PLANE BOUNDARY VALUE PROBLEMS
IN
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CHAPTER I
CHAPTER I

INTRODUCTION

1.1 BRIEF HISTORY

In nature particularly all substances are elastic to a greater or a smaller extent. No substance is ideally rigid. But its systematic analysis and study of various aspects were made in the nineteenth century by various workers like Navier, Cauchy, Poisson etc. G.Lame' formulated the theory of elasticity in displacements, while Beltrami formulated the same in stress. Although there are two fundamental problems in elasticity and they are quite different from mechanical standpoint, their mathematical formulation are almost same.

The problem of the theory of elasticity was formulated with the help of the potential function by Airy. Since the cases of plane stress or plane strain brings much simplification of calculations as compared to the general problems, Airy's initial results became very helpful and were followed by numerous workers in a great variety of problems. The two-dimensional problems are no less important than the general ones, as they cover a large variety of cases.
(i) The case of plane state of stress corresponds to a plane plate of constant thickness whose parallel faces are free from loads and the boundary is acted upon by forces parallel to the middle plane and uniformly distributed along the thickness of the plate. It covers the cases of deep beams without parallel boundaries, frame corners, junction plates for metallic structures, many machine parts, stiffening diaphragms for mean and tall buildings in seismic areas etc.

(ii) The case of plane state of strain corresponds to a long cylindrical body, supposed to be infinite, and acted upon on the lateral surface by a uniformly distributed load along the generatrix, without tangential component in the direction of the generatrix. Cases of heavy dams, supporting walls, tubes, tunnels, factory chimneys, large plate etc. come under this category.

Besides this there is the case of quasi-plane state of stress which can be reduced to the classical case of a plane state of stress. In the same way the case of a generalized plane state of strain can be reduced.

1.2 PLANE PROBLEMS

The two-dimensional problems of elasticity are classified in the following groups.
(a) **Plane state of strain**

In the case of plane strain we have

\[ e_{zz} = e_{zx} = e_{yz} = 0 \]  \( \ldots (1.2.1) \)

where \( e_{zz}, e_{zx}, e_{yz} \) are components of strain tensor. The other remaining components being different from zero are independent of the \( z \)-coordinate.

(b) **Plane state of stress**

In the case of plane stress, the stress components \( \sigma_{xz}, \sigma_{yz}, \sigma_{zz} \) all vanish, and the remaining components being different from zero and independent of the \( z \)-coordinate.

\[ \sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0 \]  \( \ldots (1.2.2) \)

(c) **Generalised plane stress**

This is the plane state of stress in a thin plate of thickness \( 2h \) in which \( \sigma_{zz} = 0 \) throughout the plate, but, \( \sigma_{xz} = \sigma_{yz} = 0 \) only on the plane surface of the plate.

\[ \sigma_{zz} = 0 \]
\[ \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at} \ z = \pm h \]  \( \ldots (1.2.3) \)

1.3 **FORMULATION OF PLANE PROBLEMS**

All the above categories of the two-dimensional problems lead to the same mathematical formulation. Stresses are given by the following equations of equilibrium.
\[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + X_1 = 0 \] 
\[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + X_2 = 0 \]  

...[1.3.1]

where \( X_1 \) , \( X_2 \) are the components of the body forces, \( X, Y \) the normal stresses and \( X_Y \) the shearing stress. The relation between the components of the strain and the displacements \( u, v \) are as follows.

\[
\begin{align*}
E_{xx} &= \frac{\partial u}{\partial x} \\
E_{yy} &= \frac{\partial v}{\partial y} \\
E_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{align*}
\] 

...(1.3.2)

The stress-strain relations

\[
\begin{align*}
E_{xx} &= \frac{1}{E} (Xx - \sigma Yy) \\
E_{yy} &= \frac{1}{E} (Yy - \sigma Xx) \\
E_{xy} &= \frac{1}{\mu} Xy
\end{align*}
\] 

...(1.3.3)

are valid for plane stress, where \( E \) is young's modulus, \( \sigma \) the Poisson's ratio; and \( \mu \), the modulus of rigidity, is given by

\[ E = 2\mu(1+\sigma) \] 

...(1.3.4)

For the plane strain the same formulations are valid if the
elastic constants $E$ and $\sigma$ are replaced by $E_o$ and $\sigma_o$ where

$$E_o = \frac{E}{1 - \sigma^2}$$

$$\sigma_o = \frac{\sigma}{1 - \sigma}$$

...(1.3.5)

Airy [2], [3] showed that in absence of body forces equations of equilibrium (1.3.1) are satisfied if we write

$$\ddot{x} = \frac{\partial^2 \chi}{\partial y^2}, \quad \ddot{y} = \frac{\partial^2 \chi}{\partial x^2}$$

$$\dddot{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$$

...(1.3.6)

where $\chi = \chi(x,y)$ is an arbitrary function. But Maxwell [24],[25] showed that the strain continuity equation

$$\nabla^2 (\ddot{x} + \ddot{y}) = 0$$

...(1.3.7)

where $\nabla^2$ is plane Laplace's operator given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

...(1.3.8)

must also be satisfied. Hence the function $\chi(x,y)$, known as Airy's function, must be biharmonic

$$\nabla^4 \chi(x,y) = 0$$

...(1.3.9)
The Airy's stress function has been interpreted in several ways and has a great significance. The tangential derivative \( \frac{\partial \chi}{\partial s} \) is the shear force \( T \) (with the sign changed) and the normal derivative \( \frac{\partial \chi}{\partial n} \) is the axial force \( N \) in an imaginary beam.

The plane problems of elasticity are further subdivided into three groups.

(I) First fundamental problem of the theory of elasticity; In this type of problems the external loads acting on the boundary are given.

(II) Second fundamental problem of elasticity; In this type the displacements on the boundary are given.

(III) Mixed type problem; In this type conditions in stresses on one side of the boundary and in displacements on other side are given.

In all these types of problems, the formulation ultimately leads to the solution of a biharmonic equation under certain given boundary conditions. In the first two types of problems, the Airy's stress function \( \chi(x,y) \) has important mechanical significance, but in the third case it loses to have such significance. S.L. Sobolev [42] studied the boundary value problem for polyharmonic equations.
For the second fundamental problem, K. Marguerre [22] gave the following formulation.

\[ u = - \frac{1+\sigma}{1-\sigma} \left( \frac{\partial^2 F}{\partial x \partial y} \right) \]

\[ v = \frac{2}{1-\sigma} \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) \]

...(1.3.10)

where \( F = F(x,y) \) is a biharmonic function and \( \sigma \) the Poisson's ratio.

1.4 METHODS OF SOLUTION

As the two-dimensional problems cover a great variety of cases, so a great variety of methods is used to solve various problems. Some of the popular and widely used methods are as follows:

(a) **Indirect and direct methods**

As is shown earlier the basic problem is the determination of the stress functions. The indirect method is to assume a certain state of stress within the body which fulfils the boundary conditions and then to check that all equations of elasticity are satisfied. If they are not satisfied then some other stress function has to be chosen and the computation repeated. Contrary to this direct method consists in approaching a problem through a general method
like finite difference method, complex variable method etc. which may lead to systematics calculation. Each of these two methods has its own advantages and disadvantages.

(b) Complex variable method

This method was developed during the middle of the nineteenth century and become very popular. Instead of real potential used by Airy, G.V.Kolosov [18],[19] gave a representation of the complex displacement (in the absence of body forces) in the following form.

$$2\mu(u+iv) = \frac{3-\alpha}{1+\alpha} \frac{\phi(z)-z\phi'(z) - \psi(z)}{z}$$ ... (1.4.1)

where $\phi(z)$ and $\psi(z)$ are analytic function of $z$ ($z=x+iy$). Similarly the stresses were given by

$$\begin{align*}
\widehat{xx} + \overline{yy} &= 4 [\phi'(z)+\overline{\phi'(z)}] \\
\overline{yy} - \widehat{xx} + 2i\widehat{xy} &= 4 [z\phi'(z)+\psi'(z)]
\end{align*}$$ ... (1.4.2)

After imposing the boundary conditions the three fundamental problems of elasticity are reduced to those of complex variable functions. A.E.H.Love [21], Muskhelishvili [31],[32] etc. made remarkable contribution in this field. Apart from Kolosov [19] and Muskhelishvili's [31] (both Russians), Stivenson [44], [45] founded the theory independently. Milne-Thomson [28] gave an exhaustive
details of the two-dimensional problems, and solved many of them in his monograph [28]. The basic tools of the complex variable technique is conformal mapping. Generally the domain in the problem together with load conditions etc. is transformed into a simpler region under conveniently changed conditions which makes the problem much simplified. The solution in the transformed state is obtained, which after the inverse conformal mapping becomes the solution to the given problem.

(c) Reduction to integral equations

Using the method of Cauchy-type integrals, the plane problems can be reduced to integral equations which can be put in the form of Fredholm-type integral equations. This method is useful for simply connected regions which is conformally transformed on a circle. Mikhlin [26], [27] modified this method for multi-connected regions using complex green functions with a logarithmic singularity. Sneddon [37] has solved several potential problems after reducing them into integral equations. Rigid punch problems on an elastic half-plane come under this category and they can be reduced to the integration of integral or integro-differential equations.
(d) **Integral transform technique**

The method of integral transform is also very convenient for solving plane problems of elasticity. I.N. Sneddon and M. Lowengrub [40] have written a monograph on crack problems in the classical theory of elasticity. Integral transforms like Fourier transform, Fourier sine and cosine transforms, Mellin-transform etc. are employed to solve various problems. Ali and Ahmed [6] discussed the problem of a pair of Griffith cracks in an infinite solid. The problem is subsequently reduced to triple integral equations and then it has been solved by "finite Hilbert transform technique". Recently M. Kurashige [20] discussed a two-dimensional crack problem for an initially stressed body. Following his theory Ali [5] discussed the problem of a crack of prescribed shape in an initially stressed body in a framework of large deformations. The problem has been solved with the help of Fourier transform technique. Cases of parabolic crack and elliptic crack have been obtained as special cases. The effect of variation in the applied force has been discussed graphically.

(e) **Numerical methods**

As the problems in nature generally do not have a closed type solution, the only choice left is that of
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CERTIFICATE

Certified that Mr. Gh. Nabi Parrey has carried out the
research work on "PLANE BOUNDARY VALUE PROBLEMS IN SOLID
MECHANICS" under my supervision and the work is suitable for
submission for the award of the degree of Master of
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ACKNOWLEDGEMENTS

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Gh. Nabi Parrey
numerical or approximate method. Recently the finite element method has become very popular in various fields of engineering mathematics. The name "finite element method" has been given by Clough [9], and the method developed very quickly in the second half of the nineteenth century and gradually became very popular. In cases where stresses vary very sharply at different points like aircraft wings, turbine blades etc., this is the most appropriate method. In this method the solution domain is divided into sub-domains which may be of any shape. In each subdomain the interpolation function is selected and the element properties are found out. The element properties are then assembled to obtain the system of equations which are then solved. After the solution the results are post-processed. For various applications of finite element method in applied mathematics, one may refer to "Huebner" [16], Reddy [33]. Recently in 1993 N. Miyazaki et al. [29] analysed the stress-intensity factor of interface crack using this method.

1.5 ABSTRACT OF THE DISSERTATION

The chapter I provides the necessary information to understand the work presented in the subsequent sections. This chapter is divided into four sections. The first section deals with the brief history of the theory of
elasticity. The second and third section consist of definitions of plane problems and their formulations. In the last section, various methods used to solve problems have been discussed.

In chapter II, some main results of complex variable techniques in solving problems of solid mechanics have been studied. The concepts of some basic equations, Airy’s stress function, complex stresses and complex displacement have been introduced and some results are discussed. Various two-dimensional problems of the plate of curvilinear boundary solved by conformal mapping have been considered. Stresses and displacement in terms of complex potentials have been determined. Some problems based on classical method have been discussed.

In chapter III, at first a direct method evolved by Sen [36] has been discussed. Although the method has a shortcoming of choosing some potential function by intuition, its efficiency can not be overlooked. Employing the method many problems have been solved. A typical problem of this type has been considered. The integral transform technique has also been considered. Several problems have been discussed which involve the use of integral transforms. The fundamental problems of cracks and punches in the two-
dimensional bodies have been considered.

In chapter IV, the modern techniques of approximate solution of solid mechanics problems have been discussed. The variational methods of Ritz, Finite difference, Weighted residuals etc. have been considered. Then the most efficient method to solve stress problems i.e. finite element method has also been discussed.

In chapter V, the recently evolved boundary element method has been discussed. Recently, in 1993, Miyazaki et al. [29] have used the method to analyse the stress intensity factor of interface crack using this method. After that in the second part of this chapter, an attempt has been made to solve an original problem of determining the stress intensity factor in an elliptic ring under concentrated forces.

Next the scope of further investigation has been discussed and at the end of the dissertation, a complete list of research papers and books which are referred to in this dissertation has been given.
CHAPTER - II
The complex variable technique has been found to be very helpful in solving problems of solid mechanics. Though it originated quite long back but later on much contributions are made to it by A. E. H. Love [21], I. N. Muskhelishvili [31], L. M. Milne-Thomson [28] etc. In cartesian coordinates the equations of motion are given by

\begin{align}
\frac{\partial \hat{\sigma}_{xx}}{\partial x} + \frac{\partial \hat{\sigma}_{yx}}{\partial y} + \frac{\partial \hat{\sigma}_{zx}}{\partial z} &= x_1 \\
\frac{\partial \hat{\sigma}_{xy}}{\partial x} + \frac{\partial \hat{\sigma}_{yy}}{\partial y} + \frac{\partial \hat{\sigma}_{zy}}{\partial z} &= x_2 \\
\frac{\partial \hat{\sigma}_{xz}}{\partial x} + \frac{\partial \hat{\sigma}_{yz}}{\partial y} + \frac{\partial \hat{\sigma}_{zz}}{\partial z} &= x_3
\end{align}

...(2.1.1)

where \(\hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\sigma}_{zz}\) are normal stresses; \(\hat{\sigma}_{xy}, \hat{\sigma}_{yz}\) etc. are shearing stresses and \(x_1, x_2, x_3\) are components of body forces. It has been found that shearing stresses are symmetric i.e.

\(\hat{\sigma}_{xy} = \hat{\sigma}_{yx}, \hat{\sigma}_{yz} = \hat{\sigma}_{zy}, \hat{\sigma}_{zx} = \hat{\sigma}_{xz}\)

...(2.1.2)

We are considering only plane systems for which there exists a plane such that a stress tensor is the same at all points.
on the normal to this plane as that at the point where the normal cuts the plane. So, we have

\[ zx = zy = zz = 0 \]  \quad \ldots (2.1.3)

We further consider the case when

\[ yz = zx = 0 \]  \quad \ldots (2.1.4)

Hence equations (2.1.1) becomes

\[ \frac{\partial x}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} \frac{\partial y}{\partial y} = x_1 \]
\[ \frac{\partial x}{\partial x} \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} \frac{\partial y}{\partial y} = x_2 \]  \quad \ldots (2.1.5)

G.V. Kolosov [19] formulated the above problem using complex variables by writing

\[ \frac{\partial}{\partial x} \left( \frac{\partial x}{\partial z} + \frac{\partial y}{\partial z} \right) = i \frac{\partial}{\partial z} - i \frac{\partial}{\partial z} \]
\[ \frac{\partial}{\partial y} + \frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \]  \quad \ldots (2.1.6)

using equations (2.1.6) in equations (2.1.5) we have

\[ \frac{\partial}{\partial z} (xx + yy) - \frac{\partial}{\partial z} (yy - xx + 2ixy) = x_1 - iX_2 \]  \quad \ldots (2.1.7)

If we set

\[ \Theta = xx + yy, \quad \hat{\Theta} = yy - xx + 2ixy \]  \quad \ldots (2.1.8)
equation (2.1.7) reduces to

$$\frac{\partial \phi}{\partial z} - \frac{\partial \bar{\phi}}{\partial \bar{z}} = x_1 - i x_2$$  \hspace{1cm} ...(2.1.9)

The functions \( \phi \) and \( \bar{\phi} \) are called "fundamental stress combinations". In terms of \( \phi \) and \( \bar{\phi} \) the stress components are given as

\[
\begin{align*}
\widehat{\sigma}_{xx} &= \frac{1}{2} \phi - \frac{1}{4} (\bar{\phi} + \phi) \\
\widehat{\sigma}_{yy} &= \frac{1}{2} \phi + \frac{1}{4} (\bar{\phi} + \phi) \\
\widehat{\sigma}_{xy} &= -\frac{1}{4} i(\bar{\phi} - \phi)
\end{align*}
\]  \hspace{1cm} ...(2.1.10)

2.2 **AIRY'S STRESS FUNCTION**

Let \( \phi_0, \bar{\phi}_0 \) be a particular solution of (2.1.9) then the appropriate solution can be written as

$$\phi = \phi_0 + 4 \frac{\partial^2 \chi}{\partial z \partial \bar{z}} , \quad \bar{\phi} = \bar{\phi}_0 + 4 \frac{\partial^2 \chi}{\partial \bar{z}^2}$$  \hspace{1cm} ...(2.2.1)

where \( \chi \) is an arbitrary real valued function of \( x \) and \( y \). But as

$$x = \frac{1}{2} (z + \bar{z}) , \quad y = -\frac{1}{2} i (z - \bar{z})$$  \hspace{1cm} ...(2.2.2)

we see that \( \chi \) is a function of \( z \) and \( \bar{z} \) also. The function \( \chi(x,y) \) or \( \chi(z,\bar{z}) \) is known as Airy's stress function. The components of stress tensor are given as
\[
\begin{align*}
\widehat{xx} &= \widehat{xx}_0 + \frac{\partial^2 \chi}{\partial y^2} \\
\widehat{yy} &= \widehat{yy}_0 + \frac{\partial^2 \chi}{\partial x^2} \\
\widehat{xy} &= \widehat{xy}_0 - \frac{\partial^2 \chi}{\partial x \partial y}
\end{align*}
\] ...(2.2.3)

where \( \widehat{xx}_0, \widehat{yy}_0, \widehat{xy}_0 \) are particular solutions. In absence of body forces equations (2.1.5) becomes

\[
\begin{align*}
\frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{xy}}{\partial y} &= 0 \\
\frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} &= 0
\end{align*}
\] ...(2.2.4)

equations (2.2.4) are identically satisfied if we take

\[
\begin{align*}
\widehat{xx} &= \frac{\partial^2 \chi}{\partial y^2} , \quad \widehat{yy} = \frac{\partial^2 \chi}{\partial x^2} \\
\widehat{xy} &= -\frac{\partial^2 \chi}{\partial x \partial y}
\end{align*}
\] ...(2.2.5)

Thus we see that two-dimensional problems are generally reduced to that of determination of fundamental stress combinations or Airy's stress functions.

A body is said to be in a state of plane deformation if

(i) one of the principal directions of deformations is
the same at every point of the material and,

(ii) apart from a rigid body displacement, particles which occupy planes perpendicular to the fixed principal directions prior to the deformation continue to occupy the same planes after the deformation.

Thus if \( u, v, w \) are the displacement components we can write for infinitesimal plane deformation:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \\
\varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} &= 0
\end{align*}
\]  

\( \ldots (2.2.6) \)

In the present dissertation the material of the body concerned is supposed to be isotropic. The boundary-value problems in elasticity are classified 'as mentioned earlier', in three groups i.e. the first fundamental problem, the second fundamental problem and the mixed boundary-value problem. The mixed boundary-value problem can be further subdivided into two groups.

(i) The problem in which at every point of the boundary either the displacement or the stress is given.

(ii) The problem in which at every point of the boundary the normal displacement and the tangential stress are given.
2.3 COMPLEX STRESSES

If the body force $X_1 - iX_2$ can be derived from a scalar potential function $V$ then

$$X_1 - iX_2 = \frac{\delta V}{\delta x} - i \frac{\delta V}{\delta y} = 2 \frac{\delta V}{\delta z} \quad \ldots(2.3.1)$$

Hence the equation (2.1.9) from its particular solution gives

$$\frac{\partial}{\partial z} (\phi - 2V) - \frac{\partial^2 V}{\partial z^2} = 0 \quad \ldots(2.3.2)$$

so that the particular solution can be taken as

$$\phi = 2V, \quad \phi = 0 \quad \ldots(2.3.2')$$

Following Milne-Thomson [28] we get

$$16 \frac{\partial^4 \chi}{\partial z^2 \partial \overline{z}^2} + 4 \frac{1 - 2\sigma}{1 - \sigma} \frac{\partial^2 V}{\partial z \partial \overline{z}} = 0 \quad \ldots(2.3.3)$$

where $\sigma$ is the Poisson's ratio given by (1.3.4). The plane laplace operator $\nabla^2$ is given by

$$\nabla^2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \overline{z}} \quad \ldots(2.3.4)$$

Then equation (2.3.3) becomes

$$\nabla^4 \chi + \frac{1 - 2\sigma}{1 - \sigma} \nabla^2 V = 0 \quad \ldots(2.3.5)$$

For the case of generalised plane stress, equation (2.3.5)
becomes valid after replacing $\sigma$ by $\sigma'$ where

$$(1-\sigma')(1+\sigma') = 1 \quad \ldots (2.3.6)$$

If we write the equation of the generalized plane stress in terms of $\sigma$ using (2.3.6), then equation (2.3.5) gives

$$\nabla^4 \chi + (1-\sigma)\nabla^2 \chi = 0 \quad \ldots (2.3.7)$$

both the equations (2.3.5) and (2.3.7) can be written in the form

$$\nabla^4 \chi + \nu \nabla^2 \chi = 0 \quad \ldots (2.3.8)$$

where

$$\nu = 2(\alpha-1)/(\alpha+1)$$

$$\alpha = 3-4\sigma \text{ for plane deformation}$$

$$\alpha = (3-\sigma)/(1+\sigma) \text{ for generalized plane stress} \quad \ldots (2.3.9)$$

Let us introduce a real-valued function $Q$ defined by

$$\nabla^2 Q = V, \text{ or } 4 \left( \frac{\partial^2 Q}{\partial z \partial \bar{z}} \right) = V \quad \ldots (2.3.10)$$

Hence equation (2.3.8) becomes

$$\nabla^4 (\chi + \nu Q) = 0 \quad \ldots (2.3.11)$$

Thus we see that the function $\chi + \nu Q$ is a plane biharmonic function. Functions $\chi$ and $Q$ being real-valued, equation (2.3.11) can be written in the form

$$\nabla^4 (\chi + \nu Q) = 4 \left( \frac{\partial^2 (\chi + \nu Q)}{\partial z \partial \bar{z}} \right) = W(z) + \bar{W}(\bar{z}) \quad \ldots (2.3.12)$$
Provide W(z) is differentiable twice. Integrating (2.3.12) with respect to $z$ we get

$$4 \frac{\partial (\chi + \nu Q)}{\partial z} = z W(z) + \int W(z) dz + w(z) \quad ...(2.3.13)$$

where $w(z)$ is the constant of integration. Integrating equation (2.3.13) with respect to $z$ we get

$$4(\chi + \nu Q) = z \int W(z) dz + z \int W(z) dz + \int w(z) dz + \int \bar{w}(z) dz \quad ...(2.3.14)$$

the last constant of integration is taken in the above form, so that both sides of the equation (2.3.14) becomes real-valued. In absence of body forces we have $Q = 0$ and so (2.3.14) becomes

$$4\chi = z \int W(z) dz + z \int \bar{W}(z) dz + \int w(z) dz + \int \bar{w}(z) dz \quad ...(2.3.15)$$

Hence the Airy's stress function $\chi$ is expressed in terms of $W(z)$ and $w(z)$ which are known as "complex stresses". The stress components in terms of complex stresses are given by (2.1.8) using (2.2.1), (2.3.2') and (2.3.15):

$$\begin{align*}
\bar{x}x + yy &= \Theta = W(z) + \bar{W}(z) + (8 - 4\nu) \frac{\partial^2 Q}{\partial z \partial \bar{z}} \\
\bar{y}y - xx + 2ixy &= \bar{z} = zW'(z) + w(z) - 4\nu \frac{\partial^2 Q}{\partial z^2}
\end{align*} \quad ...(2.3.16)$$

In absence of body forces $Q = 0$, so we get
\[
\begin{align*}
\dddot{xx} + \dddot{yy} = & \ 0 = W(z) + \bar{W}(\bar{z}) \\
\dddot{yy} - \dddot{xx} + 2i \dddot{xy} = & \ \bar{\dddot{z}} W'(z) + W(z)
\end{align*}
\]

Thus the solution of two-dimensional problems depends upon the determination of the complex stresses \( W(z) \) and \( \bar{W}(z) \) whose physical dimensions are that of stress.

2.4 **COMPLEX DISPLACEMENT**

In the isotropic case we have from equation (2.2.1)

\[
4 \frac{\partial^2 \chi}{\partial z^2} = \dddot{\phi} = 2\mu (e_{yy} - e_{xx} + 2ie_{xy}) \quad \ldots (2.4.1)
\]

Expressing the components of strains in terms of displacement components \( u \) and \( v \) by (2.2.6) we get from (2.4.1)

\[
4 \frac{\partial^2 \chi}{\partial z^2} = -4\mu \frac{\partial}{\partial z} (u - iv) \quad \ldots (2.4.2)
\]

Hence, taking the complex conjugate we get

\[
4\mu \frac{\partial}{\partial z} (u+iv) = -4 \frac{\partial^2 \chi}{\partial z^2} \quad \ldots (2.4.3)
\]

Following Milne-Thomson [28] the complex displacement, in absence of body forces is

\[
4\mu (u+iv) = \alpha \int W(z) dz - \bar{z} \bar{W}(\bar{z}) - \int \bar{w}(\bar{z}) d\bar{z} \quad \ldots (2.4.4)
\]
where $\alpha$ is given by equation (2.3.9)

2.5 CONFORMAL MAPPING

Various two-dimensional problems of thin plates of a curvilinear boundary, or thin plates having holes can be solved by mapping the boundary of the disc or the hole conformally to some simpler form, generally a circle. After solving the problem for the circle, the original solution can be obtained by inverse mapping.

Let $f(t)$ be a holomorphic function of the complex variable $t$, so that

$$z = f(t), \ t = \xi + i\eta \quad \ldots(2.5.1)$$

where $z$ is the complex variable $x+iy$ and $(\xi, \eta)$ curvilinear coordinates. Any particular value of $\xi$, i.e. $\xi = \xi_0$ gives a plane curve in the $z$-plane. Similarly $\eta = \eta_0$ is another plane curve in the $z$-plane and it can be shown that the curves $\xi = \xi_0$ and $\eta = \eta_0$ cut orthogonally in the $z$-plane. Conventionally it is supposed that a curve $\xi = \xi_0$ is described in the sense in which $\eta$ increases and the curve $\eta = \eta_0$ is described in the sense in which $\xi$ increases [see fig.(1)].

Let the given curvilinear boundary $C$ be mapped on the circumference of the circle $\Gamma, |\xi| = a$, by the holomorphic mapping function
Fig. 1 Stresses in curvilinear coordinates.
\[ z = m(\zeta), \zeta = e^{\xi + i\eta}, \ a = e^{\xi o} \]  \hspace{1cm} (2.5.2)

Let the curve \( C \) be given by \( \xi = \xi_o \), so on the curve

\[ z = m \left[ e^{\xi_o} \left( \cos \eta + i \sin \eta \right) \right] \]  \hspace{1cm} (2.5.3)

So that \( C \) corresponds point by point to the circumferences \( \Gamma, |\zeta| = e^{\xi o} \) in the \( \zeta \)-plane and \( \eta \) is the polar angle. Thus the region between the two curves \( C \) and \( C_1 \) given by \( \xi = \xi_o \) and \( \xi = \xi_1 \) respectively in the \( z \)-plane will map onto the concentric annulus bounded by \( \Gamma \) and \( \Gamma_1 \) of radii \( e^{\xi o} \) and \( e^{\xi_1} \) respectively [refer Fig.2]. In forming the mapping function \( m(\zeta) \) the following points are important:

(i) In the domain at no point \( m'(\xi) = 0 \) or \( m'(\xi) = \infty \) otherwise the reasoning will breakdown. If such points are there, they should be finite in number and must be dealt with a limiting process.

(ii) If the region under consideration is mapped onto the whole region inside the circle \( \Gamma \), then the form of \( m(\zeta) \) should be

\[ m(\zeta) = a_o + a_1 \zeta + a_2 \zeta^2 + \cdots, \ (a_1 \neq 0) \]  \hspace{1cm} (2.5.4)

The condition \( a_1 \neq 0 \) makes \( m'(0) \neq 0 \).

(iii) If the region under consideration is mapped onto the
Fig. 2 Transformations of eccentric into concentric annulus.
exterior of $\Gamma$ we must have $m(\zeta)$ of the form:

$$m(\zeta) = b_1 \zeta + b_2 \zeta^2 + \cdots + (b_1 \neq 0) \quad \ldots (2.5.5)$$

we see that $m(\zeta)$ should have no power of $\zeta$ higher than the first, otherwise $m'(\omega) = \infty$. Also $b_1 \neq 0$ ensures that $m'(\omega) \neq 0$. Other transformation function like $\zeta = ce^{\pi + i\eta}$ can also be used, if found convenient. Sometimes the mapping on the unit circle is also useful.

2.6. STRESSES AND DISPLACEMENTS

An element of arc of the curve $\zeta = \zeta_0$ is orthogonal to the direction in which $\xi$ increases. We therefore denote the normal stress across this element by $\bar{\xi}\xi$ and the shear stress by $\bar{\xi}\eta$ [see fig. 1]. Similarly across an element of arc of $\eta = \eta_0$ we have the components $\bar{\eta}\eta, \bar{\eta}\xi$. Thus we can write, following Milne-Thomson [28]

$$\sigma = \bar{\xi}\xi + \bar{\eta}\eta = xx + yy \quad \ldots (2.6.1)$$

$$\bar{\xi}_1 = \bar{\eta}\eta - \bar{\xi} + 2i\bar{\xi}\eta = \bar{\xi} e^{2i\alpha} = [\bar{\eta} - xx + 2i \bar{\xi}] f'(t)/f'(\bar{t}) \quad \ldots (2.6.2)$$

The displacement is given by

$$u + iv = (u + iu) e^{i\alpha} \quad \ldots (2.6.3)$$

If we use the mapping function (2.5.2) then the stresses are
Equation (2.6.4) is for the curve $\zeta =$constant and equation (2.6.5) for the curve $\eta =$constant. The displacement is given by

$$u + i u = \frac{\bar{\zeta} m'(\zeta)}{|\bar{\zeta} m'(\zeta)|} (u + iv) \quad \ldots (2.6.6)$$

### 2.7 STRESSES IN TERMS OF COMPLEX POTENTIALS

To get equations analogous to equation (2.3.15) for stresses in the $\zeta$-plane, we have, by writing $W_0(z)$ for $W(z)$ and $\bar{w}_0(z)$ for $\bar{w}(z)$:

$$\phi = W_0(z) + \bar{w}_0(z), \quad \bar{\phi} = z \bar{w}'(\bar{z}) + \bar{w}_0(\bar{z}) \quad \ldots (2.7.1)$$

Thus we can write

$$W_0(z) = W_0[m(\zeta)] = W(\zeta) \quad \ldots (2.7.2)$$

so we get

$$W'_0(z) = \frac{d}{dz} W_0(z) = \frac{d}{d\zeta} W(\zeta), \quad \frac{d\zeta}{dz} = \frac{W'(\zeta)}{m'(\zeta)} \quad \ldots (2.7.3)$$

Hence the complex stresses $\phi$ and $\bar{\phi}$ will be given by
\[ \sigma = W(\zeta) + \bar{W}(\bar{\zeta}) \]

\[ \bar{\phi} = \frac{m(\bar{\zeta})}{m'(\zeta)} W'(\zeta) + w(\zeta) \]  

...(2.7.4)

The stresses \( \bar{\xi}, \bar{\eta}, \) and \( \bar{\xi} \eta \) are given by

\[ 2(\bar{\xi} + i\bar{\eta}) = W(\zeta) + \bar{W}(\bar{\zeta}) - \frac{\bar{\xi} m(\zeta)}{\zeta m'(\zeta)} \bar{W}'(\zeta) - \frac{\bar{\xi} m'(\bar{\zeta})}{\zeta m'(\zeta)} \bar{W}(\bar{\zeta}) \]  

...(2.7.5)

\[ 2(\eta - i\bar{\xi}) = W(\zeta) + \bar{W}(\bar{\zeta}) + \frac{\xi m(\zeta)}{\zeta m'(\zeta)} W'(\zeta) + \frac{\xi m'(\bar{\zeta})}{\zeta m'(\zeta)} \bar{W}(\bar{\zeta}) \]  

...(2.7.6)

Similarly the complex displacement is given by

\[ 4\mu \frac{\partial}{\partial \nu} (u + iv) = [\alpha W(\zeta) - \bar{W}(\bar{\zeta})] i\xi m'(\zeta) + [m(\zeta) W'(\zeta) + m'(\bar{\zeta}) \bar{W}(\bar{\zeta})] i\xi \]  

...(2.7.7)

2.8 SOLUTION OF PROBLEMS

Let us consider the first fundamental problem in which the boundary \( C \) of the curve \( \zeta = \zeta_0 \) is under given load.

Let the curve \( C \) be transformed onto the circle \( \Gamma \) of radius \( a = e \) by the mapping function \( (2.5.2) \). The boundary value of the stresses \( \bar{\xi} + i\bar{\eta} \) is given by

\[ \bar{\xi} + i\bar{\eta} = -p(\sigma_0) + is(\sigma_0) \text{ on } C \]  

...(2.8.1)

where \( \sigma_0 \) is a point on the circle \( \Gamma \); \( p(\sigma_0) \) is the pressure
and \( s(\sigma_0) \) is the shear. The complex stress \( W(\zeta) \), following Milne-Thomson [28], is given by

\[
m'((\zeta))W(\zeta)+m'((\zeta))\bar{W}(\zeta)-\frac{2}{\zeta}m(\zeta)\bar{W}'((\zeta))\frac{1}{\zeta}m'(\zeta)\bar{W}(\zeta) = 2(\xi^2+i\eta)m'(\zeta)
\]

...(2.8.2)

Applying general continuation theorem for the circle we get

\[
m'(\zeta)W(\zeta)=\frac{1}{2\pi i}\int_{\Gamma} \frac{2[-p(\sigma_0)+is(\sigma_0)]m'(\sigma_0)d\sigma}{\sigma_0-\zeta} + \psi(\zeta) \quad ... \text{(2.8.3)}
\]

where the function \( \psi(\zeta) \) is continuous across \( \Gamma \) and

(i) the function \( w(\zeta) \) is holomorphic inside the region denoted by \( L \).

(ii) If \( L \) contains the point \( \omega \), the stresses at infinity must have given values.

(iii) The solution is non-dislocational.

We see that the stresses are expressed in terms of \( W(\zeta) \) and \( w(\zeta) \). But by the analytic continuation we can express \( w(\zeta) \) in terms of \( W(\zeta) \) and \( m(\zeta) \). By the continuation across the boundary we get from (2.8.2) by writing \( \frac{a^2}{\zeta} \) for \( \zeta \)

\[
m'(\zeta)W(\zeta) = -m'(\zeta)\bar{W}(\zeta) + \frac{a^2}{\zeta^2}m(\zeta)\bar{W}'(\zeta) + \frac{a^2}{\zeta^2}m'(\zeta)\bar{W}(\zeta)
\]

\((\zeta \text{ in } R) \quad \text{ ... (2.8.4)}
\]

From (2.8.4) by writing \( \zeta \) for \( a^2/\kappa \) and \( a^2/\zeta \) for \( \zeta \) and
taking the complex conjugate we can get \( w(\zeta) \) in \( L \). After substituting the value of \( w(\zeta) \) in equation (2.8.2) we get the stresses as

\[
2\zeta m'(\zeta)\left[\bar{\xi} \xi + i\bar{\eta} \eta\right] - \zeta m'(\zeta)w(\zeta) - \frac{a^2}{\zeta} m'(\zeta)w(\zeta) - \frac{a^2}{\zeta} w(\zeta)
\]

\[
+ \left[\zeta m'(\zeta) - \frac{a^2}{\zeta} m'(\zeta)\right] \bar{w}(\zeta) - \left[m(\zeta) - m(\zeta)\right] \bar{w}'(\zeta)
\]

...(2.8.5)

Hence the stresses are expressed in terms of only one complex stress \( W(\zeta) \). The form of the function \( \psi(\zeta) \) will be determined by considering the singularities of \( m'(\zeta)w'(\zeta) \) in the region \( R \).

Milne-Thomson [28] considered the first fundamental problem of Epitrochroidal oval under two forces. Later on Ali and Ahmed [7] solved the first fundamental problem of a Pascal's Limacon under concentrated forces by mapping the Limacon on to a circle by the transformation function

\[
z = m(\zeta) = C(\zeta + K\zeta^2)
\]

...(2.8.6)

\((C > 0, 0 \leq K \leq 1/2, z = re^{i\theta})\)

which transforms the inside region and outside region of the Limacon into inside and outside regions respectively of the circle. The whole boundary \( C \) of the Limacon is unloaded.
except the two points A and B, where standard concentrated forces act. In a very small neighbourhood of these points the stresses are unbounded. So we can think these forces to be applied as distribution of stress over small areas around these points instead of being concentrated. Consider the singularities of \( m'(\zeta)W(\zeta) \) in \( \mathbb{R} \) the function \( \psi(\zeta) \) can be taken as

\[
\psi(\zeta) = A + B\zeta 
\] 

...(2.8.7)

The function \( \psi(\zeta) \) becomes known except for the constants \( A \) and \( B \) which were ultimately determined after considering the Laurent's series expansion of \( \psi(\zeta) \). Thus the complex potential \( W(\zeta) \) was known and so the stresses were determined.

For \( K = 1/2 \) the Limacon reduces to a cardioid. It was found that the stress at the cusp was infinite and this physical impossibilities was resolved by plastic yielding of the material.

Later on the method was applied to solve the first fundamental problem of an eccentric annulus under concentrated forces. Several workers [1], [4], [8] employed this complex variable techniques.
CHAPTER - III
CHAPTER III
A DIRECT METHOD AND SOME CRACK AND PUNCH PROBLEMS

3.1. DIRECT METHOD

Systematic use of the complex variable theory developed by Muskhelishvili [31] and others gives the solution in many cases but demands a lot of calculations. In 1960 B. Sen [36] found that in many cases direct methods make the solution much simple. Although this method is direct and more simple in theory, it has a disadvantage of choosing the form of some functions. He considered the first boundary value problem of an elastic plate of isotropic material with a circular boundary. If $\bar{r}x$ and $\bar{r}y$ be the stress components in the $x$ and $y$ directions on a circle of radius $a$, then in absence of body forces, we get

$$r.\bar{r}x = \text{Re} \left[ \frac{r^2-a^2}{4} \left\{ \frac{f(z)-zf'(z)}{z} \right\} \right] + aL(z) \quad ...(3.1.1)$$

$$r.\bar{r}y = \text{Re} \left[ \frac{r^2-a^2}{4} i\left\{ \frac{f(z)-zf'(z)}{z} \right\} \right] + aM(z) \quad ...(3.1.2)$$

$L(z)$ and $M(z)$ are analytic functions of $z$ such that

$$\bar{r}x = \text{Re} L(z), \text{ on } r = a$$

$$\bar{r}y = \text{Re} M(z), \text{ on } r = a$$

...(3.1.3)
and from the relation $xx + yy = \text{Re } f(z)$, we can get in general following Sen [36]

$$f(z) = 2a \frac{L(z) + iM(z)}{z} \quad \ldots(3.1.4)$$

When the stresses $\tau_x$ and $\tau_y$ are prescribed on the circle $r=a$, the functions $L(z)$, $M(z)$ and hence $f(z)$ are known.

Using the direct method developed by Sen [36], Ali and Ahmed in 1980 [8] solved the first fundamental problem of an infinite plate having a hypotrochoidal hole. The hole has been supposed to be under uniform hydrostatics pressure $P$. The mapping function has been taken as

$$z = R(\zeta + Cr^{-m}), \quad \zeta = e^\xi + i\eta \quad \ldots(3.1.5)$$

where $R > 0$, $C > 0$ and $m$ is a positive integer. The boundary of the hypotrochoidal hole is given by $\xi = 0$ in the $z$-plane. Following Sen [36] the stresses are given by

$$\frac{\partial \xi}{\partial \eta} + \frac{\partial \xi}{\partial \eta} = \frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} + \frac{40}{h^2} + F \quad \ldots(3.1.6)$$

$$\frac{\partial \eta}{\partial \xi} + \frac{\partial \eta}{\partial \xi} = \frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} + \frac{40}{h^2} - F \quad \ldots(3.1.7)$$
\[ \frac{\partial^2 \eta}{\partial \eta^2} = -\frac{\partial^2}{\partial \zeta^2} \frac{\partial \phi}{\partial \zeta} - \frac{\partial^2}{\partial \xi^2} \frac{\partial \phi}{\partial \xi} - G \quad \text{(3.1.8)} \]

where

\[ \frac{1}{h^2} = \left( \frac{\partial x}{\partial \zeta} \right)^2 + \left( \frac{\partial y}{\partial \zeta} \right)^2 \quad \text{(3.1.9)} \]

\[ r^2 = x^2 + y^2, \quad \phi = \xi \eta + \eta \eta \quad \text{(3.1.10)} \]

and \( F \) and \( G \) are conjugate plane harmonic functions. The boundary conditions are

\[ \xi \eta = -P, \quad \xi \eta = 0 \text{ at } \xi = 0 \quad \text{(3.1.11)} \]

At this stage a suitable form of the plane harmonic function \( \phi \) along with certain constant coefficient is chosen. From the equation (3.1.8), we get the expression for \( G \). The functions \( F \) and \( G \) being conjugate and harmonic, \( F \) can also be determined. The first boundary condition (3.1.11) gives the value of unknown constant in \( \phi \).

In the present problem the expression for \( \phi \) has been taken as

\[ \phi = B \left[ 1 - \text{Re}\left( \frac{(\zeta + mc \xi^{-m})}{(\zeta - mc \xi^{-m})} \right) \right] \quad \text{(3.1.12)} \]

where \( B \) is an unknown constant and \( \text{Re} \) denotes the real part.

The function \( G \) at the boundary \( \xi = 0 \) is given by
\[ \frac{G}{h^2} = -4BR^4 m(m+1) (1+mc^2) \sin(m+1) \eta, \text{ at } \xi = 0 \quad \ldots (3.1.13) \]

It has been observed that \( G \) is the imaginary part of the function.

\[ F + iG = 4BR^4 (1+mc^2) (\xi + m^2 c \xi^{-m}) / (\xi - mc \xi^{-m}) \quad \ldots (3.1.14) \]

The imaginary part of (3.1.14) satisfies (3.1.13) and so the form of the function \( F \) is given by the real part of (3.1.14). The values of the functions \( F, G \) and \( \theta \) are put in the first boundary condition to give

\[ B = -2P \quad \ldots (3.1.15) \]

Thus all functions being completely known the stresses were determined. The stress intensity factor given by \( [\mathfrak{H}]_\xi = 0 \) was calculated for various values of \( m \) and \( c \). The case of a circular hole is given by \( c = 0 \). The stress intensity factor was found to be \( P \) at every point.

When \( m = 1 \), the boundary \( \xi = 0 \) gives an elliptic hole of major axis \( 2R(1+C) \) and minor axis \( 2R(1-C) \) in the \( z \)-plane.

When \( m = 1 \) and also \( C = 1 \), the elliptic hole degenerates into a linear crack, called a Griffith crack, of length \( 2R \). It was found that at the tips of the cracks the stress components becomes indeterminate.
Another important technique to determine the distribution of stresses in plane problems is that of integral transform. This method has been found to be very helpful, especially in the case of punch and crack problems. The important transforms which are generally used are Fourier transform, Fourier sine and cosine transforms, Mellin transform, Hilbert transform, Hankel's transform etc. The Fourier transform of a function \( f(t) \) is denoted by \( F(s) \) and is given by

\[
F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} \, dt \quad ...(3.2.1)
\]

The inverse Fourier transform of \( F(s) \) is given by

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-ist} \, ds \quad ...(3.2.2)
\]

Similarly, the Fourier sine transform is given by

\[
F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin st \, dt \quad ...(3.2.3)
\]

with the inverse transform

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The Fourier cosine transform and its inverse transform are given by (3.2.3) and (3.2.4) after changing sine to cosine.

The integral transform technique has been mainly employed to determine the stresses in the neighbourhood of a crack in an elastic body. Cracks are considered as surfaces of discontinuity of the material that is of the displacement vector. It has been found that cracks exist or develop in a solid body when it is subjected to tensile forces and that when these forces are increased beyond certain limit. In two-dimensional cases a linear crack is called a Griffith crack. But in reality it is a long flat ribbon-shaped cavity in a solid. Griffith [14], [15] calculated the distribution of stress in an infinite plate containing an elliptic crack with semi-major axis \( c \) and semi minor axis \( b \). Then the limit was taken as \( b \) tends to zero when the ellipse degenerates into a straight line of length \( 2c \). Mathematically a Griffith crack occupies the segment

\[
y = 0, \quad -c \leq x \leq c \quad \ldots (3.2.5)
\]

Generally we are interested in the case in which the
surfaces of the crack are stress free and there is prescribed tensile stress at infinity.

\[
\overline{yy} = \overline{yx} = 0 \quad \text{on} \ -c \leq x \leq c, \ y = 0 \quad \ldots \ (3.2.6)
\]

and \( \overline{yx} \to 0, \overline{yy} \to p_o, \overline{xx} \to q_o \quad \ldots \ (3.2.7) \)

where \( p_o \) and \( q_o \) are prescribed tensile stresses. These equations are used to find the stress distribution in the neighbourhood of a Griffith crack when it is opened out by the application of a constant pressure to its free surface. Using the superposition principle the solution of such a problem reduces to the discussion of the problem.

\[
\overline{yy} = -p_o, \overline{yx} = 0, \text{on} \ -c \leq x \leq c, \ y = 0 \quad \ldots \ (3.2.8)
\]

where \( \overline{yx}, \overline{yy}, \) and \( \overline{xx} \) all tend to zero at infinity. The above problem can be generalized to the case when there is an internal pressure varying along the length of the crack and which is opening it.

In this case we have

\[
\overline{yy} = -p(x), \overline{xy} = 0 \text{ on} \ -c \leq x \leq c, \ y = 0 \quad \ldots \ (3.2.9)
\]
and $y_x, y_y, x_x$ all tend to zero at infinity and $p(x)$ is a prescribed function of $x$ in the interval $[-c, c]$. Sneddon and Elliott [39] discussed the stress distribution in the neighbourhood of a crack subjected to a varying internal pressure along the length of the crack. The internal pressure $p(x)$ was supposed to be an even function of $x$, so the governing equations, can be taken as

\[
\begin{align*}
\ddot{y} &= 0, \quad \ddot{y} = -p(x), \quad 0 \leq x \leq c \\
\ddot{x} &= 0, \quad u_y = 0, \quad x \geq c
\end{align*}
\]  

...(3.2.10)

Here we have assumed that the pressure is the same for both faces of the crack, the stress field becomes symmetrical about the $x$-axis. Due to this reason $u_y$ and $\ddot{x}$ both become zero outside the crack. For convenience the unit of length has been taken to be half the width of the crack that is $c=1$. The problem was ultimately solved by employing Fourier cosine transform which reduces the problem to that of solving a pair of dual integral equations. Similarly the problem of determining the pressure distribution when the crack is opened in a prescribed shape has also been solved by Sneddon [40] by using the Fourier transforms. In this case boundary conditions will be
together with the condition that all stresses vanish at infinity. It has been found that the shape of the crack is elliptic in the case of constant internal pressure. Willimore [49] has considered the distribution of stress in the neighbourhood of two equal collinear Griffith cracks in an isotropic material when a uniform pressure \( P \) acts normally across the surfaces of each crack and there is no shearing stress. Tranter [47] discussed the case when the internal pressure varied along the length of each crack. The isochromatic lines defined by

\[
\frac{\tilde{XY}}{P} = \text{constant} \quad \ldots(3.2.12)
\]

where \( \tilde{XY} \) is the maximum shearing stress, is given in Fig.[ 3 ] for various values of some parameter. Recently Ali has discussed the problem of opening of a crack of prescribed shape in an initially stressed body. Such initial stresses are found to exit in a body by the process of preparation or by the action of body forces. For example, if a sheet of metal is rolled up into a cylinder and then the

\[
\begin{align*}
  u_y(x,0) &= w(x), \quad |x| \leq c \\
  u_y(x,0) &= 0, \quad \tilde{xy}(x,0) = 0, \quad |x| > c
\end{align*}
\]
Fig. 3 The isochromatic lines in the vicinity of a pair of collinear Griffith cracks.
edges welded together, the cylinder so formed will be in a state of initial stress and the unstressed state can not be obtained without cutting the cylinder open. These initial stresses generally cause finite deformation. The state of the body can be studied by the superposition of the latter on the former [12].

It is supposed that \( p(x) \) is the pressure on \( y=0 \) for a crack of prescribed shape given by

\[
\gamma(x,0)= w(x), |x| \leq c
\]

...(3.2.13)

where \( w(x) \) is a piecewise smooth function of \( x \) in \([-c,c]\). The boundary conditions are:

(i) \( \gamma x(x,0) = 0 \) for all \( x \)
(ii) \( \gamma y(x,0) = -p(x) \) for all \( x \)
(iii) \( \gamma y(x,0) = w(x), |x| \leq c \)
(iv) \( \gamma y(x,0) = 0, |x| > c \)

...(3.2.14)

Following Kurashige [20], the components of stresses can be found out from equations
\[ \ddot{x} = s - \mu' \frac{\partial^2 \phi}{\partial x \partial y} \]

\[ \ddot{y} = s + \mu' \frac{\partial^2 \phi}{\partial x \partial y} \]

\[ \dot{x}y = \frac{\mu'}{2} \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} \right) \]

where \( s = \frac{1}{2} (xy + yy) \)

and \( \mu' = \sigma(x^2 + y^2) \)

(\( x, y \) being extension ratios)

The problem has been solved by using Fourier transform which ultimately lead to the solution of a pair of dual integral equations

\[ \frac{2}{\pi} \int_0^\infty \frac{P(\xi)}{\xi} \cos \xi x \, d\xi = Gw(x), \ x \leq c \]

\[ \int_0^\infty \frac{P(\xi)}{\xi} \cos \xi x \, d\xi = 0, \ x > c \]

where a bar denotes the Fourier transform of the function defined by the following equations
\[ \phi(\zeta) = \int_{-\infty}^{\infty} \phi e^{i\zeta x} \, dx \] 
and 
\[ \phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi \, e^{-i\zeta x} \, d\zeta \]

Solving the equations (3.2.18), one gets

\[ \bar{p}(\xi) = \begin{bmatrix} Q^x \int_{0}^{\infty} w(x) \cos \xi x \, dx \quad , x \leq c \\ 0 \quad , x > c \end{bmatrix} \] 

or

\[ \bar{p}(\xi) = Q^x \int_{0}^{c} w(x) \cos \xi x \, dx \] 

The function \( \bar{p}(\xi) \) being known, \( \bar{\phi} \) can be found out. After inverting it the function \( \phi \) becomes known and the problem is completely solved.

If we take

\[ w(x) = \varepsilon (1 - \frac{x^2}{c^2}) \]

(\( \varepsilon \) is a small positive number) the shape of the crack is parabolic. The normal stress has been found out and the variation of the necessary pressure \( p(x) \) has been studied at various values of \( x \). It was concluded that the force applied to the faces of the crack was "Compressive" in the body of the crack but that near the tip of the crack the applied
force becomes tensile. Sneddon [40] also came to the same conclusion but for non-initially stressed body.

The elliptic shape can be obtained by taking

\[ w(x) = \varepsilon (1 - \frac{x^2}{c^2})^{1/2}, \quad 0 \leq x \leq c \quad \ldots (3.2.23) \]

Similar procedure was employed to find the normal stress. It was observed that the pressure necessary to produce a Griffith crack of elliptic shape in an initially stressed solid should also be uniform as was already found out for a non-initially stressed body [40].

Later on Ali and Ahmed [6] discussed the problem of two collinear Griffith cracks which was earlier discussed by England and Green [11], and later on by Sneddon and Srivastava [41]. The pair of collinear cracks have been assumed to occupy \(-b \leq y \leq -a\) and \(a \leq y \leq b\) on \(x=0\) and to be subjected to internal pressure \(p(y)\). All the stress components and displacements vanish at infinity.

Other boundary conditions are

\[
\begin{align*}
\tilde{xx}(o,y) &= -p(y) ; \quad a \leq y \leq b \\
\tilde{xy}(o,y) &= 0 ; \quad -\omega < y < \omega \\
u(o,y) &= 0 ; \quad y < a, y > b
\end{align*}
\]

...(3.2.24)

For the symmetry of the problem about \(y=0\), \(p(y)\) is assumed to
be an even function of $y$. When these boundary conditions are applied to the expressions of stress components in integral forms given by Sneddon [38] we get

\[
\begin{align*}
\sigma_{xx} &= -\frac{2}{\pi} \int_0^\infty \overline{p(\xi)}(1+\xi x)e^{-\xi x}\cos y \, d\xi \\
\sigma_{yy} &= -\frac{2}{\pi} \int_0^\infty \overline{p(\xi)}(1-\xi x)e^{-\xi x}\cos y \, d\xi \\
\sigma_{xy} &= -\frac{2x}{\pi} \int_0^\infty \xi \overline{p(\xi)}e^{-\xi x}\sin y \, d\xi
\end{align*}
\]

Similarly the displacement components are given by

\[
\begin{align*}
u &= \frac{2(1+\alpha)}{\pi E} \int_0^\infty \overline{p(\xi)}e^{-\xi x}[2(1-\alpha+\xi x)\frac{1}{\xi}\cos y \, d\xi \\
\nu &= -2 \frac{(1+\alpha)}{\pi E} \int_0^\infty \overline{p(\xi)}e^{-\xi x}(1-2\alpha-\xi x)\frac{1}{\xi}\sin y \, d\xi
\end{align*}
\]

where $\overline{p(\xi)}$ is the Fourier cosine transform of $p(y)$ and is an even function of $\xi$. The second boundary condition in (3.2.24) is identically satisfied and the other boundary conditions lead to the triple integral equations:
\[
\int_0^\infty \frac{1}{\xi} \tilde{p}(\xi) \cos \gamma \, d\xi = 0 \quad (0 < y < a)
\]
\[
\int_0^\infty \tilde{p}(\xi) \cos \gamma \, d\xi = \frac{\pi}{2} p(y) \quad (a \leq y \leq b)
\]
\[
\int_0^\infty \frac{1}{\xi} \tilde{p}(\xi) \cos \gamma \, d\xi = 0 \quad (y > b)
\]

... (3.2.27)

Using the results of Srivastava and Lowengrub [43] the above triple integral equations were solved to give \( \tilde{p}(\xi) \) in terms of some function which was known under the given informations and was expressed in terms of elliptic integral.

The case of uniform pressure distribution was discussed and the results were found to be in agreement with the known results. The normal stress was found out and it was observed that the nature of the normal stress \( \tilde{\sigma}_{x}(0,y) \) in the two regions \( |y| < a \) and \( |y| > b \) were of opposite nature. Also, the normal stress at the mid point of the line joining the centres of the cracks decreased as the distance between the centres of the cracks increased (the length of each crack being taken as unity). Furthermore the normal stress increases as the length of each crack increases. These problems can be generalized to the case of non-linear cracks in a two-dimensional body. For these problems one may refer to Green and Zerna [13].
3.3 **INDENTATION (OR PUNCH) PROBLEMS**

Punch problems are somewhat similar to crack problems. In the general case a perfectly rigid solid of revolution whose axis of revolution coincides with the *z*-axis is pressed normally against the plane *z*=0 of a semi-infinite elastic medium *z*>0. In the deformed state the surface of the elastic medium will fit the rigid body over a circular section of radius *a*.

The shearing stress is assumed to be zero at all points of the boundary *z*=0, the *z*-component of surface displacement is prescribed over the region *r*≤*a*, *z*=0 and the normal stress Vanishes on the remaining part of the boundary. Hence the boundary conditions on *z*=0 can be taken as

\[
\begin{align*}
    u_z &= f(r), \quad 0 \leq r \leq a, \quad z=0 \\
    z \frac{\partial u_z}{\partial z} &= 0, \quad r > a, \quad z=0 \\
    \frac{\partial^2 u_z}{\partial z^2} &= 0, \quad 0 \leq r < a, \quad z=0
\end{align*}
\]

...(3.3.1)

where the stresses and strains are expressed in cylindrical coordinates (*r*,*θ*,*z*), hence all quantities are independent of *θ*. All components of stress and displacement tend to zero at infinity.
Punch and crack problems for transversely isotropic bodies have been solved by Elliott [10] using Hankel's transform.

In plane problems, the half plane $y \geq 0$ has been assumed to be occupied by the material which is under the action of a punch by a smooth block. In the strained state, let a length 'ab' of the boundary be in contact with the pressing block whose shape is given by $y = f(x)$. So the boundary conditions are

\[
\begin{align*}
  u_y(x, 0) &= f(x), \quad \text{on } ab \quad \ldots (3.3.2) \\
  xy(x, 0) &= 0, \quad \text{everywhere on } ox \quad \ldots (3.3.3) \\
  yy(x, 0) &= 0, \quad \text{outside } ab \quad \ldots (3.3.4)
\end{align*}
\]

The strains and stresses are given by equations (1.4.1) and (1.4.2). The cases of indentation by a rectangular block and by a circular block are known [13].

Ali [4] has discussed the case of a semi-infinite plate which is indented by an infinite row of parabolic punches. These punches have been assumed to be identical and acting on the boundary $y = 0$ at equal distances. It is further assumed that the region of contact of each punch with the strained surface is $2a$ and the distance between the centres of two consecutive punches is $d$. 

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The boundary conditions are
\[
\begin{align*}
\tilde{yy}(x,0) &= 0, \quad a \leq |x-nd| \leq d-a \\
\tilde{xy}(x,0) &= 0, \quad \text{for all } x \\
v(x,0) &= \alpha - \beta x^2, \quad |x-nd| \leq a
\end{align*}
\]  
...(3.3.5)

where \( \alpha > 0, \beta > 0 \) and \( n \) is zero or integer. These boundary conditions lead to the determination of the complex potential \( \phi(z) \). The normal stress under a typical punch has been calculated. If \( P \) is the resultant thrust on a punch, the length of the region of contact \( 2a \) of each punch with the half space is given by the relation

\[
P = - \int_{-a}^{a} \tilde{yy}(x,0) \, dx \quad \ldots(3.3.6)
\]

The variation of the normal stress under a typical punch has been shown for different values of \( 2a/d \). The graph is a part of an "ellipse". It was observed that even a substantial increase in the ratio \( 2a/d \) does not appreciably affect the shape of the curve although the minor axis increases by nearly 80\%. 
CHAPTER IV
4.1. APPROXIMATE METHODS

The various techniques discussed so far in the preceding chapters were generally used to obtain the solution of certain plane problems in elasticity in a closed form. But the problems generally encountered in practical life do not have a solution in closed type or a closed type solution is difficult to be found out. In such problems the solution is obtained by some approximate method or numerical method. Some of the approximate methods are as follows:

(i) Perturbation method
(ii) Power series method
(iii) Probability schemes
(iv) Method of weighted residuals (MWR)
(v) Finite difference technique
(vi) Ritz method
(vii) Finite element method (FEM)

The perturbation method is useful in a limited number of cases because it is applicable primarily when the nonlinear terms in the equation are small in relation to the linear terms. The power series method is more powerful but
since the method required generation of a coefficient for each term in the series which is relatively tedious, it can be employed only with some success. The convergence of the power series is also a problem. The probability schemes are used for obtaining a statistical estimate of a desired quantity by random sampling and is not of much interest.

Due to high speed computers, the methods of MWR, finite difference techniques and the FEM have become very popular. These methods are of course related and in some cases the finite difference techniques and FEM can be shown to be special cases of MWR. In using the MWR, we assume the field solution in such a way that it satisfies the boundary condition exactly but the differential equations approximately. Some of important methods for this type are least square method, Galerkin method, Collocation method, moment method etc.

4.2. RITZ METHOD ( VARIATIONAL APPROACH )

The problems of solid mechanics may have different formulation but they are equivalent. They may have a differential formulation or a variational formulation. In the differential formulation, the problem is to integrate some system of differential equation with given boundary conditions. In the variational formulation, the problem is
to find the unknown function or functions that maximize (minimize) or make stationary a functional or system of functionals subjected to the same given boundary conditions. The two formulations are equivalent because the functions which satisfy the differential equations and their boundary conditions also extremize or make stationary the functionals.

In the Ritz method the form of the unknown solution is assumed in terms of known functions (trial functions) with unknown adjustable parameters. From the set of trial functions we select the function that renders the functional stationary. The trial function is substituted into the functional and thereby the functional is expressed in terms of adjustable parameters. The functional is then differentiated with respect to each parameter and resulting equation is set equal to zero. If the trial function contains n unknown parameters, there will be n simultaneous equations involving these n parameters. The unknown parameters being found out from these equations, the approximate solution is chosen from the set of assumed solutions. The Ritz method actually does nothing more than give us the best solution from the set of assumed solutions. It is obvious that the accuracy of the
appropriate solution depends on the choice of the trial function. If by chance the exact solution is contained in the set of trial solutions, the Ritz method gives the exact solution. Very often a set of trial solutions is constructed from polynomials of successively increasing degree.

4.3. **FINITE ELEMENT METHOD (FEM)**

The finite element method and the Ritz method are essentially equivalent. The major difference is that the trial functions in the FEM are not defined over the whole solution domain, and they have to satisfy no boundary conditions but only certain continuity conditions and then only some times. In the Ritz method functions are defined over the whole domain, so it can be used only for domains of relatively simple geometric shape. This limitation has been overcome in FEM by discretizing the domain into elements which can be of simple shapes. These simple shaped elements can be assembled to represent much complex geometries. Thus we see that finite element method is much more versatile than the Ritz method.

The procedure of FEM mainly consists of the following six steps:
(i) Discretization of the domain

The first step is to divide the solution domain into subdomains called elements. Different element shapes may be used and with care, different element shapes may be employed in the same solution domain. When analysing an elastic structure that has different types of components such as plates and beams, it is not only desirable but also necessary to use different types of elements in the same solution.

(ii) Selection of interpolation functions

Nodes are assigned to each element and the type of interpolation function is chosen to represent the variation of the field variable over the element. The field variable may be a scalar, a vector, or a higher order tensor. Generally polynomials are selected as interpolation functions for the field variable, because they are easy to integrate and differentiate. The degree of the polynomial chosen depends on the number of nodes assigned to the elements, the nature and number of unknowns at each node, and certain continuity conditions at the nodes and along the element boundaries.

(iii) Determination of element properties

When the elements and their interpolation functions have been selected, the matrix equations expressing the
properties of the individual elements is to be determined. For this we may use one of the four approaches viz. the direct approach, the variational approach, the weighted residual approach or the energy balance approach. Among these approaches the variational approach is generally found to be most convenient.

(iv) Assembly of the element properties to obtain the system equations

All the element properties are assembled to find the properties of the overall system. For this the matrix equations expressing the behaviour of the elements are combined to form the matrix equations expressing the behaviour of the entire system. This is done due to the fact that at a node where elements are inter-connected, the value of the field variable is the same for each element sharing that node. This is usually done by digital computers.

(v) Solution of the system equations

By the assembly of element properties we get a set of simultaneous equations that can be solved to obtain the unknown nodal values of the field variable. If the equations are linear, there are many standard solution techniques.
The solution of the system equations can be used to calculate other important parameters. For example for the nodal values of the pressure we may calculate shear stresses if these are desired.

4.4. F E M TO ELASTICITY PROBLEMS

In the F E M to solid mechanics problems, the necessary element properties or equations are derived by a variational principle. There are three most commonly used variational principles.

(i) Minimum potential energy principle - (Principle of virtual displacement)

When this variational principle is used we must assume the form of the displacement field in each element. This is sometimes called the displacement method or the compatibility method in F E M.

Let us consider an elastic body which is deformed by the action of body forces and surface tractions. The potential energy of the body is the strain energy minus the work done by the external forces. The theorem of minimum potential energy is given by Love [21] as follows.

Theorem: The displacement \((u,v,w)\) which satisfies the differential equations of equilibrium, as well as the
conditions at the bounding surface, yields a smaller value for the potential energy than any other displacement which satisfies the same conditions at the bounding surface.

Let \( \Pi(u,v,w) \) be the potential energy, \( U_P(u,v,w) \) the strain energy and \( V_P(u,v,w) \) the work done by the applied loads during displacement. Hence by this principle

\[
\delta \Pi(u,v,w) = \delta U_P(u,v,w) - \delta V_P(u,v,w) = 0 \quad \ldots (4.4.1)
\]

where the variation is taken with respect to the displacement while the other parameters are kept constant.

The strain energy is defined as

\[
U_P(u,v,w) = \frac{1}{2} \iiint [\varepsilon]^T [\sigma] \, dv \quad \ldots (4.4.2)
\]

where \( V \) is the volume of the body and \( [\varepsilon] \) is the strain row matrix given by

\[
[\varepsilon] = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{xz} & \varepsilon_{yz}
\end{bmatrix} \quad \ldots (4.4.3)
\]

and \( [\sigma] \) is the stress row matrix given by

\[
[\sigma] = \begin{bmatrix}
\sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{xz} & \sigma_{yz}
\end{bmatrix} \quad \ldots (4.4.4)
\]
([α]T is the transpose of [α]). Using Hooke's law we have

\[
V_p(u,v,w) = \frac{1}{2} \iiint [\varepsilon][C][\varepsilon]^T dv \quad \cdots (4.4.5)
\]

where C is the proportionality matrix containing 36 elements in general and is given by

\[
[\alpha]^T = [C][\varepsilon]^T \quad \cdots (4.4.6)
\]

The matrix C is known as "material stiffness matrix" and its inverse matrix D is called the "material flexibility matrix". The relations between the components of strains and displacements are

\[
\begin{align*}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} \quad, \quad e_{zz} = \frac{\partial w}{\partial z} \\
e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
e_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
e_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{align*}
\]

The equations (4.4.7) can be expressed in matrix notations as
\[ [\varepsilon]^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial / \partial x & 0 & 0 \\ 0 & \partial / \partial y & 0 \\ 0 & 0 & \partial / \partial z \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \]

or, \[ [\varepsilon]^T = [B][d]^T \quad \ldots (4.4.8) \]

where \([d] = [u\ v\ w] \). Substituting (4.4.8) in (4.4.5) we get

\[ U_p(u,v,w) = \frac{1}{2} \iiint [d][B]^T[C][B][d]^T \, dv \quad \ldots (4.4.9) \]

If initial strains, given by the row matrix \([\varepsilon_i] \), are present, the strain energy becomes

\[ U_p(u,v,w) = \frac{1}{2} \iiint \left[ [d][B]^T[C][B][d]^T - 2[d][B]^T[C][\varepsilon_i]^T \right] \, dv \quad \ldots (4.4.10) \]

The work done by the external forces is

\[ V_p(u,v,w) = \iiint [F][d]^T \, dv + \iiint [T][d]^T \, ds \quad \ldots (4.4.11) \]
where \([F]=[X \ Y \ Z],[T]=[T_\chi \ T_y \ T_z]\) and \(S_1\) is the portion of the surface of the body on which the tractions are given. Ultimately we get the general potential energy functional as

\[
\Pi(u,v,w) = \frac{1}{2} \int_V \int \left[ [d][B]^T [C][B][d]^T - 2[d][B]^T [C][\epsilon_1]^T \right] \, dv - \int_S [F][d]^T \, dv - \int_{S_1} [T][d]^T \, ds \tag{4.4.12}
\]

The displacement field \(u,v,w\) which minimizes the functional \(\Pi\) and satisfies all the boundary conditions gives the equilibrium displacement field. This approach is called the displacement method or the stiffness method.

(ii) Principle of minimum complementary energy

(Principle of virtual stress)

The minimum complementary energy principle corresponds to the compatibility condition whereas the previous principle of minimum potential energy corresponds to the equilibrium condition in an elastic body.

Let \(\Pi_c(\hat{x}x, \hat{y}y, ... \hat{z}z)\) be the complementary energy, \(U_c(\hat{x}x, \hat{y}y, ... \hat{z}z)\) the complementary stress energy and \(V_c(\hat{x}x, \hat{y}y, ... \hat{z}z)\) the work done by the applied loads during stress changes. Then according to this principle.
\( \delta c = \delta(U - V) = \delta U - \delta V = 0 \) ... (4.4.13)

where the variation is taken with respect to stress components. The complementary stress energy is defined as follows

\[ U_c(\xi x, \eta y, \ldots, \zeta z) = \frac{1}{2} \iiint [\sigma][D][\sigma]^T \, dv \] ... (4.4.14)

where \( D \) is the flexibility matrix and \( V \) is the volume of the elastic body. If \( [\epsilon_i] \) is the initial strain row matrix then

\[ U_c = \frac{1}{2} \iiint \left[ [\sigma][D][\sigma]^T + 2[\sigma][\epsilon_i]^T \right] \, dv \] ... (4.4.15)

If \( T(T_x, T_y, T_z) \) is the prescribed surface traction and \( d_p(u_p, v_p, w_p) \) the prescribed displacement, then

\[ V = \iint (T_x u_p + T_y v_p + T_z w_p) \, ds = \iint [T][d_p]^T \, ds \] ... (4.4.16)

Hence the complementary energy function becomes

\[ \Pi_c(\xi x, \eta y, \ldots, \zeta z) = \frac{1}{2} \iiint \left[ [\sigma][D][\sigma]^T + 2[\sigma][\epsilon_i]^T \right] \, dv - \iint [T][d_p]^T \, ds \] ... (4.4.17)
When the complementary energy principle is used in finite element analysis the form of the stress field is assumed in each element and then the problem is solved in the usual way. This approach is known as "force method" or the "flexibility method". The equilibrium equations are identically satisfied and the element equations are the approximate compatibility equations.

(iii) Reissner's principle

In the potential energy functional, variations of displacement are considered; and in complementary energy functional, the variations of stresses are considered. But the Reissner's functional [34] allows variations of both displacement and stress. This principle does not evolve naturally from the concept of virtual work, but may be obtained either from the potential energy or the complementary energy theorem. Thus this principle contains aspects of both the equilibrium and the compatibility conditions.

Reissner's principle states that

$$\delta_{R}(u,v,w, xx, yy, ..., zx) = 0$$

...(4.4.18)

where the functional $\Pi_{R}$ is given by
\[ \Pi_R = \iiint_V \left[ [\sigma][\varepsilon]^T - \frac{1}{2} [\sigma][D][\sigma]^T - [d][F]^T \right] \, dv - \int_{S_1} \int [d][T]^T ds_1 - \int_{S_2} \int [d-d_p][\bar{T}]^T ds_2 \quad \ldots(4.4.19) \]

The variations of \( \Pi_R \) with respect to \([d]\) and \([\sigma]\) gives neither a maximum value nor a minimum value but only a stationary value. While using Reissner’s principle in FEM, we must assume the form of both the displacement and the strain fields within each element. Application of Reissner principle is not extensive; it appears most often in the analysis of plate and shell problems.

Beside these principles, we can use Hamilton’s principle to study the dynamic behaviour of elastic structures. According to the Hamilton’s principle, the first variation of the Lagranges function, \( L \), must vanish, that is

\[ \delta \int_{t_0}^{t_1} L \, dt = 0 \quad \ldots(4.4.20) \]

where

\[ L = E_k - U_s - w_p \quad \ldots(4.4.21) \]

and

- \( E_k \) = the total kinetic energy of the body
- \( U_s \) = the internal strain energy
- \( w_p \) = the work done by the applied loads when displacement is varied.
CHAPTER V
CHAPTER V

INCLUSION PROBLEMS AND SOME NEW RESULTS

5.1. A NEW METHOD (BOUNDARY ELEMENT METHOD)

Recently the boundary element method has become popular in solid mechanics. Using this method Miyazaki et al. [29] have analysed the stress intensity factor of interface crack using boundary element method which is an application of virtual crack extension method. This is a new method for stress intensity factor analysis of two-dimensional interface crack between dissimilar materials. They have combined the method of virtual crack extension, which is a powerful tool for calculating the stress intensity factors, with the boundary element method. Employing the boundary element method they have made a stress analysis and then virtual finite elements are assumed around a crack tip. The nodal displacement of these virtual finite elements are calculated as internal points of a boundary element analysis. The method is at first applied to a centre cracked homogenous plate under tension. A bimaterial plate with a centre interface crack and then a bimaterial plate with a centre slant interface crack subjected to tension were analysed. It was found that the present method gives very accurate results whose accuracy is
Insensitive to the size of virtual finite elements.

In various engineering fields such as those dealing with composite materials, adhesive joints and electrical components, interface structures can be seen. For the structural integrity of such interface structures, the assessment of interface fracture is very important, because the origin of fracture in such structures is usually on the interface between dissimilar materials. William [48], observed, the oscillation of the stress in the immediate neighbourhood of an interface crack tip. Rice and Sih [35] proposed the stress distribution near an interface crack tip. Many workers have investigated the estimation method for actual interface structures using the finite element method [23],[46],[50]. Yauki and Cho [51] proposed an extrapolation method using the displacement field near a crack tip obtained from a boundary element analysis to determine the stress intensity factors of a bimaterial interface crack. Their method provides mixed mode stress intensity factor.

5.2 RIGID INCLUSION PROBLEMS

When a rigid inclusion is bounded to the interior of an elastic infinite medium, interface cracks are found to be developed. Recently Ishikana and Kehno [17] have analyzed the
stress intensity factor of an interface crack of a rectangular rigid inclusion. The problem has been treated as two-dimensional and the inclusion is completely bounded to the interior of an elastic infinite medium, except for a portion which is regarded as an interface crack. Employing the Muskhelishvili [31] stress function, determined for m terms of finite series of the function for the conformal mapping, the inclusion is mapped onto the unit circle. The stress intensity factors for the interface crack are then determined under the equal biaxial loading condition. Two types of interface cracks have been analysed.

(i) The crack which is located on the short side of a rectangular rigid inclusion.

(ii) The crack which is extended from the short side to the long side of a rectangular rigid inclusion.

The infinite plate into which a rectangular rigid inclusion has been embedded is taken as the xy-plane and the centre of the inclusion as the origin. A portion $L'$ of the interface between the rectangular rigid inclusion and the matrix is perfectly bounded with the matrix, and the other portion $L$ is debounded with the matrix. Using the Schwarz-Christoffel transformation, the function $w(\zeta)$ which maps the outer region of a rectangular $S$ in the $z$-plane onto
the outer region of unit circle \( \Gamma_1 \) in the \( \zeta \)-plane(\( \xi \)-\( \eta \) plane) is given as

\[
z = w(\zeta), \quad dz = aR \frac{1}{\zeta^2} \prod_{p=1}^{4} (\zeta_p - \zeta)^{\alpha p/\pi} \, d\zeta
\]  

...(5.2.1)

where \( a \) is a half length of the diagonal of a rectangular inclusion and \( R \) is real constant. The debounded interface \( L \) and the bounded interface \( L' \) are mapped onto the portions \( \gamma' \) and \( \gamma' \) respectively of the unit circle. The four corners of the rectangle in the \( z \)-plane correspond to the points \( \xi_p (p=1,2,3,4) \) on the \( \zeta \)-plane where

\[
\begin{align*}
\xi_1 &= e^{i0} = 1 ; \\
\xi_2 &= e^{i\pi} ; \\
\xi_3 &= e^{i\pi} = -1 \\
\xi_4 &= e^{i(\pi+\ln)} = -e^{i\pi}
\end{align*}
\]  

...(5.2.2)

Employing the theory of Muskhelishvili [31] the stress and displacement in the polar coordinate system \((r, \theta)\) in the \( \zeta \)-plane are given as follows:

\[
\begin{align*}
\widehat{rr} + \widehat{r\theta} &= 2 \left[ \phi(\zeta) + \phi(\xi) \right] \\
\widetilde{rr} + i\widetilde{r\theta} &= \phi(\zeta) + \phi(\xi) - \frac{\xi}{\xi w'(\xi)} \left[ w(\zeta) \phi'(\zeta) + w'(\zeta) \psi(\zeta) \right] \\
2G \frac{\partial}{\partial \theta} (u+iv) &= i\xi \psi'(\xi) \left[ \alpha \phi(\zeta) - \overline{\phi(\xi)} \right] + i\overline{\xi} \left[ w(\zeta) \phi'(\zeta) + w'(\zeta) \psi(\zeta) \right]
\end{align*}
\]

...(5.2.3)

...(5.2.4)

where \( \alpha \) is given by (2.3.9). The boundary conditions on the interface between the inclusion and the matrix are
\( \overline{rr} + i\overline{r} = 0, \ldots, \text{ on } L \) \hspace{1cm} (5.2.5)

\( u + iv = 0, \ldots, \text{ on } L' \) \hspace{1cm} (5.2.6)

Equation (5.2.5) represents the interface crack and equation (5.2.6) shows that the inclusion is perfectly bounded with the matrix. The function \( w(\zeta) \) has been expressed in the form of Taylor's series and subsequently the complex potentials \( \phi(\xi) \) and \( \psi(\xi) \) are found out, thereby giving stresses and strains. The stress intensity factors for an interface crack on the short side of a rectangular rigid inclusion and of an interface crack extended to a long side of a rectangular rigid inclusion were found out.

5.3 AN ELLIPTIC RING UNDER CONCENTRATED FORCES

The problem of determination of elastic stresses in a confocal elliptic ring under all round uniform tension has been discussed by Ahmed [1]. The method of Milne-Thomson [28] has been employed to solve the first fundamental problem of an eccentric annulus held in equilibrium under concentrated forces at points where the annulus has the extreme thicknesses. In this section an attempt is made to find the stresses and strains in a confocal elliptic ring which is kept in equilibrium by two standard concentrated forces applied at the extremities of the major axis of the exterior bounding ellipse see Fig(4).
Fig. 4 Mapping of a confocal elliptic ring on to a concentric annulus.
Solution of the problem:

The confocal mapping which transforms an ellipse in the $z$-plane onto a circle in the $\xi$-plane is given by

$$z = m(\xi) = c(\xi + \frac{1}{\xi}), \xi = e^\xi + i\eta \quad \ldots (5.3.1)$$

From equation (5.3.1), we get

$$z = m(\xi) = c \cosh(\xi + i\eta) \quad \ldots (5.3.2)$$

Separating the real and imaginary parts, we get

$$\begin{align*}
  x &= c \cosh \xi \cos \eta \\
  y &= c \sinh \xi \sin \eta
\end{align*} \quad \ldots (5.3.3)$$

Thus we see that the curve $\xi = \xi_0$ (a constant) represents an ellipse whose semi-major axis and semi-minor axis are of lengths given as follows:

$$\begin{align*}
  a &= c \cosh \xi_0 \\
  b &= c \sinh \xi_0
\end{align*} \quad \ldots (5.3.4)$$

This ellipse is transformed onto a circle of radius $e^{\xi_0}$. We assume that the elliptic ring are bounded by two ellipses say $\xi = \xi_1$ and $\xi = \xi_2$ where $\xi_1$ and $\xi_2$ are positive constants and $\xi_2 > \xi_1$. The region between these confocal ellipses is mapped onto the concentric circular annulus between circles $\Gamma_1$ and $\Gamma_2$ of radii $\alpha$ and $\beta$ where

$$\begin{align*}
  \alpha &= e^{\xi_1}, \beta = e^{\xi_2}, (\beta > \alpha)
\end{align*} \quad \ldots (5.3.5)$$

The region inside the ellipse $\xi = \xi_2$ is mapped onto the region inside the circle $\Gamma_2$ and the region outside the ellipse $\xi = \xi_1$.
onto the region outside the circle $\Gamma_1$. The circle $\Gamma_2$ is described in the same sense as the ellipse $\zeta = \zeta_2$ and similarly the circle $\Gamma_1$ in the same sense as the ellipse $\zeta = \zeta_1$.

The boundary conditions are

$$\begin{align*}
\zeta = 0 & \quad \text{on } \Gamma \\
\zeta \eta = 0 & \quad \text{on } \Gamma
\end{align*}$$  \hspace{1cm} \ldots(5.3.6)

where $\Gamma$ is the boundary of the region under consideration.

At points where concentrated forces act, stresses become infinite. From (5.3.1) we see that

$$m'(\zeta) = c(1-1/\zeta^2)$$  \hspace{1cm} \ldots(5.3.7)

Thus we see that at $\zeta = \pm 1$, $m'(\zeta) = 0$ and the mapping function is not analytic at these points. Thus at these points the transformation ceases to be conformal. However this difficulty is overcome by the fact that at these points $\zeta = \pm 1$, plastic yielding occurs in a very short neighborhood of these points due to infinity large stresses. For the sake of mathematical investigation, the force is supposed to be uniformly applied around these points extending to a small length $\epsilon$ on either side of the points. See Fig(4). The points $A_1, A, A_2, B_1, B$ and $B_2$ in the $z$-plane are mapped onto the corresponding points in the $\zeta$-plane.
In the $z$-plane $z_A = a$, $z_A = z_A + dz$ and $z_A = z_A - dz$ where $dz = i \epsilon a$. Therefore,

\[
\begin{align*}
    z_A &= a \\
    z_{A_2} &= a + i \epsilon a \\
    z_{A_1} &= a - i \epsilon a
\end{align*}
\]  

...(5.3.8)

Similarly

\[
\begin{align*}
    z_B &= -a \\
    z_{B_1} &= -a + i \epsilon a \\
    z_{B_2} &= -a - i \epsilon a
\end{align*}
\]  

...(5.3.9)

The uniformly distributed force acting on the region $A_1 A_2$ is $F/2i \epsilon a$. Similarly the uniformly distributed force on the region $B_1 B_2$ is $F/2i \epsilon a$. In the $\zeta$-plane, on the circle $\Gamma_2, \zeta = \rho e^{i \eta} = \sigma$ (say). Hence $\sigma_A = \beta$.

But from equation (5.3.1)

\[
dz = m'(\zeta)d\zeta = c(1 - 1/\zeta^2)d\zeta
\]

On $\Gamma_2, \zeta = \sigma$. Therefore

\[
dz = c(1 - \frac{1}{\sigma^2})d\sigma \quad \text{or} \quad d\sigma = \frac{\sigma^2}{c(\sigma^2 - 1)} dz
\]

At $A$ in the $\zeta$-plane, $\sigma = \beta$

Therefore

\[
d\sigma = \frac{\beta^2}{c(\beta^2 - 1)} i \epsilon a \quad \ldots(5.3.10)
\]
Hence
\begin{align*}
\sigma_A &= \beta \\
\sigma_{A_2} &= \beta + d\sigma \\
\sigma_{A_1} &= \beta - d\sigma \\
\end{align*}

...(5.3.11)

Similarly
\begin{align*}
\sigma_B &= -\beta \\
\sigma_{B_1} &= -\beta + d\sigma \\
\sigma_{B_2} &= -\beta - d\sigma \\
\end{align*}

...(5.3.12)

where \(d\sigma\) is given by (5.3.10)

following Milne-Thomson [28], the complex potential \(W(\zeta)\) is given by

\[m'(\zeta)W(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{2[-p(\sigma) + is(\sigma)]m'(\sigma)}{\sigma - \zeta} \, d\sigma + \psi(\zeta)\]  

...(5.3.13)

The form of \(\psi(\zeta)\) will be obtained by considering the singularities of \(m'(\zeta)W(\zeta)\) in the region \(R\). Hence we can write

\[\psi(\zeta) = A + \frac{B}{\zeta} + \frac{C}{\zeta^2}\]  

...(5.3.14)

The first integral of (5.3.13) is identically equal to zero as the boundary \(\Gamma_1\) is free from load. For the second integral we have

\[I = I_1 + I_2\]  

...(5.3.15)

where
Evaluating $I_1$ and $I_2$ and simplifying we get $I_1$ using the logarithmic series in the form $\log (1+\epsilon \theta) = \epsilon \theta$ Hence, the complex potential $W(\zeta)$ is given by

$$m'(\zeta)W(\zeta) = \frac{2FB}{\pi} \frac{1}{\beta^2 - \zeta^2} + A + B_0 + C_0 \frac{1}{\zeta^2}$$

...(5.3.17)

($\zeta$ in $L$ or $R$)

The unknown constants $A, B_0, C_0$ will be obtained from the "compatibility identity" of rings given as follows:

$$\left[ \frac{\beta^2}{\zeta^2} m'\left(\frac{\beta^2}{\zeta^2}\right) - \frac{\alpha^2}{\zeta^2} m'\left(\frac{\alpha^2}{\zeta^2}\right) \right] W(\zeta) - \left[ m\left(\frac{\beta^2}{\zeta^2}\right) - m\left(\frac{\alpha^2}{\zeta^2}\right) \right] W'(\zeta) = \frac{\alpha^2}{\zeta^2} m'\left(\frac{\beta^2}{\zeta^2}\right) W(\zeta) - \frac{\beta^2}{\zeta^2} m'\left(\frac{\alpha^2}{\zeta^2}\right) W(\zeta)$$

...(5.3.18)

The constants $A, B_0$ and $C_0$ being known, the complex potential $W(\zeta)$ is completely known. Hence the problem is theoretically solved. The stresses $\dot{\xi}$ and $\eta$ were determined and found to satisfy the boundary conditions

$$\begin{bmatrix} \dot{\xi} = 0 \\ \dot{\eta} = 0 \end{bmatrix}$$

on $\Gamma_2$

...(5.3.19)
Thus on the boundary $\Gamma_2$,

$$\Omega = \hat{\xi} \hat{\eta} + \hat{\eta} = \hat{W}(\zeta) + \hat{W}(\xi)$$

or

$$\hat{\eta} = \hat{W}(\zeta) + \hat{W}(\xi)$$
on $\Gamma_2$

Hence the stress intensity factor $[\hat{\eta}]_{\xi=\xi_2}$ is given as

$$[\hat{\eta}]_{\xi=\xi_2} = \left[ \frac{2F}{\pi} \beta(\beta^2+1) + 2A_0 \beta^2(\beta^2-\cos2\eta) + 2B_0 \beta(\beta^2-1)\cos\eta \right. \left. + 2C_0(\beta^2 \cos2\eta-1) \right] / \left[ c(\beta^4-2\beta^2 \cos2\eta+1) \right] \ldots (5.3.20)$$
In solid mechanics, the occurrence and propagation of cracks in structures are drawing much attention of the workers in this filed. The fundamental problem is to determine the strength of an elastic structure having a crack. Also how to prevent the propagation of crack tip is a matter of interest and importance. Generally a disc or a plate is weakened by the presence of a hole. However if the hole is stiffened in some manner, its strength is increased. The theory involved in a recent work of a rectangular inclusion by Miyazaki et al. [29] can be employed to solve problems of inclusion of other type of curvilinear boundary.

The powerful variational method and the finite element method can be employed to solve the problems of stress determination in delicate type of bodies like an eccentric annulus which is crescent-shaped. The efficiency of the method can be compared with that of the complex variable as the results are already known.
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