the report.

(Zia A. Abbasi)
CERTIFICATE

This is to certify that the Ph.D. Thesis entitled *Discrete Coded Waveforms for Signal Processing in Radar* which is being submitted by Mr. Zia Ahmad Abbasi, Reader in the Dept. of Electronics Engg., A.M.U. Aligarh for the award of the Degree of "Doctor of Philosophy" of the university, is a record of candidate's own work carried out by him under my supervision. The matter in this Thesis has not been submitted for the award of any other Degree or Diploma.

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CHAPTER - 1

INTRODUCTION
1.1 BACKGROUND

Radar is a technique for remote sensing using radio waves. Its basic purpose is to detect the presence of a target of interest and to provide information concerning the target's location, motion, size and other parameters. The problem of target detection is accomplished in a typical radar system (Fig. 1.1) by transmitting a radio signal and detecting the waveform reflected by the target in the presence of unavoidable system noise and reflections from undesired scatterers (clutter). If a return signal of adequate strength is received, it is further analysed to determine the target's range, velocity and so forth. This process is known as parameter estimation. The range of the target is determined by measuring the delay of the return signal. Similarly, the velocity of the target can be estimated, neglecting higher order effects, by measuring the shift in carrier frequency (doppler shift) of the received waveform. Furthermore, the transmitted signal can be carefully chosen and generated so as to optimize its capability for extracting the target detection and parameter estimation are difficult practical problems, particularly for small targets at great distances. In principle, however, both problems are simple when only a single target is present. Target resolution, which may be defined as the capability of a radar system to recognize particular target in the presence of others, is one of the most important but also most demanding tasks[1] & [2].

For high performance radar systems, these tasks become increasingly complex. This explains the continuing effort being directed towards improving the resolution capabilities of modern radars. Some improvements are still being made in the components which affect radar performance; for example, receivers with low noise figures, transmitters with higher output power, and antennas with more gain. Further improvement can be obtained by means of elaborate signal processing schemes. In recent years, a considerable amount of work has been done in digital processing for radar.
Fig. 1.1 A typical radar system
technical problems imposed by modern radar systems are those of processing (real time) a large number of data and the requirement of complex signal processing operations. These problems can only be solved reliably by the use of digital techniques, although in some cases modern optical processing schemes can offer an alternative.

The choice of a suitable transmit waveform is an important problem in radar design. This is so because the waveform controls resolution and clutter performance and also bears heavily on the system cost. As compactness, cheapness and computational speed of digital microcircuits continue to increase their use in signal processing applications becomes more practical. In particular, the advent of solid-state antenna arrays has its impact on radar system designers in two principal ways. First, peak power limitations of solid state array elements have necessitated the use of waveforms with long durations in order to achieve the required signal energy over a desired range. The required stability and reproducibility of such signals can only be satisfied reliably by digital signal generation and processing. Secondly, the ability to switch the beam of solid-state array at high speeds gives the radar a multi-function capability, thus requiring the flexibility to enable a variety of waveforms to be employed, [3]. These requirements have made digital signal processing with its inherent adaptability an attractive alternative to analogue processing. Theoretical studies which provide the basis for technical advances have not, so far, solved the general signal design problem. The knowledge of the properties of the pulse trains, a class of signals particularly well suited to digital processing, is therefore, of increasing practical importance.

An early suggestion for using discrete coded waveforms in radar appeared in a paper by Siebert, [4] treating the general problems of radar. Siebert noted that certain binary coded waveforms offered substantial improvement in range and velocity resolution. However, it was shown that in order to obtain these improvements, it would be necessary to employ long periodic binary sequences known as pseudo-random sequences, [5]. Later, Lerner6 suggested that the periodic sequence could be modified to form an aperiodic signal and yet retain the nearly optimum resolution property of the waveform. The use of aperiodic signals allows the construction of passive matched filter receivers. At about this time the signal design problem was approached in a slightly
different way. Assuming a matched filter receiver, the range resolution capability (in the absence of doppler shift) was found to be directly related to the autocorrelation function of the transmitted waveform. Therefore, the approach consisted of attempting to design aperiodic binary sequences having optimum autocorrelation properties, [7] & [9]. These sequences were called 'Optimum finite code groups' or Barker sequences.

Since these early evaluation a number of authors have made valuable contributions in the field of waveform design, [10], [11], ...[15]. An interesting analytical method for generating binary codes was reported by Boehmer, [10] using number theory. Another quite different approach to the problem is discussed in a paper by Vekman11 and Varakin, [12]. The authors suggest a synthesis procedure on the basis of spectral theory and the method of stationary phase.

Heimiller, [16], Frenk, [17] and Zadoff, [18] have shown that there are other suitable codes if the restriction of 0°-180° phase shifting is removed[19]. In the case of Frank codes[17], higher order poly-phase coded words can be generated by coding each sub-phase into one of M. phases. Huffman[20] considered the problem of designing amplitude and phase modulated pulse trains. He has shown that finite length signals with nearly ideal autocorrelation function can be generated. This property, however, is achieved at the expense of amplitude modulation which results in increased system complexity and lower energy utilization at the transmitter. Nevertheless, the additional expense of encoding and decoding of amplitude and/or phase modulated waveforms may be justified for radars that must cope with land clutter or operate in a dense-target environment. However, the use of amplitude modulated pulse trains is precluded in most high power applications due to the inevitable loss in energy efficiency.

In spite of considerable effort that has been devoted to the problem of designing waveforms with high range resolution there seems to be a lack of signal design techniques and theory. All present methods tend to contain an element of trial and error and, moreover, rely on the skill and ingenuity of the designer. In short, the study of the properties of pulse trains does not appear to have progressed much beyond an understanding of the types described above. The currently accepted belief that there is
no ideal waveform is not surprising considering the various different tasks modern radar systems have to perform. On the other hand, the inability to find an ideal waveform is not an excuse for failure to search for locally optimum waveforms for specific radar applications and environments.

The effort in this thesis is directed towards the improvement of a factor which constitutes a fundamental limitation to radar performance, namely the transmitted waveform. Although the ways in which the transmitted signal affects the system performance are well understood, [21], there seems to be no obvious solution to the problem of designing energy efficient pulse trains for high resolution radars. Therefore, the work presented in this thesis is concerned primarily with the study and development of design methods for improving the range resolution capability of pulse trains. The pulse sequences discussed later, besides representing an interesting mathematical area, are also of practical significance in related fields such as digital communication and navigation.

1.2 OUTLINE OF INVESTIGATION

This thesis is concerned with a number of different aspects of the pulse train design problem for radar systems. As a basis for later work, the principles of waveform and processor design are outlined in Chapter 2. This is carried out assuming matched filtering and digital signal generation and processing.

In Chapter 3, the important problem of synthesizing pulse trains from autocorrelation functions, which are specified at discrete points in phase amplitude or in magnitude only, is tackled. The resolution properties of pulse trains approximating F M type signals are also studied in this chapter.

In Chapter 4, the signal design problem is approached via numerical optimization techniques. This is done by minimizing an appropriate defined performance measure reflecting the resolution capability of a signal. The optimization method was then applied to generate the phase modulated pulse trains of various length. This chapter also treats
a new method of designing binary sequences using cyclic shifting and bit addition. Moreover, the properties of pairs of phase coded sequences having low auto-correlation sidelobes and small mutual cross-correlation are studied.

Much of the material included deals with purely phase coded sequences. For certain applications, however, the use of amplitude modulation can provide a useful means of improving range resolution and clutter rejection.

Chapter 5 presents the results of the synthesis of energy efficient amplitude and phase modulated pulse trains. In connection with Huffman Codes a new approach to the signal generation using clipping is developed.

Chapter 6 stands apart somewhat from the other chapters in that it is concerned with sidelobe reduction techniques using mismatched filters. The central problem in mismatched filtering is to consider the trade-off possibilities between resolution and degradation in signal detractability. A neural network scheme is also presented in this chapter which can suppress the sidelobes to any desired level.

Finally, Chapter 7 considers the combined range and velocity resolution properties of the pulse trains. This is done using the standard range-doppler ambiguity function of Woodward, [21].
CHAPTER 2

SIGNAL PROCESSING CONCEPTS
AND WAVEFORM DESIGN
2.1 INTRODUCTION

In this chapter some of the general principles of waveform and processor design are discussed. (The selection of the desired transmitted waveform and of the receiver choice involves in general two separable design problems) The design problem in general requires two criterion, first; the choice of transmitted waveform; secondly the design of the receiver. The waveform chosen, must be such that it optimizes the performance in the total environmental conditions such as clutter, dense target environment etc. Unfortunately, no such waveform is available which will suit for any environmental condition. Since there are a number of different ways to implement a near ideal receiver with some given waveform, therefore the design of Radar processing scheme (hardware) is generally treated as a different problem. Complexity and reliability of the hardware scheme and the corresponding cost involved are the usual criterion for the processor design.

The design of signal processing system becomes more complex if the number of input and output channels are large in number. For modern high resolution radars the complexity of the signal processing system and the amount of data to be handled easily reaches critical limits. However, in this thesis, the study is confined to the coded waveforms as modulating functions. The various modulation techniques of modulating the carrier with such function and the methods of implementing the processors will not be considered here.

In this chapter, the problem of waveform design based on the standard range Doppler ambiguity function described by Woodward, [21] is also considered. The discussion on ambiguity function is confined to the results achieved by some specific waveform. The amount of the work only depends upon the study of the zero Doppler cross-section of the ambiguity function, because the main factor here, is the range resolution capability of a signal. In the references, [22], [23], ...[26], the properties of various classes of waveforms and their ambiguity functions are given in details. The general description of the ambiguity function is, despite their wide study, not without its limitations. While with modern computers it is not difficult to derive the ambiguity
Fig. 2.1 Rect and tri functions

Fig. 2.2 Amplitude and phase modulated pulse train
function for a particular waveform it is generally not possible to derive a specific waveform starting with a given ambiguity function. It is also necessary to have a complete knowledge of target and clutter environment before selecting an appropriate waveform.

### 2.2 REPRESENTATION OF PULSE TRAINS

An important class of signals well suited to digital processing are the pulse trains. In general these pulse trains are the waveforms expressed as a finite number of contiguous. Coherent carrier pulses (sub-pulses) each modulated in amplitude, phase and frequency. Analytically this class of signals can be expressed in the form

\[ u(t) = \sum_{n=0}^{N} a(n) \text{rect}\left(\frac{t}{T} - n\right) \cos\left(2\pi f_c + f(n)\right) t + \phi(n) \]  \hspace{1cm} (2.1)

Where

\( T = \) Pulse duration

\( f_c = \) Carrier Frequency

\( a(n) = \) amplitude of the nth pulse

\( \phi(n) = \) Phase of the nth pulse

\( f(n) = \) Frequency deviation of nth pulse

and \( \text{rect}(t/T) \) is the rect. function shown in Fig. 2.1 and defined as

\[ \text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & \text{elsewhere} \end{cases} \]  \hspace{1cm} (2.2)

It can be noted that for a given carrier frequency, \( f_c \), and sub pulse duration, \( T \), the signal is completely described by the ordered sequences \( \{a(n)\}, \{\phi(n)\}, \{f(n)\} \). Depending upon these sequences the, pulse trains can be easily divided into three following categories.
Categories

Group I : \( \{ f(n) \} = 0 \)

Group II : \( \{ \phi(n) \} = \{ f(n) \} = 0, \{ a(n) \} = 1.0 \)

Group III : \( \{ a(n) \} = 1.0, \{ \phi(n) \} = 0 \)

The sub class of pulse trains considered here belong to Group I and are referred to as amplitude and phase modulated (am ph m) pulse trains, [27] and are shown in Fig 2 2

In digital signal generation the generator presents a sampled and quantized signal which can be used to modulate the given carrier either in amplitude Phase or in frequency. Therefore, a digital signal generator in general may be assumed as a time clocked unit, in which the output signal can be reconstructed by filtering or modulation or both, using the sequences of digital data produced by signal generator. In most radar application phase modulation, and particularly digital phase modulation, is the most attractive modulation method. In the case of phase modulation, the signal can be reconstructed by either of the two methods shown in Fig 2 3. The digital phase sample generator produces samples as binary numbers which are converted into \( \cos(\phi(n)) \) and \( \sin(\phi(n)) \) and thus into samples of \( S_r(t) \) and \( S_i(t) \) (Eqn 2 5). Consequently the binary number can be used to digitally phase modulate a carrier signal in a digital phase modulator.

2.2.1 Complex Envelope and Band Pass Filtering

The complex envelope concept simplifies the theory of band pass signals. Since am ph m pulse trains are band pass signals, therefore, can be explained using complex envelope representation. The band pass signal given in equation 2 1 can be expressed as

\[
u(t) = \text{Re} \left[ S(t) \exp (j2\pi f_c t) \right]
\] (2 3)
Where \( S(t) \) is the complex envelope of \( u(t) \) and is given by

\[
S(t) = \sum_{n=0}^{N} a(n) \text{rect}\left(\frac{t}{T} - n\right) \exp(j\phi(n))
\]  

(2.4)

By comparing the Equations (2.4) with Eqn. (2.1), it is clear that the absolute value of \( S(t) \) is nothing but real signal envelope. Thus the band pass signal \( u(t) \) is completely described by the knowledge of its carrier frequency, \( f_c \), and its low frequency complex envelope \( S(t) \). In terms of real and imaginary parts of \( S(t) \), the band pass signal \( u(t) \) is given by

\[
u(t) = S_r(t) \cos(2\pi f_c t) - S_i(t) \sin(2\pi f_c t)
\]

(2.5)

Where

\[
S(t) = S_r(t) + jS_i(t)
\]

The low pass signals \( S_r(t) \) and \( S_i(t) \) are called the in-phase and quadrature components, respectively, of the band pass signal. From above it is clear that the complex envelope \( S(t) \) is independent of the carrier frequency, \( f_c \). Therefore, it is sufficient to consider the complex envelope as the transmitted signal and to ignore the carrier term \( \exp(j2\pi f_c t) \). The great advantage of complex signal representation is that, the operation such as linear band pass filtering (Convolution) can be expressed directly in terms of the complex envelope. In other words, bandpass filtering of a signal can be treated simply in terms of complex lowpass signals, [27] & [28]. This filtering operation and how it can be implemented in terms of real lowpass filters is shown in Fig. 2.4. Consider the discrete coded waveform consisting of \( (N+1) \) pulses representing the impulse response of a linear filter as an ordered set of complex numbers

\[
\{h(n)\} = (h(0), h(1), h(2) \ldots h(N))
\]

(2.6)

Such sequence is often called as a time series. Alternatively the sequence \( \{h(n)\} \) with values \( h(n) = h(nT) \) can also be visualized as being generated by sampling the complex envelope \( h(t) \) of the corresponding continuous waveform every \( T \) seconds. The magnitude of the complex number \( h(n) \) represents the amplitude of the \( n^{th} \) pulse while
Input $u_2(t)$

Bandpass filter $h(t)$

Output $u_2(t) = u_1(t) * h(t)$

$u_2(f) = H(f) U_1(f)$

$S_2(f) = \frac{1}{2} \Gamma(f) S_1(f)$

Fig. 2.4 Baseband filtering.
the angle of $h(n)$ specifies its phase. The complex envelope of the filter impulse response $g(t)$ can easily be obtained by convolving the time series of Eq (2.6) with Woodward's rect function

$$
 g(t) = \sum_{n=0}^{N} h(n) \mathcal{E}(t-nT) \otimes \text{rect} \left( \frac{t}{T} \right)
$$

$$
 g(t) = \sum_{n=0}^{N} h(n) \int_{-\infty}^{\infty} \mathcal{E}(t-nT) \text{rect}(t - \frac{T}{T}) d\tau
$$

$$
 g(t) = \sum_{n=0}^{N} h(n) \text{rect} \left( \frac{t}{T} - n \right) \quad (2.7)
$$

Where $\otimes$ indicates the Convolution Operation.

The Fourier Transform of Eq (2.7) can be found by using the familiar rules of transform theory:

$$
 S(t-T) \leftrightarrow S(f) \exp(-j2\pi fT)
$$

$$
 \text{rect}(t/T) \leftrightarrow T \text{Sinc}(ft)
$$

Where

$$
 \text{Sinc}(ft) = \frac{\sin(\pi ft)}{\pi ft}
$$

The spectrum of the complex envelope $g(t)$ is therefore,

$$
 g(f) = T \text{Sinc}(ft) \sum_{n=0}^{N} h(n) \exp(-j2\pi ft)
$$

or

$$
 g(f) = T \text{Sinc}(ft) H(f) \quad (2.8)
$$

It can be seen that the spectrum $H(f)$ is periodic with a period of $1/T$ and it represents the spectrum of the coded sequence $h(n)$ as shown in Fig 2.5. The relationship between the Fourier Transforms of an analogue signal and its sampled version is thus given by.
Fig. 2.5 Effect of time domain sampling on bandlimited signal.
\[ H(f) = \sum_{n=-\infty}^{\infty} H_n(f + \frac{f}{T}) \]  

(2.9)

Where

\[ H_n(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft) \, dt \]

\[ H(f) = \sum_{n=-\infty}^{\infty} h(nT) \exp(-j2\pi fnT) \]

The above relationship essentially formulates the time domain sampling theorem which states that a continuous function of time whose spectrum is limited to the band (±W/2) is completely defined by time domain samples taken at intervals of 1/W.

Similarly, the input signal, having (M+1) pulses is represented by a(n) The complex envelope \( e(t) \) of the output signal has a FT which is, according to standard transform rules of linear filtering, the product of the FT's of the complex envelopes of the input waveform and the filter impulse response scaled by a factor \( \frac{1}{T} \).

\[ E(f) = \frac{1}{2} \left\{ \sum_{n=0}^{M} a(n) \exp(-j2\pi fnT) \left\{ \sum_{n=0}^{N} h(n) \exp(-j2\pi fnT) \right\} T^2 \text{Sinc}^2(fT) \right\} \]

\[ E(f) = \frac{1}{2} \left\{ \sum_{n=0}^{M} \sum_{k=0}^{N} a(n)h(k) \exp(-j2\pi f(n+k)T) T^2 \text{Sinc}^2(fT) \right\} \]

\[ E(f) = \frac{1}{2} \sum_{m=0}^{M+N} c(m) \exp(-j2\pi fmT) T^2 \text{Sinc}^2(fT) \]  

(2.10)

where

\[ c(k) = \sum_{n=0}^{k} a(n)h(k-n) \quad , \quad k=0, 1, 2 \ldots (M+N) \]

The complex envelope of the output signal \( e(t) \) is given by the inverse FT(IFT) of Eqn (2.10). Use of the relationship

\[ T^2 \text{Sinc}^2(fT) \leftrightarrow T \text{tri}(t/T) \]
leads to

\[ e(t) = \frac{T}{2} \sum_{m=0}^{M+N} c(m) \text{tri}(\frac{t}{T} - m) \]  

(2.11)

The function \( \text{tri}(t/T) \) is shown in Fig 2.1 and is defined as

\[ \text{tri}(\frac{t}{T}) = \begin{cases} 
1 - \frac{|t|}{T} & , |t| \leq T \\
0 & , \text{elsewhere} 
\end{cases} \]

At non-integer multiples of the sub pulse duration, \( T \), the complex envelope of the output waveform is given by linear interpolation between adjacent values. The output number sequence \( c(m) \) thus specifies, except for a constant scale factor, the output waveform at regular sampling instants \( T \).

The main conclusion from the foregoing is that complex envelope representation of pulse trains simplifies the discrete linear filtering process which can be regarded merely as multiplication of polynomials. In addition only the spectrum of the coded sequences need be considered to specify the waveform at integer multiples of \( T \).

2.2.2 The Z-Transform

It has been shown in the previous section that the FT can be used to describe the frequency properties of pulse trains. Another compact notation for the FT of such signals is the Z-transform (ZT), [31]. The ZT is also a very convenient method of representing a signal by a set of poles and zeros in the complex Z-plane. This is quite similar to Laplace transform techniques used for analogue systems which can be represented by poles and zeros in the complex S-plane.

The \( z \)-transform of an arbitrary numbers sequence, \( a(n) \), is simply a polynomial in powers of \( Z^{-1} \), and can be given by

\[ A(Z) = a(0) + a(1)Z^{-1} + \ldots + a(N)Z^{-N} \]

(2.13)
Where $Z$ is usually expressed in the polar form $Z = \exp(ST)$. In general the frequency variable $Z$ has both real and imaginary parts. Thus if,

$$S = \sigma + j2\pi f$$

$$Z = \exp(\sigma + j2\pi f)T = \exp(\sigma T) [\cos(2\pi fT) + j\sin(2\pi fT)]$$

The variable $Z$ is often referred to as a "shift operator", since $\exp(-j2\pi fT)$ implies a time delay of $T$ seconds while $\exp(j2\pi fT)$ represents a time advance of $T$ seconds.

If the $ZT$, $A(Z)$, is evaluated on the unit circle in the $Z$ plane ($|Z| = 1$), the spectrum of the time series $a(n)$ is obtained.

$$A(Z) \bigg|_{|Z|=1} = A(f) = \sum_{n=0}^{N} a(n) \exp(-j2\pi fnT) \tag{2.14}$$

It is noted that the spectrum $A(f)$ is a continuous function in frequency. In practice, however, the spectrum of discrete time series, is usually evaluated using digital computers. This means that the spectrum $A(f)$ can only be estimated at discrete points in $f$ which is generally referred to as discrete Fourier transforms (DFT). Although the spectrum can only be estimated at suitably chosen intervals in $f$, it can be shown that for time limited or periodic signals such a discrete representation of the underlying continuous function does not result in any loss of essential information, [31]. This is sometimes referred to as the frequency sampling theorem.

Using DFT Equation (2.14) can be now rewritten as a finite-length sequence.

$$A(K) = \sum_{n=0}^{N} a(n) \exp\left(-j\frac{2\pi fn}{(N+1)}\right) \tag{2.15}$$

Where $k = 0, 1, 2, \ldots \ldots, N$

Thus the frequency spacing between successive harmonics is $1/(N+1)T$ and the frequency of the $k$th harmonic is therefore $k/(N+1)T$. It can be seen from equation (2.15) that the sequence $A(k)$ is periodic with a period of $(N+1)$, i.e.
Similarly the inverse transform (IDFT) of Equation (2.15) can be written as

\[ a(n) = \frac{1}{(N+1)} \sum_{k=0}^{N} A(k) \exp \left( \frac{2\pi j n k}{N+1} \right) \quad (2.16) \]

where \( n = 0, 1, 2, \ldots, N \).

Where the multiplying factor \( 1/(N+1) \) has been included for convenience.

Alternatively the above equations can be expressed in matrix form

\[ \mathbf{a} = \frac{1}{(N+1)} \mathbf{Y}^{*} \mathbf{A} \quad (2.17) \]

Where the \( (N+1) \) element column vectors \( \mathbf{a} \) and \( \mathbf{A} \) are given by

\[ \mathbf{a} = \text{col} \{ a(0), a(1), \ldots, a(N) \} \]

\[ \mathbf{A} = \text{col} \{ A(0), A(1), \ldots, A(N) \} \]

The \( (N+1) \times (N+1) \) matrix \( \mathbf{Y}^{*} \) is the conjugate of the DFT matrix \( \mathbf{Y} \) as given by

\[
\mathbf{Y} = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & y & y^2 & \cdots & y^N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & y^N & y^{2N} & \cdots & y^{N^2}
\end{pmatrix} \quad (2.18)
\]

Where

\[ y = \exp[-j2\pi/(N+1)] \]

Equation (2.17) is the IDFT and its inverse (DFT) is therefore given by

\[ \mathbf{A} = \mathbf{Y} \mathbf{a} \quad (2.19) \]

It is noted that the DFT matrix \( \mathbf{Y} \) has the property

\[ \mathbf{YY}^{*} = (N+1) \mathbf{I} \]
Where I is the identity matrix

Although, in principle, the DFT can be evaluated using equation (2.15) or equation (2.19) in practice the fast Fourier transform (FFT) algorithm is used, [32], [33]. From the foregoing it is clear that a finite-duration sequence can be expressed exactly by samples of its ZT. Moreover, the periodic sequence obtained by sampling the ZT at (N+1) equally spaced points on the unit circle (|Z|=1) in the complex Z-plane is identical to the DFT. The sequence corresponding to these frequency samples is a periodically repeated version of the original sequence, such that if (N+1) samples, of the ZT are used no 'overlapping' or 'aliasing' occurs. Thus, in general, a finite duration sequence is represented as one period of a periodic sequence. It has also been shown that the ZT of a pulse train is simply a power series of Z\(^{-1}\) in which the coefficients of the various terms are equal to the corresponding samples when a waveform is expressed in this form, it is possible to regenerate the number sequence merely by inspection, a number having the index n is simply the coefficient of Z\(^n\) in the ZT.

The linear filtering operation of equation (2.10) can now be rewritten using the short hand notation of the ZT. If the sequence C(n) represents the convolution of the two sequence a(n) and h(n), than the ZT of C(n) is the product of the ZT's of a(n) and h(n), i.e. if,

\[
C(n) = \sum_{k=0}^{n} a(k)h(n-k) \quad n=0, 1, 2, ... (N+M)
\]

then,

\[
C(Z) = A(z) H(z) \quad \text{.................................................................} \quad .... (2.20)
\]

This can be shown considering the following expressions,

\[
C(Z) = \sum_{n=0}^{N+M} \{ \sum_{k=0}^{n} a(k)h(n-k) \} Z^{-n}
\]

Interchanging the order of summation yields
Letting \( m = n-k \) leads to

\[
C(Z) = \sum_{k=0}^{M} a(k) \sum_{n-k}^{n-k} h(n-k) Z^{-n}
\]

hence \( C(z) = A(z) H(z) \).

However, if the DFT is used to evaluate equation (2.10), the sequences \( a(n) \) and \( h(n) \) have to be modified. Taking the straight forward DFT of finite duration sequences and then inverse transforming the products of their spectra is equivalent to circularly convolving the periodic sequences created from the given sequences. To obtain the linear convolution both \( a(n) \) and \( h(n) \) must be \((N+M+1)\)-point sequences. This is achieved by appending the appropriate number of zeros to both \( a(n) \) and \( h(n) \). That is

\[
[a(0), a(1), \ldots, a(N), 0, 0, 0, \ldots, 0]
\]

\[
[h(0), h(1), \ldots, h(M), 0, 0, 0, \ldots, 0]
\]

The \((N+M+1)\) point DFT of \( a(n) \) and \( h(n) \) is then taken, multiplied and inverse transformed to obtain the correct sequence \( C(n) \). The \( ZT \), a familiar analytical technique in modern control and sampled data system, thus provides an excellent tool for studies of digital systems and signals such as pulse strains. Therefore, throughout this work use is made of the \( ZT \) representation wherever possible.

### 2.3 OPTIMUM PROCESSING OF RADAR SIGNALS

The transformation, and interference effects to which a radar signal is subjected during its path from the transmitter to the receiver will now be analyzed using Fig. 2.6. It is assumed that the signal, although generated digitally, is analogue filtered prior to transmission. The transmitted signal \( S(t) \) first passes through a time-invariant processor, which accounts for the unknown round trip amplitude attenuation \( \alpha \), time delay \( \tau \),
Fig. 2.6 Block diagram of signal model
Doppler shift $v$, and phase shift $\phi$ of the signal. Such a treatment of the transmitted signal assumes a point target (no range extent). This is a convenient assumption in analyzing system performance. In order to avoid continual repetition several general assumptions are made for subsequent discussions of radar signal processing techniques.

i) point targets are assumed

ii) Target acceleration is negligible, i.e.

$$a \ll \frac{\lambda}{T_s^2}$$

Where $a$ is target acceleration, $\lambda$ is the carrier wave length, and $T_s$ is the signal duration.

iii) Mismatch of the envelope of the received signal and the transmitted waveform due to high relative velocities is negligible, i.e.

$$2 \frac{v_r}{c} \ll \frac{1}{W T_s}$$

Where $v_r$ is the radial target velocity relative to the radar, $c$ is the velocity of light, and $W$ is the signal band-width.

iv) All signals are narrow band.

$$W \ll f_c$$

Where $f_c$ is the carrier frequency. Since radar returns are always immersed in noise and interference from all kinds of objects illuminated by the antenna beam, the receiver must be optimized in some manner. Additive interference introduced by a large number of independent target like reflections such as composite returns from an extended scattering region containing terrain, rain, sea waves etc., is usually called clutter. The return signal, immersed in clutter and noise, is now analogue filtered, digitized and processed. These approaches have been used to derive optimum processors for radar signals,[4],[21],[34].

i) Signal to Noise Ratio Criterion (SNR)

ii) Likelihood Ratio Criterion

iii) Inverse Probability Criterion.
Any of these criteria lead to the matched filter receiver, provided the signal is corrupted only by additive white Gaussian noise. Moreover, Woodward, [21] has shown that this type of receiver also preserves all the information in the radar return. Even in situations where matched filter processing is not optimum, for example when interference from clutter is significant (coloured noise), matched filters usually provide a reasonable compromise between system performance and complexity, [35] On the other hand, matched filter processing may be used simply because information about the target environment to design more optimum processors is not available. The problem of target resolution and optimum detection for a specified clutter environment has been studied by a number of authors and will not be of prime concern here, [36], [39] Therefore, unless otherwise specified, a matched filter receiver is assumed.

2.3.1 Digital Matched Filter

The characteristics of the matched filter (MF) can be designated by either a frequency response function or a time response function, each being related to the other by a F T Operation. In the frequency domain, the MF transfer function $H(f)$, is the complex conjugate function of the spectrum of the signal that is to be processed, except for an arbitrary scale factor and a linear phase shift.

$$H(f) = S^*(f) \exp(-j2\pi f T_d) \quad (2.21)$$

Where $S^*(f)$ denotes the complex conjugate spectrum of the input signal $S(t)$. The scale factor and the linear phase shift $\exp(-j2\pi f T_d)$ do not affect the signal to noise ratio (SNR) and may therefore be ignored. Thus,

$$H(f) = S^*(f) \quad (2.22)$$

In the time domain, the corresponding relationship is obtained by taking the IFT of Equation (2.22). This leads to the result that the impulse response of a MF is the mirror image of the complex conjugate of the transmitted signal $S(t)$, and the general relationship is given by

$$h(t) = S^*(T_d - t)$$
or simply

\[ h(t) = S^*(-t) \] (2.23)

in discrete form, this relation is

\[ h(n) = S^*(-n) \] (2.24)

So far it has been assumed that the spectrum of the reflected signal is completely known. However, even for the simplest case of a single point target, the return signal contains two known parameters, Doppler shift \( \nu \) and time delay \( \tau \). In general, the spectrum of the received signal is of the form

\[ S(f) = S_0(f-\nu) \exp(-j2\pi f \tau) \] (2.25)

Where \( S_0(f) \) denotes the spectrum of the transmitted signal. Therefore, for matched condition, a receiver with the frequency characteristic, (neglecting a constant phase term and amplitude factor)

\[ H(f) = S_0^*(f-\nu) \exp(j2\pi f \tau) \] (2.26)

is required for optimum detection. It is clear that for stationary or slowly moving targets \( (\nu = 0) \) a receiver matched to the transmitted waveform is optimal. However, since the doppler frequency depends on the range rate of the target and is not known beforehand, optimum reception of signals reflected from moving targets can not be accomplished by only one matched filters. An optimum receiver in this case requires a bank of matched filters with incremented frequencies, \( \nu \), of the Doppler shift \( \nu \) in the expected domain. This is illustrated in block diagram form in Fig. 2.7. The output of the matched filter bank is usually applied to a square law or envelope detector and compared with a common threshold. If any of the outputs crosses the threshold a signal is deemed detected, and from the corresponding delay and Doppler shift the target's range and velocity are estimated.
Fig. 2.7 Bank of parallel MF's
The response of the digital MF to an input signal reflected from a target, at range \( r \) and radial velocity \( V_r \), is obtained by convolving the filter impulse response with the input signal:

\[
\psi(\tau, v) = \sum_{n=-\infty}^{\infty} \{S(nT - \tau)\exp[-j2\pi v(nT - \tau)] + \epsilon(nT)\} S^*(-(m-n)T) \exp[-j2\pi v(mT-\tau)]
\]

\[
= \exp[-j2\pi v(mT-\tau)] \sum_{n=-\infty}^{\infty} \{S[(n+m)T - \tau]\exp[-j2\pi vnT]\} S^*(nT) + \sum_{n=-\infty}^{\infty} W[(n+m)T] S^*(nT)
\]

(2.27)

Where \( \tau = 2r/c \) and \( v = 2V_r f_c / c \)

\( f_c \) = Carrier Frequency, \( c \) = Velocity of Light

The noise \( W(mT) \) is assumed to be a Gaussian random variable with zero mean and variance \( \sigma^2 \). Furthermore, \( W(mT) \) and \( W(kT) \), for any integer \( k \neq m \), are uncorrelated and therefore statistically independent. Equation (2.27) can be simplified by ignoring the non-essential phase factor, \( \exp[-j2\pi v(mT-\tau)] \), and by letting \( mT-\tau = \tau' \).

Thus,

\[
\psi(\tau', v) = \sum_{n=-\infty}^{\infty} S[(nT + \tau')S^*(nT) \exp(-j2\pi vnT) + \sum_{n=-\infty}^{\infty} W[(n+m)T] S^*(nT)
\]

(2.28)

Although the input signal and the impulse response are discrete, it is noted that the function in Equation (2.28) depends on the two continuous variables \( \tau \) and \( v \). Since the radar detector at the output of the MF usually removes the phase information, the function of interest is generally:

\[
|\psi(\tau, v)|^2 = \left| \sum_{n=-\infty}^{\infty} S(nT + \tau)S^*(nT) \exp(-j2\pi vnT) \right|^2 + \left| \sum_{n=-\infty}^{\infty} W[(n+m)T] S^*(nT) \right|^2 + \text{Cross products}
\]

(2.29)

The first term in the equation above is the signal term resulting only from the target reflection:

\[
|X(\tau, v)|^2 = \left| \sum_{n=-\infty}^{\infty} S(nT + \tau)S^*(nT) \exp(-j2\pi vnT) \right|^2
\]

(2.30)
The function \(|X(\tau, \nu)|^2\) is commonly referred to as the ambiguity function. The investigation of the ambiguity function has been a field of extensive study since its introduction by Woodward, [21]. The origin of the ambiguity function (\(\tau=0, \nu=0\)) may be thought of as the output of the matched filter tuned in time delay and frequency shift to the signal reflected from the target (point source) of interest. For zero relative Doppler shift, \(|X(\tau,0)|^2\) represents the squared magnitude of the auto-correlation function (ACF) of the transmitted signal. This is the filter response to the reflections at a different range but at the same Doppler as the target. Similarly, \(|X(0,\nu)|^2\) is the response to reflections at the same range as the target but with other Doppler shifts.

Another property which reflects the fundamental constraints of radar signal design is the total volume under the ambiguity function. It is shown below that this volume is independent of the shape of the transmitted waveform

\[
V = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |X(\tau, \nu)|^2 d\tau d\nu \quad \text{(2.31)}
\]

Substituting the Equation (2.30) into the Equation (2.31) and integrating first with respect to \(\nu\),

\[
V = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} S(nT + \tau)S^*(mT + \tau)S(mT)S^*(nT) \frac{\sin[\pi(n-m)]}{T\pi(n-m)}
\]

Integrating with respect to \(\tau\) and noting that the auto-correlation of the signal is given by

\[
r[(n-m)T] = \int_{-\infty}^{\infty} S^*[(n-m)T + \tau]S(\tau)d\tau
\]

leads to

\[
V = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} S^*(nT)S(mT) r[(n-m)T] \frac{\sin[\pi(n-m)]}{T\pi(n-m)}
\]

since,

\[
\frac{\sin[\pi(n-m)]}{T\pi(n-m)} = \begin{cases} 0, & n \neq m \\ \frac{1}{\pi}, & n = m \end{cases}
\]
Fig. 2.8 Tapped delay line MF.

Fig. 2.9. DFT realization of MF.
and
\[ V = \frac{1}{T} r(0) \sum_{n=\infty}^{\infty} |S(nT)|^2 = \frac{1}{T} r(0).E \]
or
\[ V = \frac{1}{T} E^2 \] .................................................................................................................. (2.32)

Where E denotes the signal energy.

Hence the volume under the ambiguity function depends only on the total signal energy. This implies that any reduction of ambiguity anywhere in the (τ,ν)-plane will cause it to appear elsewhere. Equation (2.32) is particularly important in clutter and multiple target environments. The radar signal design problem can be considered, therefore, as a process of rearranging the undesired portions of the ambiguity function (range and Doppler ambiguity away from the origin) into region of little importance.

The second term in Equation (2.29) is due to the white noise. The computation of the mean noise power is simplified by noting that the noise power is an uncorrelated zero mean, random Gaussian variable. Thus the expected value of the various cross products in Equation (2.29) are zero. Hence the mean square noise power at the output of the MF is given by

\[ |\overline{N}(\tau,\nu)|^2 = E\{ \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} W(nT)S^* (nT+\tau)W^* (mT)S(mT+\tau) \exp[-j2\pi\nu(n-m)T] \} \]

\[ |\overline{N}(\tau,\nu)|^2 = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} E\{ W(nT)W^* (mT)S^* (nT+\tau)S(mT+\tau) \exp[-j2\pi\nu(n-m)T] \} \]

\[ |\overline{N}(\tau,\nu)|^2 = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} r_w(nT,mT)S^* (nT+\tau)S(mT+\tau) \exp[-j2\pi\nu(n-m)T] \]

Where

\[ r_w(nT,mT)=E\{ W(nT)W^*(mT) \}=\sigma^2\delta[(n-m)T] \]

24
\[ |N(\tau,v)|^2 = \sigma^2 \sum_{n=-\infty}^{\infty} |S(nT)|^2 = \sigma^2 E = N_0 E \quad (2.33) \]

Thus, for a given signal energy, the mean-square noise power at the output of the optimum receiver, is a constant over the \((\tau,v)\)-plane.

So far it has been implied that the two variables \(\tau\) and \(v\) are continuous. However, from practical considerations only signals of finite duration can be processed. It is therefore assumed that the received signal duration is \((N+1)T\) and that the delay \(\tau\) and Doppler shift \(v\) are expressed as integer multiples of the sampling period \(T\) and fundamental frequency \(1/[(N+1)T]\) respectively with these notations Equation (2.30) can now be written as

\[
|X(k,l)|^2 = \left| \sum_{n=-k}^{n+k} S(n+k)S^*(n) \exp(-j2\pi \frac{n}{N+1}) \right|^2
\]

\(k, l = 0, \pm 1, \pm 2, \ldots, \pm N \quad (2.34)\)

In subsequent chapters attention will be focused on the case for slowly moving or stationary targets. In other words, the relative Doppler spread of the targets is assumed to be negligible \((v = 0)\). The output of the MF for zero Doppler is the ACF of the transmitted signal and is given by

\[ r(k) = \sum_{n=-k}^{n+k} S(n+k)S^*(n) \quad (2.35) \]

\[ k = 0, \pm 1, \pm 2, \ldots, \pm N \]

An alternative way of representing the auto-correlation sequence \(r(k)\) as above is to use the ZT technique described in section 2.2.2. Thus, the ZT of Equation (2.35) is given by

\[ R(z) = r(-N) + r(-N+1)Z^{-1} + \cdots + r(0)Z^N + r(N-1)Z^{2N-1} + r(N)Z^{2N} \]

\[ = [S^*(N)+S^*(N-1)Z^{-1} + \cdots + S^*(0)Z^N] [(S(0)+S(1)Z^{-1} + \cdots + S(N)Z^N)] \]

25
\[
\begin{align*}
= Z^{-N}[S^*(0)+S^*(1)Z + +S^*(N)Z^N] [S(0)+S(1)Z^{-1} + +S(N)Z^N] \\
= Z^{-N} S^*(1/Z)S(Z) 
\end{align*}
\] (2.36)

The coefficients of R(z) are labeled with index values running from -N to N. The kth coefficient of the complex envelope of the ACF at a time shift is given by Equation (2.35)

From the foregoing it is clear that R(z) is an even function which has the property of complex conjugate symmetry, that is

\[
r(-n) = r^*(n) \quad (2.37)
\]

The main response peak \( r(0) \) given by

\[
r(0) = \sum_{n=0}^{N} |S(n)|^2 = E 
\] (2.38)

is always real and represents the energy contained in the sequence. Moreover, it can easily be shown that

\[
r(0) \geq |r(n)| \quad n = 0, 1, 2, \ldots, N
\]

The MF for discrete coded waveforms could be implemented using a tapped delay line as shown in Fig. 2.8. It is assumed that the input is at the RF carrier (or IF) and that each delay element is an integral number of wavelengths. In addition, the subpulse matched filter must also be centered at that frequency, [40]. Such a processor filters the input signal directly as a band-pass signal. However, for long sequences, \((N+1) > 31\), the bandwidth of the delay line presents a practical problem in that its N cascaded stages must have an overall bandwidth \( \geq 1/T \), the reciprocal of the subpulse duration. Therefore, it is often preferable to process signals at base band (zero IF or homodyne receiver) particularly when digital implementation is required.

In a typical digital processor the delay lines are replaced by digital memories, Fig 2.10. In this configuration the RF or IF signals are heterodyned to zero carrier frequency with a single-side band or quadrature mixer whose two video output represent the in-phase I, and quadrature, Q, components of the signal (sec 2.2.1). It is
even possible to add minor Doppler shifts (if known) to the Local Oscillator (LO) to prevent degradation of the matched filter output. The baseband signals are then low pass filtered, sampled and converted to digital form at high speed. The digital matched filtering operation can be carried out using a digital tapped delay line (transversal filters) as shown in Fig 2.10, or by the high speed pipeline FFT method, [41], [53], [54], [55], shown in Fig (2.9). However, in most radar applications, the FFT realization of the filter is more desirable. This is particularly the case for signals with a large time-bandwidth product, since FFT methods tend to be more efficient at performing convolution operations.

2.3.2 Pulse Compression Radar

It has been shown that, regardless of the signal shape, the MF produces one global maximum output value which is equal to the signal energy. Because of this concentration of the entire signal energy, its detection against a white noise background is enhanced. In addition, to achieve high accuracy and resolution of radar measurements, it is required that the maximum value in the MF output be as narrow as possible. Since the sharpness of the MF output signal (auto-correlation function) is inversely proportional to the rms signal bandwidth, compression of the received signal into a narrow spike can be accomplished provided the signal has a large bandwidth. Thus, the essence of pulse compression radar systems is to provide this large bandwidth without degrading radar performance in other respects such as range resolution.

An obvious way to achieve a large bandwidth is simply to reduce the duration of the transmitted pulse. However, since target detectability and measurement precision depends on the signal energy, the transmitted power must be increased proportionally, to keep the energy constant. Unfortunately, the peak power limitation of transmitters sets a lower limit on the pulse duration. Therefore, the need for a large bandwidth must be met by modulating the pulse, rather than reducing the pulse duration.

In principle, any of the three basic types of modulation could be used to increase the signal bandwidth, namely amplitude (AM), phase (PM), and frequency (FM) modulation. (Here, FM is considered as a general type of modulation where the phase of the signal is varied.) AM, however, is generally not desirable for use with radar.
waveforms due to its inherent disadvantages. First, it is an inefficient way of increasing the signal bandwidth, is that the function actually applied to the modulators must be efficiently under constant amplitude conditions. Thirdly, it is expensive and difficult to achieve good amplitude linearity throughout the radar system over the entire dynamic range of interest. Therefore, AM is of interest as a means to improve system performance, rather than as a primary method of achieving large signal bandwidth. The case of quantized FM will not be considered here as, due to complexity in frequency synthesizing, it is less practical, except in a few cases, than PM.

As implied by the term pulse compression, the objective of the receive filter is to compress the received long pulse having a time duration of $T_s$ seconds and a bandwidth of $W$ hertz into a short pulse of duration $1/w$, to allow recognition of closely spaced targets. The ratio of the duration of the long pulse to that of the short pulse is called the compression ratio. Thus the compression ratio is given by

$$m_c = T_s/(1/w) = T_sW$$

which is equal to the time-bandwidth product of the waveform. In a digital system the signal duration $T_s$ is equal to $(N+1)T$, where $T$ is the sampling interval and $(N+1)$ is the number of samples. Furthermore, if the sampling process is carried out at the Nyquist rate $T = 1/w$, then

$$m_c = (N+1)TW = N+1$$

that is, the compression ratio or time-bandwidth product is equal to the total number of samples.

The main conclusion from the foregoing is that in all cases where the transmitted signal spectrum is substantially widened by modulation, decompression of signals can be accomplished during reception.

### 2.3.3 Range Resolution in a Matched Filter Radar

The estimation of target resolution performance is probably the most difficult problem to solve in modern high performance radar systems. In some cases the interfering objects may themselves be targets of interest, whereas in others they may be
undesirable scatterers introducing a type of noise, known as clutter, into the system. As mentioned in section 2.3.1 the optimum receiver for maximum resolution is not necessarily a MF. In practice, however, the typical target situation is too complex and not enough prior information is available to implement anything but a MF receiver or an approximation. For good target resolution, it has thus been necessary to retain a MF processor but to optimize the signal waveform so as to reduce the mutual interference (self-clutter) between targets.

The ways in which the transmitted waveform limits the radar performance in white noise are well known, [21] Three parameters are of prime importance. These are the bandwidth $W$, the time duration $T_s$, and the total signal energy $E$. Specifically

i. Range resolution for a MF receiver and stationary targets is determined by the spectrum envelope of the. For a given spectral shape, range resolution is proportional to $1/W$. Therefore, good range resolution is achieved with a spectrum for which the total occupied frequency band is large.

ii. Making use of the time-frequency duality, velocity resolution (radial) for targets having the same range is determined by the time structure or envelope of the signal. For a given envelope, velocity resolution is proportional to $1/T_s$. Hence low velocity ambiguity requires a waveform that occupies, with significant energy, a large total time interval.

iii. Target detectability is determined by the ratio of received signal energy to received noise power (SNR). For given system parameter, signal detectability, and thus range, can only be improved by increasing the transmitted energy. Therefore, it is desirable to transmit a waveform which has both a rectangular envelope as well as a rectangular spectrum.

In order to appreciate the resolution problem consider two stationary targets slightly separated in range. Neglecting an amplitude attenuation factor the combined received signal is of the form

$$S(nT) = S_0(nT - \tau_0) + S_0(nT - \tau_0 - \tau)$$  \hspace{1cm} (2.41)
where

\[ S_{o}(nT) = \text{transmitted waveform} \]

\[ \tau_{o} = \text{position of first target} \]

\[ \tau = \text{target separation} \]

To be able to distinguish the two target returns it is necessary to select a suitable waveform \( S_{o}(nT) \). A measure of distinguishability, given by the sum of the square difference of the two signals, can be expressed as

\[
\varepsilon^{2} = \sum_{n=-\infty}^{\infty} \left[ \left| S(nT - \tau_{o}) \right|^{2} + \left| S(nT - \tau_{o} - \tau) \right|^{2} - 2 \text{Re}[S(nT - \tau_{o})S^{*}(nT - \tau_{o} - \tau)] \right]
\]

The first two terms represent the signal energy \( E \) and are therefore constants. The last term is recognized as the ACF \( r(\tau) \). For the two signal returns to be as different as possible it is required to maximize the equation above, that is

\[
\text{max } \varepsilon^{2} = \{ E - \text{Re}[r(\tau)] \}
\]

(2.42)

Hence, in a MF radar the optimum waveform to use would be one whose receiver response, or ACF, has an envelope consisting of a single spike at \( \tau = 0 \) of a width smaller than the spreading of the targets in range (delay). However, in practice such a waveform cannot usually be realized. Actual waveforms have ACF's whose envelopes show one or more of the properties indicated in Fig 2. 11. Basically there are three types of MF responses, a single lobe (a), a narrow main lobe accompanied by relatively large side-lobes, (b), and a single lobe surrounded by a noise-like, low-level response spread out in time (c). Each one of these response types presents its own resolution problems, suppose the MF output consists of a single lobe (a). If the separation of the targets is larger than the width of the lobe there will be no problem in resolving them. However, for closely spaced targets the responses overlap and the envelope of the combined output will depend on the phase relationship of the echoes, Fig 2. 12. Thus, in the region of overlap, target resolution is difficult to achieve. Therefore, to distinguish two or more targets their returns must be separated by at least the half power (3dB)
Fig. 2.11 Forms of range ambiguities.
Fig. 2.12 Overlap of two target returns.
response width. A different type of resolution difficulty is caused by the MF response shown in Fig. 2.11(b). Although the main lobe may be sufficiently small to meet the required target resolution, target returns separated at multiples of $T_d$ are completely masked. Problems of yet a different nature are introduced by the response of type (c). Again, the main lobe may be narrow enough for the desired close target resolution. However, for targets with widely varying cross-sections, it still may not be possible to distinguish them. The pedestal-like extension of the response due to a strong target may have an amplitude strong enough to obscure the main response peak of weaker targets. This effect is aggravated particularly in a multiple-targets environment where the combined side lobes from many returns may build up to a level that even relatively strong targets can no longer be recognised.

As shown by Woodward, [21], the MF receiver utilizes the full information available from the return signal. The width of the main response lobe can be regarded as a measure of uncertainty about the exact target range, while the spread of the response introduces ambiguity of the target location. Both effects, although conceptually different, are lumped together in a figure of merit known as the time resolution constant, [21]

$$\Delta \tau = \frac{\int |r(\tau)|^2 d\tau}{|r(0)|^2}$$  \hspace{1cm} (2.43)

The resolution problem has been discussed here in a purely qualitative manner. Analytical treatments of the resolution in parameters can be found in many excellent radar books, [22], [23], [26], [42]. Nevertheless, the preceeding discussion allows the general formulation of the requirements which have to be met for target resolution. First, the output signal to white noise ratio must be large enough for reliable detection. Secondly, the target of interest must be separated sufficiently from any other target of comparable or larger cross-section to prevent overlap of the main response lobes. Thirdly, the combined interference from other targets must not be so strong as to mask the target return of interest.
Interference from other targets due to response sidelobes acts like clutter caused by undesirable scatterers. However, the term clutter implies that the interference causing reflectors are so dense that they cannot be resolved. The type of clutter due to side lobes is often called self-clutter to distinguish it from the effect of undesired objects.

In summary, the resolution performance of a radar thus depends not only on the width of the main response lobe but also on the low level response surrounding the main peak. In the published literature resolution is often referred to as the 3dB points of the main response lobe. In the present context, however, target resolution means the ability to recognize a target in the presence of others.
CHAPTER-3

SYNTHESIS OF PULSE TRAINS FROM SPECIFIED AUTOCORRELATION FUNCTIONS
3.1 INTRODUCTION

The problem of synthesizing signals which realize the desired auto-correlation function (ACF) can be divided into a number of inter-related problems, each of which has an independent practical significance. The separate problems could be formulated as follows, [42]

1. Determine the class of functions which are realizable ACF's for arbitrary signals

2. Determine the subclass of ACF's for various signal structures such as discrete coded waveforms

3. Synthesis of signals whose ACF is a close approximation to the desired ACF not belonging to the class of realizable functions

4. Synthesis of signals (pulse trains) which satisfy a given set of requirements, e.g., range resolution, energy utilization, etc.

So far, a comprehensive analytical treatment of these problems and their solution has not been formulated. For example, a simple criterion for determining the realizability of an ACF has not yet been found.

To appreciate the nature of the problem arising here consider the ACF

\[ r(t) = \sum_{n=-\infty}^{\infty} S(nT + \tau)S^*(nT) \]  \hspace{1cm} (3.1)

or its equivalent expression

\[ r(\tau) = \int_{-w/2}^{w/2} |S(f)|^2 \exp(j2\pi f \tau) df \]  \hspace{1cm} (3.2)

Where \( |S(f)|^2 \) is the power spectrum of the signal \( S(nT) \) and is assumed to be band limited. In other words, \( S(f) \) is zero outside some range (-w/2, w/2). Strictly speaking, the requirement for a finite band width \( W \) is incompatible with a finite duration signal. However, an approximation to the finite spectrum condition can be reached if a major portion of the signal energy is concentrated within a specified frequency band.
From Eq (3.2) it can be seen that the ACF and the power spectrum form a Fourier transform pair. Hence it follows that for \( r'(\tau) \) to be a realizable ACF its spectrum \( R(f) \), must be real and non-negative. Even if the given ACF is realizable the synthesis problem cannot be solved uniquely. Since the phase information is lost in the power spectrum, it is not possible to determine \( S(f) \) itself which is necessary to find \( S(nT) \). Therefore, all signals whose spectra differ only in phase will have the same ACF. Thus the synthesis problem may be divided into the following two steps:

1. The power spectrum \( |S(f)|^2 \) is determined from the given ACF. (It is assumed that the ACF is realizable, i.e., \( R(0) = \text{FT}\{r(\tau)\} > 0 \))

2. From the determined power spectrum one signal having such a spectrum is derived by assigning an arbitrary phase function \( \theta(f) \), i.e.,

\[
S(nT) = \text{IFT}\{|S(f)| \exp(j\theta(f))\}
\]

For digital application, however, only finite length sequence can be processed. The next section will, therefore, be devoted to the problem of factorising the power spectrum using ZT techniques.

### 3.2 SYNTHESIS OF PULSE TRAINS IF THE ACF IS KNOWN IN MAGNITUDE AND PHASE

If the ACF is given in phase and magnitude at discrete points its ZT can be written as Eq (2.36)

\[
R(Z) = Z^{-N}S(Z)S^*(1/Z)
\]

As mentioned previously the ZT provides a convenient method to represent a signal in the form of its zero pattern which is obtained by factorizing its polynomial in \( Z \). In factorized form the equivalent representation of a polynomial \( S(z) \) of order \( N \) in power of \( Z^{-1} \) can be written as

\[
S(Z) = S(0) \prod_{i=1}^{N} \left(1 - \frac{Z_i}{Z}\right)
\]  

(3.3)
Fig. 3.1  (a) Factorized ACF of 11 element Barker Code  
(b) Resulting sequence when choosing the zero patterns 1, 2, 3, ..., 10.
Where Z, are the zeros of S(Z), i.e., S(Z,) = 0, i = 1, 2, ..., N. Similarly, S*(1/Z) can be represented as

\[ S^*(1/Z) = S^*(0) \prod_{i=1}^{N} (1 - Z_i \cdot Z*) \quad (3.4) \]

\[ S^*(1/Z) = S(0)(-1)^N (\prod_{i=1}^{N} Z_i^*) \prod_{i=1}^{N} (Z - \frac{1}{Z_i}) \]

\[ S^*(1/Z) = S^*(N) \prod_{i=1}^{N} (Z - \frac{1}{Z_i}) \]

\[ R(Z) = S(0) \cdot S^*(N) \prod_{i=1}^{N} (1 - \frac{Z}{Z_i})(Z - \frac{1}{Z_i}) \]

Where the unessential delay factor Z^N has been neglected. Since R(Z) essentially represents a power spectrum, the above equation can be regarded as the factorized power spectrum, [43].

The equivalent expressions above allow the study of pulse trains using their zero patterns in the complex Z-plane. The conditions S(0) \neq 0, and S(N) \neq 0, are clearly equivalent to

S(0) \neq 0 and S^*(0) \neq 0

It is easy to verify that if S^0(Z) devotes the polynomial

\[ S^0(Z) = Z^N S^*(1/Z) \]

then

\[ (S^0(Z))^0 = S(Z) \quad (3.5) \]

and

\[ (S(Z) P(Z))^0 = S^0(Z) P^0(Z) \quad (3.6) \]

In addition, it follows from Eq (3.3) and Eq (3.4) that for

\[ S(Z) = |S^*(1/Z)| \quad (3.7) \]

Moreover, if a polynomial S(Z) of degree N has P_1 zeros inside the unit circle, |Z| = 1 (counting multiples), P_2 on the unit circle, and P_3 zeros outside, where P_1 + P_2 + P_3 = N, it is referred to as of the type (P_1, P_2, P_3). Since it has been assumed
that $S(0) \neq 0$, it is clear that $Z_j$ is a zero of $S(Z)$ if $1/Z_j^*$ is a zero of $S^*(1/Z)$. The zeros $Z_j$ and $1/Z_j^*$ have the same angle in the $Z$-plane but reciprocal magnitude as indicated in Fig 3.1. It is clear that $S(Z)$ is of the type $(P_1, P_2, P_3)$ if $S^*(1/z)$ is of the type $(P_3, P_2, P_1)$.

The class of polynomials, $P$, for which $P(Z)$ and $P^*(1/Z)$ have the same set of zeros are known as self-inversive polynomials, [46]. It is apparent that a polynomial $P(Z)$ is self-inversive if its zeros are symmetric with respect to inverse on the unit circle from the foregoing it should be clear that

1. A self-inversive polynomial of degree $N$ is of type $(P, N-2P, P)$ for $P \geq 0$

2. Since the polynomial $R(z)$ consists of the product of the two factors $S(z)$ and $S^*(1/Z)$ it is of type $(P_1+P_3, 2P_2, P_1+P_3)$, where $2(P_1+P_2+P_3) = 2N$. Consequently, $R(Z)$ is self-inversive and its zeros must occur in reciprocal conjugate pairs.

Thus for a finite pulse train to be an ACF it has to satisfy condition (ii). The design technique for pulse trains from a given realizable ACF can now be summarized as follows:

1. Factorization of the ZT polynomial which represents the ACF

2. Selection of a suitable zero pattern to obtain the signal after multiplication.

The design procedure is probably best illustrated by an example. Consider the 11-element Barker Code where ACF has the ZT representation

$$R(Z) = -1Z^{-1}Z^{-2}Z^{-4}Z^{-6}Z^{-8}+11Z^{-10}Z^{-12}Z^{-14}Z^{-16}Z^{-18}Z^{-20}$$

This polynomial can now be factorized on a digital computer using a standard root-finding algorithm. The resulting zeros are given in Fig 3.1(a). All the zeros occur in reciprocal conjugate pairs. In addition, as a consequence of the coefficients of $R(z)$ being real, all complex zeros must occur in conjugate pairs. The selection of zeros for $S(z)$ and multiplying them out completes the design procedure. The resulting sequence choosing the zeros labeled as 1, 2, 3, ..., 10 is shown in Fig 3.1(b)
3.3 SYNTHESIS OF PULSE TRAINS IF ONLY THE MAGNITUDE OF THE ACF IS KNOWN.

In the previous section the magnitude and phase of the ACF at discrete points was required to find a solution to the synthesis problem. However, from practical considerations only the magnitude of the function is usually known, since the phase does not effect the accuracy and resolution of the range measurements. In this section the design procedure is extended to the case where only the magnitude of a realizable ACF is given at discrete points. The basic underlying idea of the method presented here is due to Vakman and is also implied by Voclicker, in a different context.

Before proceeding any further it is necessary to recall the convolution theorem derived from basic Fourier transform theory:

\[ r(t)^* r^*(-t) \leftrightarrow |R(f)|^2 \]  

(3.8)  
\[ |r(t)|^2 \leftrightarrow R(f)^* R^*(-f) \]  

(3.9)

The above relation show the duality between the ACF of the spectrum \( R(f) \) and the ACF of the time signal \( r(t) \). As \( r(f) \) is assumed to be band-limited it can be represented by its Nyquist samples. However, due to the convolution process in the frequency domain, the squared envelope, \( |r(t)|^2 \), will have twice the band-width of \( r(t) \). In other words, if \( |r(t)|^2 \) is sampled at Nyquist rate, \( r(t) \) is sampled at twice that rate. Hence it is assumed that the squared envelope of the ACF is known at integer multiples of \( T/2 = T' \).

Since \( |r(t)|^2 \) is of finite duration it is completely defined by frequency domain samples taken at intervals \( 1/T' \). Such a signal has a finite Fourier representation of \( N \) terms:

\[ m(t) = |r(t)|^2 = \sum_{k=0}^{N-1} c(k) \exp(j2\pi k T_s t) \]  

(3.10)

where \( T_s \) is the duration of the signal and \( N = WT_s \).

The samples of the envelope are thus given by...
\[ m(z) = \sum_{k=0}^{N-1} c(k) \exp(j2\pi k \frac{n}{N}) \] ................................. (3.11)

\[ n = 0, 1, 2, \ldots, (N-1) \]

By substituting the symbol \( Z \) for \( \exp(j2\pi t/T_0) \) the finite Fourier series can be rewritten as a polynomial in powers of \( Z \)

\[ m(Z) = C(0) + C(1)Z + \ldots + C(N-1)Z^{N-1} \]

\[ m(Z) = \sum_{k=0}^{N-1} c(k)Z^k \] ................................. (3.12)

This transformation can be regarded as the dual of the ZT discussed in Chapter 2.

Consider now the polynomial of order \( L \) representing the ACF, \( r(t) \).

\[ r(Z) = R(0) + R(1)Z + \ldots + R(L)Z^L \] ................................. (3.13)

clearly, for \( r(Z) \) to be realizable all \( R(n) \) must be real and non-negative (\( R(n) \geq 0 \)), since the coefficients of the polynomial are the power spectral samples. The squared modulus of \( r(t) \), \( |r(t)|^2 \) can thus be represented as a polynomial multiplication.

\[ m(Z) = [R(0) + R(1)Z + \ldots + R(L)Z^L][R^*(L) + R^*(L-1)Z + \ldots + R^*(0)Z^L] \]

\[ m(Z) = Z^L[r(Z) + R(1)Z + \ldots + R(L)Z^L][R^*(0) + R^*(1)Z + \ldots + R^*(L)Z^L] \]

\[ m(Z) = Z^L r(Z) r^*(1/Z) \]

Where the coefficients of \( m(Z) \) are given by

\[ C(k) = \sum_{n=0}^{L-|k|} R(n)R(n+k) \]

\[ k = 0, \pm 1, \ldots, \pm L \]

Since \( r(z) \) is a polynomial with real coefficients

\[ m(Z) = Z^L r(Z) r^*(1/Z) \]

Thus the operation of convolution in the frequency domain reduces to a multiplication of two polynomials. This clearly reflects the duality of time and frequency as pointed out earlier.
Since \( m(z) \) has the form of an ACF it is possible to proceed in a similar manner to that described in section 3.1 in order to find the power spectral components \( R(n) \) given \( m(z) \). The properties of \( m(z) \) are revealed by studying the zeros in the complex \( Z \)-plane. The coefficients of \( m(z) \) specify the ACF of the spectrum of \( r(z) \). Its \( 2^k \) zeros must therefore occur in reciprocal conjugate pairs. In addition, since all coefficients are real (and in particular non-negative) they all occur in complex conjugate pairs. Thus if \( Z_j \) is a zero of \( m(Z) \), then \( Z_j^* \), \( 1/Z_j \) and \( 1/Z_j^* \) also must be zeros of \( m(z) \). This relationship is illustrated in Fig. 3.1(a). Consequently, if \( m(z) \) is to represent a realizable power spectrum ACF the following conditions must be satisfied.

i) \( m(Z) \) is finite and its zeros occur in complex conjugate reciprocal pairs.

ii) The coefficients, \( R(n) \) of \( r(Z) \) must be real and non-negative.

If these conditions are met then at least one and in general a whole set of ACF’s having the same magnitude can be found. The steps in the design procedure can be summarized and best illustrated by using the 7-element Barker code as an example.

1. Given the samples of \( r(t) \), where \( r(t) \) is assumed to be band limited and sampled at twice the Nyquist rate, Fig. 3.2(a), one computes the DFT of the sequence \( |r(0)|^2, |r(1)|^2 \ldots |r(13T')|^2 \ldots |r(T')|^2 \).

\[
\text{DFT}[|r(0)|^2, |r(T')|^2 \ldots |r(13T')|^2 \ldots |r(T')|^2]
\]

This gives, except for a scale factor, the \( (2L+1) \) Fourier coefficients of the periodically repeated square envelope of the ACF, Fig. 3.2(b).

2. Factorize the polynomial whose coefficients are these Fourier components. The \( 2^L \) roots should occur in reciprocal complex conjugate pairs, Fig. 3.2(c).

3. Select \( L \) roots from each reciprocal conjugate pair and its conjugate. Now, multiply out to obtain a set of \( (L+1) \) Fourier coefficients which are, neglecting a scale factor, the DFT of the samples of \( r(t) \). Verify that \( r(t) \) is indeed an ACF. This is done simply by making sure that the coefficients obtained are all real and
Fig. 3.2 (a) Samples of the ACF magnitude
(b) Spectrum samples of squared envelope of (a)
(c) Zero pattern of (b)
(d) Spectrum samples of the ACF
(e) ACF in magnitude and phase.
non-negative. If the test fails, select a new zero pattern and repeat the procedure from step 3, until a realizable ACF is obtained, Fig. 3.2(d). From the L roots only L/2 can be chosen independently, since the zero must be selected in complex conjugate pairs. Hence, there are in general 2^{L/2} possible zero patterns. However, not all zero combinations will result in realizable ACF's. The zeros chosen in this case are labelled 1, 2, ..., 13, Fig. 3.2(c).

4. The final stage of the synthesis procedure is to take the IDF of the Fourier Transform coefficients to obtain the sampled values of r(t) Fig. 3.2(e). Once the ACF has been found in magnitude and phase, it is then straightforward to synthesize a pulse train which realizes this ACF by following the procedure outlined in section 3.2.

3.4 DISCRETE PHASE APPROXIMATION TO LINEAR FM SIGNALS

The Linear FM (LFM) or (Chirp) waveform is probably the principal type of signal transmitted by a radar or sonar system. The main advantage in using such waveform lies in their ease of generation and insensitivity to small doppler shifts. The description and properties of analogue chirp techniques are well documented in the literature and will not be repeated here, [22], [25], [41].

In general the complex envelope of a LFM signal can be expressed as

\[ S(t) = \exp(j \mu t^2/2) \] ................................. (3.15)

where

\[ \mu = 2\pi w/T_s \]

\[ W = f_2 - f_1 = \text{Frequency Change During Sweep} \]

\[ T_s = \text{time duration of the sweep.} \]
As implied by the term LFM, the instantaneous frequency is swept linearly from \( f_1 \) at \( t=0 \), to a maximum value of \( f_2 \) at \( t=T \). The complex envelope can be written in terms of the pulse compression ratio (time bandwidth product) denoted as \( m_c \).

\[
S(t) = \exp[j\pi(\omega t)^2/m_c]
\]

If the waveform is sampled at uniform time interval of \( T \) seconds

\[
S(nT) = \exp[j\pi(\omega nT)^2/m_c]
\]

and with

\[
m_c = NTW
\]

\[
S(nT) = \exp[j\pi W T n^2/N] \quad (3.16)
\]

\[n = 0, 1, 2, \ldots, (N-1)\]

The total phase change over the signal duration is

\[
\phi = \pi W T (N-1)^2/N
\]

Furthermore, if \( T \) is set equal to the Nyquist rate, then

\[
T = 1/W
\]

and

\[
S(nT) = \exp(j\pi n^2/N) \quad (3.17)
\]

Since each segment is coded into one of \( N \) possible phases these sequences are some time referred to as polyphase codes. In particular the sequences whose phase follows a quadratic progression will be called Quadratic Phase (QP) codes. An interesting property of QP codes is their periodic ACF which is zero for all non-zero time lags, [50], provided the sequence is coded as in eqn (3.17) if \( N \) is even and is modified to

\[
S(nT) = \exp(j\pi n(N+1)/N) \quad (3.18)
\]

for \( N \) odd
Higher order poly-phase codes with zero circular ACF \((k \neq 0)\) have been described by Frank Zadoir and Heimiller, [16], [17] & [18]. The length \(N\) of such codes, however, is restricted to perfect squares. While the QP sequences and Frank codes have ideal cyclic auto-correlation, they do not have of course perfect a periodic auto-correlation.

Another property of practical importance is the simple generation of QP pulse trains if the sequence length is chosen properly. This can be demonstrated by expanding the expression \(n^2/N\) as

\[
n^2/N = q + q_1 + \alpha_n/N
\]

where \(q = 0, 1, 2, \ldots\)

\(q_1 = 0\) or 1

\(\alpha_n = \text{remainder}\)

\[
\exp(jqn) = \begin{cases} 
-1; \ldots \quad \text{q is odd} \\
+1; \ldots \quad \text{q is even}
\end{cases}
\]

and

\[
\exp(j\pi/2) = j
\]

The QP code can be also written as

\[
S(nT) = (\pm 1)(1/j) \exp(j\pi\alpha_n/N) \quad \begin{align*} & \text{..........................} (3.19) \end{align*}
\]

The number of different samples to be generated is thus a function of the number of distinct remainders of \(n^2/N\). Roy and Lowenschurs, [51] have shown that for a proper choice of \(N\) the number of different samples can be kept very small indeed. For example for \(N = 16\) only three different values must be generated, \(\exp(j\pi/16), \exp(j\pi/4)\) and 1. Incidentally, this property has also been exploited in the Bluestein algorithm which computes the DFT using a Chirp Filter, [52].
3.4.1 Properties of the Compressed Pulse Train

The exact expression for the ACF can be obtained by substituting eqn (3.16) into Eqn (3.1):

\[ r(kT) = \sum_{n=0}^{N-1-k} \exp(j\pi n^2 T^2_k/N) \cdot [n^2 - (n + k)^2] \]

\[ = \exp(j\pi n^2 T^2_k/N) \cdot \sum_{n=0}^{N-1-k} \exp(-j2\pi n T_k) \]

\[ k = 0, 1, 2, \ldots, (N-1) \]

The summation in the last expression is of the form of a geometric progression and can be written in closed form. By rewriting the sum term as \( r^n \), where \( r = \exp(-j2\pi WT/N) \) it is recognised that the series containing a total of \((N-K)\) terms has a sum of, \( S = \frac{r^{N-K} - 1}{r - 1} \).

Therefore,

\[ r(kT) = \exp(-j\pi nk^2 T^2_k/N) \cdot \frac{\exp[-j2\pi nT(N - k)] - 1}{\exp(-j2\pi nT^2_k/N) - 1} \]

\[ r(kT) = \exp[-j\pi nkT(N - k)] \cdot \frac{\sin[\pi nk(N - k)/N]}{\sin(\pi nT^2_k/N)} - 1 \] (3.20)

Since only the magnitude of the ACF is of interest

\[ |r(kT)| = \left| \frac{\sin[\pi nk(N - k)/N]}{\sin(\pi nT^2_k/N)} \right| \]

If \( T \) is equal to the Nyquist sampling rate, i.e. \( WT = 1 \), Eqn (4.16) becomes

\[ |r(kT)| = \left| \frac{\sin[\pi k(1 - k)/N]}{\sin(\pi k^2/N)} \right| \] (3.21)

Because \( r(KT) \) exhibits complex conjugate symmetry with respect to \( k = 0 \), it is sufficient to consider only positive time lags. The nature of the function (3.21) in the vicinity of \( k = 0 \) has the form of a sine function with a peak value of \( N \). Because of the periodicity of the expression this characteristic will be repeated at intervals \( 1/N \). For even length sequences the function is symmetrical with respect to \( N/2 \). The effect of the term \((1-K/N)\) can be explained, considering that
|Sin[πK(1-K/N)]| = |Sin(πK^2/N)|

Thus it modulates the frequency of the ripples in a 'chirp like' fashion. Whenever a sequence is used as a modulating function it is always of interest to consider its spectrum. Since the power spectrum contain the relevant information, it is sufficient to consider the amplitude spectrum only. For convenience the spectrum is assumed to be zero outside some range (0, W). This is no restriction, since any band limited signal can be brought into this form by a suitable frequency translation.

The magnitude of the ACF is shown in fig. 3.3 for a QP sequence of length N = 128 when sampled at the Nyquist rate, WT = 1. The characteristics of the QP pulse trains when sampled at a lower or faster rate than the Nyquist rate (under or over sampling) are depicted in Fig. 3.3 & 3.4. These graphs reveal some interesting properties. First, if sampled at the Nyquist rate (WT = 1), the ACF consists of a sharp narrow spike with low residue side lobes. Secondly if WT < 1, over sampling occurs and side lobes near the main peak appear. This is not surprising since increased sampling rate implies a closer approximation to the analogue FM signal whose maximum side-lobes are immediately adjacent to the main lobe, as shown in Fig. 3.4. In other words, the sampling points do not miss these large side-lobes as is the case for WT = 1. Thirdly, for under-sampling WT > 1, the aliased versions of the ACF will produce significant range ambiguity at time shifts K = N/WT from the compressed pulse i.e. for WT = 2, K = N/2 as illustrated in Fig. 3.7. The cause of the spurious response peaks can be avoided provided the waveform is sampled at the Nyquist rate. The resulting sequences have low side-lobes and low side-lobe energy and thus are suitable in a multiple-target environment. However, there may be specific cases where WT is made slightly greater than one, to attain somewhat better range resolution. By comparison of Fig. 3.6 with 3.3, it can be seen that a small increase in WT tends to reduce the side-lobes close to the main
lobe at the expense of an increase towards the end of the response. The important properties of QP pulse trains where $WT = 1$ may now be summarized as follows:

i) If sampled at the Nyquist rate QP codes have virtually all the properties of LFM signals. Their ACF consists of a single spike with low level side-lobes.

ii) The codes have zero periodic ACF's for $K \neq 0$.

iii) For $WT < 1$, large close-in side-lobe appears, whereas for $WT > 1$ spurious response peaks occur further away from the main-lobe.

iv) The major side-lobes occur in two bands approximately centered at time shifts $K = \sqrt{(N/2)}$ and $K = [N - \sqrt{(N/2)\}]$ respectively. The width of the bands is about $2[\sqrt{N} - \sqrt{(N/2)\}]$. Moreover, for sequences of even length $N$, the side-lobe structure is symmetrical with respect to $K = N/2$.

v) The maximum side-lobes increase approximately as $0.5 \sqrt{N}$ and the energy ratio $E_r < 5\%$ for $N > 40$.

vi) Relative simple generation of such pulse trains with a suitable choice of $N$.

Hence QP sequences have good range resolution properties which make them suitable for a dense-target environment. For example, the peak range side-lobe for $N = 128$ is $-27.5$ dB down on the main response and the rms side-lobes are about $-36.6$ dB down.

In subsequent chapters an essentially different method of synthesizing discrete coded signals with desired auto-correlation properties is described. The method is based on numerical optimization techniques. Such an approach has a number of advantages. First, no restriction on the class of admissible phase functions is necessary. Secondly, no information of the signal's phase structure is usually required. Thirdly, these methods are flexible in a sense that it is possible to control particular side-lobes or side-lobe regions.
Fig. 3.3 — (a) ACF, (b) amplitude spectrum of 128-element QP code for $WT = 0.5$

(values in $dB$).
Fig. 3.4 ACF of a LFM signal.
close-in sidelobes are reduced due to slight under-sampling

(a) ACF, and (b) the amplitude spectrum.

Fig. 3.6 Effects of slight under-sampling on the
(a) ACF, and (b) the amplitude spectrum.
Fig. 3.7 Aliasing effects on the ACF (a), and amplitude spectrum (b), due to
CHAPTER - 4

DESIGN OF BINARY CODED PULSE TRAINS
4.1 INTRODUCTION

A relatively large group of signals which have received special attention are certain binary sequences (Group I, Chapter 2). Such signals have been extensively considered for improving ambiguity and resolution in radar and solving special problems in the field of communications, [5], [7], [8], [14] and control system, [75].

The attractive feature of binary coding is that a number of simple, efficient and flexible decoders can be built (a pair of shift registers can be used as a tapped delay line pulse compressor). The aperiodic ACF of a N-element binary sequence C(n) can be written as:

\[ r(k) = \sum_{n=0}^{N-1} c(n)c(n+k) \] \hspace{1cm} (4.1)

Where \( c(n) = \pm 1, \ n = 0, 1, 2, \ldots, N-1 \)

A class of binary sequences whose ACF's satisfy the conditions

\[ r(k) = \begin{cases} +N; & \text{if } k = 0 \\ 0; & \text{if } (N-k) \text{ is even} \\ \pm 1; & \text{if } (N-k) \text{ is odd} \end{cases} \] \hspace{1cm} (4.2)

are called Barker sequences or perfect words, [8]. Barker Sequences exist for length \( N = [2, 3, 4, 5, 7, 11, 13] \). Turyn, [9] has shown that there are no other binary sequences with this property for \( 13 < N \leq 6084 \) and that it is unlikely any will exceed 6084. The limitations to a maximum length of 13 is a serious one in radar detection. Consequently considerable effort has been devoted to the problem of finding longer binary sequences which, if not optimum, are at least satisfactory for a given application, [6], [10], [11], [14]. For a binary sequence of length \( N \) there may be \( 2N \) possible combinations and for the binary sequence of length \( N \) will best possible ACF, the computer has to search \( 2N \) these possible combinations. For large values of \( N \) the searching problem becomes enormous. J. Linder, [56] has given sequences of best possible auto-correlation function up to the length 40.

It is well known that by choosing a large number of elements \( C(n) \) randomly, sequences where rms side lobe levels are of the order \( \sqrt{N} \) can be found. However, it
can be expected that in the statistical synthesis a large number of side lobes will exceed \(\sqrt{N}\). It will be shown that by proper choice of the sequence both average and peak side lobe can be held at a lower value and thus yield a better range resolution and clutter rejection. However, nothing is known about how small the maximum peak side lobe might be in the best cases. At present there is apparently no solution, other than an exhaustive search, to this general problem and good binary sequences which approach the Barker codes have been found only by trial and error.

The major difficulty in synthesizing binary sequences is the discrete nature of the amplitude and phase. Consequently, the major objective in subsequent sections will be to describe the various ways of obtaining binary coded waveforms using Numerical methods, to study their properties, and to consider where they are suitable.

### 4.2 THE OPTIMIZATION PROBLEM

The discrete nature of pulse trains complicates their synthesis considerably. No ordinary methods can be used in an attempt to design such sequences unless a discrete phase approximation to analogue signals is made. For example, the methods based on the stationary phase principle result in relatively large side lobes (> -30 dB) and moreover, limits the class of admissible waveforms. Instead of synthesizing pulse trains from a given power spectrum one can try to find the actual signal itself which, if not ideal, has at least a satisfactory approximation to the desired ACF. In general, approximation is essential, since the specified ACF is rarely realizable for a given set of constraints on the signal waveforms. After an initial solution has been obtained. It is compared with the required MF response. The result of the comparison is the approximation error and the objective is to reduce the error by modifying the initial solution.

Before presenting the optimization techniques used on the class of signal design problems considered here it is necessary to state the design problem in a suitable form.
4.2.1 Formulation of the Synthesis Problem

The most important steps in the design procedure may be outlined with reference to Fig 4.1 as follows.

First, at the outset the designer should have a clear understanding of the functions to be performed by the signal. This step includes an evaluation of the specific radar tasks such as multiple target resolution capability, transmitter power limitations, etc.

Next, the principal waveform type has to be selected. Here the designer has to decide whether to use signals of Group I, II, or III (Chapter 2). The signal under consideration may be characterized by means of a set of design variables in amplitude, \((a_n)\), phase \(\phi_n\), and frequency \(f_n\). It is often convenient to replace ordered sequences by vectors to permit the rotation of linear vector-spaces. Thus

\[
x = (|a_1|, |a_1|, |a_{N-1}|, \phi_1, \phi_{N-1}, f_1, \ldots, f_{N-1})^T
\]

where \(x\) is the design vector.

The next step is the mathematical formulations of the system. This is expressed as a set of equations

\[
h_f(x) = 0, \quad f = 1, 2, \ldots, l
\]

In most practical cases there are restrictions on the permissible values of the design variables which may be written in the form

\[
g_i(x) \geq 0, \quad i = 1, 2, \ldots, m
\]

These constraints exclude undesirable solutions for a particular application. All solutions which simultaneously satisfy Eq (4.2) and (4.3) correspond to acceptable or feasible designs. At this point a criterion or objective function must be adopted to determine when the 'best' solution is arrived at for a given degree of complexity. There are several alternative ways to define a measure of best performance. For example, one may choose to minimize the sum of the squares of the errors, the absolute error, the sum of the absolute errors and so forth. The choice of any of these criteria is usually dependent.
Formulation of system requirements

Selection of waveform type

Definition of design variables: $\mathbf{x} = (x_0, \ldots, x_{N-1}, f_0, \ldots, f_{N-1})^T$

Mathematical formulation of the behaviour of the system:

Mathematical formulation of the behaviour of the system:

Constraints:

Selection of the performance index: $F(x)$

Choice of an initial design: $x^{(0)}$

Control of the constraints

Evaluation of $F(x)$

$F(x) = \min$

Final evaluation: practical considerations (complexity)

Optimal solution

Selection of new design: $x^{(k)}$

Fig. 4.1 Flow diagram of the design procedure.
on many factors, the principal ones being the tasks the system has to perform. In fact if the optimization technique is adequate, the performance index will completely determine the final system.

At this stage a method to search for the optimum solution remains to be chosen. Several techniques exist for minimizing non-linear functions, [64], through [73]. All these methods have been investigated in details and their convergence efficiency has been studied, [72].

The function of interest is the sampled squared magnitude of the MF response

\[ |r(k)|^2 = \left| \sum_{n=0}^{N-1-k} a_n a_{n+k}^* \right|^2 \] (4.6)

The aim is to find the design vector \( X = (|a|\phi)^T \) which minimizes the objective function or performance index (also known as error criterion or cost function)

\[ F(x) = \sum_{k=0}^{N-1-k} f(|r(k)|^2 - |\tilde{r}(k)|^2) \] (4.7)

Where \( \tilde{r}(k) \) is the desired ACF and \( f \) is an arbitrary but suitable function of interest. For most practical applications the signal will be subjected to the constraints

\[ |a_n| = 1, \quad n = 0, 1, 2, \ldots, (N-1) \] (4.8)

Eqn (4.5) and the above conditions represent a constrained non-linear optimization problem.

For pulse compression sequences two characteristics are of particular concern (Chapter 2). One is the total side lobe energy given by

\[ E_s = \sum_{k=1}^{N-1} |r(k)|^2 \] (4.9)

In a dense target environment the self clutter power at the MF output is proportional to this quantity (Since the auto-correlation is an even function, only one half is considered, i.e., the actual side lobe energy would be twice the value given here). Another property of interest is the peak side lobe level, \( \max_k |r(k)| \), which represents a source of mutual interference that can obscure weaker targets. Therefore, it is required...
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to minimize a suitable measure which incorporates these characteristics. Naturally, any such measure is to some extent arbitrary, but in general will be of the form $F(|r(k)|)$.

Specifying the desired ACF as being zero ($k = 0$), the objective may be redefined, [67] as

$$F_p(x) = \sum_{k=1}^{N-1} w(k)|r(k)|^p$$  \hspace{1cm} (4.10)

Where $p > 1$ and $w(k) > 0$. The weighting sequence $w(k)$ allows control of the side lobe structure of the resulting ACF. A good compromise is to set $w(k) = 1$ for all $k$, if no, or little, prior information about the radar environment is known. At this point the value of $p$ remains to be chosen. It should be noted that for $p = 2$, minimization of

$$F_p(x) = \sum_{k=1}^{N-1} |r(k)|^p$$  \hspace{1cm} (4.11)

results in minimizing the side lobe energy $E_s$. However, for $p = 2$ and sequence length $N > 20$, the measure reacts weakly to large peak values, while for large $p$ it will not respond to the energy criterion, since for well behaved functions

$$\lim_{p \to \infty} \left\{ \sum_{k=1}^{N-1} |r(k)|^p \right\} = \max_k |r(k)| \hspace{1cm} (4.12)$$

The measure $(F_p)^{1/p}$ are called Chebyshev or Uniform Norms because of the consequences of Eqn (4.12). Minimization with respect to $(F_p)^{1/p}$ for large $p$ is often referred to as minimax approximation, [59]. In other words, the least $p$-th approximation tends to the minimax approximation as $p \to \infty$. Hence as suggested by Eqn (4.12), a Chebyshev solution could be approached in principle, by successively minimizing $F_p$ each time incrementing the index $p$, i.e. $p = 2, 4, 6$, etc. In most cases acceptable minimax approximations can be obtained with relatively moderate values of $p$ ($p = 10$).

Incidentally, certain optimization problems can only satisfactorily be described using several error criterion. A suitable overall performance index could then consists of a linear combination of functions of the form (4.10) i.e.

$$F = \lambda_1 F_p^{(1)} + \lambda_2 F_p^{(2)} + \lambda_3 F_p^{(3)} +$$

Where the $\lambda_i$ is weighting factors, would be given values, according to the importance of $F_p^{(1)}, F_p^{(2)}, F_p^{(3)}$, etc.
It is not clear which characterizes the 'goodness' of a sequence more fully, the maximum peak value or the energy criterion. Therefore, for short sequences \((N < 20)\) \(p = 2\) should be adequate, while for longer sequences a larger value for \(p\), for example, \(p = 4\) has the desirable effect of reducing large single sidelobe peaks.

### 4.2.2 Synthesis Using Element Complementation

Probably the most obvious method of synthesizing binary sequences is to choose a number of sequences, perhaps at random, and to observe their ACF's. This method can be improved significantly by adopting a search strategy which, starting from an initial code, produces a succession of progressively better codes. This involves minimization of a measure of the sidelobes such as that given Eq (4.11) over a set of discrete points in multidimensional space. To be more specific, the problem is to minimize a function of \(N\) discrete variables over a set, \(S\), of \(2^N\) points in \(N\)-dimensional space. Any attempt to compute all possible functional values becomes unfeasible, even for moderately large \(N\) \((N > 20)\), as \(2^N\) increases exponentially. Hence, the search will have to be restricted to a subset \(S1\) of \(S\) i.e. \(S1 \subset S\). In addition iterative methods must take the discrete nature of the variables into account and, moreover, must be economical with respect to the volume of computations.

A simple search strategy is to take the current sequence, changing one of its elements to either +1 or -1, and to evaluate the ACF, [54]. If the measure of the sidelobes is reduced, the modification is retained and the new code is subjected to further modifications. This is done iteratively until changes of the elements do not yield a further reduction in the objective function. Such a method is known as the element complementation technique, [11], [74].

A search program based on the element complementation method was written in fortran to find sequences with minimum side lobe peaks. The initial starting points were randomly chosen sequences. In addition the sequences were also tested to see if cyclic shifts would improve the objective function, [82], [83], [84], [85], since such an operation results in merely removing and adding bits of the sequence.
Table 4.1 shows the binary sequences obtained when optimizing sequences, which has been generated using various methods such as maximal length sequences or Vakman, [11] sequences with respect to the criteria $\sum_{k} |r(k)|^2$, $\sum_{k} k |r(k)|^2$, $\sum_{k} |r(k)|^4$, $\sum_{k} k^2 |r(k)|^2$

As expected minimization of $F_4$ measure yields better results particularly for large compression codes. The S/L energy ratio $E_s$ varies between 10% and 16% of the main lobe energy, which gives a r.m.s. sidelobe of approximately 0.3 $\sqrt{N}$ - 0.4 $\sqrt{N}$. As mentioned previously a non-negative weighting of the objective function enables control of the sidelobe distribution along the delay axes of the ACF. A designer may feel that it is worthwhile reducing the sidelobes which are further away from the main peak, at the expense of an increase in the close-in side lobes, by minimizing a functional of the form, [67], [69]. $F_p = \sum_{k=1}^{N-1} k |r(k)|^p$. This is illustrated in Fig. 4.2

On the other hand, where a binary sequence is used for resolving closely spaced targets or observing missiles in the presence of tank fragments or decoys for example, it is desirable to have low residues near the main lobe, [26], [49]. However, ambiguities can be tolerated in range if they are sufficiently distant from the expected target position. In this situation a weighting function of the form $w(k) = N - k$

$$w(k) = \begin{cases} +1; & \text{ if } .k \leq N_1 \\ 0; & \text{ if } .N_1 < .k \leq (N - 1) \end{cases}$$

Where $N_1$ denotes the desired clear region, could be used. For example sequences of length $N > 200$ whose mainlobe to side lobe ratio is 200 1 or 46 dB within 10 segments length of the main peak were found. Thus with a prefer choice of the weighting function it is possible to obtain sequences with clear regions. However, an increase in the maximum and average sidelobe levels is usually observed.

An extensive computer search has shown that starting with a randomly chosen initial sequence usually results in improvement in the initial code. In Table 4.2 the results obtained with this method are compared with that of best available codes. It can be seen from Table 4.2 that results obtained are comparable with best available codes obtained.
TABLE 4.2

Best Binary Sequences Obtained with Random Initial Starting Point

<p>| Code | Length | Functional : $E[|r(k)|^4]$ | Best known Sequences |
|------|--------|----------------------------|----------------------|
|      |        | $\max_k |r(k)|$ | $Es(%)$ | $N |r(k)|$ |
| 13   | 1      | 3.55         | 1                    |
| 19   | 3      | 11.36        | 2                    |
| 23   | 3      | 11.91        | 2                    |
| 31   | 4      | 10.72        | 3                    |
| 37   | 4      | 11.83        | 3                    |
| 41   | 4      | 12.61        | 4                    |
| 43   | 5      | 12.17        | 4                    |
| 47   | 4      | 8.28         | 4                    |
| 59   | 6      | 14.62        |                      |
| 63   | 6      | 11.96        |                      |
| 67   | 6      | 12.77        | 5                    |
| 71   | 6      | 9.26         | 5                    |
| 79   | 7      | 16.15        | 6                    |
| 91   | 7      | 16.15        |                      |
| 95   | 8      | 13.60        |                      |
| 97   | 8      | 13.52        |                      |
| 99   | 8      | 15.89        |                      |
| 103  | 8      | 14.58        |                      |
| 105  | 7      | 13.61        |                      |
| 109  | 9      | 12.40        |                      |
| 115  | 8      | 14.65        |                      |
| 119  | 9      | 15.62        |                      |
| 123  | 9      | 11.72        |                      |
| 125  | 9      | 15.83        |                      |
| 251  | 14     | 14.41        |                      |
| 255  | 14     | 14.44        |                      |
| 259  | 13     | 14.66        |                      |
| 299  | 15     | 15.09        |                      |</p>
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<th>Maximum Side-lobe Value</th>
<th>R.M.S. Value</th>
<th>Starting sequence is Barker sequence of Length 11 &amp; 2 bits are added at the end of the sequence</th>
<th>Maximum Side-lobe Value</th>
<th>R.M.S. Value</th>
<th>Starting sequence is Barker sequence of Length 13 &amp; 2 bits are added at the end of the sequence</th>
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<td>9</td>
<td>4.23</td>
<td>9</td>
<td>4.65</td>
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</tbody>
</table>
### TABLE 4.4

<table>
<thead>
<tr>
<th>Sequence Length</th>
<th>Starting sequence is Barker sequence of Length 11 &amp; 4 bits are added at the end of the sequence</th>
<th>Sequence Length</th>
<th>Starting sequence is Barker sequence of Length 13 &amp; 4 bits are added at the end of the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Side-lobe Value</td>
<td>R.M.S. Value</td>
<td>Maximum Side-lobe Value</td>
<td>R.M.S. Value</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------------</td>
<td>-----------------</td>
<td>-------------</td>
</tr>
<tr>
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<td>10</td>
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<td>119</td>
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</tr>
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<td>127</td>
<td>12</td>
<td>5.36</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Objective function of the form:

$$\sum_{k} k |x(k)|^4$$

Fig. 4.2 - ACF of 255-element binary sequence for linear weighing sequence.

Fig. 4.3 - ACF of optimum 101-element binary sequence with random initial starting point.
with complex mathematical operation with extensive search. The r.m.s. level is about 0.4 $\sqrt{N}$ and $\max_k |r(k)|$ ranges from 0.6 $\sqrt{N}$ to $\sqrt{N}$ but generally does not exceed $\sqrt{N}$ as illustrated in Fig. 4.4. A typical ACF of such a sequence is depicted in Fig. 4.3.

It has been observed that there are a number of good binary sequences which have the lowest sidelobe known for a given length. In general, however, they will differ in sidelobe energy performance.

The element complementation method requires a computation time which increases almost linearly with the code length $N$. In most cases the optimum was found in about 3 to 4 iterations. (An iteration consists of complementing the $N$ bits of the sequences plus $N$ cyclic shifts). On an VAX-11/780 digital computer good binary sequences of length as large as $N = 900$ were found in less than 30 minutes. On a time sharing computer a search program of this kind runs at low priority very cheaply, since both input and output data are extremely small.

### 4.2.3 Synthesis using Cyclic Shift and Bit Addition

In this section a different approach of minimizing the objective function $F_P$ based on cyclic shifting, [82], [83], [84], [85] of a particular sequence is discussed. A sequence is taken randomly and all its cyclic shifts are tested with respect to $\text{Max} |r(k)|$ and S.L. energy ratio $E_{SL}$. The sequence for which maximum side lobe and S.L. energy ratio is minimum is retained.

This is the optimum sequence for that length. To increase the dimension or length of the sequence 2-bits or 4-bits are added at the end of the optimum sequence of length $n$ obtained by shifting procedure. Therefore, the new sequence generated would be of length $N = n + 4$ or $N = n + 2$, while adding the additional bits all the combinations are tried. In the 4-bit case, there would be $2^4 = 16$ combination as against $2^2 = 4$ combination in the case of 2-bits addition. Each sequence formed by increasing the length from $n$ to $N$, is shifted cyclically and the ACF of each shift is evaluated. The sequence for which the MSL and RMS is minimum is taken as the optimum sequence. The procedure is repeated till the desired length of sequence is obtained. The results are shown in Table 4.3 & 4.4. In this method the initial or
Fig. 4.5
Peak side-lobe as a function of the sequence length $N$ for adding hits.
starting sequence is the Barker sequence. The Table 4.4 shows the results when the starting sequence is the Barker Sequence of length 11 and 13 respectively and 4-bits are added at the end of each sequence. Table 4.3 shows the results when the starting sequences are the Barker Sequences of length 5, 11 and 13 respectively and 2-bits are added at the end of each sequence. As can be seen from these tables, the sequences obtained in Table 4.3, where only 2-bits are added at the end of each sequence, are better in most of the cases in comparison to the sequences obtained in Table 4.4, where 4-bits are added at the end of each sequence. The time and computation involved are also less in the case of sequences given in Table 4.3, because of less number of combination ($2^2 = 4$ combination only in comparison to $2^4 = 16$ combination). Figure 4.4 shows the variation of MSL with sequence length. As can be seen from the figure, MSL is below the upper bound specified by $\sqrt{n}$ in all the cases. The comparison of results obtained in Table 4.3 with those given in Table 4.2, clearly shows that, the results obtained in the proposed method are comparable to those given in Table 4.2 and in some cases there is improvement. It is also observed that the computer time taken by this method is also lesser than the methods given in Table 4.1 & 4.2. This is because of less number of computation involved. Figure 4.5 shows the variation of maximum sidelobe level with sequence length. It can be seen from the figure that the maximum sidelobe levels remain within the limit $\sqrt{N}$ for each sequence length.

4.3 UNIFORM SEQUENCES WITH LOW AUTOCORRELATION SIDELOBES AND SMALL CROSSCORRELATION

Much attention has been paid by many authors to the construction of binary sequences having ACF's as small as possible away from the coincidence peak. However, little is known about sequences with small cross-correlation. Such sequences have many practical applications. For example, they may be used as address codes in a time division multiple-access (TDMA) system, where information from several data sources is to be transmitted over a channel, [61], [86], [89]
It has been shown that for most good binary sequences of length \( N \) (\( N > 13 \)), the attainable sidelobe levels are approximately \( \sqrt{N} \). The mutual cross-correlation peaks, however, of sequences of the same length tend to be much larger and are usually in the order of \( 2\sqrt{N} \) to \( 3\sqrt{N} \). Consequently, the objective in this section is to find a set of binary sequences of length \( N \) with auto-correlation side lobes and cross-correlation peak values both of approximately \( \sqrt{N} \).

### 4.3.1 Statement of the Problem

The more complicated problem of finding a set of uniform pulse compression codes which besides having small ACF sidelobes also have small cross-correlation, can be approached using optimization techniques. The ACF and cross-correlation of \( L \) arbitrary sequences of length \( N \) are given by:

\[
r_1(k) = \sum_{n=0}^{N-1-|k|} a(n) a^*(n+k)
\]

\[
r_2(k) = \sum_{n=0}^{N-1-|k|} b(n) b^*(n+k)
\]

\[
r_3(k) = \sum_{n=0}^{N-1-|k|} c(n) c^*(n+k)
\]

\[
r_4(k) = \sum_{n=0}^{N-1-|k|} l(n) l^*(n+k)
\]

\[
r_{12}(k) = \sum_{n=0}^{N-1-|k|} a(n) b^*(n+k)
\]

\[
r_{23}(k) = \sum_{n=0}^{N-1-|k|} b(n) c^*(n+k)
\]

\[
r_{13}(k) = \sum_{n=0}^{N-1-|k|} a(n) c^*(n+k)
\]

Where \( k = 0, \pm 1, \pm 2, \ldots, \pm (N-1) \).
### TABLE 4.5

Correlation Properties Obtained Using Numerical Minimization

<table>
<thead>
<tr>
<th>Sequence Length</th>
<th>Peak Side Loops</th>
<th>Peak Cross Corr.</th>
<th>Side Lobe Energy Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>r1(k)</td>
<td>max</td>
</tr>
<tr>
<td>N</td>
<td>k</td>
<td>k</td>
<td>k</td>
</tr>
<tr>
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<tr>
<td>43</td>
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</tr>
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<td>61</td>
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<td>6</td>
</tr>
<tr>
<td>67</td>
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<td>6</td>
<td>7</td>
</tr>
<tr>
<td>73</td>
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<td>7</td>
</tr>
<tr>
<td>79</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>91</td>
<td>9</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>101</td>
<td>11</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>109</td>
<td>10</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

Note: The table entries include values for different sequences or properties, and the values are shown with decimal places for precision.
In the binary case, sequences which result in the best possible auto-correlation and cross-correlation must satisfy the conditions

\[
\begin{align*}
\overset{\text{even}}{r_1(k), r_2(k), \ldots, r_{(N-k)}(k), r_{(N-k)}(k), r_{(N-k)}(k)} &= \begin{cases} 0;\ldots\ldots (N-k). \text{ is even} \\
\pm 1; \ldots\ldots (N-k). \text{ is odd} 
\end{cases} 
\end{align*}
\] (4.20)

The above conditions form a system of non-linear equations in the unknown \(a(1), a(2), \ldots, a(N-1), b(1), b(2), \ldots, b(N-1), l(1), l(2), \ldots, l(N-1)\). It can be easily observe that it is impossible to satisfy (4.18) simultaneously, except for the two trivial cases of \(N=1, 2\). Again an approximate solution to this set of equations is sought using numerical methods.

As mentioned previously one of the most important steps in the optimization of any design or process is the choice of the optimization criterion. For pulse compression sequences the properties of concern are the total sidelobe energy and the peak side lobe. For a set of sequences, however, an additional important criterion is the peak magnitude and energy of their mutual cross-correlation function. This may be of concern in satellite communication systems such as TDMA where the problem of unique word synchronization requires sequences which not only have good auto-correlation properties, but also should have as little cross-correlation as possible, [61], [86].

Therefore, what is required is the minimization of a suitable measure which characterizes 'goodness' of a pair of sequences. Any such measure is to some extent arbitrary but in general will be of the form \(F\{|r(k)|\}\). For any particular set of sequences the performance index for each individual correlation function is defined as

\[
\begin{align*}
F_1 &= \sum_{k=1}^{N-1} |r_1(k)|^4 \\
F_2 &= \sum_{k=1}^{N-1} |r_2(k)|^4 \\
F_t &= \sum_{k=1}^{N-1} |r_t(k)|^4
\end{align*}
\] (4.21) (4.22) (4.23)
\[ F_{12} = \sum_{k=(N-1)}^{N-1} |r_{12}(k)|^4 \]  
(4.24)

\[ F_{23} = \sum_{k=(N-1)}^{N-1} |r_{23}(k)|^4 \]  
(4.25)

Again only one half of the ACF is considered, since it is an even function. The optimum sequences \(a(n), b(n), c(n)\) etc are determined from the condition

\[ F_1 + F_2 + + F_L = \min \]  
(4.26)

and

\[ F_{12} + F_{13} + F_{23} = \min \]  

This can be accomplished by minimizing a linear combination of the performance indices

\[ \min F = \lambda_1 F_1 + \lambda_2 F_2 + + \lambda_L F_L + \lambda_1' F_{12} + \lambda_2' F_{13} + \lambda_3' F_{23} \]  
(4.27)

Where \(\lambda_i\) and \(\lambda_i'\) are weighting factors chosen according to importance of \(F_1, F_2, F_L\) and \(F_{12}, F_{13}, F_{23}\) etc.

The problem of minimizing \(F\) is one of minimizing of \(2(N-1)\) discrete variables, which can assume only two values \(\pm 1\), over a set of \(2^{2(N-1)}\) points. In addition a train of values of weighting parameters are required. If \(\lambda_i' > \lambda_i\), minimization results in sequences with good cross-correlation but poor auto-correlation. For small \(\lambda_i\), and \(\lambda_i' = 0\) sequences with good ACF's are obtained.

The choice for the best value of \(\lambda_i\), and \(\lambda_i'\) depend on the specific applications as well as the knowledge of the behaviour of the functional \(F\), including the interaction, between the performance indices. Obviously \(F_1, F_2, F_L\) etc are independent of each other. This is, however, not the case for \(F_{12}, F_{13}, F_{23}\) etc.

4.3.2 Results of the Synthesis
The element complementation method is used to minimize the functional $F$ of the form

$$F = F_1 + F_2 + F_3 + F_{12} + F_{13} + F_{23}$$  \hspace{1cm} (4.28)$$

Taking a set of three sequences $a(n), b(n) \& c(n)$ functional $F$ from Eqns 4.13 - to- 4.18 is given by

$$F_1 = \sum_{k=1}^{N-1} |r_1(k)|^4 + \sum_{k=1}^{N-1} |r_2(k)|^4 + \sum_{k=1}^{N-1} |r_3(k)|^4 + \sum_{k=-(N-1)}^{N-1} |r_{13}(k)|^4 + \sum_{k=-(N-1)}^{N-1} |r_{23}(k)|^4 \ldots$$  \hspace{1cm} (4.29)$$

For the simplest case, $\lambda_n, = \lambda_{n'} = 1$ is chosen initially. There are number of ways of applying the optimization procedure. One approach is first to change all the elements of sequence $a(n)$, keeping $b(n) \& c(n)$ fixed that is minimization of

$$F = A + \sum_{k=1}^{N-1} |r_1(k)|^4 + \sum_{k=-(N-1)}^{N-1} |r_{13}(k)|^4 + \sum_{k=-(N-1)}^{N-1} |r_{23}(k)|^4$$  \hspace{1cm} (4.30)$$

is carried out, where

$$A = \sum_{k=1}^{N-1} \{ |r_2(k)|^4 + |r_3(k)|^4 \} + \sum_{k=-(N-1)}^{N-1} |r_{23}(k)|^4$$  \hspace{1cm} (4.31)$$

is now a constant term. The next step is to repeat the process for sequence $b(n)$ while $a(n) \& c(n)$ remain unchanged, i.e.

$$F = B + \sum_{k=1}^{N-1} |r_2(k)|^4 + \sum_{k=-(N-1)}^{N-1} \{|r_{12}(k)|^4 + |r_{23}(k)|^4 \}$$  \hspace{1cm} (4.32)$$
Fig. 46  Correlation functions of two binary sequences of length $N = 101$:

(a) $\max_k |x_1(k)| = -21$ dB
(b) $\max_k |x_2(k)| = -19$ dB
(c) $\max_k |x_{12}(k)| = -18$ dB
where

\[
B = \sum_{k=1}^{N-1} \left| r_1(k) \right|^4 + \left| r_2(k) \right|^4 + \sum_{k=-(N-1)}^{N-1} \left| r_3(k) \right|^4 \tag{4.33}
\]

is a constant term.

The next step is to repeat the process for sequence \( c(n) \) while \( a(n) \) & \( b(n) \) remain unchanged. This is done iteratively until (local) minimum is reached.

An extensive computer search was carried out using the element complementation techniques. The search was started with a randomly chosen set of binary sequences or sequences generated in section 4.2.3. Some of the results are summarized in Table 4.5 for various values of \( N \), where \( e_1, e_2 \) and \( e_{12} \) are the normalized side lobe energy ratios given by

\[
e_{1,2,3} = 10^2 E_{1,2,3}/N^2
\]

\[
e_{12} = 10^2 E_{12}/N^2, \quad e_{13} = 10^2 E_{13}/N^2, \quad e_{23} = 10^2 E_{23}/N^2
\]

For \( \lambda_i'=0 \), the r.m.s. and peak values of the auto-correlation sidelobes are around \( 0.45\sqrt{N} \) and \( \sqrt{N} \) respectively. These values correspond to those obtained in section 4.2.2 and 4.2.3. It should be noted that the cross-correlation energy for this case is approximately \( N^2 \). The results in Table 4.5 show quite clearly that sequences having optimum ACF do not give smallest cross-correlation values and thus appear to be correlated. Table 4.5 also shows good agreement with the sought after energy distribution when a measure given by Eqn. 4.29 is minimized. Moreover, the cross-correlation peak values have decreased considerably to approximately \( 1.4\sqrt{N} \) as compared to \( 2\sqrt{N} \) to \( 3\sqrt{N} \) for. However, as expected, this improvement is achieved at the expense of an increase of the maximum auto-correlation side lobes which are, except, for small values of \( N \), usually of the same order. In Fig. 4.6 a representative graph of two sequences of length \( N = 101 \) and the magnitude of their correlation functions is shown. In all cases the minimum cross-correlation peak values obtained have been found to be greater than the minimum bound estimated as shown in Fig. 4.7. For \( \lambda_3 = 0 \) and \( \lambda_2'=\lambda_3'=0 \) the synthesis will give pair of sequences with good auto and cross-correlation properties.
It has been shown that binary sequences whose largest auto-correlation side lobes and cross-correlation do not exceed unity do not exist for $N>2$. However, adopting a numerical optimization technique it is possible to find sequences which are satisfactory for most practical applications. With a proper choice of $\lambda$, the weighting parameter, a significant improvement of the cross-correlation peak value can be achieved at the expense of only a relatively small increase in peak side lobe level. It can be observed that minimization merely results in a redistribution of the energies contained in both the auto-correlation and cross-correlation functions.

### 4.4 TERNARY SEQUENCES

The performance of a binary sequence can be improved significantly using numerical technique. The attainable energy ratios and peak side lobes are of the order of 20% and $\sqrt{N}$ respectively. The still relatively large residues might be the limiting factor for particular applications such as precision trackers. One approach to try and improve the range resolution and yet to maintain the binary nature of the sequence is to introduce zeros, that is by setting a number of elements in the sequence equal to zeros. The resulting sequence can be regarded as having three possible levels, namely ±1 or 0, and is sometimes referred to as interrupted binary sequence or ternary code.

The loss in transmitted signal energy associated with ternary codes can be kept small provided that the number of zeros $L$, is small compared to the code length $N$. For $L = 0 \leq N$, a reduction of $0.4$ dB in SNR is obtained. If $L$ is large ($L = 3N/4$) the code becomes energy inefficient and approaches the staggered pulse train properties, [22]. Hence the performance of ternary code is somewhere between these two extremes, depending on the number of zeros, $L$.

The question which arises now is how much can the performance of a code be improved by allowing the sequence elements to take on values $+1$, $0$? For an estimation of the rms side lobes level consider a random sequence $c(n)$ whose elements
can assume the values -1, 0, +1, with probabilities p(-1), p(0), p(1) Assuming a symmetric probability distribution, that is

\[ p(-1) = p(1) = w \]

and

\[ p(0) = 1 - 2w \]

The ACF is given by

\[ r(k) = \sum_{n=0}^{N-1-k} c(n)c(n+k) \quad (4.34) \]

The elements c(n) are independent random variables. Making the substitution \( N_k = N-k \), Eqn (4.34) can be written as,

\[ r(k) = \sum_{n=0}^{N_k-1} q(n). \quad (4.35) \]

Where \( q(n) = c(n)c(n+k) \) is a random variable taking on values -1, 0, 1 with probabilities \( w, 1-2w, w \) Each \( q(n) \) is independent \( (k \neq 0) \) with zero mean and variance \( \sigma^2 = 2w \) The probability distribution of the sum of two random variables is the convolution of their individual distribution. Having \( N_k \) terms in (4.35) this operation has to be performed \( N_k \) times. For \( N_k \) reasonably large, the central limit theorem applies, [44] The distribution will be gaussian with zero mean and variance \( \sigma_k^2 \), where \( \sigma_k^2 \) is simply the sum of the variances \( \sigma^2 \), that is

\[ \sigma_k^2 = N_k \sigma^2 = 2wN_k = 2w(N-k) \]

and

\[ P_r(k) = (2/\pi\sigma^2)^k \exp[-r^2(k)/2\sigma_k^2] \quad (4.36) \]

where

\[ \sigma_0 = (2wN)^2 \]
The rms value of kth side lobe is

$$\sigma_k = \sigma_0 (1-K/N)^{1/2}$$

It is clear that the rms value decreases with distance from the main peak

$$\sigma_k = \sigma_0 \text{ for } k \text{ small}$$

$$\sigma_k = \sqrt{2w} \text{ for } k \text{ large } \text{ (}k < (N-1)\text{)} \quad (4.37)$$

For strictly bipolar case $w = 1/2$ and $\sigma_0 = \sqrt{N}$, which is familiar result. To obtain small rms side lobe values Eqn (4.36) indicates that $w$ should be small. However, the smaller with greater the loss in transmitted signal energy. For $w = 1/4$ the decrease in SNR is about 3 dB. The rms side lobe on the other hand will be reduced to $0.7N$. This is not so significantly better in comparison to $\sqrt{N}$ for the strictly binary case and in general to obtain low side lobes one must be prepared to introduce more than $N/2$ zeros. However, it is shown that with a proper choice of the zero positions some improvement in peak side lobe is obtained with little loss in energy performance.

The problem is to find the optimum zero positions which yield the maximum reduction in side lobe energy and side lobe level. The search procedure can be carried out in a similar manner as described in section 4.2.2 by a simple modification of the basic element complementation method.

The results are given in Table 4.6 for a relatively small number of zeros. In many cases the peak side lobe is reduced by three units while suffering a loss in SNR of less than 1 dB. For further improvements more zeros have to be introduced at the expense of the energy performance of the code.

An interesting feature of ternary codes is their use in a multiplex pulse-compression system, [82]. If one of the zero elements of a ternary code, $C$, is changed to $+1$ or $-1$ resulting in the sequences $C+$ and $C-$ respectively, it can easily be seen from Eqn (4.34) that the coherent summation of their individual ACF's is given by

$$r_c(k) = 1/2 \{r_c(k) + r_c(k)\} \quad (4.38)$$
<table>
<thead>
<tr>
<th>Code Length</th>
<th>Binary Sequences</th>
<th>Ternary Sequence</th>
</tr>
</thead>
<tbody>
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<td>N</td>
<td>max $</td>
<td>r(k)</td>
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<td>3</td>
<td>11.36</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
<td>11.5</td>
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For a ternary code with \(L\) zeros, \(2^L\) sequences with this property can be found.

### 4.5 SUMMARY

In this chapter the application of numerical methods to the design of discrete coded waveform (pulse trains) has been investigated. The objective was to study the range resolution and clutter rejection performance of these discrete coded waveforms. Sequences which optimize the correlation properties, defined by suitable cost functionals, were considered. Without prior information about the radar environment the choice of such a measure of 'best' is to some extent arbitrary. It has been shown that minimizing an \(F_4\)-measure has the desirable effect of reducing the peak sidelobes as well as the side lobe energy. However, such a measure is highly non-linear and multimodal and at present there is no criterion to tell whether the obtained extremum is a global minimum.

The design of binary sequences using cyclic shifting and bit addition also yields sequences with good auto-correlation function properties. These sequences can also be used as a starting point for the synthesis of pairs or set of binary sequences having low auto-correlation side lobes and small mutual cross-correlation.

Finally, set of phase coded sequences having low auto-correlation side lobes and small mutual cross-correlation have been designed. These sequences are particularly useful in spread spectrum multiple-access systems. It has been shown, however, that improved cross-correlation properties can only be obtained at the expense of an increase in the auto-correlation side lobes.
Although, numerical methods have proved to be successful they are not without their weaknesses. From the theoretical point of view the most serious objection is that the results are not unique. Thus for a given design objective, the sequence to which the procedure converges is dependent on the initial choice of the starting sequence. Thus each individual applications the initial point would have to be chosen judiciously. Furthermore, when very large pulse trains with hundreds or thousands of pulses are required difficulties of a computational nature arise. These difficulties can be overcome to some extent by trying to generate longer sequences by combining short ones, [15]. Unfortunately, the ACF's of good short sequences do not reveal any pattern which would suggest a plausible rule for their construction. Although various techniques to obtain combination codes exist, [15], little efforts has been made to found construction method which diminishes maxₖ|r(k)|.
CHAPTER - 5

AMPLITUDE AND PHASE MODULATED PULSE TRAINS
5.1 INTRODUCTION

Most modern high performance radars use traveling-wave tube amplifiers to obtain coherent transmission. As pointed out previously these tubes work most efficiently under constant amplitude conditions. Moreover, good amplitude modulation (AM) is difficult (and very expensive) to achieve with these devices. Therefore, only purely phase modulated pulse trains have been considered so far. However, with the emergence of solid state microwave sources, efforts are being made to replace the relatively large and expensive vacuum devices by low power solid state elements and the waveguide elements by planar circuits. With these new components the size and costs are reduced and a number of commercial applications become feasible.

The contribution of solid-state devices has also been significant in areas where performance was previously inadequate. For example, it is much easier to use any form of modulation with solid state components. Although, AM is not an efficient method to provide the large time-band width required for good radar performance, it does provide a excellent means of improving the resolution capability. Consequently, this chapter treats the problem of finding energy efficient amplitude and phase modulated (a m ph m) or multilevel pulse train.

5.2 HUFFMAN SEQUENCES

Huffman, [20] has shown that it is possible to derive sequences a(n) of any arbitrary length (N+1) whose ACF's have the property

\[ r(k) = \sum_{n=0}^{N-|k|} a(n)a^*(n+k) = \begin{cases} E; & k = 0 \\ 0; & |k| \neq 0, N \\ r(N); & |k| = N \end{cases} \]  

(5.1)

Where E is the energy of the sequence.

This ACF is zero for all time shifts excepts for the unavoidable end sidelobes. Without loss of generality the end side lobes r(N) can be set to unity. Using the familiar ZT notation, the ACF of a Huffman code reduces to

\[ R(Z) = Z^N A(Z) A^*(1/Z) = 1 + EZ^N + Z^{2N} \]  

(5.2)
where
\[ A(Z) = \sum_{n=0}^{N} a(n)Z^{-n} = a(0) \prod_{i=1}^{N} (1 - Z^{-1} \cdot Z_i) \]

For a pulse train \( A(Z) \) to have the property (5.1) Huffman, [20] showed that the roots \( Z_i \) of \( A(Z) \) must lie at equal angular intervals \((2\pi/N)\) in the complex \( Z \)-plane on either of two origin centered circles whose radii are given by
\[ X = (E/2 + [(E/2)^2 - 1]^{1/2})^{1/N} \]
\[ 1/X = (E/2 - [(E/2)^2 - 1]^{1/2})^{1/N} \] ............................... (5.3)

Since the polynomial \( A(Z) \) has \( N \) roots, there are \( 2^N \) possible root patterns and thus \( 2^N \) Huffman codes with the same ACF that can be derived for a given Energy \( E \) and sequence length \((N+1)\). Although some inefficiency in the use of transmitter power may be acceptable in order to obtain the impulse like property (5.1), it would certainly be wasteful not to seek the most efficient sequence for an application. A figure of merit for the energy distribution of a sequence is the energy ratio or energy efficiency defined by
\[ E.R = E/\{(N) \max_a |a(n)|^2\} \leq 1 \] ............................... (5.4)

The maximum value of E.R. is attained for purely phase modulated pulse trains such as binary sequences. The energy ratio depends on two independent variables, namely, the total energy \( E \) of the coded waveform and the magnitude of the largest coefficient of \( A(Z) \) or the largest pulse of the code, denoted by \( \max_a|a(n)| \). By referring to eqn. (5.3), it can be easily verified that the energy \( E \) is given by
\[ E = X^N + X^{-N} \] ............................... (5.5)

and thus is a function of the radius \( X \) only. The maximum amplitude \( \max_a|a(n)| \), however, depends on the particular choice of the \( N \) zeros for \( A(Z) \) as well as the radius. Unfortunately, a mathematical method has not been found which leads, without trial and error, to the most efficient Huffman code.
The design of a Huffman sequence in general requires the choice of the code length \((N+1)\), the circle radius \(X\) and the zero pattern for which the magnitude sequence \(|a(n)|\) is most uniformly distributed. This, however, remains an unsolved problem. Direct evaluation of all \(2^N\) possible root patterns, for a given radius \(X\) and \(N\), is not feasible if \(N\) is large \((N > 20)\). This remains so even if account is taken of the zero patterns formed from others by rotation in the \(Z\)-plane through an angle \(\phi\) or by other transformations for which the energy ratio is invariant, \([74]\). A general trial and error procedure is to choose a root pattern, perhaps at random, and to compute the sequence for a succession of values for the radius \(X\). Other methods devoted to this problem have been suggested by Ackroyd, \([75]\), using the stationary phase principle, which is applicable for an arbitrary length sequence. Another method which immediately comes to mind is a random solution of the zero pattern. In fact Huffman suggested that in order to maximize the energy ratio, the roots should be chosen in a random fashion with approximately half the zeros on each circle. He then concluded further that the optimum energy ratio should be proportional to \((N+1)^{1/2}\). In general correlation properties are used to judge the randomness of a sequence, that is a sequence is uncorrected with itself, i.e. random, if its ACF has uniformly low side lobes.

Although various methods described above may be acceptable for designing Huffman codes of moderately large length \((N=100)\), major difficulty arise for longer sequences because of difficulty of the computational nature due to factorizing polynomials of degrees larger than 100. Following sections describe some useful methods for designing the multilevel sequences.

5.3 DESIGN BASED ON OPTIMIZATION

The numerical minimization method discussed in Chapter 4 (for the design of binary sequences) is extended here for the design of multi-level (or a.m.ph.m.) sequences. The objective function or optimization criterion is the same as presented in section 4.3 and 4.5. However, for multi-level sequences where the sequence can have any number of levels, the choice of starting point becomes difficult as the no. of levels increasing. The minimization algorithm searches the minimum with respect to phases.
### TABLE 5.1

**SEQUENCES OF HUFFMAN CODES**

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<th>SEQUENCE</th>
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TABLE 5.3

Sequence derived from window function for a=3
Optimization criterion is sidelobe energy

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TABLE 5.4

Sequence derived from window function for a=2
Optimization criterion is "max. sidelobe level
and sidelobe energy"

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<th>SIDELOBE ENERGY</th>
<th>MAXIMUM SIDELOBE LEVEL</th>
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Sequence derived from window function for $a=3$
Optimization criterion is "max. sidelobe level and sidelobe energy"

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<th>SIDELOBE ENERGY</th>
<th>MAXIMUM SIDELOBE LEVEL</th>
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The Alpha in figures is denoted by "a" in the text.
Fig. 5.2 Autocorrelation Function v/s Time
(alpha=3)

Fig. 5.3 Autocorrelation Function v/s Time
(alpha=2)
Fig. 5.4 Autocorrelation Function v/s Time (alpha=3)

Fig. 5.5 Autocorrelation Function v/s Time (alpha=2)
Fig. 5.6 Sidelobe energy vs Sequence length

Fig. 5.7 Max. sidelobe vs Sequence length
Fig. 5.8(a) Autocorrelation Function vs Time

Fig. 5.8(b) Autocorrelation Function vs Time
Fig. 5.9(b) Autocorrelation Function vs Time

Fig. 5.9(c) Autocorrelation Function vs Time
Fig. 5.3(a) Autocorrelation Function vs Time

Fig. 5.3(b) Autocorrelation Function vs Time
Fig. 10a Autocorrelation Function vs Time

Fig. 10b Autocorrelation Function vs Time
Fig. 5.11(a) Autocorrelation Function vs Time

Fig. 5.11(b) Autocorrelation Function vs Time
only as for binary or uniform sequence $|a_n| = 1$ Therefore, the starting sequences in the present work is taken as the sampled versions of the various window functions such as Hann or Hamming window function and Kaiser window functions, [106]) (Kaiser Window Function is selected for generating sequences because it realizes all other window functions by varying its parameter $a$. When $a=0$ Kaiser window function corresponds to rectangular window function, $a=5.444$ it represents hamming window function, and for $a=8.885$ it corresponds to Blackman Window Function.) It is being observed that the Kaiser window gives better results in comparison to other window functions. The three parameters for comparison are

(i) Energy Ratio or Energy Efficiency,

(ii) Side Lobe Energy,

(iii) Maximum Side Lobe Level

The Table 5.1 gives Huffman code along with its energy efficiency auto-correlation properties i.e. side lobe energy and max side lobe level. The sequences generated from Kaiser window function are given in Table 5.2, 5.3, 5.4 & 5.5. The sequences in Table 5.2 (for $a=2$) and in Table 5.3 (for $a=3$) are optimized with sidelobe energy. Sequences in Table 5.4 (for $a=2$) and Table 5.5 (for $a=3$) are optimized with optimization criterion which combined effect of max. sidelobe level and side lobe energy. The different results of the above-said tables have also been plotted in figures so as to make the comparison among them easier.

Figure 5.1 is the plot of energy ratio verses the sequence length. The energy ratio of the sequences for $a=2$ is always less than the energy ratio of the sequences for $a=3$, if the same length of the sequences is considered. But the trend of variation of the energy ratio with the sequence length is same i.e., energy ratio decreases gradually with increase in the sequence length up to a length of approximately 21 and then shows no appreciable deterioration.

Figures 5.2, 5.3, 5.4 are plots of the side lobe energy and the max. side lobe level verses sequence length. Figures 5.2 & 5.3 are for the sequences which are optimized with side lobe energy and Figs 5.4 & 5.5 are for the sequences which are optimized.
with both max side lobe level and the side lobe energy. The parameter $\alpha=3$ for figures 5.2 & 5.4, and $\alpha=2$ for figs 5.3 and 5.5. When all the figures are compared, it is found that the side lobe energy given in figures 5.2 & 5.3 is lower than the side lobe energy given in figures 5.4 & 5.5 at almost all sequence lengths. The reason for lower side lobe energy of figures 5.2 & 5.3 is that it is optimized for minimum side lobe energy whereas the figures 5.4 & 5.5 are optimized with max side lobe level.

The auto-correlation function has also been plotted for the different sequences of lengths $N = 5$, $N = 13$, $N = 21$ and $N = 49$. Fig 5.8(b) & 5.8(c) represent sequences of length $N = 5$, which are derived from Kaiser window for $\alpha=3$ & $\alpha=2$ respectively.

The comparison of the sequences obtained with the above method and those given by Huffman, shows that the results are not very good with respect to the side lobe energy and the maximum side lobe. This is so because of the fixed values of the amplitudes of the sequence obtained by the window functions. Search is confined to changing the sign of the sequence element i.e. if sequence is given by $a_0, a_1, a_2, \ldots a_{N-1}$, then in the search (using element complementation) the sequence is tested by changing the sign of $a_0$ to $-a_0$ if the change decreases the functional then new value is retained and the next element $a_2$ is changed from $a_2$ to $-a_2$ and so on till the minimum is reached. A more general problem may be to search not only with respect to phase but magnitude also, i.e. $|a_i|$ is also changed and the functional observed. However, as the sequence length $N$ increases searching in both the directions phase and magnitude (in fact magnitude for $\{a_i\}$ may assume very large number of values ideally infinite) becomes quite cumbersome and one is not sure about the degree of improvements of the result. In next section however, a simple approach for improving the energy efficiency of the multi-level sequences is presented.

5.4 CLIPPING METHOD

A figure of merit for the energy distribution of a sequence is the energy efficiency defined by

$$\eta = \frac{E}{N \max_n |a_n|^2} \leq 1$$

(5.6)
It is evident that the maximum value of $\eta$ will be attained for purely phase modulated pulse trains such as binary sequences, having all element values equal to unity. In this case the energy efficiency will be 100% as $E$ will be equal to $N$ (the length of the sequence). The energy efficiency depends on two independent variables, namely, the total energy $E$ of the pulse train and the magnitude of the largest pulse denoted by $\max_n|a(n)|$. As can be seen from Eqn. (5.6) the energy efficiency of the Huffman sequences can be increased by decreasing the energy contained in the largest pulse $\max_n|a(n)|$. This is done by process of clipping where the largest pulse is clipped to some level or in other words to limit the total energy contained in the largest pulse. The clipping level is defined as

$$CLIP = \frac{\max(Pulse.Height) - \min(Pulse.Height)}{100} \cdot x$$

It is the amount of clipping done in each step. The $x$ is the percentage at which the clipping is required, in the present case it is 5%. The clipped sequence so obtained would not be a true Huffman sequence and its ACF would depart from being pulse-like. Furthermore, the side lobe energy in the ACF of the clipped sequence will be greater than that of the unclipped sequence. As the clipping level is increased, the energy efficiency of the resulting sequence increases, while the side lobe energy also increases. In the limit when the energy ratio is maximum, the resulting sequence will be a binary sequence, but still having an ACF with small side lobe levels. The process of clipping thus provides a means to generate sequences where energy ratio and side lobe energy is a compromise between those of Huffman and binary sequences. What level of clipping should be chosen would however, depend upon the actual requirements and test conditions.

The process of clipping is applied to the Huffman sequences of length 13 and 32 and 128. These sequences were obtained using the method given in Section 5.2. The values of these sequences are given in appendix. The results are given in Tables 5.6, 5.7 and 5.8 respectively. The lobes give the variation of energy efficiency and side lobe energy for various levels of clipping. For the Huffman sequence of length 13, 32 & 128 these variations are also shown graphically in Fig. 5.12a, b, c. It can be seen from the lobes and figures that as the clipping level is increased the energy efficiency of the resulting clipped sequence increases while the side lobe energy also increases. For full
Energy-ratio, Side lobe Energy for different levels of clipping of Huffman Sequence of Length=128

Fig. 5.12(c)
<table>
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<th>Energy Ratio</th>
<th>Side-lobe Energy</th>
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clipping such that all the pulses have same value, the resulting sequences are the binary sequences of corresponding length. It is interesting to mention here that full clipping of the Huffman sequence of length 13 results in the Barker sequence of same length. The binary sequence of length 32 obtained by clipping the Huffman sequence of length 32 and 128 (Table 5.7 & 5.8) also has low side lobe energy.

Thus the method of clipping can be used to advantage for generating either binary sequences or non-binary sequences having low side lobe energy and high energy efficiency. The main requirement of this method is that the starting sequence which is a non-binary sequence, must have very low side lobe energy. This method can, therefore, be used only in those cases where non-binary sequences having good auto-correlation properties are available or a method is available for the generation of such sequences.

5.5 ITERATIVE PROCESS FOR THE GENERATION OF MULTILEVEL SEQUENCES

5.5.1 Introduction

In this section the multilevel sequences are generated iteratively. It is observed that the inverse of a sequence having low side-lobe energy has good ACF better than the sequence itself. This fact is being utilized to generate the sequences of low side-lobe energy by repeating the process of inversion a number of times. The inverse of a sequence is obtained by obtaining the inverse filter of the same length. The values of the top gains of the inverse filter are taken as the new sequence which would have lower side-lobe energy as compared to the first. Inverse filter coefficients are again obtained using the new sequences to yield a sequence which would have still lower side-lobe levels. Thus new sequences having lower side-lobe energy are generated iteratively by successively taking the coefficients of the inverse filter as a sequence and computing the corresponding inverse filter coefficients. In the following sections the problem of inversion and the selection of the initial sequences, is considered in more details.
5.5.2 Optimum Inverse (Least Error Energy) Filter

The optimum inverse (or least error energy or spike) filter, [43] problem for discrete time system can be formulated with reference to Fig 5.13. The error sequence \( \{e_t\} \) represents the difference between the desired output sequence \( \{d_t\} \) and the actual output sequence \( \{c_t\} \); \( \{a_t\} \) is the input sequence of length \( n+1 \). The desired output sequence in the present case, consists of a unit 'spike' at some time index \( k \). The optimization problem is to select the filter weighting sequence \( \{f_t\} \) of length \( m+1 \) such that the error energy of the error sequence \( \{e_t\} \) is minimized. The error energy \( I \) is given by

\[
I = \sum_{t=0}^{m+n} e_t^2 = \sum_{t=0}^{m+n} [d_t - \sum_{s=0}^{m} f_s a_{t-s}]^2
\]  

(5.7)

The solution to this optimization problem is given by Robinson, [43], [77]. According to him, if there is no constraint on the admissible values of \( f_j, j = 0, 1, 2, \ldots, N \), then the optimum filter coefficient must satisfy the relation \( \partial I / \partial f_j = 0 \), for \( j = 0, 1, \ldots, m \). The result of performing the indicated operation is

\[
\sum_{r=0}^{N} f_r r_{j-r} = g_j
\]  

(5.8)

\( j = 0, 1, 2, \ldots, m \)

where \( r_{j-r} = \sum_{t=0}^{m} a_{t-r} a_{r-j} \), is the auto-correlation function (for index, \( j \)'s) of the input sequence \( \{a_t\} \), and \( g_j = \sum_{r=0}^{2N-2} d_r a_{r-j} \), \( j = 0, 1, \ldots, m \), is the cross-correlation of the sequence \( \{d_t\} \) with sequence \( \{a_t\} \) since \( r+j = r-j \ \forall \ j > 0 \). Eqn (5.8) provides a set of \( (m+1) \) linearly independent expressions in \( (m+1) \) unknowns. The minimum value of \( I \), obtained by using \( f_0, f_1, \ldots, f_m \), determined from the solutions of Eqn (5.8) is

\[
I_{\text{min}} = \sum_{t=0}^{m+n} d_t^2 - \sum_{s=0}^{m} f_s g_s
\]  

(5.9)

The finite length optimum inverse filter for a given input sequence obtained by solving the set of Eqn (5.8) is not an exact inverse. An exact inverse would be of infinite length, [77], [78]. The filter, however, is optimum for the chosen length \( m+1 \). If the desired length of the filter in Eqn (5.8) is increased, it will approach the exact inverse.
Input Sequence \( \{a_i\} \)

Digital Filter \( \{f_i\} \)

Actual output Sequence \( \{c_i\} \)

Desired Output Sequence \( \{d_i\} \)

Error Sequence \( \{e_i\} \)

Fig. 5.13
Fig 5.14 (d) Variation of energy ratio and side-lobe energy at different iterations for the Barker sequence of length 13.
Fig. 5.14(b) Variation of energy ratio and side-lobe energy at different iterations for the Barker sequence of length 11.
Fig. 5.14(c) Variation of energy ratio and side-lobe energy at different iterations for the binary sequence of length 31.
Fig 5.15 Variation of side-lobe energy at different iterations for the starting sequence chosen at random.
## TABLE 5.9: Energy ratio and side-lobe energy at different iterations for Binary and Huffman Sequences of Different Lengths

<table>
<thead>
<tr>
<th>Comp-</th>
<th>Barker sequence of length 13</th>
<th>Huffman sequence of Length 13</th>
<th>Binary sequence of Length 31</th>
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<td>7.36</td>
<td>0.0086</td>
<td>7.42</td>
</tr>
<tr>
<td>6</td>
<td>6.92</td>
<td>0.0079</td>
<td>7.22</td>
</tr>
<tr>
<td>7</td>
<td>6.94</td>
<td>0.0072</td>
<td>7.02</td>
</tr>
<tr>
<td>8</td>
<td>6.66</td>
<td>0.0067</td>
<td>6.87</td>
</tr>
<tr>
<td>9</td>
<td>6.71</td>
<td>0.0062</td>
<td>6.71</td>
</tr>
<tr>
<td>10</td>
<td>6.50</td>
<td>0.0059</td>
<td>6.59</td>
</tr>
<tr>
<td>11</td>
<td>6.48</td>
<td>0.0055</td>
<td>6.46</td>
</tr>
<tr>
<td>12</td>
<td>6.26</td>
<td>0.0052</td>
<td>6.36</td>
</tr>
<tr>
<td>13</td>
<td>6.29</td>
<td>0.0049</td>
<td>6.29</td>
</tr>
<tr>
<td>14</td>
<td>6.10</td>
<td>0.0046</td>
<td>6.17</td>
</tr>
<tr>
<td>15</td>
<td>6.1</td>
<td>0.0044</td>
<td>6.08</td>
</tr>
<tr>
<td>Computer Run</td>
<td>Barker sequence of length 5</td>
<td>Huffman sequence of Length 7</td>
<td>Binary sequence of Length 11</td>
</tr>
<tr>
<td>-------------</td>
<td>-----------------------------</td>
<td>-------------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td></td>
<td>Energy Ratio</td>
<td>Side-lobe energy</td>
<td>Energy Ratio</td>
</tr>
<tr>
<td>1.</td>
<td>5.0</td>
<td>0.16</td>
<td>7.0</td>
</tr>
<tr>
<td>2.</td>
<td>2.97</td>
<td>0.05</td>
<td>5.22</td>
</tr>
<tr>
<td>3.</td>
<td>3.5</td>
<td>0.03</td>
<td>4.43</td>
</tr>
<tr>
<td>4.</td>
<td>3.2</td>
<td>0.023</td>
<td>4.03</td>
</tr>
<tr>
<td>5.</td>
<td>3.35</td>
<td>0.020</td>
<td>3.95</td>
</tr>
<tr>
<td>6.</td>
<td>3.32</td>
<td>0.016</td>
<td>3.93</td>
</tr>
<tr>
<td>7.</td>
<td>3.69</td>
<td>0.013</td>
<td>3.96</td>
</tr>
<tr>
<td>8.</td>
<td>3.38</td>
<td>0.012</td>
<td>3.99</td>
</tr>
<tr>
<td>9.</td>
<td>3.42</td>
<td>0.010</td>
<td>4.02</td>
</tr>
<tr>
<td>10.</td>
<td>3.45</td>
<td>0.009</td>
<td>4.06</td>
</tr>
<tr>
<td>11.</td>
<td>3.33</td>
<td>0.008</td>
<td>4.10</td>
</tr>
<tr>
<td>12.</td>
<td>3.20</td>
<td>0.007</td>
<td>4.14</td>
</tr>
<tr>
<td>13.</td>
<td>3.10</td>
<td>0.0067</td>
<td>4.18</td>
</tr>
<tr>
<td>14.</td>
<td>3.00</td>
<td>0.0061</td>
<td>4.22</td>
</tr>
<tr>
<td>15.</td>
<td>2.92</td>
<td>0.0056</td>
<td>4.26</td>
</tr>
</tbody>
</table>
### TABLE-5.11: Energy-Ratio and Side-Lobe Energy of the Sequence of Length 20, Generated from the Barker Sequence of Length 13.

<table>
<thead>
<tr>
<th>Computer Run</th>
<th>Energy-Ratio</th>
<th>Side-Lobe Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 13 (Barker sequence of Length 13)</td>
<td>5.71</td>
<td>0.071</td>
</tr>
<tr>
<td>2.</td>
<td>7.18</td>
<td>0.268</td>
</tr>
<tr>
<td>3.</td>
<td>7.87</td>
<td>0.41</td>
</tr>
<tr>
<td>4.</td>
<td>7.17</td>
<td>0.126</td>
</tr>
<tr>
<td>5.</td>
<td>7.56</td>
<td>0.32</td>
</tr>
<tr>
<td>6.</td>
<td>7.37</td>
<td>0.095</td>
</tr>
<tr>
<td>7.</td>
<td>7.46</td>
<td>0.27</td>
</tr>
<tr>
<td>8.</td>
<td>7.61</td>
<td>0.081</td>
</tr>
<tr>
<td>9.</td>
<td>7.41</td>
<td>0.23</td>
</tr>
<tr>
<td>10.</td>
<td>7.41</td>
<td>0.072</td>
</tr>
</tbody>
</table>
**TABLE 5.12:** Energy-Ratio and Side-Lobe Energy of Sequences of Length 13 and 5 obtained by Choosing the initial sequence at Random (Optimum Pulse Position not at nT second)

<table>
<thead>
<tr>
<th>Computer Run</th>
<th>(a) Optimum Pulse position at T sequence-All the elements having values +1</th>
<th>(b) Optimum Pulse position at 15T-sequence 1,-1, 1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1.</th>
<th>(c) Sequence of Length 5, having optimum Pulse position at 4T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Energy Ratio</td>
<td>Side-Lobe energy</td>
<td>Energy Ratio</td>
</tr>
<tr>
<td>1.</td>
<td>13</td>
<td>7.69</td>
<td>13</td>
</tr>
<tr>
<td>2.</td>
<td>2.01</td>
<td>0.50</td>
<td>3.15</td>
</tr>
<tr>
<td>3.</td>
<td>5.03</td>
<td>5.71</td>
<td>6.3</td>
</tr>
<tr>
<td>4.</td>
<td>1.85</td>
<td>0.50</td>
<td>3.13</td>
</tr>
<tr>
<td>5.</td>
<td>4.65</td>
<td>5.2</td>
<td>5.59</td>
</tr>
<tr>
<td>6.</td>
<td>1.85</td>
<td>0.5</td>
<td>3.1</td>
</tr>
<tr>
<td>7.</td>
<td>4.51</td>
<td>4.97</td>
<td>5.4</td>
</tr>
<tr>
<td>8.</td>
<td>1.85</td>
<td>0.50</td>
<td>3.2</td>
</tr>
<tr>
<td>9.</td>
<td>4.44</td>
<td>4.86</td>
<td>5.1</td>
</tr>
<tr>
<td>10.</td>
<td>1.85</td>
<td>0.50</td>
<td>3.12</td>
</tr>
<tr>
<td>11.</td>
<td>4.39</td>
<td>4.79</td>
<td>4.89</td>
</tr>
<tr>
<td>12.</td>
<td>1.85</td>
<td>0.50</td>
<td>3.07</td>
</tr>
<tr>
<td>13.</td>
<td>4.36</td>
<td>4.24</td>
<td>4.7</td>
</tr>
<tr>
<td>14.</td>
<td>1.85</td>
<td>0.50</td>
<td>3.04</td>
</tr>
<tr>
<td>15.</td>
<td>4.33</td>
<td>4.70</td>
<td>4.61</td>
</tr>
</tbody>
</table>
and, therefore, the error energy in Eqn (5 7) and the minimum mean square error in Eqn (5 9) would decrease. Robinson, [43], [77] has also mentioned this. He further comments that, for a given sequence the least error energy depends very critically upon the lag of the spike in the desired output, relative to the input. The optimum lag depends upon the delay properties (maximum, minimum, or mixed delay) of the input sequence.

If the length of the spike filter (m+1), is made equal to the length of input sequence (n+1), then the (n+1) equations (Eqn 5 8) can be written in matrix form as

\[ RA = G \]  

(5 10)

Where A is a column vector containing (n+1) weighting sequences of the filter, g is a vector containing (n+1) discrete cross correlation function of the input sequence \( \{a_i\} \) and the desired output \( \{d_i\} \). R is the (n+1) x (n+1) auto-correlation matrix, each row of which consists of a shifted version of the auto-correlation of the input sequence \( \{a_i\} \) and is given by

\[ R = \begin{bmatrix} r_0 & r_1 & \cdots & r_n \\ r_1 & r_0 & \cdots & r_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_{n-1} & \cdots & r_0 \end{bmatrix} \]  

(5 11)

If the input sequence has near an ideal auto-correlation function (such as the Huffman sequence), then \( r_j = 0 \) for \( j \neq 0 \) or \( n \), and also \( r_0 \gg r_n \), therefore, neglecting \( r_n \), the auto-correlation matrix R becomes a diagonal matrix and the filter coefficients \( f_j \) from Eqn (5 10) are given by

\[ f_j = g_j/r_0, j = 0, 1, \ldots, n \]  

(5 12)

If the spike in the desired output sequence lies at time index (n+1), the cross-correlation coefficients are

\[ g_j = a_{nj}, j = 0, 1, 2, \ldots, n \]  

(5 13)
From Eqn (5 12) the spike filter coefficients are
\[ f_j = a_{n-j}/r_0, \quad j = 0, 1, \ldots, n \] (5 14)

Which shows that in this special case the spike filter is the filter matched to the input sequence \( \{a_n\} \), Hence, with sequences having near ideal ACF the spike filter is almost identical to a matched filter Eqn (5 12) further suggests that to make the filter energy small, i.e., better performance in the presence of noise, \( r_0 \) (the total energy of the sequence) should be large. This can be done by choosing the sequence of greater length.

Robinson, [43], [77] gives a most useful set of Fortran programmes for efficiently solving such matrix equation and computing the coefficients of the least error energy filter for optimum spike position. The listing of the programme is given in Appendix.

5.5.3 Results and Discussion

The side lobe energy of the sequence \( \{a_n\} \)

\[ (SLE)_A = 2 \sum_{i=1}^{n} r_i^2 \] (5 15)

If \( f_j \) is taken as the input sequence (eqn 5 14) then the side lobe energy of the sequence \( \{f_j\} \)

\[ (SLE)_F = 2 \sum_{i=1}^{n} r_i^2 = 2 \sum_{i=1}^{n} \frac{a_j^2}{r_0^2} \] (5 16)

Where \( r_i^2 \) are the side lobes of the auto-correlation function of the sequence \( \{f_j\} \)

\[ (SLE)_F = \frac{1}{r_0^2} 2 \sum_{i=1}^{n} r_i^2 = \frac{1}{r_0^2} (SLE)_A \]

for \( n > 1 \), \( r_0^2 > 1 \), therefore

\[ (SLE)_F < (SLE)_A \] (5 17)

This condition is derived when \( r_j = 0 \), for \( j \neq 0 \) or \( n \). In practice \( r_j \neq 0 \) for \( j \neq 0 \) or \( n \), i.e., the auto-correlation function always has some side-lobes, however, if the main lobe is of considerably greater magnitude than the side-lobes, the condition derived in
Equ. (5.17) will still hold good, though the filter coefficients $f_j$ will not be exactly equal to the value given by Equ. (5.14), [98].

From the above discussion it can be concluded that the inverse of a sequence having low sidelobe energy should have a good auto-correlation function, better than the sequence itself. This fact is being utilized to generate a sequence of low side-lobe energy by repeating the process of inversion a number of times. The inverse of a sequence is obtained obtaining the inverse filter of the same length. The values of the tap gains of the inverse filter are taken as the new sequences which would have lower side-lobe energy as compared to the first. Inverse filter coefficients are again obtained using the new sequences to yield a sequence which would have still lower side-lobe levels. Thus new sequences having lower side-lobe energy are generated iteratively by successively taking the coefficients of the inverse filter as a sequence and computing the corresponding inverse filter coefficients. The initial or the starting sequence for this iterative process can by any sequence of desired length, having low side-lobe energy.

As an example of the above process, the Barker sequence of length 13 is taken as the starting sequence. The side-lobe energy of this sequence is 0.071 whereas the side-lobe energy of the corresponding inverse filter coefficients is 0.017 which is much smaller than the side-lobe energy of the original Barker sequence. The sequences are here normalized to have unit energy Table 5.9 gives the energy ratio and the side-lobe energy for the Barker and Huffman sequences of length 13 and the binary sequence of length 31, at different iterations or computer runs. It can be seen from the table that indeed the side-lobe energy in all three cases decreases with the number of iterations or computer runs. Undesirable effect is that the energy ratio of the resulting sequences also decreases. However, the decrease in energy ratio is not much as compared to the decrease in the side-lobe energy. For example, in the case of Barker sequence of length 13, and for 15th computer run the value of the side-lobe energy decreases from 0.071 to 0.004 while the energy ratio reduces from 13 to 6.12. The side lobe energy of the 31 length binary sequences is reduced from 0.28 to 0.007. Hence the sequences having very low side-lobe energy can be generated using this method.
In all the cases shown in Table 5.9 & 5.10, the side-lobe energy is decreasing with the number of iterations. In other words the iterations process is said to be converging. In the present work the word 'converge' would be used in this sense and would mean that the side-lobe energy decreases with the increase in the number of iterations.

For the Barker sequence of length 13 and 11 and for the binary sequence of length 31, the variation in the side-lobe energy and the energy ratio with the number of computer runs (iteration) is shown in Figs. 5.14 a, b & c respectively. It can be seen from these figures that the iterative process converges in each case. Both the side-lobe energy and energy ratio tend to become constant after about 15th iteration. The maximum improvement in the side-lobe energy takes place during the first few iterations after which there is only a slight improvement.

In all the above cases the length m of the optimum inverse filter was kept equal to the length of the initial sequence n, thus the new sequence generated has the same length as that of the starting sequence. If the length of the optimum inverse filter is increased so that it is greater than the length of the starting sequence, a sequence of length greater than the original sequence and equal to the length of the filter will be generated. This new sequence can now be taken as the starting sequence for iterative process of section 6.2. This would thus, provide a method to generate sequences of lengths other than that of the chosen starting sequence. As an example, coefficients of 20 length optimum I.F., were computed for the Barker sequence of length 13. These 20 coefficients of the filter formed the elements of the new starting sequence of length so. The results of the iterative process are shown in Table 5.11. It can be seen from the table that the resulting sequences of length so, do not have 100 side-lobes levels. Furthermore, the convergence rate of the process is very small. This suggests that in order to obtain sequences having good correlation properties the length of the filter should be the same as the length of the starting sequence.
5.6 CHOICE OF STARTING SEQUENCE

The present section is devoted to study the consideration necessary in choosing the starting sequence for the iterative process. In other words to find out necessary condition to be fulfilled by a sequence so that it can be taken as the initial sequence.

In section 6.2 and 6.3 Barker sequence of different length and binary sequence of length 31 were taken as the initial sequences. The iterative process converged in all these cases. These sequences possess good ACF properties in that their side-lobe energy is low. It would not be out of place here to check, if convergence is achieved with any sequence chosen at random. Also if this does not works, what other constraints must be imposed on the starting sequence so that the iterative process converges.

5.6.1 Choice at Random

A sequence of length 13 having all elements values equal to +1 was taken as starting sequence for the iterative process. The results are given in Table 5.12. It is observed from the table that the process does not converge but oscillates between two values. Similarly a few other sequences were chosen at random. The results are also given in Table 5.12. Again convergence is not obtained in these cases. The variation of side-lobe energy will the number of computer runs for these sequences are shown in Fig 5.15. It can be seen from the figure that the process does not converge in any of the cases, and even diverges in some cases.

5.6.2 Effect of Optimum Pulse Position

Comparison of results obtained with Barker sequence and these given in Table 5.13 where the choice of the starting sequence was made at random, show that convergence is achieved in all these cases where the optimum pulse position in the response of the IF, to the sequence lies at nT second. It, therefore, appears that sequences for which the optimum pulse position in the output of the inverse filter occurs at nT seconds, should provide better results. This argument seems to be more logical since the main lobe in the ACF of a sequence occurs at a delay of nT seconds where n is the length of the sequence and 2n-1 is the length of the ACF of the sequence. Thus the
initial sequence should be such that the optimum pulse position in the output of the inverse filter should coincide with the main lobe in the ACF of the initial sequence.

For the initial sequences chosen at random in section 6.5.1 the optimum pulse does not occur at the instant nT in the output of I.F. The optimum pulse position in these cases occurs at T and 15 T. In both these cases convergence is not proper.

For Barker Sequence taken as initial sequences for the iterative process, the optimum pulse position in the output of I.F., occurs at nT. For Binary sequence of Length 31 and Huffman sequence of length 13, the optimum pulse also occurs at the instant nT. The iterative process converges in all these cases. It, therefore, appears that the sequence for which the optimum pulse position in the output of the I.F. occurs at the instant nT, do converge and other sequences do not converge. It is possible while designing the optimum inverse filter, to deliberately fix the position of the pulse in the output, at the instant nT. For example, in designing the I.F. for the sequence of length 13 having all values +1, the pulse position in the output of the IF is fixed at 13 T. The results of the operation are shown in Table-5.12. It is observed from the table that the results obtained are not satisfactory. The side-lobe energy reduces to zero, while the energy ratio reduces to one, i.e. a single pulse of unit value is left at the input.

In the light of the above discussion it appears that a good initial sequence for the iterative process is the one for which the optimum pulse position in the response to the corresponding I.F. naturally lies at the time instant nT, where n is the length of the sequence.

5.6.3 Conditions for Convergence

From the discussion in the preceding sections, it is clear that for the iterative process to converge, the starting sequence should be such that the optimum pulse position in the response of the optimum I.F. should occur at the time instant nT. It may be because of the fact that the main lobe in the ACF of sequence occurs at the instant nT. Furthermore, the weighting sequences of digital matched and optimum I.F are almost identical when the ACF of the sequence is near ideal. It is also necessary for convergence to be achieved that the length of the optimum I.F. be made equal to the
length of the sequence. As has been observed in the results presented in the previous sections, the iterative process converge in all such cases when the starting sequence fulfills the above mentioned conditions.

5.7 AVERAGING THE SPIKING FILTER COEFFICIENT WITH THE INPUT CODE

In this section non-uniform or multi-level sequences having pulse like auto-correlation function are generated by applying the linearity property of the Fourier transform to a code and the inverse of the code in order to average the spectrum. The inverse of the code is obtained by inverse or least square energy filter (section 5.4). The following well known properties of the Fourier Transform are used:

A) Linearity Property (Superposition)

\[ a \cdot g_1(t) + b \cdot g_2(t) \iff a \cdot G_1(f) + b \cdot G_2(f) \]

where \( a \) & \( b \) are constants.

B) The energy density spectrum and auto-correlation function form the FT pairs

\[ R_g(t) \iff |G(f)|^2 \]

C) \( F[\delta(t)] = 1 \) or constant i.e. \( \delta(t) \iff 1 \)

As can be seen from the relation B, the auto-correlation function and energy density spectrum form the Fourier Transform pair. For an ideal auto-correlation function, it should have a narrow peak at the centre and zero elsewhere as shown in the Fig 5.16. With reference to relation C, the Fourier Transform or spectrum of this type of function is a constant i.e. flat or uniform. To obtain such type of auto-correlation function, the code should also be a impulse. However, this is not possible, the transmitted energy can not be concentrated in a single pulse, it has to be distributed over
some interval of time. Therefore, the spectrum of the actual transmitted signal departs from being a uniform or flat one. If the spectrum of the input code \{a_i\} is \(A(f)\), the spectrum of its inverse code \{f_i\} will approximate \(1/A(f)\) in the least square sense. The resultant or average spectrum of these two codes will be:

\[ \frac{1}{2} A(f) + \frac{1}{2} \cdot \frac{1}{A(f)} \] ................................. (5.18)

This spectrum will be closer to being flat or uniform spectrum in comparison to the spectrum \(A(f)\) of the input code \{a_i\}. Hence using the linearity property of the F.T. from the relation \(A\), following result is obtained:

\[ \frac{1}{2} \{a_i\} + \frac{1}{2} \{f_i\} \leftrightarrow \frac{1}{2} A(f) + \frac{1}{2 \cdot \frac{1}{A(f)}} \] ................................. (5.19)

The average code \{b_i\} given by \(\frac{1}{2} \{a_i\} + \frac{1}{2} \{f_i\}\) will have spectrum which will be closer to being a uniform spectrum in the least square sense. Since the energy density spectrum and the auto-correlation function form the Fourier Transform pair, therefore, the average code \{b_i\} corresponding to the LHS of the eqn. (5.19) will have better auto-correlation function than the input code \{a_i\}. This can also be shown with reference to Eqn. (5.14) by the analysis given below. The average code \{b_i\} is given by the following equation

\[ \frac{1}{2} \{a_i\} + \frac{1}{2} \{f_i\} \]

\[ \frac{1}{2} \{a_i\} + \frac{1}{2} \frac{\{a_i\}_{\infty} + \{a_i\}_{-\infty}}{\infty} = \frac{\{a_i\}_{\infty} + \{a_i\}_{-\infty}}{2} \] ................................. (5.19)

The side-lobe energy of the code \{a_i\} is

\[(SLE)_A = 2 \sum_{i=1}^{n} r_i^2\]

and the side-lobe energy of the average code \{b_i\} is

\[(SLE)_B = 2 \sum_{i=1}^{n} r_i^2\]

where \(r_i\) are the auto-correlation side lobes of the code \{b_i\}
<table>
<thead>
<tr>
<th>Initial Code (a_i)</th>
<th>Corresponding Average Code (b_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1) Binary Code of Length 5</strong>&lt;br&gt; All elements (+1)&lt;br&gt; Energy Ratio = 5&lt;br&gt; SLE = 2.39&lt;br&gt; Max SL = 0.8</td>
<td>Energy Ratio = 2.5&lt;br&gt; SLE = 1.20&lt;br&gt; Max SL = 0.43</td>
</tr>
<tr>
<td><strong>2) Barker Code of Length 5</strong>&lt;br&gt; Energy Ratio = 5&lt;br&gt; SLE = 0.16&lt;br&gt; Max SL = 0.2</td>
<td>Energy Ratio = 3.72&lt;br&gt; SLE = 0.05&lt;br&gt; Max SL = 0.15</td>
</tr>
<tr>
<td><strong>3) Barker Code of Length 7</strong>&lt;br&gt; Energy Ratio = 7&lt;br&gt; SLE = 0.122&lt;br&gt; Max SL = 0.143</td>
<td>Energy Ratio = 6.0&lt;br&gt; SLE = 0.09&lt;br&gt; Max SL = 0.137</td>
</tr>
<tr>
<td><strong>4) Barker Code of Length 11</strong>&lt;br&gt; Energy Ratio = 11&lt;br&gt; SLE = 0.082&lt;br&gt; Max SL = 0.09</td>
<td>Energy Ratio = 9.74&lt;br&gt; SLE = 0.071&lt;br&gt; Max SL = 0.08</td>
</tr>
<tr>
<td><strong>5) Barker Code of Length 13</strong>&lt;br&gt; Energy Ratio = 13&lt;br&gt; SLE = 0.071&lt;br&gt; Max SL = 0.077</td>
<td>Energy Ratio = 9.08&lt;br&gt; SLE = 0.017&lt;br&gt; Max SL = 0.063</td>
</tr>
<tr>
<td><strong>6) Huffman Code of Length 13</strong>&lt;br&gt; Energy Ratio = 8.5&lt;br&gt; SLE = 0.02&lt;br&gt; Max SL = 0.099</td>
<td>Energy Ratio = 8.32&lt;br&gt; SLE = 0.017&lt;br&gt; Max SL = 0.091</td>
</tr>
<tr>
<td><strong>7) Huffman Code of Length 32</strong>&lt;br&gt; Energy Ratio = 10.35&lt;br&gt; SLE = (0.23 \times 10^{-4})&lt;br&gt; Max SL = 0.099</td>
<td>Energy Ratio = 10.35&lt;br&gt; SLE = (0.19 \times 10^{-5})&lt;br&gt; Max SL = 0.091</td>
</tr>
<tr>
<td><strong>8) Binary Code of Length 31</strong>&lt;br&gt; Energy Ratio = 31&lt;br&gt; SLE = 0.28&lt;br&gt; Max SL = 0.129</td>
<td>Energy Ratio = 20.0&lt;br&gt; SLE = 0.13&lt;br&gt; Max SL = 0.095</td>
</tr>
<tr>
<td><strong>9) Binary Code of Length 31</strong>&lt;br&gt; Energy Ratio = 31&lt;br&gt; SLE = 0.39&lt;br&gt; Max SL = 0.193</td>
<td>Energy Ratio = 13.87&lt;br&gt; SLE = 0.18&lt;br&gt; Max SL = 0.185</td>
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<td><strong>10) Binary Code of Length 21</strong>&lt;br&gt; Energy Ratio = 21&lt;br&gt; SLE = 0.245&lt;br&gt; Max SL = 0.143</td>
<td>Energy Ratio = 11.2&lt;br&gt; SLE = 0.107&lt;br&gt; Max SL = 0.113</td>
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<td>Binary Code of Length 31</td>
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<td>SLE</td>
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<td>0.052</td>
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<tr>
<td></td>
<td></td>
<td>Max SL</td>
<td>0.097</td>
<td>0.065</td>
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<tr>
<td></td>
<td></td>
<td>SLE</td>
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<td>0.119</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Max SL</td>
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<td>0.091</td>
</tr>
<tr>
<td>Barker code of length 13</td>
<td>Average code ${b_j}$ derived from Barker code of length 13</td>
<td>Huffman code of length 13</td>
<td>Average code ${b_j}$ derived from Huffman code of length 13</td>
<td></td>
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<td>---------------------------------------------------------------</td>
<td>--------------------------</td>
<td>---------------------------------------------------------------</td>
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<td>1.0 0 0.96 0</td>
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<tr>
<td>1</td>
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<td>1.093 0 1.10 0</td>
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<tr>
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<td>1</td>
<td>0 1.05 0</td>
<td>0.791 0 0.80 0</td>
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<tr>
<td>1 180</td>
<td>1.05 180</td>
<td>0.905 180 0.92 0</td>
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<tr>
<td>1</td>
<td>0 0.90 0</td>
<td>1.0 0 0.96 0</td>
<td></td>
<td></td>
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</tbody>
</table>
Fig. 5.16
\[
(SLE)_B = \left(\frac{x_{n+1}}{2^n}\right)^2 \cdot 2 \sum_{i=1}^{n} r_i^2
\]

\[
(SLE)_B = \left(\frac{x_{n+1}}{2^n}\right)^2 \cdot (SLE)_A
\]

for \( n > 1 \); \( \frac{x_{n+1}}{2^n} < 1 \)

Therefore

\[
(SLE)_B < (SLE)_A
\]

This condition is derived when \( r_j = 0 \) for \( j \neq 0 \) or \( n \). In practice, \( r_j \neq 0 \) for \( j \neq 0 \) or \( n \) i.e. auto-correlation function always has some side lobes, however, if the main lobe is of considerably greater magnitude than the side lobes, the condition derived in Eqn. (5.20) will still hold good, though the filter coefficients, \( \{ f_i \} \) will not be exactly equal to the value given by Eqn. (5.14).

Table 5.14 gives the energy ratio, side-lobe energy and the maximum side-lobe for the various codes of different lengths. Table 5.15 gives the excitation values for some of the initial codes and the codes derived from them using proposed method. The Code No. 6 and 7 in Table 5.14 are the Huffman codes of length 13 and 32 given by Ackroyd, [75], [79]. Code No. 8 is a binary code of length 31 given by Chakraborty, [63]. Code No. 9 is a maximal length shift register sequence. Whereas the codes from No. 10 to No. 13 are taken from Chapter 4. As observed from the table the side-lobe energy and the value of maximum side-lobe of the derived average code \( \{ b_i \} \) is smaller than the codes from which they are derived i.e. code \( \{ a_i \} \). This improvement is achieved in all the cases presented in Table I. In some cases there is significant improvement. For example, the binary codes of length 31 and 41 has side-lobe energy 0.14 and 0.133 respectively, whereas the derived average code \( \{ b_i \} \) has side-lobe energy 0.052 in both the cases. Similarly the maximum side-lobe of these initial codes are 0.096 and 0.097, whereas the derived average codes have maximum side-lobe equal to 0.067 and 0.068 respectively. The improvement in auto-correlation function can also be seen with reference to Fig. 5.17, which depicts the auto-correlation function of binary code of length 31 and the auto-correlation function of the corresponding code \( \{ b_i \} \). The value of the auto-correlation function at each time shift is smaller in the case of code \( \{ b_i \} \). The same is the
case of Barker code of length 13 as the initial code whose auto-correlation function is given in Fig 5.19. The corresponding average code \{b,\} has smaller auto-correlation side-lobes as can be seen in Fig 5.20. Fig 5.21 shows the auto-correlation function of Huffman code of length 13. The maximum side-lobe of this code is 0.099 whereas the corresponding code \{b,\} has the maximum side-lobe equal to 0.091.

From the above discussion, it can be concluded that the basic idea of averaging the input code with the coefficients of the spiking filter provides a useful method of generating the non-uniform codes of any desired length with good auto-correlation properties.

5.8 SUMMARY

In some circumstances, the use of pulse trains, other than purely phase modulated ones, may be precluded due to the expense incurred in providing AM. However, the additional expense of encoding and decoding in amplitude and phase may be justified for radars that must cope with land clutter or operate in a dense-target environment.

The very low side-lobes that can be achieved with AM pulse trains presented in this chapter make their use particularly attractive in systems requiring a large dynamic range. Moreover, the excellent self-clutter rejection performance is obtained without sacrificing close target separability (no main lobe widening). This property is, however, achieved at the expense of a significant reduction in power utilization at the transmitter i.e., low energy efficiency of these pulse trains. The loss in energy efficiency of these codes or pulse trains is mainly attributed to the high degree of side-lobe suppression. In some applications, a complete suppression of the side-lobes may not be required, instead, they can be kept to a specified low level. Consequently, the clipping method or averaging method of spiking filter coefficients with the input sequence may be used to advantage for improving the energy efficiency.
CHAPTER-6

SIDEBLOBE REDUCTION
6.1 INTRODUCTION

In a dense target environment or in situations where there are large undesired scatterers (point clutter), it is often desirable to reduce the time sidelobes of phase coded sequences to a prescribed low level. In principle, there is no difference between the problems of resolving a target in the interference from other targets and the detection of target in clutter. In previous chapters the reduction of the range sidelobes of the compressed pulse has been of much concern in the application of matched filter techniques to radar systems. In fact the mutual interference between targets or self clutter imposes rather fundamental limitations on resolution performance. So far, the reduction of the range sidelobes has been approached via waveform design. However, the attainable sidelobes levels for phase coded pulse trains might be inadequate for specific applications. Although it is possible to use a.m.ph.m. pulse trains such as Huffman Codes (or codes discussed in previous chapter) to obtain the desired degree of discrimination, in most high power radar systems this method is not readily available. Thus, the designer has to resort to other sidelobe reduction techniques.

In principle sidelobe reduction can be achieved by either

(i) Amplitude-or Phase weighting in the Frequency Domain

(ii) Amplitude-or Phase weighting in the Time Domain

The weighting may be accomplished at the transmitter or receiver or at both. Furthermore, the shaping can be performed at the RF, IF or video stages.

Sidelobe suppression in the frequency domain requires the design of a filter such that the spectrum of the filtered waveform has a linear phase-frequency relationship and that the spectrum magnitude be proportional to one of the many available weighting functions such as Taylor or Chebyshev, [22], [31]. The rapid advance of digital hardware, alongwith the pipeline FFT configuration, does permit practical realization of the required transfer function as depicted in Fig. 2.9.

Amplitude weighting may be introduced into a pulse-compression system either entirely at the receiver, entirely at the transmitter, or at both simultaneously. Equal weighting at both the transmitter and receiver is equivalent to altering the transmitted
waveform. In this case the system is still considered as matched. However, it can be shown that in the peak power limited case, the SNR which assumes weighting at the receiver alone is greater or equal to the SNR which assumes matched weighting at the transmitter and receiver, [35]. An additional reason for unilateral weighting at the receiver is due to the advantage of operating the transmitter at its peak power limit (no expensive amplitude modulators required). Furthermore, amplitude weighting solely at the receiver can be maintained conveniently, due to the accessibility of the components and the low power levels involved. For these reasons it is henceforth assumed that weighting is performed solely at the receiver at the expense of a lower SNR.

A convenient way of weighting is at the IF stage in the time domain. Most of the sidelobe reduction techniques which have been proposed depend on cascading a weighting filter (tapped delay line) after the MF or by providing a suitable band shaping network, [76], [96]. However, instead of placing a sidelobe reduction filter after the MF it is probably more straightforward to design a mismatched filter (MMF) under some conditions of optimality, [78]. The amount of mismatch from the matched conditions is usually characterized by the loss factor $L_s$, given by, [76]

$$L_s = \frac{\text{SNR(weighted)}}{\text{SNR(matched)}}$$

For an input sequence $a(n)$ of length $(N+1)$ and a filter weighting sequence $h(n)$ of length $(M+1)$, $L_s$ becomes

$$L_s = \frac{\sum_{n=0}^{M} h(n) \cdot a(j - n)}{\sum_{n=0}^{N} |a(n)|^2 \sum_{n=0}^{M} |h(n)|^2} \leq 1 \quad \text{...(6.1)}$$

Where $j \geq N$ denotes the time delay for which the output is maximum. (It is assumed that $(M+1) \geq (N+1)$. The basis of Eqn.6.1 is that for coherent summation signal components add as voltage levels while the noise components add as power levels.

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6.2 INVERSE FILTERS

The reduction of the sidelobe interference can be accomplished through the use of inverse or deconvolution filters. These filters (equalizers) have been of much concern in removing inter symbol interference in communication and they are also of interest in spectrum analysis. What is required ideally is a digital filter which is inverse to the code sequence. That is, the response of the filter to the code would ideally be a sequence of which all the elements except one are zero. The position, \( L \), of this nonzero element, which can be chosen to have unity magnitude, is immaterial. If the code sequence is \( (a_0, a_1, \ldots, a_M) \) its Z-transform is given by

\[
A(Z) = a_0 + a_1 Z^{-1} + \ldots + a_M Z^{-M}
\]

The filter will have the desired response if the transfer functions \( Z^L / A(Z) \). If \( A(Z) \) has all its zeroes within the unit circle, \( Z^L / A(Z) \) will have all its poles there and the filter will be stable and the transfer function will be realizable by a recursive filter or it can be closely approximated by a transversal filter. However, it has been shown in the preceding chapter that sequences with high energy efficiency (such as binary sequences) must have approximately half of their zeroes inside and half outside the unit circle. The inverse filter will thus have poles outside the unit circle and consequently will be unstable. Thus \( Z^L / A(Z) \) will not be physically realizable for the signals which are of interest as it will have poles outside the unit circle. Nevertheless, provided that \( L \) is large enough, a good approximation to \( Z^L / A(Z) \) can be obtained which is physically realizable.

One approach is to factor \( 1/A(Z) \) into the product of a non-physically realizable part \( 1/A_(Z)_\), whose poles are all outside the unit circle, and a realizable part \( 1/A_(Z) \), whose poles are all within the unit circle. This factorization can be done by the use of a polynomial root finding routine and a digital computer. The task, however, becomes onerous for sequences of length more than about forty. \( 1/A(Z) \) can be expanded by polynomial division into a convergent series in powers of \( Z \) and if the expansion is truncated after the term in \( Z^L \), then the result, after multiplication by \( Z^L \), is a physically realizable transfer function. \( 1/A_+(Z) \), having all its poles within the unit circle,
can be directly realized as the transfer function of a recursive filter. Alternatively, it can be approximately realized by expanding it as a convergent series in powers of $Z^{-1}$ which is truncated at some point and then approximately realized as the transfer function of a non-recursive filter.

Another approach to obtaining, to a degree of approximation, the weighting sequence of an ideal inverse filter is to augment the sequence $(a_0, a_1, \ldots, a_M)$ to $(P+1)$ points with a large number of zeroes and to compute the discrete Fourier transform $(A_0, A_1, \ldots, A_P)$ of the augmented sequence. The sequence $(1/A_0, 1/A_1, \ldots, 1/A_P)$ represents the sample values of the transfer function of the inverse filter. By computing its inverse discrete Fourier transform an aliased or folded version of the ideal inverse filter weighting sequence is obtained.

A filter designed in one of these ways has a weighting sequence which is either a truncated or a folded version of the weighting sequence of the ideal non-physically realizable inverse filter. However, for a given weighting sequence length it is possible to design filters with a lower mean square sidelobe level than can be achieved by this method.

Robinson and Trietel, [77] have comprehensively studied the problem of designing a non-recursive digital filter so that its response to a given input sequence approximates a desired output sequence in the least squares sense. In the present case the ideal output is a sequence whose elements are all zero except for one unity element occurring at some position $L$. The actual response of a filter having the weighting sequence $(f_0, f_1, \ldots, f_N)$ to the input $(a_0, a_1, \ldots, a_M)$ will be some sequence $(C_0, C_1, \ldots, C_{M+N})$. What is required is that the sum

$$V = C_0^2 + C_1^2 + \cdots + (C_L - 1)^2 + \cdots + C_{M+N}^2$$

should be minimized by proper choice of the weighting sequence. The sum $V$ represents the 'energy' of the difference between the actual response and the ideal response sequences. Robinson and Treitel show that $V$ is minimized when the weighting sequence is given by the solution of the vector matrix equation.

$$Rf = g$$
Where \( f \) is a column vector whose elements are the elements of the weighting sequence that is to be calculated and \( g \) is a vector containing the discrete cross-correlation function of the input sequence and the desired output sequence. \( R \) is a matrix each row of which consists of a shifted version of the auto-correlation of the input sequence (Section 5.5.2).

Robinson, [43], [62] gives a most useful set of Fortran programs for efficiently solving such matrix equations. In addition he gives a routine which computes the optimum filter and evaluates \( V \) for each position \( L \) of the response peak. His programs thus enable one to find the response peak position \( L \) for which the least squares inverse filter of weighting sequence length \( M \) produces the best approximation to the desired response and to compute the filter weighting sequence.

There are three characteristics of a sidelobe suppressing pulse compression filter which are of particular interest. One is the total response sidelobe energy given by

\[
V_s = \sum_{k=0}^{M+N} C_k^2 - 1
\]

Where filter weighting sequence has been normalized so that the peak of its response to input sequence is unity. In a dense uncorrelated target environment, the self clutter power at the filter output is proportional to this quality. Another property of concern is the peak sidelobe level, \( \max |C_k| \), which determines the ability of the system to resolve a weak target which is adjacent to a strong one. Finally, the total 'filter energy', given by

\[
V_x = \sum_{k=0}^{N} f_k^2
\]

is important since with white noise at the receiver input the noise power at the filter output is proportional to this quantity.

As the optimum filter weighting sequence length is increased, its behaviour approaches that of the ideal inverse filter (i.e. response sidelobes are suppressed), [78]. The Table 6.1 shows the decrease in signal to noise ratio (i.e. loss factor \( L_n \), equ 6.1) that results when the matched filter is replaced by the ideal inverse filter for the Barker
sequences of Length from four to thirteen. It is to be remarked that a loss of only 0.2dB in SNR is incurred when the sidelobes of the 13-element Barker sequence are completely suppressed a fact which was first pointed out by Key et al, [76] and Messe et al, [96].

6.2.1 An Iterative Method for Range Side Lobe Suppression for Binary Codes

In section 5.5.2, it has been shown that the inverse filter weighting sequence or coefficients obtained in the manner as discussed above, has better autocorrelation function, smaller than that of the input sequence itself, [98], [99]. The same concept is used here for sidelobe suppression by taking the filter coefficients as the input sequence, evaluating its inverse filter and each time evaluating its response.

Table 6.2 and Fig. 6.1, show the variations of three quantities of a sidelobe suppressing filter (V_n, max C(k) and |V_n|) for each iteration or computer run, when the input sequence is the Barker sequence of length 13. The corresponding matched filter output is also shown for comparison. It can be seen from the table and the fig. that quantities V_n and max|C(k)| are decreasing with the number of iterations. The quantity V_n is approximately constant. It can also be observed that as the iteration is increased the performance of inverse filter and the matched filter becomes almost identical. The advantage of the method proposed for sidelobe suppression is that the desired suppression is achieved without increasing the length of the filter. The proposed scheme can also be used for codes other than Barker codes.

<table>
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<td>Barker Sequence Length</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>11</td>
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<tr>
<td>13</td>
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Fig. 44 Variation of $V_s$, $V_e$, and $\max |C_1|$ at different iterations for Barker code of length 13

- $\bullet$ sidelobe energy (IF output); $V_s$
- $\square$ sidelobe energy (corresponding MF)
- $\times$ max sidelobe (IF); $\max |C_1|
- $\Delta$ filter energy (IP); $V_e$
| Iteration | IF output SL energy $V$, | max SL (IF) $\max |C_k|$ | Matched filter | max SL (MF) | Filter energy $V_n$ (IF) |
|-----------|--------------------------|-----------------|----------------|---------------|-----------------|
| 1         | 0.026                    | 0.063           | 0.071          | 0.077         | 0.980           |
| 2         | 0.010                    | 0.055           | 0.017          | 0.063         | 0.985           |
| 3         | 0.00858                  | 0.056           | 0.0117         | 0.061         | 0.985           |
| 4         | 0.0075                   | 0.052           | 0.0097         | 0.052         | 0.986           |
| 5         | 0.00717                  | 0.052           | 0.0086         | 0.053         | 0.987           |
| 6         | 0.0067                   | 0.049           | 0.0079         | 0.050         | 0.987           |
| 7         | 0.0062                   | 0.048           | 0.0072         | 0.049         | 0.988           |
| 8         | 0.0059                   | 0.047           | 0.0067         | 0.048         | 0.988           |
| 9         | 0.0055                   | 0.046           | 0.0062         | 0.046         | 0.989           |
| 10        | 0.0052                   | 0.045           | 0.0059         | 0.045         | 0.989           |
| 11        | 0.0050                   | 0.044           | 0.0055         | 0.044         | 0.990           |
| 12        | 0.0047                   | 0.043           | 0.0052         | 0.043         | 0.990           |
| 13        | 0.0045                   | 0.042           | 0.0049         | 0.042         | 0.991           |
| 14        | 0.0043                   | 0.041           | 0.0046         | 0.041         | 0.991           |
| 15        | 0.0041                   | 0.040           | 0.0044         | 0.040         | 0.991           |
| 16        | 0.0039                   | 0.039           | 0.0042         | 0.040         | 0.992           |
| 17        | 0.0038                   | 0.038           | 0.0040         | 0.039         | 0.992           |
| 18        | 0.0036                   | 0.038           | 0.0038         | 0.038         | 0.992           |
| 19        | 0.0035                   | 0.037           | 0.0037         | 0.037         | 0.993           |
| 20        | 0.0034                   | 0.036           | 0.0035         | 0.037         | 0.993           |
6.3 SIMPLE SCHEME FOR SIDE LOBE ELIMINATION

In this section, a simple approach is developed which completely eliminates the effects of the sidelobes.

Let \( \{a_i\} \) is the input sequence to a matched filter, matched to the input sequence \( \{a_i\} \) with discrete weighting sequence \( \{f_i\} \). With reference to Fig. 6.2, the output of the matched filter \( \{C_i\} \) is the convolution of \( \{a_i\} \) with \( \{f_i\} \). In terms of Z-transform.

\[
A(Z) = a_0 + a_1 Z^{-1} + \ldots + a_{N-1} Z^{N-1} \quad (6.2)
\]
\[
F(Z) = a_{N-1} + a_{N-2} Z^{-1} + \ldots + a_0 Z^{N-1} \quad (6.3)
\]
\[
C(Z) = A(Z)F(Z) = C_0 + C_1 Z^{-1} + C_2 Z^{-2} + \ldots + C_{N-1} Z^{N-1} + \ldots + C_{2N-2} Z^{2N-2} \quad (6.4)
\]

\( C(Z) \) is also the Z-transform of the auto-correlation function of the sequence \( \{a_i\} \) which is of the length \((2N-1)\). The main lobe in the output of the filter occurs at a delay of \((N-1) T\) seconds or in the centre, \(1/T\) being the clock frequency. The output should ideally be a pulse at \(t = NT\) seconds and zero elsewhere. However, it is not the actual case. The output usually contains some side-lobes in addition to the main lobe. Suppose, the ideal or desired output is denoted by \( D(Z) \) and the undesired output (having only the side-lobes) is denoted by \( B(Z) \), such that

\[
C(Z) = B(Z) + D(Z) \quad (6.5)
\]

where

\[
B(Z) = C_0 + C_1 Z^{-1} + \ldots + 0.Z^{N+1} + \ldots \quad (6.6)
\]

and

\[
D(Z) = 0 + 0.Z^{-1} + \ldots + C_{N-1} Z^{N+1} + 0.Z^{N+2} + \ldots \quad (6.7)
\]

Where the length of the \( B(Z) \) and \( D(Z) \) is the same as that of \( C(Z) \) i.e. equal to \((2N-1)\).
The undesired effects represented by $B(Z)$, can now be eliminated after subtracting it from the output $C(Z)$, as shown in fig. 6.3 so that only the desired output is left.

$$
D(Z) - C(Z) - B(Z) = A(Z) F(Z) - B(Z) \quad \quad \quad \quad \quad (6.8)
$$

The undesired effects represented by $B(Z)$ can be generated after subtracting $D(Z)$ from $C(Z)$ as shown in Fig. 6.4 i.e.

$$
B(Z) = C(Z) - D(Z) \quad \quad \quad \quad \quad (6.9)
$$

The overall scheme for side-lobe elimination is shown in Fig. 6.5 where both the matched filters are identical. As can be seen from Fig. 6.5 the output contains no side-lobes. If the auto-correlation function of the input sequence $\{a_i\}$ (output of the matched filter) is normalized such that the main lobe has a value equal to unity, then the $C_{N-1}$ in equation 6.7 is equal to unity.

The overall transfer function of the scheme proposed for side-lobe elimination can be evaluated with reference to figures 6.6 to 6.8. As can be seen from these figures the overall transfer function of the scheme is $1/A(Z)$ which is same as that of an ideal inverse filter.

From the above discussions it can be concluded that the scheme proposed for side-lobe elimination may be regarded as some kind of ideal inverse filtering operation, without actually using an inverse filter.

However, the noise performance of above scheme Fig. 6.5 comes out to be poor, because at the output of the subtracter of the upper channel both the noise terms will be added. If the noise is white Gaussian with zero mean and variance $\sigma^2$. Then noise output at both the channels will be proportional to $\sigma^2 \sum_{i=0}^{n-1} f_i^2 = \eta$. Therefore after subtracting the outputs of both the channels the noise terms will fail to cancel, therefore overall effect of noise will increase. However, the effect of noise can be decreased by inserting a squarer circuit in both the channels as shown in Fig. 6.9.
Fig. 6.2

Fig. 6.3

Fig. 6.4
Fig. 6.5

Fig. 6.6
Fig. 6.7

\[ F(Z) - F(Z) + \frac{D(Z)}{A(Z)} \]

Fig. 6.8

\[ \frac{D(Z)}{A(Z)} \]
Fig. 6.9
The input to the squarer is $C(Z) + \eta$. Therefore the output will be

$$[C(Z) + \eta]^2 = C^2(Z) + \eta^2 + 2\eta C(Z) \quad \text{................................. (6.10)}$$

Modified $B(Z)$ will be

$$B(Z) = C^2(Z) + \eta^2 + 2\eta C(Z) - D(Z) \quad \text{................................. (6.10)}$$

The output of the subtracter of the upper channel will be the difference of Equ. 6.10 & 6.11 which will be equal to $D(Z)$, which is free from undesired sidelobes and noise.

**6.4 NEURAL NETWORK FOR SIDELOBE SUPPRESSION**

### 6.4.1 Introduction

In this section various aspects of using Artificial Neural Networks (ANN) for sidelobe reduction are explored. After two decades of eclipse, interest in artificial neural networks has grown rapidly over the past few years. This resurgence of interest has been fired by both theoretical and application successes. ANN has found wide application in pattern recognition. The pattern recognition is applicable in radar signal detection or classification processes. In the present case Radar signals can be treated as data patterns. The basic theories of statistical hypothesis, testing, decision theory etc. apply in these situations. Indeed, many pattern recognition techniques employ likelihood ratios, when the statistics are known apriori, and pattern recognition or classification is then merely an application of multiple hypothesis testing. Nevertheless, some interesting techniques presented under the name of pattern recognition, learning machines, artificial intelligence etc. are applicable to radar system and should be available for the radar designers consideration. In the following sections brief theory of neural networks and their use for sidelobe suppression is discussed.
4.2 Fundamentals of Artificial Neural Networks

The basic implementation mechanism can be explained with reference to Figure 6.10(a). Here, a set of inputs labeled $x_1, x_2, \ldots, x_n$ is applied to the artificial neuron. These inputs, collectively referred to as the vector $X$, correspond to the signals into the synapses of a biological neuron. Each signal is multiplied by an associated weight $w_1, w_2, \ldots, w_n$, before it is applied to the summation block, labelled $\Sigma$. Each weight corresponds to the 'strength' of a single biological synoptic connection. (The set of weights is referred to collectively as the vector $W$). The summation block, corresponding roughly to the biological cell body, adds all of the weighted inputs algebraically, producing an output that is called NET. This may be compactly stated in vector notation as follows:

$$\text{NET} = x_1w_1 + x_2w_2 + \ldots + x_nw_n = XW$$

Activation Functions

The NET signal is usually further processed by an activation function $F$ to produce the neuron's output $OUT$. This may be a simple linear function

$$\text{OUT} = K(\text{NET})$$

Where $K$ is a constant, a threshold function,

$$\text{OUT} = 1 \text{, if } \text{NET} > T$$

$$\text{OUT} = 0 \text{ otherwise}$$

Where $T$ is a constant threshold value, or a function. That more accurately simulates the non-linear transfer characteristics of the biological neuron and permits more general network functions.
Fig. 6.10 (a) Artificial Neuron

Fig. 6.10 (b) Artificial Neuron with Activation Function

Fig. 6.10 (c) Sigmoidal Logistic Function

Fig. 6.11 Single Layer Neural Network
In figure 6.10(b) the block labeled F accepts the NET output and produces the signal labeled OUT if the F processing block compresses the range of NET, so that OUT never exceeds some low limits regardless of the value of NET, F is called a 'squashing function'. The squashing function is often chosen to be the logistic function or 'sigmoid' (meaning S-shaped) as shown in Fig. 6.10(c). This function is expressed mathematically as \( F(x) = \frac{1}{1+e^{-x}} \). Thus

\[
OUT = \frac{1}{1 + e^{-\text{NET}}}
\]

By analogy to analog electronic systems, the activation function may be considered as a non-linear gain for the artificial neuron. This gain is calculated by finding the ratio of the change in OUT to a small change in NET. Thus gain is the slope of the curve at a specific excitation level. It varies from a low value at large negative excitations to a high value at zero excitation, and it drops back as excitation becomes very large and positive. Grossberg (1973) found that this non-linear gain characteristic solves the noise saturation dilemma that he posed; that is, how can the same network handle both small and large signals, small input signals require high gain through the network if they are to produce usable output; however, a large number of cascaded high gain stages can saturate the output with the amplified noise (random variation) that is present in any realizable network. Also, large input signals will saturate high gain stages, again eliminating any usable output. The central high gain region of the logistic function solves the problem of processing small signals, while its regions of decreasing gain at positive and negative extremes are appropriate for large excitations. In this way, a neuron performs with appropriate gain over a wide range of input levels.

Although a single neuron can perform certain simple pattern detection functions, the power of neural computation comes from connecting neurons into networks. The simplest network is a group of neurons arranged in a layer as shown in Figure 6.11. The set of inputs \( X \) has each of its elements connected to each artificial neuron through a separate weight.
Training the Network

The objective of training the network is to adjust the weights so that application of a set of inputs produces the desired set of outputs. These input-output sets can be referred to as vectors. Training assumes that each input vector is paired with a target vector representing the desired output. Before starting the training process, all of the weights must be initialized to small random numbers. This ensures that the network is not saturated by large values of the weights and prevents certain other training pathologies. Training the back propagation network requires the steps that follow:

1. Select the next training pair from the training set, apply the input vector to the network input.
2. Calculate the output of the network.
3. Calculate the error between the network output and the desired output.
4. Adjust the weights of the network in a way that minimizes the error.
5. Repeat steps 1 through 4 for each vector in the training set until the error for the entire set is acceptably low.

6.4.3 Results and Discussion

The above method is applied in the present case for suppressing the undesired sidelobes at the output of the matched filter. Three possible schemes have been proposed and studied; they are as follows.

Scheme 1: The neural network is used after the matched filter for further suppressing the response sidelobes. Neural network can suppress the sidelobe to any desired value (ideally zero) but as the sequence length increases, number of neurons and weights also increases making it practically difficult to realise the scheme for longer codes. For example in the case of Barker sequence of length 13, the matched filter output (which is also the auto-correlation function of the code) is of length 25. The number of weights of
the network becomes $25 \times 25 = 625$ which is quite large and obviously have practical limitations for implementing the scheme using appropriate hardware. As the sequence length increases, number of weights also increases accordingly.

**Scheme II**: In this scheme the matched filter is replaced by a single neural network. Input is the received code sequence and output is the auto-correlation function of the received sequence the desired output is ideal i.e. $[0 \ 0 \ 0 \ 0 \ ... \ 0 \ 1 \ 0 \ 0 \ ... \ 0 \ 0]$ (if the input sequence is normalized). This scheme is used with Barker Code of length 5. The results are given in Table 6.4(a). Table (a) shows the sidelobe energy (SLE) and Peak sidelobe levels with number of iterations for various value of $n$ (the learning rate of ANN). With the increase in $n$ the number of iterations are decreasing, while SLE and PSL almost remains constant. Table 6.4(b) shows the results when the input code sequence is of length 5 will all element values given as +1. Table 6.4(c) shows the number of iterations with respect of $\beta$ (Degree of non-linearity used). As $\beta$ increases number of iterations decreases, sidelobe energy and peak sidelobe level also decreases. This table also shows the variation of summation of the square of weights corresponding to the nth output, which also decreases as $\beta$ increases. Table-6.5 gives these results when the input sequence is of length 13 (all element values +1). Table (a) shows the variation w.r.t. $\eta$ and Table (b) shows the variation w.r.t $\beta$. Fig. 6.12 shows the variation of SLE, PSL, and number of iteration corresponding to various values of $\eta$ & $\beta$. From Tables and Figures it is clear that the process is converging. However, as the sequence length increases number of weights of the neural network also increases. When the sequence length is quite large say 100 then number of weights required are $100 \times 199 = 19900$ which may be quite large putting serious limitations for practically implementing this scheme. Therefore, in order to reduce the number of weights on alternate scheme is being suggested and is being given as follows.
TABLE 6.4

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<th></th>
<th>S.No.</th>
<th>No. of iterations</th>
<th>SLE (Nor)</th>
<th>PSL (Nor)</th>
<th>E w_{ni}</th>
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<td>( E_{w1} )</td>
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<td>PSL (Nor)</td>
<td>$E_{w_{ni}}$</td>
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<td>0.046</td>
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<td>6.28</td>
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</table>

SLE = Side Lobe Energy = Total output Energy - Main Lobe Energy

PSL = Peak Side Lobe Level

$E_{w_{ni}}$ = Summation of the Squares of Weights Corresponding to the nth output

Tolerance = Actual output - Desired Output

n = Learning rate of the Artificial Neural Network

$\beta$ = Degree of non-linearity used.
Scheme III: In this scheme the output of the neural network is kept as 5 in all the cases, independent of the length of the input code (instead of the ACF of the code as given in Scheme II). This reduces the number of weights considerably. Table 6.6 gives the variation of SLE, PSL, $\sum w^2_{ni}$ and number of iterations for various values of $n$ for a code of length 13 (all element values +1). Comparing these results with the results obtained in Scheme II (Table 6.5(a)) it is clear that the results are almost similar except the case of SLE which is much smaller in this case which is quite obvious as the length of the output is reduced from 25 to 5 only. This shows that the results are better in this case, and this achievement is obtained using much less number of weights (13X5) in this case as compared to (13x25) used in Scheme II. Tolerance in all the cases is kept at 0.1.

6.5 SUMMARY

If processor complexity is not of overriding concern, the use of a MMF may be justified in those cases where improvements in clutter performance are of significant magnitude. In general, there exists a trade-off between resolution and detection performance which sets practical limits on the sidelobe suppression that can be obtained. However, the degradation in SNR is usually small if the input waveform is optimized for matched conditions. Moreover, as printed out by Rihaczek, [35] the approach of optimum waveform design for an MF receiver also implicitly solves the problem of waveform design for the optimum filter in the presence of clutter. Therefore, whenever possible it is preferable to transmit a wider spectrum to achieve the desired resolution rather than widening the spectrum of the receiver filter.
CHAPTER 7

COMBINED RANGE AND RANGE RATE RESOLUTION
7.1 INTRODUCTION

The principal aim throughout this work has been to design waveforms for a radar environment where the relative Doppler spread of the target is negligible. However, when there is significant Doppler shift, the reflections from a target are no longer replicas of the transmitted waveform and the matched filter (MF) response in time and frequency has to be considered.

The following discussions of combined range and range rate (Doppler) resolution of a signal is included for completeness. Detailed analysis of resolution in range and range rate of general types of waveforms can be found in many contemporary books, [22], [23]. Moreover, consideration will only be given to properties pertinent to the types of waveforms described in previous chapters.

Although, in principle, the various methods developed to design pulse trains having good range resolution can be extended to the more general case of range and Doppler resolution, the computational efforts involved for longer sequences is quite formidable even for modern computers. For this reason waveform synthesis for range and velocity resolution is commonly done by trial and judicious use of available information (e.g. ambiguity function).

It has been shown (Chapter 2) that the range resolution property of a signal depends on the shape of its spectrum envelope. Based on the time frequency duality it can be argued, therefore, that resolution in range rate depends only on the envelope of the signal in the time domain. Consequently, combined range and velocity resolution depends on the complete waveform structure in time and frequency. Hence signals with good resolution in one parameter may perform very poorly when combined resolution in both parameters is required.

For combined resolution in range and velocity the waveform must be investigated in terms of the complete MF response in delay and Doppler. This generalized response is given by Woodward's Ambiguity Function (ABF) which has already been introduced in Chapter 2 and is repeated here for convenience.
It is noted that in the literature, the terms \( x(\tau, v) \), \( |x(\tau, v)| \) and \( |x(\tau, v)|^2 \) are often used synonymously for the ABF.

The ABF plays a central part in the analysis of combined resolution. This is so because the width of the main response peak of the ABF serves as a measure for close-target visibility in range-Doppler, while the low-level response and subsidiary spikes give an indication of the self clutter and target masking problem by mutual interference. Since the volume of the ABF over the entire \((\tau, v)\)-plane is constant, the signal design problem for combined range and velocity resolution may therefore be regarded as shifting the unavoidable ambiguity (volume) to those parts of the \((\tau, v)\)-plane where it causes least interference for a given environment and application. Some of the general resolution properties of the various types of pulse trains considered previously are discussed subsequently using the ABF description.

### 7.2 AMBIGUITY FUNCTION OF PULSE TRAINS

One derivative of the ambiguity function involves computing the signal output of the receiver filter. The ambiguity function is the complex modulation of this output signal. The transmitted signal is

\[
S_\tau(t) = a_\tau \nu(t)e^{j2\pi f_0 t} \tag{7.2}
\]

where \( a_\tau \) = amplitude factor

\( \nu(t) \) = complex transmitted modulation

\( f_0 \) = carrier frequency
Depending on the radar cross section, range, and Doppler velocity of the target, the return signal differs from the transmitted signal by

(i) having a different amplitude \( a_r \) determined by the radar range equation.

(ii) being delayed by \( \tau \) and

(iii) being Doppler shifted by \( \nu \)

\[
S_r(t) = a_r \nu(t - \tau)e^{j2\pi(\nu X(t) - \nu r)} \tag{7.3}
\]

In the general case the receiver filter is matched to some signal \( S_m(t) \) with modulation \( \nu(t) \), and thus has the impulse response

\[
h_m(t) = KS^*(T_0 - t)
\]

\[
= K\alpha_m^* (T_0 - t - \tau_m)e^{-j2\pi(f_0 - \nu_m)X(T_0 - t - \tau_m)} \tag{7.4}
\]

where, \( \tau_m \) and \( \nu_m \) are the delay and the Doppler shifts to which the filter is matched. If \( \nu(t) = \nu(t) \), then the filter is matched filter. If \( \nu(t) \neq \nu(t) \), the filter is called mismatched filter. The signal at the filter output in case of a matched filter is given by

\[
Z(t) = \frac{K\alpha_m a_m^* e^{-j2\pi(f_0 - \nu_m)X(T_0 - t)}}{2} \left[ \int_{-\infty}^{\infty} \nu(t) \nu^*(t + \tau)e^{-j2\nu t} dt \right] \tag{7.5}
\]

The quantity within brackets is known as ambiguity function

\[
x(\tau, \nu) = \int_{-\infty}^{\infty} \nu(t) \nu^*(t + \tau)e^{-j2\nu t} dt \tag{7.6}
\]

A discrete function, the same is given by equation 7.1

Pulse trains whose ACF decreases at a faster rate for small Doppler shifts are the non-linear FM type approximation. It can be seen from figures 7.1 to 7.3 and the ABF's of these waveforms basically exhibits the ridge line structure which suggests a relatively strong range Doppler coupling. However, the linear FM property is more and more eliminated as the order of the spectrum tapering increases.
For certain applications the inability to resolve targets in range and velocity along the ridge might be unacceptable. In these circumstances a signal whose ABF approaches that of a single strong spike (thumbtack) as shown in Figs. 7.4 and 7.8 might be adequate. However, the expense of implementing many Doppler channels may be prohibitive.

The choice of a thumbtack ABF may be justified for high close target resolution in the absence of any prior information of the target environment. The close-target resolvability is, however, achieved at the expense of introducing self clutter. Therefore, if the target space is confined to a narrow region, there is no reason to spread the volume of the ABF uniformly over the (τ, v) plane. Moreover, if visibility of small targets is of overriding importance, it is preferable to choose an ABF whose volume is concentrated in strong spikes or a narrow ridge. Such an ABF trades uniform poor visibility for weak targets (thumbtack ABF) against good visibility for most targets and extremely poor visibility for some targets. Fig. 7.7 illustrates the relatively large increase in sidelobe levels off the delay axes for an optimum binary sequence of length 101. It can clearly be seen that noise-like waveforms are inherently suited to approximate thumbtack ABF's. In general, however, binary sequences are usually better suited to improving range resolution rather than velocity resolution.

In summary, all of the waveforms discussed in previous chapters are optimum for some particular clutter environment. The different pulse trains yield a wide variety of ABF shapes. The contours may be of the diagonal ridge structure as for linear FM by signals, or may consists of a single strong spike surrounded by a low level pedestal for noise like waveforms, in various combination, of these basic structures. The waveforms have different tolerances to Doppler shifts. This may be exploited for hardware savings if ambiguity in the range-Doppler coverage can be tolerated.

Another advantage of discrete coding which has not been mentioned is the flexibility of eliminating the range Doppler ambiguity of codes, for example, by simply scrambling the order of the sub-pulses. Moreover, the variations possible with discrete coded pulse trains are virtually unlimited in that the phase, amplitude, frequency and time of transmission of each sub-pulse can be varied. The resulting multi-function capability and adaptability to a particular target environment is clearly one of the most attractive features discrete coding.
Fig. 7. (a) ABF of 128-element non-linear FM type pulse train for $n = 1$.
(b) Peak response as a function of doppler shift $v$. 
Fig. 72  (a) ADF of non-linear FM type pulse train for $n = 2$
(b) Peak response as function of doppler shift $v$. 

- $n = 2$
- $c = 0$
- $WT = 1$
Fig. 7.3  
(a) ADF of 128-element non-linear FM type pulse train for $n = 3$
(b) Peak response as a function of doppler shift $v$. 
Fig. 7.4  (a) ABF of optimum 128-element binary sequence when the initial sequence is chosen randomly.

(b) Peak response as a function of doppler shift $v$. 
Fig. 7.5 (a) ABF of 100-element Huffman code derived from a random zero pattern.

(b) Peak response as a function of doppler shift $v$. 

\[ |x(\tau, v)| \]

\[ 1 \]

\[ 0 \]

\[ -1/2T \quad 0 \quad 1/2T \]
Fig. 7.2  (a) ABR of optimum 128-element binary sequence when the initial sequence is chosen randomly.
(b) Peak response as a function of doppler shift $v$. 
Figure 7.7: Plot of autoambiguity diagram for 31 element 31\(\times\)31 code.
Figure 7.3: Plot of autoambiguity diagram for 53 element binary code.
Figure 7a: Plot of autoambiguity diagram for 91 element binary code.
Figure 7.6: Plot of autoambiguity diagram for 101 element binary code.
CHAPTER 8

DISCUSSION AND CONCLUSION
This thesis has reported investigations into discrete coding techniques for improving range resolution and clutter performance of radar systems. The waveform considered in this work besides representing an interesting mathematical area, are also of practical significance in related fields such as sonar, navigation, and digital communications. The study is focused on the properties of the coded waveforms as modulating functions of a carrier signal. The design of hardware structures of radar processors has not been considered, since there are generally a number of ways available to implement near optimum receivers for a given waveform. Cost, complexity and reliability are usually the bounds set on processor design rather than physical realizability. These problems would have to be considered for each application individually.

The waveform design approach using discrete coding offers a degree of flexibility and has many advantages in terms of waveform shaping and processor implementation. The variations possible with such waveform is virtually unlimited in that the phase, amplitude, frequency and time of transmission of each sub-pulse can be varied. This is clearly in contrast to analogue waveforms which depend on one or possibly two parameters. The inherent multifunction capability of discrete coded waveforms is compatible with the requirements of modern phased array radars. In addition amplitude and phase modulated pulse trains are particularly well suited to digital implementation.

Throughout this work digital processing has been assumed. The application of digital processing techniques to radar becomes more practical as compactness, cheapness and operational speed of digital micro circuits continue to increase. Although modern optical processing techniques sometimes provide an attractive alternative, the use of digital method with its inherent flexibility and reliability offers many advantages. To mention but a few, it simplifies pulse compression and real time multi-dimensional analysis of input data in range, Doppler, bearing, etc. Furthermore, it also offers considerable
advantages in post detection and display processing. In addition the use of digital processor will in many cases reduce future system modifications to easy and inexpensive software changes, rather than requiring costly hardware replacements.

An attempt has been made in this work to present results of various design objectives. The assumption of a matched filter receiver underlying most of the work, is not serious limitations on the applicability of the results, since in practice very little prior information about the target environment is usually available. Therefore, the investigations were concentrated on the auto-correlation function properties of the various types of pulse trains considered. The principal aim has been to design pulse trains whose auto-correlation side lobes are as low as possible. The design methods presented in this thesis have shown that phase coded pulse trains can improve the range, resolution and clutter ejection performance of a radar system. Self clutter interference introduced by inadequacies in the matched filter response impose rather fundamental limitations on weak target visibility. Moreover, the resolution problems caused by self clutter and undesired objects are in principle no different in that both impede resolution in the same manner. If processor complexity is not of utmost concern, the use of amplitude and phase modulated pulse trains may offer an improved performance. Alternatively, self clutter suppression may be achieved by side lobe reduction filters or methods proposed in this thesis. Although mismatching the receiver filter may be useful means of adopting the waveforms to a particular target environment, it should not be used as a primary method to improve resolution. Hence, the proper approach to clutter suppression, except in few special cases, is via waveform design and matched filtering.

In situations where the relative Doppler spread of the targets can not be neglected, the matched filter response has to be investigated in terms of the ambiguity function. In principle, the methods developed could be applied to the more general problem of designing pulse trains suitable for resolving targets in range and range rate. However, even with modern computers, the search for phase codes having good resolution properties in both range and range rate is so expensive that it would have to be
restricted to relatively short sequence and small areas of the range Doppler plan. For these reasons waveform synthesis for range and range rate resolution usually consists of a trial and error procedure and a judicious use of available information.

In summary, the signal problem has in general defied solution by all means other than exhaustion. In particular, no concise set of necessary and sufficient conditions has been formulated by which signals with specified properties can be synthesized. Signal theory, the basis for many technical advances is far from being complete and its further development is, therefore, of fundamental importance. As a final remark the design methods developed in this thesis are of general interest. With appropriate modifications they can be applied to a variety of signal and filtering problems.
APPENDIX-A

RECURSIVE SOLUTION OF SIMULTANEOUS EQUATIONS INVOLVING AN AUTOCORRELATION MATRIX
It is possible to take advantage of the special form of an auto-correlation matrix in order to reduce the computational work figured for the solution of a set of simultaneous equations involving such a matrix. More specifically, an auto-correlation matrix, say

\[
R_n = \begin{bmatrix}
    r_0 & r_1 & \cdots & r_n \\
    r_1 & r_0 & \cdots & r_{n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_n & r_{n-1} & \cdots & r_0
\end{bmatrix}
\]

has the property that all the elements on any given diagonal are the same. For example, \( r_0 \) is the element which appears on the main diagonal; \( r_1 \) is the element which appears on the first super-diagonal as well as on the first sub-diagonal; etc. Hence the entire \((N+1) \times (N+1)\) auto-correlation matrix \( R_n \) does not involve \((n+1)^2\) distinct elements, but only \(n+1\) distinct elements, namely \( r_0, r_1, r_2, \ldots, r_n \). Let us now see how we can take advantage of this structure.

Suppose that we wish to solve the set of simultaneous equations given by

\[
\begin{align*}
    f_0 r_0 + f_1 r_1 + \ldots + f_m r_m &= g_0 \\
    f_0 r_1 + f_1 r_0 + \ldots + f_m r_{m-1} &= g_1 \\
    \vdots & \vdots \\
    f_0 r_m + f_1 r_{m-1} + \ldots + f_m r_0 &= g_m
\end{align*}
\]

(1)

Where the auto-correlation coefficients \( r_0, r_1, \ldots, r_m \) as the right hand side coefficients \( g_0, g_1, \ldots, g_m \) represent the known quantities, and the filter coefficients \( f_0, f_1, \ldots, f_m \) represent the unknown quantities. The structure of the auto-correlation matrix makes possible the following recursive method for the solution of these equations.

The recursive procedure solves the equations in a step-wise manner. The step \( n = 0 \) is given as an initial condition, and then the steps \( n = 1, n = 2, \ldots, n = m \) are done successively in a recursive manner. The desired filter coefficients are the ones which
result on the completion of the final step, namely step \( n = m \). A description of one of
these steps provides the necessary information to enable one to program the method for
digital computer. Let us suppose therefore that we have completed step \( n \), and we wish to
do step \( n + 1 \).

The completion of step \( k = n \) requires that we have computed and have retained in
machine storage the numerical values of the following quantities:

\[
\alpha_n, \alpha_{n+1}, \ldots, \alpha_m, \beta_n
\]

and

\[
f_n, f_{n+1}, \ldots, f_m, \gamma_n
\]

By definition, these quantities satisfy the matrix equations:

\[
\begin{align*}
(a_{n0}, a_{n1}, \ldots, a_{nm}, 0) R_{n+1} &= (\alpha_n, 0, \ldots, 0, \beta_n) \\
(f_n, f_{n+1}, \ldots, f_m, 0) R_{n+1} &= (g_0, g_1, \ldots, g_n, \gamma_n)
\end{align*}
\]

Here the parentheses enclose \( 1 \times (n+2) \) row vectors and where \( R_{n+1} \) is the \((n+2) \times (n+2)\)
auto-correlation matrix.

Because of the special structure of the auto-correlation matrix, equation (2)
may be manipulated into the equivalent from given by

\[
(0, a_{nm}, \ldots, a_{n1}, a_{n0}) R_{n+1} = (\beta_m, 0, \ldots, 0, \alpha_n)
\]

Let us now multiply equation (4) by a constant \( k_n \), as yet undetermined and
then add the result to equation (2). We obtain

\[
(\alpha_n + k_n a_m, 0, \ldots, 0, \beta_n + k_n \gamma_n)
\]
We want equation (5) to be identical to the equation

\[(a_{n+1}, 0, a_{n+1,1}, \ldots, a_{n+1,n}) R_{n+1}\]

\[= (a_{n+1}, 0, \ldots, 0, 0) \quad \text{.......................................................... (6)}\]

In order for equations (5) and (6) to be identical we must first of all require that

\[\beta_n + k_n \alpha_n = 0 \quad .\]

This equation allows us to determine the constant \(k_n\), that is, we compute \(k_n\) by the formula

\[k_n = -\frac{B_n}{a_n} \quad \text{.......................................................... (7)}\]

The identity of equations (5) and (6), together with our knowledge of \(k_n\), then allows us to compute the quantities

\[a_{n+1,0} = a_n0\]

\[a_{n+1,1} = a_{n,1} + k_n a_n \quad \text{.......................................................... (8)}\]

\[\ldots\]

\[a_{n+1,n} = a_{n,n} + k_n a_n1\]

\[a_{n+1,n+1} = k_n a_n0\]

as well as

\[\alpha_{n+1} = \alpha_n + k_n \beta_n \quad \text{.......................................................... (9)}\]

If we define the polynomials \(A_n(z)\) and \(A_{n+1}(z)\) as

\[A_n(z) = a_{n0} + a_{n1}z + \ldots + a_{nn}z^n\]

and

\[A_{n+1}(z) = a_{n+1,0} + a_{n+1,1}z + \ldots + a_{n+1,n}z^n + a_{n+1,n+1}z^{n+1}\]

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then we see that the equations appearing in (8) may be compassed within the confines of one equation, namely

\[ A_{n+1}(Z) = A_n(Z) + k_n Z(a_m + \ldots + a_{n-1}Z^{n-1} + a_n Z^n) \]

which is

\[ a_{n+1}(Z) = A_n(Z) + k_n Z^{n+1} A_n(1/Z) \]

This equation demonstrates the recursion from the known polynomial \( A_n(Z) \) to the unknown polynomial \( A_{n+1}(Z) \).

Next, if we compute

\[ \beta_{n+1} = a_{n+1,0} r_{n+2} + a_{n+1,1} r_{n+1} + \ldots + a_{n+1,n+1} r_1 \]

then we will have computed all the quantities in the equation

\[ (a_{n+1,0}, a_{n+1,1}, \ldots, a_{n+1,n+1}, 0) R_{n+2} = (0, 0, \ldots, 0, \beta_{n+1}) \]

where the parentheses enclose \( (n+3) \times (n+3) \) row vectors and where \( R_{n+2} \) is the \( (n+3) \times (n+3) \) auto-correlation matrix. Equations (2) and (12) are counterparts; equation (2) pertains to \( \gamma_n \) whereas equation (12) pertains to step \( n+1 \).

Because of the structure of the auto-correlation matrix, equation (6) is the same as the equation

\[ (a_{n+1,n+1}, a_{n+1,n}, \ldots, a_{n+1,1}, a_{n+1,0}) R_{n+1} = (0, 0, \ldots, 0, \alpha_{n+1}) \]

Let us multiply equation (13) by a constant \( q_n \), as yet undetermined, and then add the result to equation (3). We obtain

\[ (f_0 + q_n a_{n+1,n+1}, f_1 + q_n a_{n+1,n}, \ldots, f_{n+1} + q_n a_{n+1,1}, q_n a_{n+1,0}) R_{n+1} = (g_0, g_1, \ldots, g_n, \gamma_n + q_n a_{n+1}) \]

We want equation (14) to be identical to the equation

\[ (f_{n+1,0}, f_{n+1,1}, \ldots, f_{n+1,n+1}) R_{n+1} = (g_0, g_1, \ldots, g_n, q_{n+1}) \]
In order for equations (14) and (15) to be identical we must first of all require that

\[ \gamma_n + q_n \alpha_{n+1} = g_{n+1} \]

This equation allows us to determine the constant \( q_n \); that is, we compute \( q_n \) by

the formula

\[ q_n = \left( g_{n+1} - \gamma_n \right) / \alpha_{n+1} \]

(16)

The identity of equations (14) and (15), together with our knowledge of \( q_n \), allows us
to compute the quantities

\[ f_{n+1,0} = f_{n0} + q_n \alpha_{n+1,0} \]

\[ f_{n+1,1} = f_{n1} + q_n \alpha_{n+1,1} \]

\[ f_{n+1,n+1} = q_n \alpha_{n+1,0} \]

(17)

If we define the polynomials \( F_n(z) \) and \( F_{n+1}(z) \) as

\[ F_n(z) = f_{n0} + f_{n1}z + \ldots + f_{nn}z^n \]

and

\[ F_{n+1}(z) = f_{n+1,0} + f_{n+1,1}z + \ldots + f_{n+1,n}z^n + f_{n+1,n+1}z^{n+1} \]

then the equations appearing in (17) may be encompassed within the confines of one
equation, namely

\[ F_{n+1}(z) = F_n(z) + q_n z \alpha_{n+1} (1/z) \]

(18)

This equation demonstrates the recursion from the known polynomials \( F_n(z) \)
and \( \alpha_{n+1}(z) \) to the unknown polynomial \( F_{n+1}(z) \).

Next if we compute

\[ \gamma_{n+1} = f_{n+1,0} r_{n+2} + f_{n+1,1} r_{n+1} + \ldots + f_{n+1,n+1} r_1 \]

(19)
Then we will have computed all the quantities in the equation

\[(f_{n+1,0}, f_{n+1,1}, \ldots, f_{n+1,n+1}, 0) \quad R_{n+2} = (g_0, g_1, \ldots, g_{n+1}, \gamma_{n+1})\]  

(20)

Where the parentheses enclose \(1\times(n+3)\) row vectors and where \(R_{n+2}\) is the \((n+3)\times(n+3)\) auto-correlation matrix. Equations (3) and (20) are counterparts; equation (3) pertains to step \(n\) and equation (20) pertains to step \(n+1\). We have thus completed our task; we have done step \(n+1\) given that we had step \(n\).

Let us now summarise the computations required. We are given the quantities.

\[r_0, r_1, \ldots, r_m\]
\[g_0, g_1, \ldots, g_m\]

the completion of step \(n\) we have the quantities

\[a_{n0}, a_{n1}, \ldots, a_{mn}, \alpha_n, \beta_n\]

\[f_{n0}, f_{n1}, \ldots, f_{nn}, \gamma_n\]

machine storage. To obtain step \(n+1\) we do the following computations. We compute \(k_n\) by means of equation (7). Then we compute

\[a_{n+1,0}, a_{n+1,1}, \ldots, a_{n+1,n}, a_{n+1,n+1}, \alpha_{n+1}, \beta_{n+1}\]

means of equations (8), (9) and (11). We compute \(q_n\) by means of equation (16). Then compute

\[f_{n+1,0}, f_{n+1,1}, \ldots, f_{n+1,n}, f_{n+1,n+1}, \gamma_{n+1}\]

means of equations (17) and (19). This completes step \(n+1\).

To obtain the final result, namely the solution of the simultaneous equations we must start at step \(n = 0\), that is start with initial quantities which we take as
Then we do the recursions for $n = 1$ to $n = m$; the final values obtained for the filter coefficients, namely

$$f_{m,0}, f_{m,1}, \ldots, f_{m,m}$$

represent the solution $f_0, f_1, \ldots, f_m$ of the normal equations (1). Instead of having the value of $m$ fixed in advance, it is also possible to monitor the recursions according to the value of the error associated with the filter at each step of the recursions and the recursions can be stopped when this error reaches a certain limit.

The machine time required to solve the simultaneous equations (1) for a filter with $m$ coefficients is proportional to $m^2$ for the recursive method, as compared to $m^3$ for conventional methods of solving simultaneous equations. Another advantage of using this recursive method is that it only required computer storage space proportional to $m$, rather than $m^2$ as in the case of the conventional methods.

This recursive method can also be extended to the matrix valued case in order to compute the coefficients of multi-channel filters. Also there is a sideways recursion which corresponds to shifting the time index of the right-hand side of the simultaneous equations. This side ways recursion is valuable in applications for it represents a relatively index-pensive way of determining the filter that incorporates the optimum time-delay between the input signal and the desired output signal. For example, the sideways recursion can be used to determine the optimum spike position for a spiking filter and the optimum positioning of the desired output for a shaping filter.
APPENDIX-B

LIST OF SEQUENCES
### B.1 Binary Sequences \((+ = 1, - = -1)\)

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### B-3 Multilevel Sequences

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