CONTRIBUTIONS TO
THE THEORY OF ENTIRE AND MEROMORPHIC FUNCTIONS

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By
Jodh Pal Singh

DEPARTMENT OF MATHEMATICS AND STATISTICS,
ALIGARH MUSLIM UNIVERSITY,
ALIGARH.
I wish to place on record my deep sense of gratitude and indebtedness to Professor M. A. Kazim, Head, Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh, India, for his esteemed and helpful supervision of my research work.
CONTRIBUTIONS

TO

THE THEORY OF ENTIRE AND MEROMORPHIC FUNCTIONS
The present thesis entitled "Contributions to the theory of Entire and Meromorphic functions", embodies research work carried out by me at this university during the period 1966 to 1970 under the supervision of Dr. Shankar Hari Dwivedi, at present visiting professor in Oklahoma university, U.S.A., and later on under Prof. M.A. Kazim, Head, Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh. I started the research work under Dr. Shankar Hari Dwivedi and had done most of the work with him, but I could get admission as a research scholar only in December, 1967, and had to clear M.Phil. All these things delayed its submission.

The thesis consists of eight chapters in all. Chapter I is mainly a general introduction consisting of a brief historical background of the subject.

Chapter II is devoted to the study of the mean values of an entire function represented by Taylor series. Firstly, we have found some properties of geometric means of an entire function and, secondly, have investigated the behaviour of the means $M_{\delta}(r)$ and $m_{\delta,k}(r)$ in relation to growth, and also expressed the order $\rho$ and lower order $\gamma$, in terms of $\{M_{\delta}(r,f); M_{\delta}(r,f^{(m)})\}$ and $\{m_{\delta,k}(r,f); m_{\delta,k}(r,f^{(m)})\}$. Some material of this chapter appears as a paper entitled "On the Means of an Entire function and its Derivatives", published in Rev. de la Faculte des Sciences de l' Univ. d' Istanbul Ser A 33 (1968), 63 - 67.
Chapter III deals with the mean values of an entire function represented by a Dirichlet series. Theorem 3.1 generalises a result of Juneja [31]. Our Theorem 3.4 is not only more general than the theorem of Juneja and Awasthi [32], but has a proof different from theirs as well as shorter and more widely applicable. Theorem 3.8 includes a theorem of Jain [27] which in turn includes a theorem of Juneja and also a theorem of Gupta [21]. The method of proof of our results is quite different technically from that of Jain [ibid]. We have also established certain relationships between two or more entire functions represented by Dirichlet series. The last theorem of this chapter supplies some information regarding the comparative growth of mean values. A portion of this chapter will appear in The Aligarh Bulletin of Mathematics Vol. I as a paper entitled " On the mean values of an Entire function represented by a Dirichlet series ".

In Chapter IV, we have defined a new mean value of an entire function of two or more complex variables and then studied the comparative growth of this mean value relative to certain auxiliary functions. For simplicity, we have considered only two variables, though our considerations can be extended to the case of several complex variables. A portion of this chapter has been accepted for publication in " Riv. Mat. Univ. Parma " as a paper entitled " On the mean values of an entire function of several complex variables ".

(II)
In chapter V, we have obtained some growth relations of entire functions. Our theorem 5.3 includes the well known results of Shah [72],[73], Shah and Khanna [85] and Clunie [16]. We have also obtained a theorem concerning the proximate order $B$ of an entire function. In Theorem 5.5, we have extended a result of Mandelbrojt and Gergen [50] to lower order in a horizontal strip. Again, our method of proof is entirely different from those of Rahman [50], [51], Srivastava [95] and Sunyer i Balaguer [101].

A portion of this chapter has been published in "Rev. de la Faculte des Sciences de l' Univ. d' Istanbul Ser A 33 (1968)," as a paper entitled "A theorem on proximate order $B$ of an entire function", while another portion has been communicated for publication in revised form in Tohoku Math. J. under the title "A theorem on entire function of Infinite order".

The object of chapter VI is to study the relationship of growth of an entire function $f(z)$ with the rate of growth of $E_n^{1/n}(f)$. Our theorems 6.1 and 6.2 extend the result of Varga [107] to lower order $\lambda$. In Theorem 6.3, we consider the growth of $E_n^{1/n}(f)$ in case $f(x)$ has an analytic extension $f(z)$ which is an entire function of irregular growth. In Theorem 6.4, we obtain a relationship of $t_{\lambda}$ with the rate of growth of $E_n^{1/n}(f)$. The last theorem supplies information on how the kth proximate type $T_k$ is related to the rate of growth of $E_n^{1/n}(f)$.

In Chapter VII, we have studied the relations among entire functions of finite non-zero orders and types and relations among...
the coefficients in the Taylor expansion of an entire function of several complex variables. Our Theorem 7.5 extends a result of S.K. Singh [89] to two complex variables. The case of arbitrary finite number of variables can be examined in the same way. A portion of this chapter appears as a paper entitled "On the order and type of entire functions of several complex variables" in Riv. Mat. Univ. Parma (2) 10 (1969), 1-11.

The last chapter of the thesis is devoted to the study of the rate of growth of Ahlfor-Shimizu characteristic function $T_{o}(r)$ of a meromorphic function $f(z)$ and the area function $S(r)$ of the image of the disc $|z| < r$ on the Riemann sphere under the mapping $f(z)$. Our Theorem 8.1 extends the results of S.K. Singh [90]. We have used for this purpose the general kind of comparison function which on specialization gives the results of S.K. Singh [ibid]. Towards the end of this chapter, we have established a result on simultaneous convergence and divergence of two integrals involving $T_{o}(r)$ and $S(r)$, respectively. The results of this chapter, as also, most of the results of all the previous chapters, have been submitted for publication.

A vivid picture of how the thesis grows, can be had by looking into the table of contents. We have also supplied short introductions to each chapter to meet the requirements of Clause VIII of Chapter XXV of Academic Ordinances. Every chapter is followed by relevant references with a comprehensive
bibliography at the end of the thesis. The notations used throughout are standard and customary.

It is my pleasant duty to express my deep sense of gratitude to Prof. M.A.Kazim, who granted me admission as a bonafide research scholar in this department. He was also kind enough to award me a university fellowship since September 1969. Moreover, Prof. M.A.Kazim has been constantly encouraging me and has looked thoroughly into the whole of my thesis.

Dr. Shankar Hari Dwivedi has been a great source of inspiration and a pillar of support to me throughout my research work. He gave helpful suggestions and constructive criticisms on my work even after he left for Oklahoma University, U.S.A. I am greatly indebted to him for his generous kindness.

I must add that this work could not have been possible but for my parents who continued to support me financially during the three and half years of my research work. It is their patience and boundless affection which enabled me to complete this research programme. My friends have also been a source of strength and encouragement to me. Dr. Niranjan Singh of the Kurukshetra University, has been particularly kind. I heartily thank him and all others.

I am grateful to the Librarian, Mathematics Seminar, for his most helpful attitude and to Mr. Mukhtar Nabi Khan for his careful and excellent typing of this thesis.

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Jodh Pal Singh

JODH PAL SINGH
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CHAPTER I

INTRODUCTION

1.1 ORIGIN. The general theory of entire functions (Integral functions), the functions which are regular throughout the finite part of the complex plane, starts with the work of Weierstrass [109] 1876. Weierstrass [109] extended the fundamental theorem concerning the factorization of a polynomial to cover the case of such functions. He further pointed out the following:

' The range of the values taken by a non-constant entire function is everywhere dense in the plane \( |f(z)| < \infty \).'

It was this result which was subsequently developed in 1879 by Picard, known as classical theorem of Picard [52]. The earliest work in this direction was due to Borel [10] and subsequent introduction of new methods has thrown much light on obscure points in the theory of analytic functions.

1.2 THE CONSTRUCTION OF AN ENTIRE FUNCTION WITH GIVEN ZEROS.

An important contribution of Weierstrass in the study of entire functions, is the following:

' If \( z_1, \ldots, z_n \) be any sequence of numbers whose only limiting point is at infinity, it is possible to construct an entire function which vanishes at each of these points and nowhere else.'

The construction involves the use of Weierstrass’s primary factors:
Each primary factor vanishes when \( u = 1 \), but the behaviour of \( E(u,p) \) as \( u \) approaches zero depend on \( p \). For \( u \leq \frac{1}{k} \), \( k > 1 \), we get
\[
\left| \log E(u,p) \right| \leq \frac{k}{k-1} u^{p+1}.
\] .... (1.2.2)

It is this inequality which determines the convergence of a product of primary factors.

Let us suppose that the sequence of zeros \( \{z_n\} \) of \( f(z) \) be arranged in the order of non-decreasing sequence of their moduli, multiple zeros being repeated in the set according to their orders. Then, since \( |z_n| = r_n \) increases indefinitely with \( n \), we can always find a sequence of positive integers \( \{p_n\} \) such that the series
\[
\sum_{n=1}^{\infty} \left( \frac{r}{r_n} \right)^{p_n}
\] .... (1.2.3)
converges for all values of \( r \).

Let
\[
f(z) = \prod_{n=1}^{\infty} E\left( \frac{z}{z_n}, p_{n-1} \right)
\] .... (1.2.4)

Where \( p_n \) is chosen so that the series (1.2.3) is convergent for all values of \( r \). This determination of \( f(z) \) in this way is not unique because of a wide choice of \( \{p_n\} \). This difficulty was overcome by the introduction of canonical product.
Let
\[ f(z) = \sum_{n=1}^{\infty} E \left( \frac{z}{z_n} \right) \] .... (1.2.5)

Where \( p \) is the smallest integer for which the series
\[ \sum \left( \frac{r}{r_n} \right)^{p+1} \] is convergent, the product (1.2.5) is called the Canonical product formed with the zeros of \( f(z) \) and \( p \) is called its genus.

However, this is suggested that it is possible to factorize any entire function \( f(z) \) in the form
\[ f(z) = z^m e^{g(z)} P(z) \] .... (1.2.6)

where \( m \) is a positive integer or zero, and \( g(z) \) is an entire function. Such a factorization, however, is not of much use, as very little is known about the function \( g(z) \). A more precise result in this direction relating to entire function of finite order has been obtained by Hadamard [22] in 1893.

An entire function \( f \) is said to be of **finite order** if there is a positive number \( A \) such that as \( |z| = r \to \infty \)
\[ f(z) = \mathcal{O} \left( e^{r^A} \right). \] .... (1.2.7)

The lower bound \( \rho \) of such number \( A \) is called the order of the function \( f(z) \). We now state Hadamard's factorization theorem.

If \( f(z) \) is an entire function of finite order \( \rho \) with \( m \)-fold zeros at the origin, then
\[ f(z) = z^m e^{Q(z)} P(z) \] .... (1.2.8)
Where \( Q(z) \) is a polynomial of degree \( q \leq p \) and \( P(z) \) is the Canonical product of genus \( p \) formed with the zeros (other than \( z = 0 \)) of \( f(z) \) (Chandrasekharan [15] and Rajagopal [62] have given the different proofs of their factorization theorem).

The greater of the two integers \( p \) and \( q \) is called the genus of \( f(z) \).

1.3 THE GROWTH SCALE.

To characterize the growth of an entire function \( f(z) \) we consider the maximum modulus of a function \( f(z) \) defined as

\[
M(r) = \max_{|z| = r} |f(z)|.
\]

If \( f(z) \) is not constant \( M(r) \) grows more monotonically as \( r \) increases. The rate of growth of this function is an important characteristic of an entire function. It is known [45] that

'For an entire function which is not a polynomial, the maximum modulus \( M(r) \) grows faster than any power of \( r^t \).

The order \( \rho \) of an entire function \( f(z) \) is defined as

\[
\rho = \lim_{r \to \infty} \sup \frac{\log \log M(r)}{\log r}, \quad (0 \leq \rho \leq \infty). \quad \ldots \quad (1.3.1)
\]

Whittaker [110] in 1933, defined the lower order of \( f(z) \) as:

\[
\lambda = \lim_{r \to \infty} \inf \frac{\log \log M(r)}{\log r}, \quad (0 \leq \lambda \leq \infty). \quad \ldots \quad (1.3.2)
\]

If \( \rho = \lambda \), we say that \( f(z) \) is of regular growth.
For functions of a given order, a more precise characterization of the growth of $f(z)$ is given by the numbers $T$ and $t$, the type and the lower type respectively of $f(z)$ defined as:

$$T = \limsup_{r \to \infty} \inf_{r^\rho} \log M(r)$$

$$t = \limsup_{r \to \infty} \inf_{r^\rho} \log M(r)$$ \hspace{1cm} (0 < \rho < \infty) \hspace{1cm} \ldots \hspace{1cm} (1.3.3)

The function $f(z)$ is said to be of maximal, mean or minimal type according as $T = \infty$, $0 < T < \infty$ or $T = 0$ respectively.

The function $f(z)$ is said to be of very regular growth if

$$0 < A \leq \liminf_{r \to \infty} \frac{\log M(r)}{r^\rho} < \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} \leq B < \infty,$$

perfectly regular growth, if

$$T = t \hspace{1cm} (0 < T < \infty).$$

1.4 MAXIMUM TERM

Since $f(z)$ is regular in the finite $z$-plane

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \hspace{1cm} \ldots \hspace{1cm} (1.4.1)$$

has infinite radius of convergence.

The maximum term $\mu(r)$ in the Taylor expansion is defined as

$$\mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| r^n.$$

The rank $\nu(r, f)$ of the maximum term is the value of $n$ for which the maximum is attained, $\nu(r)$ is a non-decreasing unbounded function of $r$ and has only ordinary discontinuities.
Valiron [106] obtained the maximum term by constructing a Newton's polygon and showed that two functions with the same polygon have the same $\mu(r)$ and $\nu(r)$. Choosing a suitable comparison function, he has found the following integral representation of $\log \mu(r)$ ( [106], p. 31 ),

$$\log \mu(r) = \log \mu(r_0) + \int \frac{\nu(x)}{r} \, dx.$$ \hspace{1cm} (1.4.2)

A well known property of entire functions of finite order is that ([106], p. 32 - 33 ),

$$\log \mu(r) \sim \log M(r) \hspace{1cm} \ldots (1.4.3)$$

and

$$\limsup_{r \to \infty} \frac{\log \nu(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \mu(r)}{\log r} = \rho \hspace{1cm} \ldots (1.4.4)$$

In analogy with these results, Whittaker [110] has proved

$$\liminf_{r \to \infty} \frac{\log \nu(r)}{\log r} = \liminf_{r \to \infty} \frac{\log \mu(r)}{\log r} = \lambda \hspace{1cm} \ldots (1.4.5)$$

In 1942, Shah [72] has obtained the following result,

'If $f(z)$ be of order $\rho$, $0 \leq \rho \leq \infty$, then

$$\liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} . \hspace{1cm} \ldots (1.4.6)$$

In 1948, again Shah [77] proved

'If $\limsup_{r \to \infty} \frac{\nu(r)}{r^\rho} = c \hspace{1cm} \text{, then}$

$$d \leq \frac{c}{e} \leq \rho t \leq c, \hspace{1cm} \ldots \hspace{1cm} (i) \hspace{1cm} \ldots (1.4.7)$$

$$d \leq \rho t \leq d(1 + \log \frac{c}{d}) \leq c \hspace{1cm} \ldots (ii)$$
Further in 1950, Shah [78] proved

For every entire function of infinite order

$$\liminf_{r \to \infty} \frac{\log M(r)}{\nu(r)} = 0.$$  \hspace{1cm} (1.4.8)

This result was further generalised by Shah and Khanna [35], Clunie [16], Rahman [56], and Ahmad [3]. Our investigation in this direction is included in Chapter V.

Shah and Singh [86] have constructed examples to exhibit other possibilities of the result (1.4.5) in case when \( f \) is zero.

Further interesting cases are due to Shah ([79], [82],[83]) and Singh ([91],[92]).

1.5 DISTRIBUTION OF ZEROS. The investigations of the zeros of an entire function, their behaviour and properties have enriched the general theory of entire functions. Among the early results connecting the moduli of its zeros with the modulus of a function, we have one due to Jensen [29].

Let \( f(z) \) be analytic for \( |z| < R \), suppose that \( f(0) \) is not zero, and let \( r_1, r_2, \ldots, r_n \ldots \) be the moduli of the zeros of \( f(z) \) in the circle \( |z| < R \), arranged as a non-decreasing sequence. Then, if \( r_n \leq r \leq r_{n+1} \)

$$\log \left\{ \frac{1}{r_1 r_2 \cdots r_n} \right\} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad \ldots \quad (1.5.1)$$

The left hand side can also be written as \( \int_0^r \frac{n(x)}{x} \, dx + \log |f(0)| \), where \( n(r) \) denotes the number of zeros of \( f(z) \).
for $|z| \leq r$. Next, we define the exponent of convergence of the zeros of $f(z)$ as

$$\limsup_{r \to \infty} \frac{\log n(r)}{\log r} = \rho_1,$$

the series $\sum r^{-\alpha}$ being convergent for $\alpha > \rho_1$ and divergent for $\alpha < \rho_1$.

It was found that the order of an entire function is never less than the exponent of convergence of zeros (See [8], p.17). Thus we have the following relation among the order, the exponent of convergence, and genus of the set of zeros:

$$\rho_1 - 1 \leq p \leq \rho_1 \leq \rho$$

in all cases. Examples, illustrating the various possibilities have been constructed by Shah [71].

Polya [53] and Valiron [104] established a result regarding the magnitude of $n(r)$, which was subsequently generalised by Shah [70] and is as follows:

'If $f(z)$ has at least one zero (but $f(0) \neq 0$), then

$$\liminf_{r \to \infty} \frac{n(r)}{\log M(r)} \leq \liminf_{r \to \infty} \frac{\log n(r)}{\log r} = \lambda_1 \leq \rho_1.$$'

The number $\lambda_1$ is sometimes called the lower order of zeros (or lower exponent convergence of $f(z)$) and is defined as

$$\liminf_{r \to \infty} \frac{\log n(r)}{\log r} = \lambda_1.$$
1.6 GROWTH RELATIONS IN TERMS OF COEFFICIENTS.

The entire function $f(z)$ is regular in the whole finite $z$-plane and, therefore, it can be expanded by Taylor Series, with radius of convergence equal to infinity, as given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \ldots \ldots \quad (1.6.1)$$

where $a_n$'s are complex numbers. A systematic study of the growth properties have been done by the help of Taylor Coefficients. In this connection we may mention the following important result by Borel

$$\frac{1}{\rho} = \liminf_{n \to \infty} \frac{\log |a_n|}{n \log n}. \quad \ldots \ldots \quad (1.6.2)$$

The principal result regarding the entire functions of regular growth is the following:

'The necessary and sufficient condition that an entire function of order $\rho$ is of regular growth is that there exists a increasing sequence of positive integers $\{n_h\}$, satisfying the conditions:

$$\lim_{h \to \infty} \frac{\log n_{h+1}}{\log n_h} = 1, \quad \lim_{h \to \infty} \frac{\log |a_{n_h}|}{n_h \log n_h} = \frac{1}{\rho}. \quad \ldots \ldots \ (1.6.3)$$

(10) and (47).

'An entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of order $\rho$ $(0 < \rho < \infty)$ is of type $T$ $(0 < T < \infty)$, iff

$$\limsup_{n \to \infty} \left\{ n |a_n|^{\rho/n} \right\} = (e^\rho T). \quad (3) \quad \ldots \ldots \quad (1.6.4)$$
Shah pointed out that the lower limits in (1.6.2) and (1.6.4) need not necessarily be equal to the lower order and lower type respectively, while it is possible by imposing some extra conditions on \( \{a_n\} \). He ([75], [80]) has proved that if \( |a_n| \) is a non-decreasing function of \( n \), then

\[
\liminf_{n \to \infty} \frac{n \log n}{\log |a_{n+1}|} = \liminf_{n \to \infty} \frac{\log n}{\log |a_n|} = \lambda \quad \cdots (1.6.5)
\]

and

\[
\liminf_{n \to \infty} \frac{\log |a_n|}{\log n} = t. \quad \cdots (1.6.6)
\]

The result (1.6.4) has been extended by Levin ([45], p.42) to the more precise description of the growth as given by the proximate orders ([25], p.42).

Recently, Roux ([69]) has proved

\[
\frac{1}{\lambda} = \min_{\{n_h\}} \limsup_{h \to \infty} \frac{\log \frac{1}{\log |a_{n_h}|}}{n_h \log n_{h-1}} \quad \cdots (1.6.7)
\]

where \( \{n_h\} \) is an increasing sequence of positive integers \( (n_0 < n_1 < n_2 < \ldots )' \).

Quite recently Juneja and Singh ([33], [34]) have proved some results of type (1.6.7). Unfortunately their results are implied in the result of Roux ([69], Theorem 1).

1.7 APPROXIMATION OF ANALYTIC FUNCTIONS. We now give a very brief introduction to approximation of an analytic function connected with the work done in this thesis. A real or complex-valued function \( f \) defined on \( I = [-1,1] \) is called
analytic on I if there exists an analytic extension of $f$ onto some open set $G$ of the complex plane that contains $I$. We mean by this that there must exist on $G$ a single-valued analytic function that coincides with $f$ on $I$. If this extension exists, it is unique. Example of open sets that contain $I$, are the open elliptic discs $D_\sigma$, $\sigma > 1$ bounded by the ellipses $E_\sigma$ with the foci $\pm 1$.

From the properties of analytic functions, it follows that for each $f$ analytic on $I$, there exists a $\sigma_0 > 1$ characterized by the property that $f$ has an analytic extension onto the disc $D_{\sigma_0}$, but not onto any of the $D_\sigma$ for $\sigma > \sigma_0$. We must however, admit the possibility $\sigma_0 = \infty$, which is realised for the functions $f$ analytic in the whole plane (entire functions).

Let $f(x)$ be a real valued continuous function on $[-1,1]$ let

$$E_n(f) = \inf_{p \in \pi_n} \| f - p \|, \quad \text{for } n = 0,1,2,\ldots \quad \ldots(1.7.1)$$

where the norm is the maximum norm on $[-1,1]$ and $\pi_n$ denotes the set of all polynomials with real coefficients of degree at most $n$. Bernstein [6] proved the following:

"A function $f$, defined on $[-1,1]$ is analytic on this interval if and only if $\limsup_{n \to \infty} E_{1/n}(f) < 1$; and more exactly

$$\limsup_{n \to \infty} \frac{E_{1/n}(f)}{n} = \frac{1}{\sigma_0}. \quad \ldots(1.7.2)$$"
In particular, \( f \) has an analytic extension onto the whole plane if and only if

\[
\limsup_{n \to \infty} \frac{1}{n} E_n(f) = 0 \quad \cdots (1.7.3)
\]

Recently, Varga [107] has proved that

\[
\limsup_{n \to \infty} \frac{n \log n}{\log \frac{1}{E_n(f)}} = \rho \quad \cdots (1.7.4)
\]

where \( \rho \) is the non-negative real number if and only if \( f(x) \) is the restriction to \([-1,1]\) of an entire function \( f(z) \) of order \( \rho \). "

1.8 PROXIMATE ORDERS. The theory of proximate orders is a very powerful method in the study of entire and meromorphic functions.

Lindelöf [46] in 1902, proved the existence of a proximate order for an entire function \( f(z) \) known as Lindelöf proximate order, defined as:

(a) \( \rho(r) \) is real continuous and piecewise differentiable for \( r \geq r_0 \).
(b) \( \rho(r) \to \rho \) as \( r \to \infty \) (\( \rho \) is the order of the function and \( 0 < \rho < \infty \)).
(c) \( r \rho(r) \log r \to o \) as \( r \to \infty \), where \( \rho'(r) \) is either the right hand or the left hand derivative at points where they are different.
(d) \( \limsup_{r \to \infty} \frac{\log M(r,f)}{r \rho'(r)} = 1 \). (\[4\], p. 54)
When the order of the function is not an integer, we have Boutroux's proximate order [13].

In 1946, Shah [74] gave an alternative proof of the existence of Lindelöf proximate order. He also replaced the condition (d) by

\[(d') \quad \log M(r) \leq r^{\rho(r)} \quad \text{for all } r \geq r_0 \text{ and } \log M(r) = r^{\rho(r)} \quad \text{for a sequence of values of } r \to \infty.\]

Using the same method and technique, Shah [76] in 1948, proved the existence of another proximate order (which he named as Lower Proximate order \(\lambda(r)\) of \(f(z)\)) of lower order satisfying conditions analogous to (a), (b), (c) and in place of (d), and (d') the conditions

\[(e) \quad \log M(r,f) \geq r^{\lambda(r)} \quad \text{for } r \geq r_0, \quad (0 < \lambda < \infty),\]

\[(e') \quad \log M(r,f) = r^{\lambda(r)} \quad \text{for a sequence of values of } r \to \infty.\]

Utilising the properties of these proximate orders one may get some easy assertions:

(i) \(r^{\rho(r)}\) is an increasing function of \(r\) for \(r > r_0\).

(ii) \(\frac{(kr)^{\rho(kr)}}{r^{\rho(r)}} \to k^\rho\) uniformly as \(r \to \infty\).

(iii) \(\int_{r_0}^{r} t^{\rho(t)-1} dt \sim \frac{r^{\rho(r)}}{\rho}\).

(iv) \(r^{\lambda(r)}\) is an increasing function of \(r\) for \(r > r_0\).

(v) \(\frac{(kr)^{\lambda(kr)}}{r^{\lambda(r)}} \to k^\lambda\).
1.9 ENTIRE FUNCTION EXPRESSED AS A DIRICHLET SERIES.

A series of the form

$$\Phi(s) = \sum a_n e^{-s\lambda_n}$$  \hspace{1cm} (1.9.1)

$$a_n \neq 0 \quad (n = 0, 1, 2, \ldots), \quad 0 \leq \lambda_0 < \lambda_1 < \ldots \lim_{n \to \infty} \lambda_n \to \infty,$$

$$s = \sigma + it$$ is known as Dirichlet series. If the series is convergent for all finite $$s$$, $$\Phi(s)$$ is an entire function of $$s$$. For any real $$\sigma$$, suppose

$$M(\sigma) = M(\sigma, \Phi) = \sup_{-\infty < t < \infty} |\Phi(\sigma + it)|;$$

$$M(\sigma)$$ is non-decreasing function of decreasing $$\sigma$$, for $$\sigma \to -\infty$$, $$M(\sigma) \to +\infty$$. The following limits

$$\rho_R = \rho_R(\Phi) = \limsup_{\sigma \to -\infty} \frac{\log \log M(\sigma, \Phi)}{-\sigma},$$

$$\lambda_R = \lambda_R(\Phi) = \liminf_{\sigma \to -\infty} \frac{\log \log M(\sigma, \Phi)}{-\sigma},$$

are called the order and lower order of $$\Phi(s)$$, respectively, (in the sense of Ritt).

The Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n}$$  \hspace{1cm} (1.9.2)

has been considered for the first time by Ritt [68] in 1928 by introducing positive exponent in place of negative ones. This heightened the similarity between his results and corresponding known results on functions defined by Taylor Series.
The series (1.9.2) defined in its half plane of convergence is analytic function. Let \( \sigma_c \) and \( \sigma_a \) be respectively the abscissa of convergence and absolute convergence of \( f(s) \). Mandelbrojt [50] and Valiron [105] have obtained the following relations:

\[ 0 \leq \sigma_c - \sigma_a \leq E \]  
... (1.9.3)

and

\[ -E \leq \sigma_a + \limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n} \leq 0 \]  
... (1.9.4)

where

\[ \limsup_{n \to \infty} \frac{\log n}{\lambda_n} = E < \infty \]  
... (1.9.5)

For \( f(s) \), we define

\[ M(\sigma) = M(\sigma, f) = \max_{-\infty < t < \infty} |f(\sigma + it)| \]

and

\[ \mu(\sigma) = \max_{n \geq 1} \log e^{(\sigma + it)\lambda_n} \]

The function \( \log M(\sigma) \) tends to \( +\infty \) as \( \sigma \) tending to infinity ([68], p. 77) and is convex ([17], p. 237).

Ritt [68] has defined the finite linear order* \( \rho \) of an entire function \( f(s) \), namely

\[ \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho \quad (0 \leq \rho \leq \infty) \]  
... (1.9.6)

and corresponding lower order (R) \( \lambda \) as follows

\[ \liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \lambda \]  
... (1.9.7)

Sigimura [100] introduced the other definitions of order and lower order, as

*According to Mandelbrojt ([50], p. 216) we call \( \rho \) as the Ritt order of \( f(s) \).
While it is obvious that \( p_* < p \) and \( \lambda_* < \lambda \), there are entire Dirichlet Series for which \( p_* < p \) and \( \lambda_* < \lambda \) ([100] Satz 4 which gives such a series with \( p_* < p \)). Hence, we have generally to distinguish between the limits of (1.9.6), (1.9.7) and those of (1.9.8), as well as between the types of \( f(s) \) belonging to the same order \( p (0 < p < \infty) \) and types of \( f(s) \) belonging to the same order \( p_* (0 < p_* < \infty) \). The types \( T, t \) associated with \( p \) and types \( T_*, t_* \) associated with \( p_* \) are defined in the usual way as follows:

\[
\limsup_{\sigma \to \infty} \inf \frac{\log M(\sigma)}{e^{\rho \sigma}} = T \quad (0 < \rho < \infty),
\]

\[
\limsup_{\sigma \to \infty} \inf \frac{\log M(\sigma)}{e^{\rho_* \sigma}} = T_* \quad (0 < \rho_* < \infty).
\]

The function \( f(s) \) is said to be of linearly regular or linearly irregular growth according as \( \rho = \lambda \) or \( \rho \neq \lambda \) respectively.

The maximum term \( \mu(\sigma) \) of an entire Dirichlet Series \( f(s) \) and rank of the maximum term plays an important role in the development of the theory of entire functions represented by Dirichlet Series. Yung ([111], p. 67) has established

\[
\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_*(t) \, dt \quad \ldots \quad (1.9.11)
\]

valid for all \( \sigma \geq \sigma_0 \), and for functions of finite order (R) he has, further, proved that
\[
\log \mu(\sigma) \sim \log M(\sigma) \quad \ldots \quad (1.9.12)
\]

provided
\[
\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = 0. \quad \text{Sugimura ([100], p. 73) pointed out that a sufficient condition for the truth of the asymptotic equality is known only in the form}
\]
\[
o \leq \limsup_{n \to \infty} \frac{\log n}{\log \lambda_n} = E \leq \infty.
\]

1.10 MEAN VALUES OF ENTIRE FUNCTIONS. There is another approach to study the growth of an entire function by the mean values of the modulus of the function. In 1915, Hardy ([25], p. 269) introduced the mean values of the modulus of an analytic function \( f(z) \) and defined as
\[
I(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta, \quad 0 < r = |z| < R \quad \ldots \quad (1.10.1)
\]

He also showed that \( I(r) \) is an increasing convex function of \( r \) and \( \log I(r) \) is a convex function of \( \log r \).

In 1956, Rahman [57] extended a result of Polya and Szegö ([55] problem 66, p. 10) and proved that
\[
\text{If } f(z) \text{ be an entire function of order } \rho \text{ and lower:}
\]
order 2 and
\[
I_{\rho}(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta, \quad \ldots \quad (1.10.2)
\]
\[
m_{\rho, k}(r) = \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} |f(re^{i\theta})|^\delta x^k \, dx \, d\theta, \quad \ldots \quad (1.10.3)
\]
\[
\limsup_{r \to \infty} \inf \left\{ \frac{I_{\rho}(r)}{m_{\rho, k}(r)} \right\}^{1/\log r} = I_{\rho,k}, \quad \ldots \quad (1.10.4)
\]
where $\delta$ and $k$ being any positive numbers, then

$$I_{\delta,k} = e^\delta \text{ and } I_{\delta,k} = e^{\lambda k}.$$ 

However, the above results for $\delta = 2, k = 1$ were earlier obtained by Shah [87].

To prove (1.10.5), he used the following two lemmas:

(i) If $f(z)$ is regular in $|z| < R$ and if $z = re^{i\theta},$

$$|f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2-r^2)|f(Re^{i\theta})|^{\delta}}{R^2-2rrR \cos(e^{-\theta})+r^2} \, d\theta. \quad ... (1.9.6)$$

(ii) Let

$$\lim_{r \to \infty} \sup \frac{\log \log I_{\delta}(r)}{\log r} = A$$

$$\lim_{r \to \infty} \inf \frac{\log \log m_{\delta,k}(r)}{\log r} = \beta,$$

then

$$A = \alpha = \beta, \quad B = \beta = \gamma. \quad ... (1.10.7)$$

Srivastava ([94], p. 230) has proved that for every entire function of order $\rho$ and lower order $\lambda$

$$\lim_{r \to \infty} \inf \frac{\log[r \{I_{\delta}(r,f^{(1)})/I_{\delta}(r,f)\}^{\delta}]}{\log r} = \rho \quad ... (1.10.8)$$

where $\delta > 1$ and $I_{\delta}(r,f^{(1)})$ is the mean value of $|f^{(1)}(z)|,$

the derivative of $f(z)$ for $|z| = r.$

In 1963, Lakshminarasimhan [4] pointed out that the result (1.10.8) is incorrect for $\delta = 1,$ as far as its proof is
concerned. He ([43], [43]) has supplied the proof of (1.10.8).

Rahman [59] further defined the mean values of \(|f(z)|\) for \(|z| = r\) as

\[
M_\delta(r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \right\}^{1/\delta}, \quad 0 < \delta < \infty
\]

... (1.10.9)

and

\[
m_{\delta,k}(r) = \frac{1}{r^{k+1}} \int_0^r x^k M_\delta(x) \, dx, \quad -1 \leq k < \infty
\]

... (1.10.10)

Lakshminarasimhan [42] proved that
1. \(r^{k+1} M_\delta(r)\) is a convex function of \(r^{k+1}\)
2. \(m_{\delta,k}(r)\) is an increasing function of \(r\).

He further proved that

"For an entire function \(f(z)\) of order \(\rho\) and lower order \(\lambda\), \(0 \leq \lambda, \rho \leq \infty\)

\[
\lim_{r \to \infty} \sup \left\{ \frac{M_\delta(v)}{m_{\delta,k}(r)} \right\}^{\log r} = e^\rho
\]

... (1.10.11)

Bose and Stivastava [12] extended (1.10.5) to the product of two or more entire functions but Shah pointed out that the result are not correct and constructed a simple example*.

The result (1.10.8) was further generalised for \(m\)-th derivative of \(f(z)\) by Juneja [30]. He also proved (1.10.8) for \(0 < \delta < 1\) for the case of limit superior. Our further investigations in this direction appear in Chapter II.

* See MR [33 \# 7537].
1.11 GEOMETRIC MEAN VALUES OF AN ENTIRE FUNCTION.

The geometric means of an entire function provide another approach to the study of growth of function. Polya and Szegö ([54], p.144) in 1925, defined the geometric mean values of the modulus of an entire function as:

\[ G(r) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \right\} \]  
\[ \ldots (1.11.1) \]

and

\[ g(r) = \exp\left\{ \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \log |f(xe^{i\theta})| \, x \, dx \, d\theta \right\} \]  
\[ \ldots (1.11.2) \]

respectively, for \(|z| = r\) and \(|z| \leq r\).

G(r) is an increasing function of r and log G(r) is convex function of log r.

Problems concerning geometric means were first considered by Polya and Szegö ([55], p.10). Shah [81] later sharpened some of these results, in fact he has proved

\[ \exp(-\frac{1}{2}) \leq \liminf_{r\to\infty} \left( \frac{g(r)}{G(r)} \right)^{\frac{1}{n(r)}} \leq \exp\left( -\frac{1}{2+\lambda_1} \right) \]
\[ \leq \exp\left( -\frac{1}{2+\lambda_1} \right) \leq \limsup_{r\to\infty} \left( \frac{g(r)}{G(r)} \right)^{\frac{1}{n(r)}} \leq 1 . \]  
\[ \ldots (1.11.3) \]

In 1966, Kumar [40] has defined the geometric mean \( g_k(r) \) of \(|f(z)|\) for \(|z| \leq r\) as

\[ g_k(r) = \exp\left\{ \frac{k+1}{2\pi k+1} \int_0^{2\pi} \int_0^r \log |f(xe^{i\theta})| \, x^k \, dx \, d\theta \right\}, 0 < k < \infty. \]  
\[ \ldots (1.11.4) \]
We have studied some properties of these mean values of $f(z)$. The results are included in Chapter II.

1.12 Mean Values of Entire Functions Represented by Dirichlet Series.

The study of growth of entire function defined by Dirichlet Series, by way of mean is much more complicated. The general mean values of $f(s)$ are defined as:

$$
\left\{ I_\sigma (\sigma ) \right\} = A_\sigma (\sigma, f) = A_\sigma (\sigma )
$$

$$
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta dt, \; 0 < \delta < \infty, \ldots (1.12.1)
$$

and

$$
V_{\delta, k}(\sigma ) = \lim_{T \to \infty} \frac{1}{2T e^{kT}} \int_{-T}^{T} |f(x + it)|^\delta e^{kx} dx dt, \; 0 < k < \infty
$$

$$
= \frac{1}{e^{k\sigma}} \int_{0}^{\infty} A_\sigma (x) e^{kx} dx . \ldots (1.12.2)
$$

The mean values given by (1.12.1) and (1.12.2), and their properties, for a particular case if $\delta = 2, \; 0 < k < \infty$, were first considered by Kamthan ([37], [38]). The mean value given by (1.12.1) for $\delta = 2$ was almost simultaneously discussed by Gupta [21].

In 1966, Reddy [64] pointed out that Kamthan's additional condition on $\lambda_n$ needlessly restrictive, while Gupta's additional condition on $\gamma_n$ can be justified though his reasoning is faulty.
Later on, developing an analogous of Poisson formula for Dirichlet Series Kamthan [39] proved that order \( R \) of \( f(s) \) can be expressed in terms of \( A_\delta(\sigma, f) \) for all \( \delta > 1 \). In fact, he has established the following results

\[
A_\delta(\sigma, f) \leq \left\{ M(\sigma) \right\}^\delta < (1)A_\delta(\sigma+\gamma, f) \sigma \geq \sigma_0, \gamma > \sigma,
\]

and

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log A_\delta(\sigma, f)}{\sigma} = \lambda, (\sigma \leq \lambda \leq \rho \leq \infty).
\]

Quite recently Jain [26] has proved that \( \log A_\delta(\sigma) \) is a convex function of \( \sigma \). He has deduced some interesting results.

In Chapter III, we have obtained a number of properties of mean values of entire functions represented by Dirichlet Series.

1.13 ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

We now give a very brief introduction to entire functions of several complex variables connected with the work done in this thesis. Let us consider \( m \)-dimensional vector space \( \mathbb{C}^m \) over the field \( \mathbb{C} \) of complex numbers \( m > 1 \).

Given a point \( a = (a_1, \ldots, a_m) \) and real number \( r_1, \ldots, r_m > 0 \) we call the set \( \mathcal{P} = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_j-a_j| < r_j, j=1,\ldots,m\} \), the open polydisc with polycentre \( a \) and polyradius \( r = (r_1, \ldots, r_j, \ldots, r_m) \). Similarly the closed polydisc with
polycentre a and polyradius \( r = (r_1, \ldots, r_j, \ldots, r_m) \) is the set
\[
\overline{P} = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_j - a_j| \leq r_j, \ j = 1, \ldots, m\}
\]
The set
\[
\Gamma = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid |z_j - a_j| = r_j, \ j = 1, \ldots, m\}
\]
is called the edge or distinguished boundary of \( P \) (and \( \overline{P} \)).

A complex-valued function \( f \), defined on an open subset of \( U \) of \( \mathbb{C}^m \) is holomorphic in \( U \), if for every point \( b \in U \), there exists an open polydisc \( P \subset U \) with polycentre \( b \), and a power series
\[
\sum_{k_1, \ldots, k_m = 0}^{\infty} a_{k_1, \ldots, k_m} (z_1 - b_1)^{k_1} \cdots (z_m - b_m)^{k_m}
\]
converging to \( f(z) \) at every point \( z \in P \).

Let
\[
f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}
\]
be a function of two complex variables \( z_1 \) and \( z_2 \), where coefficients \( a_{m_1, m_2} \) are complex numbers and is holomorphic in the closed polydisc \( \overline{P}(z_1, z_2) \subset \mathbb{C}^2 \mid |z_j| \leq r_j, \ j = 1, 2\).

The series (1.13.2) represent an entire function of two complex variables \( z_1, z_2 \) if it converges absolutely for all values of \( |z_1| < \infty \) and \( |z_2| < \infty \). M.M. Džrbašyan ([18], p.1) has shown that the necessary and sufficient condition for the series (1.13.2) to represent an entire function of variables \( z_1 \) and \( z_2 \) is
\[ \limsup_{m_1+m_2 \to \infty} \frac{1}{m_1+m_2} |a_{m_1,m_2}| = 0. \] \hfill (1.13.3)

Let \( \mathcal{G}_r \) be the family of closed polycircular domains in space \( (z_1,z_2) \) dependent on parameter \( r > 0 \) and possess the property that \( (z_1,z_2) \in \mathcal{G}_r \) if and only if \( \left( \frac{z_1}{r}, \frac{z_2}{r} \right) \in \mathcal{T}_1 \). The maximum modulus of the entire function \( f(z_1,z_2) \) is denoted by

\[ M_G(r,f) = \max_{(z_1,z_2) \in \mathcal{G}_r} |f(z_1,z_2)| \]

and the function will be called \( G \)-order and \( G \)-type respectively, if

\[ \phi_G = \limsup_{r \to \infty} \frac{\log \log M_G(r)}{\log r}, \hfill (1.13.4) \]

and

\[ \tau_G = \limsup_{r \to \infty} \frac{\log M_G(r,f)}{r \phi_G}. \hfill (1.13.5) \]

Denote

\[ \Phi_g(m_1,m_2) = \max_{(z_1,z_2) \in \mathcal{G}_r} \frac{1}{z_1^{m_1} z_2^{m_2}}. \]

A.A. Gol'dberg ([20], p. 146) has proved the following:

All orders \( \phi_G \) are equal and

\[ \phi = \phi_G = \limsup_{m_1+m_2 \to \infty} \frac{(m_1+m_2) \log(m_1+m_2)}{- \log \left| a_{m_1,m_2} \right|}, \hfill (1.13.6) \]

and

\( G \)-type \( T_G \) satisfies the correlation

\[ (e \phi \tau_G)^\frac{1}{\phi} = \limsup_{m_1+m_2 \to \infty} \left[ (m_1+m_2)^\frac{1}{\phi} \left\{ \Phi_g(m_1,m_2) |a_{m_1,m_2}| \right\} \right]^\frac{1}{m_1+m_2}. \]

\hfill (1.13.7)
In Chapter VII, we have obtained the relations between two or more entire functions and studied the relations between the coefficients in the Taylor expansion of entire functions and their orders and types.

1.14 MEAN VALUES OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

Srivastava and Kumar [97] have introduced the idea of the mean values in the case of functions of two complex variables. Their definitions for the mean values of function \( |f(z_1, z_2)| \) are:

\[
I(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| d\theta_1 d\theta_2, \quad ...(1.14.1)
\]

\[
m_{l_1, k_1, k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_{0}^{2\pi} \int_{0}^{2\pi} I(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1 dx_2, \quad ...(1.14.2)
\]

where \( k_1 \) and \( k_2 \) are any positive numbers.

They have also proved some results on the mean values of two or more entire functions. Unfortunately, the results are incorrect as pointed out by Shah.*

In 1968, Agarwal ([1], p. 51) has defined the mean value of the function \( f(z_1, z_2) \) as

\[
I_{\phi}(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^\phi d\theta_1 d\theta_2, 1 \leq \phi < \infty,
\]

and deduced some interesting results.

* See [MR 39 # 3022].
In Chapter IV, we have introduced a new mean value of of $f(z_1,z_2)$, namely, $J_{\delta},k_1,k_2(r_1,r_2;f)$ where $\delta \geq 1$ and $0 < k_1,k_2 < \infty$, and investigated some interesting properties of $I_{\delta}(r_1,r_2;f)$ and $J_{\delta},k_1,k_2(r_1,r_2;f)$.

1.15 MEROMORPHIC FUNCTIONS. Until now I have considered only entire functions. In this section we give a very brief introduction to the theory of meromorphic functions, i.e. functions whose only singularities, except at infinity, are poles.

In discussing meromorphic functions $f(z)$, we can no longer use the maximum modulus as a convenient tool for expressing the rate of growth of the function. Instead we use Nevanlinna's Characteristic $T(r,f)$. Let $f(z)$ be meromorphic and non-constant in the entire complex plane. Let

$$ \log^+ u = \log u \quad \text{if } u \geq 1 $$

$$ \log^+ u = 0 \quad \text{if } 0 < u \leq 1. $$

Thus, if $u > 0$, we have $\log u = \log^+ u - \log^+ \frac{1}{u}$. Hence, we put

$$ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta $$

$$ = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})|} \, d\theta. $$

Let

$$ N(r,f) = \int_0^r \frac{n(t,\infty) - n(o,\infty)}{t} \, dt + n(o,\infty) \log r $$
where $n(t,\infty)$ is the number of poles of $f(z) = f(re^{i\theta})$ in $|z| \leq t$, and

$$N(r,a) = N(r, \frac{1}{f-a}) = \frac{r}{f-a} \int_0^t \frac{n(t,a)-n(o,a)}{t} dt + n(o,a) \log r$$

where $n(t,a)$ denotes the number of zeros of $f(z)-a$ in $|z| \leq t$, $(o \leq |a| < \infty)$.

Also, let

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log + \frac{1}{|f(re^{i\theta})|} d\theta,$$

and

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log + \frac{1}{|f(re^{i\theta})-a|} d\theta.$$

The function

$$T(r,f) = m(r,f) + N(r,f)$$

is called the Navanlinna characteristic function of $f(z)$.

It is known [51A] that $T(r,f)$ is an increasing convex function of $\log r$.

The order $\rho$ and lower order $\gamma$ of the function $f(z)$ are defined by the following relations:

$$\limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} = \rho$$

$$\liminf_{r \to \infty} \frac{\log T(r,f)}{\log r} = \gamma$$

Since an entire function is a particular case of meromorphic function, the above formula for $\rho$ and $\gamma$ provides an
alternative definition for order and lower order of an entire function.

While concluding this brief introduction, we may point out that the only results which are connected with the work done in this and worthwhile to mention, have been referred.

REFERENCES

AGARWAL [1]; AHMAD [3]; HEBRSTEIN [6]; BOAS [8]; BOREL [10];
BOSE AND SRIVASTAVA [12]; BOUTROUX [13]; CARTWRIGHT [14];
CHANDRASEKHARAN [15]; CLUNIE [16]; DOETSCH [17]; DŽERBAŞYAN [18];
GOLDBERG [20]; GUPTA [21]; HADAMARD [22]; HARDY [23]; JAIN [26];
JENSEN [29]; JUNEJA [30]; JUNEJA AND SINGH [33],[34]; KAMTHAN [37],
[38]; KUMAR [40]; LAKSHMINARASIMHAN [41], [42], [43]; LEVIN [45];
LINDLÖF [46]; [47]; MANDELBROJT [50]; NEVANLINNA [51A];
PICARD [52]; POLYA [53]; POLYA AND SZEGÖ [54],[55]; RAHMAN [56],
[57],[58]; RAJAGOPAL [62]; REDDY [64]; RITT [68]; ROUK [69];
SINGH [91],[92]; SRIVASTAVA [94]; SRIVASTAVA AND KUMAR [97];
SUGIMURA [100]; VALIRON [104], [105], [106]; VARGA [107];
WEIERSTRASS [109]; WHITTAKER [110]; YUNG [111].

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CHAPTER II
ON THE MEAN VALUES OF AN ENTIRE FUNCTION

2.1 INTRODUCTION. Our aim in this chapter is to prove some results on the mean values of an entire function \( f(z) \) represented by Taylor series. In this chapter we have firstly obtained some results on geometric means \( G(r) \) and \( g_k(r) \) of an entire function and growth relations of these mean values relative to the counting function \( n(r) \) and certain other auxiliary functions. It is pointed out that the order and lower order of functions \( \log G(r) \) or \( \log g_k(r) \) do not exceed the order \( \sigma \) and the lower order \( \lambda \) of \( f(z) \). A number of relations regarding the comparative growth of these geometric means relative to each other have been found.

Secondly, we have investigated some properties of the means \( M_0(r) \) and \( m_{0,k}(r) \). The order \( \sigma \) and lower order \( \lambda \) of \( f(z) \) have also been expressed in terms of \( (M_0(r,f), M_0(r,f^{(m)})) \) and \( (m_{0,k}(r,f), m_{0,k}(r,f^{(m)})) \).

2.2 GEOMETRIC MEANS OF AN ENTIRE FUNCTION.

Let \( f(z) \) be an entire function of order \( \sigma \). Let \( \sigma_1 \) and \( \lambda_1 \) respectively, be the exponent convergence and lower exponent convergence of the zeros of \( f(z) \), so that

\[
\lim_{r \to \infty} \sup \inf \frac{\log n(r)}{\log r} = \frac{\sigma_1}{\lambda_1}, \quad (0 \leq \lambda_1 \leq \sigma_1 \leq \infty), \quad \cdots (2.2.1)
\]

where \( n(r) \) represents the number of zeros of \( f(z) \) in the disc

\[
D = \{ z : |z| \leq r \}.
\]

Further, let

\[
N(r) = \int_0^r \frac{n(x)}{x} \, dx. \quad \cdots (2.2.2)
\]
Define the following mean values of $f(z)$:

\[
G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right\}, \quad \ldots \tag{2.2.3}
\]

\[
g(r) = \exp \left\{ \frac{1}{2\pi r} \int_0^{2\pi} \int_0^r \log |f(xe^{i\theta})| dx d\theta \right\}.
\]

Further, let

\[
g_k(r) = \exp \left\{ \frac{k+1}{2\pi r^{k+1}} \int_0^r \int_0^r \log |f(xe^{i\theta})| x^k dx d\theta \right\}, \quad 0 < k < \infty. \quad \ldots \tag{2.2.4}
\]

Clearly if $T(r)$ denotes the Nevanlinna's characteristic function, then

\[
G(r) \leq \exp \{ T(r) \}
\]

\[
g_k(r) \leq \exp \left\{ \frac{k+1}{2\pi r^{k+1}} \int_0^r T(x) x^k dx \right\},
\]

and so the orders of $g_k(r)$ and $G(r)$ do not exceed the order of $f(z)$.

We remark that the result of Srivastava [98]

\[
\limsup_{r \to \infty} \frac{\log \log G(r)}{\log r} = \frac{f}{\lambda} \quad \ldots \tag{2.2.5}
\]

and the result of Kumar [40]

\[
\liminf_{r \to \infty} \frac{\log \log g_k(r)}{\log r} = \frac{f}{\lambda}, \quad \ldots \tag{2.2.6}
\]

can be negated as $G(r)$ and $g_k(r)$ are expressible in terms of the zeros, and we consider an entire function with large $M(r)$ and small number of zeros (for instance $f(z) = e^z$ and
\[ f(z) = z^p \cos z, \text{ where } p \text{ is any positive integer } \]

In view of Jensen's theorem on the zeros of \( f(z) \), we note that

\[
\lim_{r \to \infty} \inf \limsup_{r \to \infty} \frac{\log \log G(r)}{\log r} = \lambda_1, \quad (0 < \lambda_1 \leq \lambda_1 \leq \infty),
\]

and

\[
\lim_{r \to \infty} \inf \limsup_{r \to \infty} \frac{\log \log g_k(r)}{\log r} = \lambda_1, \quad (0 < \lambda_1 \leq \lambda_1 \leq \infty).
\]

\[\text{(2.2.7)}\]

\[\text{(2.2.8)}\]

2.3 GROWTH OF \( G(r) \) RELATIVE TO \( n(r) \). We obtain the following theorem:

**THEOREM 2.1.** If \( f(z) \) is an entire function of exponent convergence \( \lambda_1 \) and lower exponent convergence \( \lambda_1 \), then

\[
\liminf_{r \to \infty} \frac{\log G(r)}{n(r)} \leq \frac{1}{\lambda_1} \leq \lambda_1 \leq \limsup_{r \to \infty} \frac{\log G(r)}{n(r)} \quad \text{...(2.3.1)}
\]

**PROOF:** We first prove the latter half involving \( \lambda_1 \), supposing that \( \lambda_1 > 0 \). If this is not true, there will be a positive number \( J \) such that, for all sufficiently large \( r \),

\[
\frac{\log G(r)}{n(r)} < \left( \frac{1}{\lambda_1} - J \right).
\]

\[\text{...(2.3.2)}\]

By Jensen's theorem

\[
\log G(r) = \int_{r^0}^r \frac{n(x)}{x} \, dx + O(1).
\]

* This fact has not been pointed out in the reviews of Srivastava's paper [98] MR [28 2216] and Kumar's paper [40]MR [33 4280].
Substituting for log G(r) in (2.3.2), we have

\[ \frac{\int_{r_0}^{r} \frac{n(x)}{x} \, dx}{n(r)} < \left( \frac{1}{\lambda_1} - J \right) + O(1) \quad (r \to \infty) \]

or

\[ \frac{n(r)}{\int_{r_0}^{r} \frac{n(x)}{x} \, dx} > \left( \frac{1}{\lambda_1} - J \right)^{-1} + O(1) \quad (r \to \infty). \quad \ldots (2.3.3) \]

Therefore, by the integration of (2.3.3),

\[ \log N(r) > \left( \frac{1}{\lambda_1} - J \right)^{-1} \log r + O(\log r) \]

which in virtue of Lemma 1.4 [25], leads to the contradiction

\[ \lambda_1 = \liminf_{r \to \infty} \frac{\log N(r)}{\log r} \geq \left( \frac{1}{\lambda_1} - J \right)^{-1}. \]

Similarly we prove the rest part of the theorem.

**Theorem 2.2** For an entire function f(z) of exponent convergence \( \lambda_1 \) and lower exponent convergence \( \lambda_1 \), we have

\[ \limsup_{r \to \infty} \frac{\log G(r)}{n(r) \log r} \leq 1 - \frac{\lambda_1}{f_1} \quad \ldots \quad (2.3.4) \]

**Proof.** When \( \lambda_1 = 0 \) or \( f_1 = \infty \) (i.e. \( f_1^{-1} = 0 \)), it is obvious from Jensen's theorem. Hence we suppose that \( \lambda_1 > 0 \), \( f_1 < \infty \) and deduce from Jensen's theorem

\[ \log \frac{G(r)}{n(r) \log r} = O(1) + 1 - J(r)P(r), \quad (r \to \infty). \quad (2.3.5) \]

where

\[ J(r) = \frac{\int_{r}^{\infty} \log x \, dn(x)}{\left\{ \int_{r_0}^{r} \log n(x) \, dn(x) \right\}} \]

\[ \left[ \int_{r_0}^{r} \log n(x) \, dn(x) \right] \]
\[ P(r) = \frac{\int_{R_0}^{r} \log n(x) \, dx}{\{ n(r) \log r \}} \]
\[ = \frac{n(r) \log n(r) - n(r) + \text{a constant}}{n(r) \log r} \]

Now
\[ \liminf_{r \to \infty} P(r) = \liminf_{r \to \infty} \frac{\log n(r)}{\log r} = \lambda_{r} \quad \ldots (2.3.7) \]

(2.3.6) and (2.3.7) in conjunction with (2.3.5) prove the theorem.

2.4 Growth of \( G(r) \) Relative to \( r^f \).

Theorem 2.3. If \( f(z) \) is an entire function of finite non-zero order \( f \), then
\[ \limsup_{r \to \infty} \frac{\log G(r)}{r^f \log r} \leq \limsup_{r \to \infty} \frac{n(r)}{r^f} \leq e^{f \, T} - \beta \quad \ldots (2.4.1) \]
\[ \liminf_{r \to \infty} \frac{\log G(r)}{r^f \log r} \leq \liminf_{r \to \infty} \frac{n(r)}{r^f} \leq \beta \quad \ldots (2.4.2) \]

and
\[ \liminf_{r \to \infty} \frac{\log G(r)}{r^f \log r} \leq \liminf_{r \to \infty} \frac{n(r)}{r^f} \leq \lambda_1 T \quad \ldots (2.4.3) \]

For the proof of the theorem we require the following lemmas.

Lemma 2.1 For any entire function \( f(z) \) of finite non-zero order
\[ e^{f \, T} \geq \Delta + \beta t \quad \ldots (2.4.4) \]
where \[ \Delta = \limsup_{r \to \infty} \frac{n(r)}{r^f} \]
PROOF. We suppose $f(0) \neq 0$. By Jensen's theorem

$$\log M(r) \geq O(1) + \int_{\mathbb{R}} \frac{n(x)}{x} \, dx.$$ 

Let $\Delta' > 0$ such that $\Delta = \Delta' = \epsilon > 0$. Suppose

$$\frac{n(r_1)}{r_1} > \Delta'$$

where $r_1 = r_1(\Delta')$. Then for all $r > r_1$,

$$\log M(r) > O(1) + \log M(r_1) + \Delta' \int_{r_1}^{r} \frac{dx}{x},$$

$$= O(1) + \log M(r_1) + \Delta' \int_{r_1}^{r} \frac{\log \left( \frac{r}{r_1} \right)}{x},$$

$$\ldots(2.4.5)$$

Also, it is possible to choose $r_1$ such that

$$\frac{\log M(r_1)}{r_1^f} > t'$$

with $t - t' = \epsilon$. Therefore, from (2.4.5) for all $r > r_1$, we obtain

$$\frac{\log M(r)}{r^f} > \left( \frac{r_1}{r} \right)^f \left\{ t' + \Delta' \log \left( \frac{r}{r_1} \right) \right\} + O(r^f).$$

$$\ldots(2.4.6)$$

Now, by the usual methods of calculus, we maximise the first term of the right hand side of (2.4.6). We find its maxima which is attained for that value of $r$ which satisfies the relation

$$\left( \frac{r}{r_1} \right) = \exp \left\{ \frac{(\Delta' - ft')}{\Delta'} \right\}$$

$$\ldots(2.4.7)$$

and that maximum value is $\left( \frac{\Delta'}{r} \right) \exp \left\{ \left( \frac{ft' - \Delta'}{\Delta'} \right) \right\}$. 


Therefore from (2.4.6), we get

$$\frac{\log M(r)}{r^f} \geq \left(\frac{\Delta'}{r^f}\right) \exp\left\{\left(\frac{\int t'-\Delta'}{\Delta'}\right)\right\} + o(r^f) \quad \ldots(2.4.8)$$

for \(r\) satisfying (2.4.7). We see that

$$\int T \geq \Delta' \exp\left\{\left(\frac{\int t'}{\Delta'}\right) - 1\right\} \quad \ldots(2.4.9)$$

Now, since \(\Delta'\) can be fixed arbitrary close to \(\Delta\) and \(t'\) arbitrary close to \(t\), we immediately deduce from (2.4.9) the following result

$$\int T \geq \Delta \exp\left\{\left(\frac{\int t}{\Delta}\right) - 1\right\} \quad \ldots(2.4.9)$$

Since for every real \(x\), \(e^x \geq 1 + x\), we finally get

$$e \int T \geq \Delta + \int t$$

or

$$\limsup_{r \to \infty} \frac{n(r)}{r^f} \leq e \int T - \int t.$$ 

**Lemma 2.2 [63]**

$$\liminf_{r \to \infty} \frac{n(r)}{r^f} \leq \int t \quad \ldots(2.4.10)$$

**Lemma 2.3**

$$\liminf_{r \to \infty} \frac{n(r)}{\log M(r)} \leq \lambda_1 T \quad \ldots(2.4.10)$$

**Proof.** It is known (Shah [70]) that

$$\liminf_{r \to \infty} \frac{n(r)}{\log M(r)} \leq \lambda_1 \quad \ldots(2.4.10)$$

Let

$$\frac{n(r)}{r^f} = \frac{n(r)}{\log M(r)} \cdot \frac{\log M(r)}{r^f}$$

It is well known that \(J(x)\) and \(P(x)\) are two non-negative functions then
have $\liminf J(x), \limsup P(x)$

\[ \liminf \frac{J(x) \cdot P(x)}{\text{log } M(r)} \leq \liminf J(x), \limsup P(x). \]

Here \( \frac{n(r)}{\text{log } M(r)} \) and \( \frac{\text{log } M(r)}{r^j} \) are non-negative, so we have

\[ \liminf_{r \to \infty} \frac{n(r)}{r^j} \leq \liminf_{r \to \infty} \frac{n(r)}{r^j} \cdot \limsup_{r \to \infty} \frac{\text{log } M(r)}{r^j} \]

which along with (2.4.10) and definition of type, give us

\[ \liminf_{r \to \infty} \frac{n(r)}{r^j} \leq \lambda_1 T. \]

**Proof of Theorem 2.3.** From (2.2.3), we have

\[ \log G(r) = O(1) + \int_{r_1}^r \frac{n(x)}{x} \, dx \leq n(r) (\log r - \log r_0) + O(1) \]

or

\[ \frac{\log G(r)}{r^j \log r} \leq \frac{n(r)}{r^j} + o(r^j). \]

Proceeding to limits and making use of Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get the required inequalities.

**Remark.** Our inequalities are better than those of Srivastava's inequalities [99].

2.5 **Growth of G(r) Relative to N(r).**

**Theorem 2.4.** If \( f(z) \) is an entire function having no zeros in the unit circle, then,

\[ \frac{G(r_2)}{G(r_1)} \geq \left( \frac{r_2}{r_1} \right)^{N(r_2)} \log r_2, \quad (0 < r_1 < r_2). \quad (2.5.1) \]
PROOF: Using Jensen's formula, we have

\[ \log G(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \]

\[ = \int_0^r \frac{n(x)}{x} \, dx + \log |f(0)|. \]

Hence

\[ \log G(r_2) - \log G(r_1) = \int_0^{r_2} \frac{n(x)}{x} \, dx - \int_0^{r_1} \frac{n(x)}{x} \, dx. \]

... (2.5.2)

N(x) is a convex function of log x. If we draw the graph of the function N(x), it will pass through the origin. Let 0 be origin and J(log r_2; N(r_2), P(log r_1; N(r_1)), be two points on the graph. Then the slope of OJ is greater than the slope of OP. Hence

\[ \frac{N(r_2)}{\log r_2} \geq \frac{N(r_1)}{\log r_1} \]

and it follows that

\[ \frac{N(r_2) - N(r_1)}{\log r_2 - \log r_1} \geq \frac{N(r_2)}{\log r_2}. \]

Thus (2.5.2) gives us

\[ \frac{G(r_2)}{G(r_1)} \geq \left( \frac{r_2}{r_1} \right)^{\frac{N(r_2)}{\log r_2}}, \]

which is what we require.

2.6 COMPARATIVE GROWTH OF log G(r) AND log g_k(r). This section is devoted mainly to study the growth of the mean values G(r) and g_k(r) relative to each other.
Inspite of the fact that the function $\log G(r)$ and $\log g_k(r)$ have the same order and same lower order it is to be noted that for an entire function $f(z)$ of exponent convergence $\gamma_1$ ($0 < \gamma_1 < \infty$), the asymptotic relation

$$\log G(r) \sim \log g_k(r) \quad \text{as } r \to \infty,$$

need not be true always, as in the case of ordinary means ([42] Theorem 2). Consider for instance an entire function

$$f(z) = \prod_{\lambda = 1}^{\infty} \left(1 + \frac{z}{n^2}\right)$$

for which

$$\log G(r) \sim r^2 \quad \text{and} \quad \log g_k(r) \sim \frac{2(k+1)r^2}{(2k+3)}.$$

We give below a theorem which gives us information as to how the function $\log G(r)$ and $\log g_k(r)$ grow relative to each other as $r \to \infty$.

**Theorem 2.5.** If $f(z)$ be an entire function of finite lower exponent convergence $\lambda_1$, then

$$\liminf_{r \to \infty} \frac{\log G(r)}{\log g_k(r)} \leq \left(\frac{\lambda_1}{k+1}\right) \left(1 + \frac{\lambda_1}{\lambda_1}ight), \quad \lambda_1 > 0$$

$$\leq 1, \quad \lambda_1 = 0.$$  \hspace{1cm} \text{(2.6.1)}

**Proof.** Since

$$\limsup_{r \to \infty} \inf \frac{\log \log G(r)}{\log r} = \frac{\gamma_1}{\lambda_1}, \quad (0 \leq \lambda_1 \leq \gamma_1 \leq \infty).$$

Then, following Shah ([76], p.31) there exist a lower proximate
order \( \lambda_1(r) \), \((0 \leq \lambda_1 \leq \infty)\) relative to \( \log G(r) \), satisfying the following conditions:

(i) \( \lambda_1(r) \) is a non-negative continuous function of \( r \) for \( r \geq r_0 > 0 \).

(ii) \( \lambda_1(r) \) is differentiable for all \( r > r_0 \) except at isolated points at which \( \lambda_1'(r-0) \) and \( \lambda_1'(r+0) \) exist.

(iii) \( \lim_{r \to \infty} \frac{r}{\log r} \lambda_1'(r) \log r = 0 \).

(iv) \( \lim_{r \to \infty} \lambda_1(r) = \lambda_1 \).

(v) \( \frac{\lambda_1(r)}{r} \leq \log G(r) \) and \( \liminf_{r \to \infty} \frac{\log G(r)}{r \lambda_1(r)} = 1 \).

From (i) - (iv), we deduce that

\[
\lim_{r \to \infty} \frac{J(hr)}{J(r)} = h \lambda_1, \quad h > 1 \quad ([93], \text{p. 565})
\]

where \( J(r) = r \lambda_1(r) \).

From (2.2.4), we have

\[
\log g_k(r) = \frac{k+1}{r^{k+1}} \int_0^r \log(x) \cdot x^k \, dx \leq \log G(r) \quad \text{...(2.6.3)}
\]

Further,

\[
\log g_k(R) = \frac{k+1}{R^{k+1}} \int_0^R x^k \log G(x) \, dx \geq \frac{k+1}{R^{k+1}} \int_0^R x^k \log G(x) \, dx \quad (2.6.4)
\]

\[
\geq \frac{R^{k+1}}{R^{k+1}} \cdot \log G(r).
\]
Let $R = xr$, $x > 1$. Then

$$\log G(r) \leq \frac{x^{k+1}}{x^{k+1} - 1} \log g_k(xr)$$

and

$$1 = \liminf_{r \to \infty} \frac{\log G(r)}{J(r)} \leq \frac{x^{k+1}}{x^{k+1} - 1} \liminf_{r \to \infty} \frac{\log g_k(xr)}{J(r)}.$$ 

From which it follows that

$$\liminf_{r \to \infty} \frac{\log g_k(xr)}{J(r)} \geq \frac{x^{k+1} - 1}{x^{k+1}}.$$ ...

(2.6.5)

Put

$$\log g_k(xr) = \frac{\log g_k(xr)}{J(xr)} \cdot \frac{J(xr)}{J(r)}.$$ 

Here $\frac{\log g_k(xr)}{J(xr)}$ and $\frac{J(xr)}{J(r)}$ are non-negative and so that

$$\liminf_{r \to \infty} \frac{\log g_k(xr)}{J(r)} \leq \liminf_{r \to \infty} \frac{\log g_k(r)}{J(r)} \cdot x^1$$

by (2.6.2). This inequality with (2.6.5) gives us

$$\liminf_{r \to \infty} \frac{\log g_k(r)}{J(r)} \geq \frac{x^{k+1} - 1}{x^{k+1} + x}.$$ 

Using this inequality and from the equality

$$\frac{\log G(r)}{\log g_k(r)} = \frac{\log G(r)}{J(r)} \cdot \frac{J(r)}{\log g_k(r)},$$

we get

$$\liminf_{r \to \infty} \frac{\log G(r)}{\log g_k(r)} \leq \liminf_{r \to \infty} \frac{\log G(r)}{J(r)} \cdot \limsup_{r \to \infty} \frac{J(r)}{\log g_k(r)}.$$
Now, by the usual method of calculus we minimize the right hand side of (2.6.6). We find that its minima is attained for that value of $x$ which satisfies the relation

$$x = \left\{ \frac{k+1 + \lambda_1}{\lambda_1} \right\}^{1/k+1}, \quad \lambda_1 > 0.$$  

Substituting this value of $x$ in (2.6.6), we get

$$\liminf_{r \to \infty} \frac{\log G(r)}{\log g_k(r)} \leq \left( \frac{\lambda_1}{k+1} \right) \left\{ (1 + \frac{k+1}{\lambda_1}) (1 + \frac{\lambda_1}{k+1}) \right\}.$$  

In the case when $\lambda_1 = 0$, the minimum value of the right hand side of (2.6.6) is one as $x \to \infty$. This completes the proof of Theorem 2.5.

2.7 A FURTHER RESULT ON COMPARATIVE GROWTH OF $\log G(r)$ AND $\log g_k(r)$.

The preceding section has been devoted to study the growth of the mean values $\log G(r)$ and $\log g_k(r)$ relative to each other. In this section, we give further result in the same direction.

Let $f(r)$ be Lindelöf proximate order satisfying the following conditions:

(i) $f(r)$ is real, continuous and differentiable for $r > r_0$.
(ii) $f(r) \to \rho$ as $r \to \infty$. 

(iii) \( (r \log r) f'(r) \to 0 \) as \( r \to \infty \), where \( f'(r) \) is either the right hand or left hand derivatives at points where they are different.

Further, let \( 0 < \beta < \infty \),

\[
J(r) = \exp \left( \int_{r_0}^{r} \frac{\rho(x)}{x} \, dx \right), \quad S(r) = \frac{1}{r^{k+1}} \int_{r_0}^{r} x^k \rho(x) \log G(x) \, dx.
\]

Then \((k+1) S(r) \sim \rho \log g_k(r)\), as \( r \to \infty \). Define:

\[
\limsup_{r \to \infty} \frac{S(r)}{J(r)} = \frac{J}{p}, \quad (0 < p < J < \infty), \quad \ldots \ (2.7.1)
\]

and

\[
\limsup_{r \to \infty} \frac{\log G(r)}{J(r)} = \frac{L}{M}, \quad (0 < M \leq L < \infty). \quad \ldots \ (2.7.2)
\]

We wish to prove the following:

**THEOREM 2.6.** Let \( f(z) \) be an entire function of finite non-zero order \( \rho \). Then

\[
\frac{\rho M}{(\rho + k + 1)L} \leq \limsup_{r \to \infty} \frac{S(r)}{\log G(r)} \leq \frac{\rho}{(\rho + k + 1)} \frac{L}{M} \quad \ldots \ (2.7.3)
\]

where \( L \) and \( M \) are as given by \((2.7.2)\).

Following lemmas are required to prove the theorem.

**LEMMA 2.4**

\[
\int_{r_0}^{r} x^{k+1} J'(x) \, dx \sim \frac{\rho}{(\rho + k + 1)} \int_{r_0}^{r} J(r) \, dx, \quad r \to \infty.
\]

**PROOF.** We have

\[
\frac{d}{dr} \left\{ x^{k+1} J(r) \right\} = x^{k+1} J'(r) + (k+1)x^{k+1} J(r),
\]

\[
= x^{k+1} J'(r) \left\{ 1 + \frac{k+1}{\rho(r)} \right\}.
\]
by (ii) and so the lemma follows.

**Lemma 2.5** For any finite \( \gamma \geq 0 \),

\[
\int_{r}^{r+\eta} x^k \varphi(x) \, dx \sim \frac{\varphi}{k+1} \left\{ \left(1 + \frac{\eta}{k+1}\right) - 1 \right\} r^{k+1}
\]

Proof is straight forward.

**Lemma 2.6.** For any finite \( \eta \geq 0 \),

\[
\log \left\{ \frac{J(r+\eta r)}{J(r)} \right\} \rightarrow \log (1 + \eta)^{f}, \quad r \to \infty.
\]

**Proof.** We have

\[
\log \left\{ \frac{J(r+\eta r)}{J(r)} \right\} = \int_{r}^{r+\eta r} \frac{\varphi(x)}{x} \, dx
\]

\[
= \left[ \varphi(x) \log x \right]_{r}^{r+\eta r} = \int_{r}^{r+\eta r} \varphi(x) \log x \, dx.
\]

But

\[
\left| \int_{r}^{r+\eta r} \varphi(x) \log x \, dx \right| < \varepsilon \log (1 + \eta), \quad r > r_0(\varepsilon).
\]

Hence for sufficiently large \( r \)

\[
\log \left\{ \frac{J(r+\eta r)}{J(r)} \right\} = \log\left[ \frac{(r+\eta r)^{f(r+\eta r)}}{r^{f(r)}} \right] + O(\log(1 + \eta)).
\]

Therefore,

\[
\log \left\{ \frac{J(r+\eta r)}{J(r)} \right\} \rightarrow \log (1 + \eta)^{f} \quad \text{as} \quad r \to \infty.
\]

**Proof of Theorem 2.6.** We have for any finite \( \eta \geq 0 \),

\[
S(r+\eta r) = \frac{1}{(r+\eta r)^{k+1}} \int_{0}^{r+\eta r} x^k \varphi(x) \log G(x) \, dx \quad \ldots (2.7.4)
\]
\[ = O \left( r^{-(k+1)} \right) + \frac{1}{(r+\eta r)^{k+1}} \int_{r_0}^{r} x^{k+1} \log G(x) \frac{J'(x)}{J(x)} \, dx \]

\[ + \frac{1}{(r+\eta r)^{k+1}} \int_{r}^{r+\eta r} x^{k+1} \log G(x) \frac{J'(x)}{J(x)} \, dx. \]

\[ > O \left( r^{-(k+1)} \right) + \frac{M-C}{(r+\eta r)^{k+1}} \int_{r_0}^{r} x^{k+1} J'(x) \, dx \]

\[ + \frac{\log G(r)}{(r+\eta r)^{k+1}} \int_{r}^{r+\eta r} x^{k+1} \frac{J'(x)}{J(x)} \, dx. \]

\[ \sim \frac{(M-C)}{(1+\eta)^{k+1}} \left( \frac{\rho}{(\rho+k+1)} \cdot \frac{J(r)}{J(r+\eta r)} \right) + \frac{\log G(r)}{(1+\eta)^{k+1}} \cdot \frac{J(r)}{J(r+\eta r)} \cdot \frac{\rho}{(1+\eta)^{k+1}(k+1)} \left\{ (1+\eta)^{k+1} - 1 \right\}. \]

Now dividing both the sides by \( J(r+\eta r) \), we obtain

\[ \frac{S(r+\eta r)}{J(r+\eta r)} > \frac{(M-C)}{(1+\eta)^{k+1}} \frac{\rho \cdot J(r)}{(\rho+k+1) J(r+\eta r)} + \frac{\log G(r)}{J(r)} \cdot \frac{J(r)}{J(r+\eta r)} \cdot \frac{\rho}{(1+\eta)^{k+1}(k+1)} \left\{ (1+\eta)^{k+1} - 1 \right\}. \]

On making use of lemma 2.6 and proceeding to limit inferior, we get

\[ P \geq \frac{M}{(1+\eta)^{\rho+k+1}} \cdot \frac{\rho}{(\rho+k+1)} + \frac{\rho M}{(1+\eta)^{\rho+k+1}} \left\{ (1+\eta)^{k+1} - 1 \right\}. \ldots (2.7.5) \]

The above inequality will be best in its form if right hand side is replaced by its maximum value for some \( \eta \). It can, therefore, be easily seen that the maxima of the right hand expression of (2.7.5) occurs at \( \eta = 0 \). Substituting this value
of $\eta$ in (2.7.5), we get

$$P \geq \frac{M^\rho}{(\rho+k+1)} \quad \ldots(2.7.6)$$

Next, from (2.7.4), we have

$$S(r+\eta r) < 0 \left( \begin{array}{c} r^{-(k+1)} \\ (r+\eta r)^{k+1} \end{array} \right) + \frac{(L+\rho)}{(r+\eta r)^{k+1}} \int_r^\infty x^{k+1} J'(x)dx$$

$$+ \frac{\log G(r+\eta r)}{(r+\eta r)^{k+1}} \int_r^\infty \frac{x^{k+1} J'(x)}{J(x)}dx.$$}

$$\sim \frac{(L+\rho)}{(1+\eta)^{k+1}} \cdot \frac{\rho}{(\rho+k+1)} J(r) + \frac{\rho}{(1+\eta)^{k+1}} \frac{(1+\eta)^{k+1}-1}{(1+\eta)^{k+1}} \log G(r+\eta r) \quad \ldots(2.7.7)$$

Dividing both the sides by $J(r+\eta r)$ and taking limit superior, we get

$$J \leq \left[ \frac{1}{(\rho+k+1)(1+\eta)^{\rho+k+1}} + \frac{1}{(1+\eta)^{\rho+k+1}} \left\{ \frac{(1+\eta)^{k+1}-1}{(1+\eta)^{k+1}} \right\} \right] \rho L. \quad \ldots(2.7.8)$$

To obtain the best possible value, we substitute, the value of $\eta$ for which the right hand expression in (2.7.7) is minimum, which is zero and hence

$$J \leq \frac{\rho L}{(\rho+k+1)} \quad \ldots(2.7.8)$$

From (2.7.1) and (2.7.2), we obtain

$$\frac{P - \varepsilon}{L + \varepsilon} < \frac{S(r)}{\log G(r)} < \frac{J+\varepsilon}{M-\varepsilon}.$$}

Taking limits and using (2.7.6) and (2.7.8), we prove the theorem.

**Corollary 2.1** If $M = L$, then

$$(\rho+k+1) \log g_k(r) \sim (k+1) \log G(r).$$
2.8 MEAN VALUES OF AN ENTIRE FUNCTION AND ITS DERIVATIVES.

Let us define

\[ M_\delta(r,f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta, \delta > 0 \]

and

\[ M_\delta(r; f^{(m)}) = \frac{1}{2\pi} \int_0^{2\pi} |f^{(m)}(re^{i\theta})|^\delta d\theta \]

where \( f^{(m)}(z) \) denotes the \( m \)th derivative of \( f(z) \). It has been shown that

\[ \limsup_{r \to \infty} \frac{\log \log M_\delta(r,f)}{\log r} = \rho \quad [57] \quad (2.8.3) \]

and, "if \( \delta \geq 1 \), then

\[ \limsup_{r \to \infty} \inf \frac{\log [r \frac{M_\delta(r,f^{(m)})}{M_\delta(r,f)}]}{\log r} = \rho \quad [30] \quad (2.8.4) \]

for every entire function \( f(z) \) of finite order \( \rho \) and lower order \( \lambda > 1 + \frac{1}{m} \).

The case \( \delta = 1, m = 1 \), was first considered in [96] for the upper limit. The method [96], in fact fails, to give the lower limit of the left hand expression in (2.8.4). It was generalised in [94] for lower limit and \( \delta > 1 \), which is incorrect if \( \delta = 1 \), as far as its proof is concerned.

We extend here the result of Juneja [ibid], (2.8.4), to wider class of entire functions and prove the following:
THEOREM 2.7 If \( f(z) \) is an entire function, other than a polynomial, of order \( p \) and lower order \( \lambda \), then

\[
\lim_{r \to \infty} \sup \inf \log \left[ \frac{M_\delta(r,f(m))}{M_\delta(r,f)} \right] \frac{1}{m} \log r = \rho \nu,
\]

where \( \delta \geq 1 \) and \( m = 1, 2, \ldots, m \).

PROOF. We have

\[
f^{(m)}(z) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(w) \, dw}{(w-z)^{m+1}}.
\]

So,

\[
|f^{(m)}(z)| \leq \left( \frac{m}{2\pi} \right) \frac{R}{(R-r)^{m+1}} \int_0^{2\pi} |f(\text{Re}^i\theta)| \, d\theta.
\]

By Hölder's inequality

\[
|f^{(m)}(z)| \leq \left( \frac{m \cdot R}{2\pi} \right) \frac{1}{(R-r)^{\delta(m+1)}} \int_0^{2\pi} |f(\text{Re}^i\theta)|^{\delta-1} \, d\theta (2\pi),
\]

\[
\leq \left\{ \frac{m}{(R-r)^{\delta(m+1)}} \right\} \delta \left[ M_\delta(R,f) \right]^{\delta-1}.
\]

Integrating both the sides of this inequality with respect to \( \theta \) from \( 0 \) to \( 2\pi \), we have

\[
M_\delta(r,f^{(m)}) \leq \frac{A}{(R-r)^{\delta}} M_\delta(R,f), \quad \ldots \quad (2.8.6)
\]

where \( A \) is a constant.

Since by a well-known theorem of Hardy [23], \( \log M_\delta(r,f) \) is an increasing convex function of \( \log r \), we have
\[
\log M_\circ(r,f) = \log M_\circ(r,f) + \int_{r}^{R} \frac{S(t)}{t} \, dt \quad \ldots(2.8.7)
\]

where \( S(t) \) is a non-decreasing function of \( t \). Therefore for \( R > r \),

\[
\log M_\circ(R,f) = \log M_\circ(r,f) + \int_{r}^{R} \frac{S(t)}{t} \, dt \quad \ldots(2.8.8)
\]

(2.8.6) gives us

\[
\log M_\circ(r,f(\text{m})) \leq O(1) + m \log \frac{1}{R-r} + \log M_\circ(R,f),
\]

whence we get, using (2.8.8) to express \( \log M_\circ(R,f) \) on the right in terms of \( \log M_\circ(r,f) \),

\[
\log M_\circ(r,f(\text{m})) \leq O(1) + m \log \frac{1}{R-r} + \log M_\circ(r,f) + \int_{r}^{R} \frac{S(t)}{t} \, dt.
\]

\ldots(2.8.9)

In (2.8.9), we choose \( R \) in terms of \( r \) as follows. Let the expression in (2.8.9) be considered as a function of \( R \) in the first instance (with \( r \) fixed for the time being). This expression, for varying \( R \), is least when

\[
\frac{S(R)}{R} = \frac{1}{R-r}
\]

i.e.

\[
R = \frac{r}{1 - \frac{1}{S(R)}} \leq r \{ 1 + O(1) \}.
\]

Hence,

\[
\log M_\circ(r,f(\text{m})) < \log M_\circ(r,f) + m \log \frac{S(R)}{R} = S(R) \log(1 - \frac{1}{S(R)}) + O(1),
\]

\[
= \log M_\circ(r,f) + m \log \frac{S(R)}{R} + O(1),
\]

where \( R < R^* = \frac{r S(r)}{S(r) - 1} = r \{ 1 + O(1) \} \).
From this, we get
\[ M_\delta(r,f^{(m)}) < M_\delta(r,f) \left\{ \frac{S(R^*)}{r} \right\}^m \cdot O(1). \quad \ldots \quad (2.8.10) \]

It can be easily seen from (2.8.7) that
\[ \lim_{r \to \infty} \sup \frac{\log \log M_\delta(r,f)}{\log r} = \lim_{r \to \infty} \inf \frac{\log S(r)}{\log r}. \]

This together with (2.8.3) will give us
\[ S(r) < r^\rho + \epsilon, \quad r > r_0, \quad \epsilon > 0, \quad (\rho < \infty). \]

Using this in (2.8.10), we get for \( r > r_0(\epsilon) \)
\[ M_\delta(r,f^{(m)}) < \left\{ M_\delta(r,f) r^{(\rho - 1 + \epsilon)m} \right\} \cdot O(1). \quad \ldots \quad (2.8.11) \]

Further, if \( f(z) \) is an entire function of finite lower order \( \lambda \), then we get similarly, for a sequence of values of \( r \) tending to infinity
\[ M_\delta(r,f^{(m)}) < \left\{ M_\delta(r,f) r^{(\lambda - 1 + \epsilon)m} \right\} \cdot O(1). \quad \ldots \quad (2.8.12) \]

Consequently, we get
\[ \lim_{r \to \infty} \sup \log \left[ \frac{M_\delta(r,f^{(m)})}{M_\delta(r,f)} \right]^m / \log r \leq \frac{\rho}{\lambda}. \quad \ldots \quad (2.8.13) \]

Next, we consider the inverse of the above inequality. We have
\[ M_\delta(r,f^{(m)}) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f^{(m)}(r e^{i\theta})| \, d\theta \right\}^{\frac{1}{\delta}}, \]
\[ = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \lim_{\epsilon \to 0} \left| \frac{f^{(m-1)}(r e^{i\theta}) - f^{(m-1)}(r(1-\epsilon)e^{i\theta})}{\epsilon \, r e^{i\theta}} \right| \, d\theta \right\}^{\frac{1}{\delta}}, \]
\[ \geq \left\{ \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{|f^{(m-1)}(r e^{i\theta})| - |f^{(m-1)}(r(1-\epsilon)e^{i\theta})|}{\epsilon r} \right) \, d\theta \right\}^{\frac{1}{\delta}}. \]

Now, applying the Minkowski's inequality ([103], p. 384),
we get,

\[ M_0(r, f^{(m)}) \geq \lim_{\varepsilon \to 0} \frac{M_0(r, f^{(m-1)}) - M_0(r, r^{1/\varepsilon} f^{(m-1)})}{\varepsilon r} \geq \frac{d}{dr} M_0(r, f^{(m-1)}) \]  

(2.8.14)

Let 

\[ S(r) = \frac{\log M_0(r, f^{(m-1)})}{\log r} \]

As \( S(r) \) is an increasing function for all \( r > r_0 \).

We have \( S'(r) > 0 \) and hence

\[ \frac{d}{dr} \log M_0(r, f^{(m-1)}) = \frac{S(r)}{r} + S'(r) \log r \]

\[ > \frac{S(r)}{r} = \frac{\log M_0(r, f^{(m-1)})}{r \log r} \]

This together with (2.8.14) will give us

\[ M_0(r, f^{(m)}) > M_0(r, f^{(m-1)}) \left\{ \frac{\log M_0(r, f^{(m-1)})}{r \log r} \right\} \]  

(2.8.15)

Putting \( m = 1, 2, \ldots, m \) and multiplying \( m \) inequalities thus obtained, we get

\[ \frac{M_0(r, f^{(m)})}{M_0(r, f)} > \prod_{1}^{m} \left\{ \frac{\log M_0(r, f^{(m-1)})}{r \log r} \right\} \]  

(2.8.16)

It also follows from (2.8.15), taking the logarithm of both the sides that

\[ \log M_0(r, f^{(m-1)}) > O(1) \log M_0(r, f) \]

for sufficiently large \( r \) and \( m \geq 2 \).
This together with (2.8.16) give us for large \( r \),
\[
\frac{M_6(r, f^{(m)})}{M_6(r, f)} > \left\{ \frac{\log M_6(r, f)}{r \log r} \right\}^m \cdot O(1). \quad \ldots(2.8.17)
\]

This leads to
\[
\frac{\log [r \left\{ \frac{M_6(r, f^{(m)})}{M_6(r, f)} \right\}^{\frac{1}{m}}]}{\log r} > \frac{\log \log M_6(r, f)}{\log r} \cdot \frac{\log \log r}{\log r} + o(1).
\]

Proceeding to limits and using (2.8.3), we get
\[
\lim_{r \to \infty} \sup \inf \frac{\log [r \left\{ \frac{M_6(r, f^{(m)})}{M_6(r, f)} \right\}^{\frac{1}{m}}]}{\log r} \geq \lambda. \quad \ldots(2.8.18)
\]

Combining (2.8.13) and (2.8.18), we complete the proof of the theorem.

2.9 AN EXPRESSION FOR \( \rho \) AND \( \lambda \) FOR THE CASE \( 0 < \rho < 1 \).

In this section, we generalize another result of Juneja [30]. His Theorem 3 [30] is included in our Theorem 2.8. Our Theorem 2.8 is not only more general than Juneja's theorem, but has the different proof and more widely applicable.

First of all we point out an important point* over looked by Juneja. Juneja has proved

"For every entire function, other than a polynomial,

* This point is not observed in the review of Juneja's paper [30] MR [33 #2 2814].
\[
\limsup_{r \to \infty} \frac{1}{\log r} \log \left[ r \left\{ \frac{I_0(r, f(m))}{I_0(r, f)} \right\}^{m \delta} \right] = \rho, \quad 0 < \delta < 1,
\]
where \( r \) tends to infinity through values excluding an exceptional set of at most finite measure "•

In the proof of the theorem he has used two lemmas. His Lemma 3 is based on the following result of Valiron ([106], p. 105)

\[
f^{(m)}(z) = \left( \frac{\nu(r)}{z} \right)^m (1 + h_m \nu(r)^{\frac{1}{16}}), \quad (|h_m| < k).
\]

This is valid at the points on the circle \( |z| = r \) at which one of the functions \( f(z), \ldots, \left( \frac{z}{r} \right)^m f^{(m)}(z) \) is greater in modulus than \( M(r) \left\{ \nu(r) \right\}^{\frac{1}{8}} \). At remaining points, the validity of this result is not known. Now, \( I_0(r, f) \) is the mean of \( |f(re^{i\theta})|^6 \) taken over the circle \( |z| = r \). We can carry out integration in the above equation (2.9.1) to form the means of \( f(z) \) and \( f^{(m)}(z) \) over the circle \( |z| = r \), provided of course the set of omission points has a measure zero.

Now, consider the entire function \( e^z \). For this function \( M(r) = \exp(r), |\exp(z)| = \exp(r \cos \theta) \) and \( \nu(r) = n, \) for \( E_n = n \leq r < n+1 \). Let

\[
J = \{re^{i\theta} \mid r \in E_n \ ; \ \theta \leq \cos^{-1} \left( 1 - \frac{\log r}{8r} \right) \}
\]

according as \( 0 \leq \theta \leq \pi \), or \( \pi \leq \theta \leq 2\pi \).
Clearly, the total variation of log $r$ in $\mathbb{E}_n$ tends to infinity with $n$. Also, at all points of the above set $J$, we have both the numbers $|\exp(z)|$ and $|\left(\frac{z}{\nu(r)}\right)^m \exp(z)|$ are less than or equal to $M(r) \left\{\nu(r)\right\}^{-\frac{1}{8}}$ and $m(J) > 0$. This establishes the fact that entire functions with $|f(z)|$, $|\left(\frac{z}{\nu(r)}\right)^m f^{(m)}(z)|$ less than or equal to $M(r) \left\{\nu(r)\right\}^{-\frac{1}{8}}$ over a set of measure greater than zero exist. Thus, for all entire functions the integration carried out by Juneja is not justified. Hence, his proof is wrong.

Lemma 4 of Juneja [30] is based on the following unproved assertion

$$\left|\frac{f^{(1)}(z)}{f(z)}\right| \leq O(r^{\rho+\epsilon-1}).$$

Now, we prove our theorem.

**THEOREM 2.8.** For every entire function, other than a polynomial,

$$\lim_{r \to \infty} \sup \inf \log \left[ \frac{M_{\delta}(r,f^{(m)})^{\frac{1}{m}}}{M_{\delta}(r,f)} \right] \log r = \frac{\rho}{\gamma} \quad \cdots (2.9.2)$$

where $m = 1, 2, \ldots, m$ and $0 < \delta < 1$.

**PROOF.** We know that for $\epsilon > 0$ and large $r$

$$\frac{M_{\delta}(r,f^{(1)})}{M_{\delta}(r,f)} \leq r^{(\rho+\epsilon-1)}, \quad 0 < \delta < 1, \rho < \infty. [44]$$
Since order of function is invariant under differentiations, therefore we have

\[ \frac{M_0(r, f'(p))}{M_0(r, f'(p-1))} \leq r^p (p+\epsilon-1). \]

Giving \( p \) the values 1, 2, 3, ..., \( m \) and multiplying the \( m \) inequalities thus obtained, we get,

\[ \frac{M_0(r, f(m))}{M_0(r, f)} \leq r^{(p+\epsilon-1)m}. \quad \ldots (2.9.3) \]

Further, if \( f(z) \) is an entire function of finite lower order \( \lambda \), then we can get similarly, for a sequence of values of \( r \) tending to infinity,

\[ \frac{M_0(r, f(m))}{M_0(r, f)} \leq r^{(\lambda+\epsilon-1)m}. \quad \ldots (2.9.4) \]

Consequently, we get

\[ \lim_{r \to \infty} \sup \log \left[ r \left\{ \frac{M_0(r, f(m))}{M_0(r, f)} \right\}^{\frac{1}{m}} \right] \leq \lambda. \quad \ldots (2.9.5) \]

By the property of a derivative, we have

\[ |f'(p-1)(re^{i\theta})| < |f'(p)(re^{i\theta})| \epsilon r + \epsilon^2 r + |f'(p-1)(r - r\epsilon e^{i\theta})| \]

\[ \sim |f'(p)(re^{i\theta})| \epsilon r + |f'(p-1)(r - r\epsilon e^{i\theta})|. \]
As \((a+b)^{\delta} \leq a^{\delta} + b^{\delta}\) if \(0 < \delta < 1\), \(a - b \leq \frac{1}{\delta} (a-b)\) if \(\frac{1}{\delta} \geq 1\) and \(a \geq 0, b \geq 0\), we have

\[|f(p)(re^{i\theta})|^{\delta} > |f(p-1)(re^{i\theta})|^{\delta} = |f(p-1)\overline{r-re^{i\theta}}|^{\delta}\]

Integrating both the sides of the above inequality with respect to \(\theta\) from 0 to \(2\pi\), we get

\[\frac{1}{2\pi} \int_{0}^{2\pi} |f(p)(re^{i\theta})|^{\delta} \, d\theta > \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |f(p-1)(re^{i\theta})|^{\delta} \, d\theta \right\} / (\varepsilon \delta).

Hence,

\[I_{\delta}(r, f(p)) \geq \frac{\delta r}{\{I_{\delta}(r, f(p-1))\}^{1-\delta}} \frac{d}{dr} M_{\delta}(r, f(p-1)). \quad (2.9.6)

Let

\[S(r) = \frac{\log M_{\delta}(r, f(p-1))}{\log r},

As \(S(r)\) is an increasing function of \(r\), we have \(S'(r) > 0\) and hence

\[\frac{d}{dr} \log M_{\delta}(r, f(p-1)) = \left( \frac{S(r)}{r} \right) + S'(r) \log r \]

\[> \frac{S(r)}{r} = \frac{\log M_{\delta}(r, f(p-1))}{r \log r}

This together with (2.9.6) gives us

\[I_{\delta}(r, f(p)) > \frac{\delta r (1-\delta)}{\{I_{\delta}(r, f(p-1))\}^{1-\delta}} \cdot \left\{ \frac{M_{\delta}(r, f(p-1)) \log M_{\delta}(r, f(p-1))}{r \log r} \right\}.

\]
Taking logarithms of both the sides of (2.9.7) and recalling that \( \log M_0(r, f^{(p-1)}) / \log r \) tends to \( \infty \) with \( r \), we find that

\[
\log M_0(r, f^{(p)}) > \log \alpha + \{1 + O(1)\} \log M_0(r, f^{(p-1)}) - \{1 + O(1)\} \log r.
\]

Hence, by repeated use of above inequality, we obtain for all sufficiently large \( r \) and any \( p > 2 \),

\[
\log M_0(r, f^{(p-1)}) > O(1) \log M_0(r, f).
\]

This together with (2.9.7) gives us

\[
M_0(r, f^{(p)}) > O(1) \left\{ \frac{M_0(r, f^{(p-1)})}{r} \right\} \left( \frac{\log M_0(r, f)}{\log r} \right).
\]

Giving \( p \) the values \( 1, 2, 3, \ldots, m \) and multiplying the \( m \) inequalities thus obtained, we get

\[
M_0(r, f^{(m)}) > O(1) \left\{ \frac{M_0(r, f)}{r^m} \right\} \left( \frac{\log M_0(r, f)^m}{\log r} \right) \text{ for all } r > r_0.
\]

Combining (2.8.3) and (2.9.9), we have
Comparing (2.8.5) and (2.9.10), we prove our theorem viz.

\[
\lim_{r \to \infty} \sup \frac{\log r}{\log M \left( r, f, m \right)} \geq \rho . \quad \ldots(2.9.10)
\]

2.10 \( m_{\delta, k}(r) \) VALUES OF AN ENTIRE FUNCTION AND ITS DERIVATIVES.

Let us define

\[
m_{\delta, k}(r, f) = \frac{k+1}{r^{k+1}} \int_{0}^{r} x^{k} M_{\delta}(x, f) \, dx \quad \ldots(2.10.1)
\]

and

\[
m_{\delta, k}(r, f, m) = \frac{k+1}{r^{k+1}} \int_{0}^{r} x^{k} M_{\delta}(x, f, m) \, dx , \quad \ldots(2.10.2)
\]

where \( \delta > 0 \) and \(-1 < k < \infty\). It is known that

\[
\lim_{r \to \infty} \sup \frac{\log \log m_{\delta, k}(r, f)}{\log \log m_{\delta, k}(r, f)} = \frac{\rho}{\lambda}, \quad 0 \leq \lambda, \quad \rho \leq \infty, [57] \quad \ldots(2.10.3)
\]

In the preceding sections, we have obtained two expressions for \( \rho \) and \( \lambda \) in terms of \( M_{\delta}(r, f) \) and \( M_{\delta}(r, f, m) \). In this section our aim is to obtain certain expression for \( \rho \) and \( \lambda \) in terms of means defined in (2.10.1) and (2.10.2). In what follows, we shall prove the following:

THEOREM 2.9. If \( f(z) \) is an entire function, other than a polynomial, of order \( \rho \) and lower order \( \lambda \), then
\[
\lim_{r \to \infty} \sup \left\{ \frac{\log \left[ r \left( \frac{m_0, k(r, f(m))}{m_0, k(r, f)} \right)^{\frac{1}{m}} \right]}{\log r} \right\} = \frac{\rho}{\lambda}, \quad \sigma \geq 1, \quad k \geq 0.
\]

**(PROOF.** From (2.8.6), we get

\[
M_0(x, f(m)) \leq \frac{A}{(j)^m} M_0(x+j, f), \quad (x > 0, \quad j > 0).
\]

Multiplying both the sides by \(x^k (k \geq 0)\) and integrating with respect to \(x\) between the limits \(0\) to \(r\), we get

\[
\int_0^r x^k M_0(x, f) dx \leq \frac{A}{(j)^m} \int_0^r x^k M_0(x+j, f) dx, \quad k \geq 0.
\]

\[
< \frac{A}{(j)^m} \int_0^r (x+j)^k M_0(x+j, f) dx,
\]

\[
< \frac{A}{(j)^m} \int_0^r u^k M_0(u, f) du,
\]

\[
< \frac{A}{(j)^m} \int_0^r x^k M_0(x, f) dx,
\]

Or

\[
\frac{r^{k+1}}{k+1} m_0, k(r, f) < \frac{A^2}{(R-r)^m} \frac{r^{k+1}}{k+1} m_0, k(R, f), \quad (R = r+j).
\]

Now, following the proof of Theorem 2.7, we have

\[
\frac{m_0, k(r, f(m))}{m_0, k(r, f)} < \mathcal{O}(1) r^{(\ell + \epsilon - 1)m} \quad \text{for} \quad r > r_0, \quad \epsilon > 0,
\]

\[
\ell < \infty.
\]

Further, if \(f(z)\) is an entire function of finite lower order \(\lambda\), then we get similarly, for a sequence of values of \(r\) tending to infinity,
Combining (2.10.3), (2.10.5) and (2.10.6), we get

\[
\lim_{r \to \infty} \sup \frac{\log [r \left( \frac{m_0, k(r, f(m))}{m_0, k(r, f)} \right) \frac{1}{m}]}{\log r} \leq \beta.
\]

Next, we consider the inverse of the above inequality.

Following Kamthan [35], we get,

\[
m_0, k(r, f(p)) > m_0, k(r, f(p-1)) \left( \frac{\log m_0, k(r, f(p-1))}{\log r} \right).
\]

Giving \( p \) the values 1, 2, 3, ..., \( m \) and multiplying the \( m \) inequalities thus obtained, we get

\[
m_0, k(r, f(m)) > m_0, k(r, f) \prod_{p=1}^{m} \left( \frac{\log m_0, k(r, f(p-1))}{r \log r} \right)
\]

for \( r > r'_0 = r_0(f) \) and \( \delta \geq 1 \).

Taking logarithms of both the sides of (2.10.8) and recalling that \( \frac{\log m_0, k(r, f(p-1))}{\log r} \) tends to \( \infty \) with \( r \), we find that

\[
\log m_0, k(r_0 f(p-1)) > O(1) \log m_0, k(r, f), \text{ for } p \geq 2.
\]

Substituting the value of \( \log m_0, k(r, f(p-1)) \) in (2.10.9),
we get
\[ m_{0,k}(r,f(m)) > m_{0,k}(r,f) \left( \frac{\log m_{0,k}(r,f)}{r \log r} \right)^m \cdot O(1). \]
This leads to
\[ \limsup_{r \to \infty} \sup_{\inf} \log \left[ r \left\{ \frac{m_{0,k}(r,f(m))}{m_{0,k}(r,f)} \right\}^{\frac{1}{m}} \right] / \log r \]
\[ \geq \lim_{r \to \infty} \sup \frac{\log \log m_{0,k}(r,f)}{\log r} \]
\[ \geq \lim_{r \to \infty} \inf \frac{\log \log m_{0,k}(r,f)}{\log r} \]
Combining this with (2.10.3), we have
\[ \log \left[ r \left\{ \frac{m_{0,k}(r,f(m))}{m_{0,k}(r,f)} \right\}^{\frac{1}{m}} \right] \]
\[ \limsup_{r \to \infty} \inf \frac{\log \log m_{0,k}(r,f)}{\log r} \geq \gamma. \]
Comparing (2.10.7) and (2.10.11), we get (2.10.4).
This completes the proof of Theorem 2.9.

REFERENCES

HARDY [23]; HAYMAN [25]; JUNEJA [30]; KAMTHAN [35]; KUMAR [40];
LAKSHMINARASIMHAN [42],[44]; RAHMAN [57]; RAO [63]; SHAH [70],[76];
SINGH AND DWIVEDI [93]; SRIVASTAVA [94]; SRIVASTAVA [96];
SRIVASTAVA [98],[99]; TITCHMARSH [103]; VALIRON [106].
CHAPTER III

ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY

A DIRICHLET SERIES

3.1 INTRODUCTION. In the second chapter, we have discussed some properties of the mean values of an entire function represented by Taylor series. In this chapter, we confine ourselves to the mean values of an entire function represented by a Dirichlet Series. In the usual notation,

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \]  

(s = \sigma + it), 0 < \lambda_n < \lambda_{n+1}  

(n \geq 1), \lim_{n \to \infty} \lambda_n = \infty,  

... (3.1.1)

is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite \( s \) and possesses two generally different pairs of orders:

\[ \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\sigma} = \rho, \quad \ldots \ (3.1.2) \]

\[ \lim_{\sigma \to \infty} \sup \frac{\log \log \mu(\sigma)}{\sigma} = \lambda_*, \quad \ldots \ (3.1.3) \]

where \( 0 \leq \lambda, \rho \leq \infty \), \( 0 \leq \lambda_*, \rho_* \leq \infty \), and \( M(\sigma), \mu(\sigma) \) have their usual meanings, viz.

\[ M(\sigma) = \max_{-\infty \leq t \leq \infty} |f(\sigma + it)|, \mu(\sigma) = \max_{n \geq 1} |a_n e^{(\sigma + it) \lambda_n}|. \]

It is obvious that generally \( \lambda_* \leq \lambda \) and \( \rho_* \leq \rho \), there are entire Dirichlet series for which \( \lambda_* < \lambda \) and \( \rho_* < \rho \) (see e.g. [100] Satz 4 which gives such a series with \( \rho_* < \rho \)). Hence we have
generally to distinguish between the limits of (3.1.2) and those of (3.1.3), as well as types of \( f(s) \) belonging to the same order \( \rho \) \((0 < \rho < \infty)\) and types of \( f(s) \) belonging to the same order \( \rho_* \) \((0 < \rho_* < \infty)\). The types \( T, t \) associated with \( \rho \) and types \( T_*, t_* \) associated with \( \rho_* \) are defined in the usual way as follows:

\[
\lim_{\sigma \to \infty} \sup_{\nu} \frac{\log M(\sigma)}{e^{\rho \sigma}} = T \quad (0 < \rho < \infty), \quad (3.1.4)
\]

\[
\lim_{\sigma \to \infty} \inf_{\nu} \frac{\log M(\sigma)}{e^{\rho \sigma}} = t \quad (0 < \rho < \infty), \quad (3.1.5)
\]

The mean values of \( f(s) \) are defined as follows:

\[
\left\{ I_2(\sigma, f) \right\}^2 = \left\{ I_2(\sigma) \right\}^2 = A_2(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt,
\]

\[
\ldots (3.1.6)
\]

\[
m_{2,k}(\sigma, f) = m_{2,k}(\sigma) = \lim_{T \to \infty} \frac{1}{2\pi k \log_{\infty} - T} \int_{-T}^{T} |f(x + it)|^2 e^{kx} \, dx \, dt
\]

\[
= \frac{1}{e^{kx}} \int_{-\infty}^{\infty} A_2(x)e^{kx} \, dx, \quad 0 < k < \infty \ldots (3.1.7)
\]

Kamthan [37] has obtained a few properties of the mean \( V_k(\sigma, f) \) of \( f(s) \), where \( V_k(\sigma, f) \) is defined as

\[
V_k(\sigma, f) = \frac{1}{e^{k\sigma}} \int_0^{\infty} A_2(x)e^{kx} \, dx = m_{2,k}(\sigma) - J, \quad 0 < k < \infty,
\]

\[
\ldots (3.1.8)
\]

where \( J \) is a real constant depending on \( k \) and \( f \). It easily follows from (3.1.8) that for all large \( \sigma \) the behaviour of \( m_{2,k}(\sigma, f) \) is the same as that of \( V_k(\sigma, f) \), and all results that have been derived for \( V_k(\sigma, f) \), can be obtained for \( m_{2,k}(\sigma, f) \).
3.2 GROWTH OF $m_{2,k}(\sigma)$. In this section, we investigate into a few properties of $m_{2,k}(\sigma)$.

**THEOREM 3.1** If $f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n}$, is an entire function of Ritt order $\rho$ and lower order $\lambda$, then

$$\rho_* \leq \lim_{\sigma \to \infty} \sup_{\sigma} \log \log m_{2,k}(\sigma) \leq \rho \quad \cdots(3.2.1)$$

In particular, when $(\lambda_n)$ satisfies the additional condition

$$0 \leq \lim_{n \to \infty} \sup_{\lambda_n} \frac{\log n}{\lambda_n} = D < \infty \quad \cdots(3.2.2)$$

(3.2.1) becomes

$$\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log m_{2,k}(\sigma)}{\sigma} = \rho = \rho_* \quad \cdots(3.2.3)$$

In fact for the truth of "limsup" part of the above conclusion the following additional condition on $(\lambda_n)$ is sufficient.

$$\lim_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0 \quad \cdots(3.2.2')$$

**PROOF.** The definition of $A_2(\sigma)$ and Parseval's identity for Dirichlet series, viz.

$$A_2(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma \lambda_n},$$

together give us
\[
\left\{ \mu(\sigma) \right\}^2 \leq A_2(\sigma) \leq \left\{ M(\sigma) \right\}^2.
\] \quad \cdots (3.2.4)

Also, since \( M(\sigma) \) is an increasing function of \( \sigma \),

\[
m_{2,k}(\sigma) = \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} e^{kx} A_2(x) dx,
\]

\[
\leq \left\{ M(\sigma) \right\}^2 / k.
\] \quad \cdots (3.2.5)

This leads to

\[
\limsup_{\sigma \to \infty} \inf \loglog m_{2,k}(\sigma) < \limsup_{\sigma \to \infty} \inf \loglog M(\sigma).
\] \quad \cdots (3.2.6)

Comparing (3.1.2) and (3.2.6), we get

\[
\limsup_{\sigma \to \infty} \inf \loglog m_{2,k}(\sigma) \leq \lambda.
\] \quad \cdots (3.2.7)

From (3.1.7), we have for \( h > 0 \)

\[
m_{2,k}(\sigma + h) \geq \frac{1}{e^{k(\sigma + h)}} \int_{\sigma}^{\sigma + h} e^{kx} A_2(x) dx.
\]

This with (3.2.4) will give us

\[
m_{2,k}(\sigma + h) \geq \frac{\left\{ \mu(\sigma) \right\}^2}{k} \left\{ 1 - e^{-kh} \right\}.
\] \quad \cdots (3.2.8)

Consequently, we get

\[
\limsup_{\sigma \to \infty} \inf \frac{\loglog \mu(\sigma)}{\sigma} \leq \limsup_{\sigma \to \infty} \inf \frac{\loglog m_{2,k}(\sigma)}{\sigma}.
\]

Now, using (3.1.3), we get

\[
\lambda_\ast \leq \limsup_{\sigma \to \infty} \inf \frac{\loglog m_{2,k}(\sigma)}{\sigma}.
\] \quad \cdots (3.2.9)
Combining (3.2.7) and (3.2.9), we have

\[ \rho_* \leq \lim_{\sigma \to \infty} \sup_{\sigma} \log \log m_{2,k}(\sigma) \leq \lambda. \]

To prove (3.2.3), we use the known result ([111], p. 68) that under the condition (3.2.2)

\[ M(\sigma) < K \mu(\sigma + D + \epsilon) \] ...(3.2.10)

where \( \epsilon \) is an arbitrary small positive number and \( K \) is a constant depending on \( D, \epsilon \). (3.2.5) in conjunction with (3.2.10) gives

\[ \sup_{\sigma} \log \log M(\sigma) \leq \lim_{\sigma \to \infty} \inf_{\sigma} \log \log m_{2,k}(\sigma) \]

From this the particular case, stated as part of Theorem 3.1, now follows immediately.

It is known that under the condition

\[ \lim_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0, \]

\[ \rho = \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}. \] [5] *

* This result is included in a theorem of Tanaka' ([102], p. 68).
Further, from the result of Reddy [66] we conclude that
\[
\rho_* = \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}.
\]
Combining these two we have
\[
\rho_* = \rho. 
\]
Thus, we have completed the proof of the theorem.

**Remark.** Juneja [31] has proved the particular case of our theorem under the condition (3,2,2) with \(D = o\). His argument, however, is faulty at the point where he has used the asymptotic equality \(\log M(\sigma) \sim \log \mu(\sigma), \sigma \to \infty\). For, a sufficient condition for the truth of asymptotic equality is known only in the form ([111], p.73)
\[
0 \leq \limsup_{n \to \infty} \frac{\log n}{\log \lambda_n} = \mathbb{E} < \infty,
\]
and this condition is not necessarily implied by Juneja's assumption of (3,2,2) with \(D = o\).

**Theorem 3.2** If \(f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n}\) is an entire function of Ritt order \(\rho (0 < \rho < \infty)\), type \(T\) and lower type \(t\), then
\[
2T_* \leq \limsup_{\sigma \to \infty} \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \leq 2T \leq 2T_* e^{\rho D} \leq 2t \leq 2t_* e^{\rho D}
\]
under the condition (3.2.11).
In the particular case \( D = 0 \) of condition (3.2.2)

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} = \frac{2T}{2t} = \frac{2T^*}{2t^*}. \] …(3.2.12)

**Proof.** From (3.2.5) and (3.2.10), we get

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \leq 2 \lim_{\sigma \to \infty} \sup \inf \frac{\log M(\sigma)}{e^{\rho \sigma}} \]

\[ \leq 2 \lim_{\sigma \to \infty} \sup \inf \frac{\log \mu(\sigma+D+\epsilon)}{e^{\rho(\sigma+D+\epsilon)}} \cdot \frac{\rho(\sigma+D+\epsilon)}{e^{\rho(\sigma+D+\epsilon)}}. \]

… (3.2.13)

Combining (3.1.4), (3.1.5) and (3.2.13), we have

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \leq \frac{2T}{2t} \leq \frac{2T^*}{2t^*} e^{D+\epsilon} \cdot \]

…(3.2.14)

Also, (3.2.8) leads to

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \geq \frac{2}{e^{\rho h}} \lim_{\sigma \to \infty} \sup \inf \frac{\log \mu(\sigma)}{e^{\rho \sigma}}. \]

Since left hand side is independent of \( h \), therefore making \( h \to 0 \), we get

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \geq 2 \lim_{\sigma \to \infty} \sup \inf \frac{\log \mu(\sigma)}{e^{\rho \sigma}}. \]

…(3.2.15)

Combining (3.1.5) with (3.2.15), we get

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log m_{2,k}(\sigma)}{e^{\rho \sigma}} \geq \frac{2T^*}{2t^*}. \]

…(3.2.16)

* Condition (3.2.2) ensures \( \rho = \rho^* \).
Comparing (3.2.14) and (3.2.16), we get

\[
2T_s \leq \lim_{\sigma \to \infty} \sup_{\varepsilon} \frac{\log m_{2,k}(\sigma)}{e^{\rho_x}} \leq \frac{2T}{2t} \leq \frac{2T e^{\rho(D+\varepsilon)}}{2t e^{\rho(D+\varepsilon)}},
\]

which is (3.2.11), \( \varepsilon \) being arbitrary. If \( D = 0 \),

\[
\lim_{\sigma \to \infty} \inf_{\varepsilon} \frac{\log m_{2,k}(\sigma)}{e^{\rho_x}} = \frac{2T}{2t} = \frac{2T_s}{2t}.
\]

This is the conclusion sought.

3.3 MEAN VALUES OF AN ENTIRE FUNCTION AND ITS DERIVATIVES.

In this section, we study the growth properties of \( m_{2,k}(\sigma) \) and \( m_{2,k}(\sigma, f(m)) \). Let us define

\[
m_{2,k}(\sigma, f(m)) = \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} A_2(x, f(m)) e^{kx} dx
\]

where \( m = 1, 2, 3, \ldots, m \).

Now, we prove

**Theorem 3.3** If \( f(s) \) is an entire function, then

\[
\lambda_* \leq \lim_{\sigma \to \infty} \sup_{\varepsilon} \log \left( \frac{m_{2,k}(\sigma, f)}{m_{2,k}(\sigma, f)} \right)^{\frac{1}{2}} / \sigma \leq \rho \cdot \lambda_* \leq \lambda.
\]

... (3.3.1)
In particular, when \( \{\lambda_n\} \) satisfies the additional condition (3.2.2), we have
\[
\lim_{r \to \infty} \sup \frac{\log \left\{ \frac{m_2,k(\sigma,f(1))}{m_2,k(\sigma,f)} \right\}^{1/2}}{\log \left( \frac{1}{\rho} \right)} = \frac{\rho}{\rho_*} \quad \text{...(3.3.2)}
\]

In fact, for the truth of "limsup" part of the above conclusion the following additional condition is sufficient
\[
\lim_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0 .
\]

The following lemmas are needed in the proof of Theorem 3.3.

**Lemma 3.1** Given \( \epsilon = \epsilon(\sigma) > 0 \), which may be a constant, we have
\[
m_2,k(\sigma,f(1)) \leq \frac{m_2,k(\sigma+\epsilon)}{\epsilon^2} . \quad \text{...(3.3.3)}
\]

**Proof.** By the definitions, we have
\[
A_2(\sigma) = \left( \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} \right)
\]
and
\[
m_2,k(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|^2 e^{2\sigma\lambda_n}}{2\lambda_n + k} .
\]

Similarly, it can be shown that
\[
m_2,k(\sigma,f(1)) = \sum_{n=1}^{\infty} \frac{2l_n |a_n|^2 e^{2\sigma\lambda_n}}{2\lambda_n + k}
\]

By using the fact that
which is always true, we can have finally

\[ m_{2,k}(\sigma + \varepsilon) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \left| a_n \right| e^{2 \lambda_n (\sigma + \varepsilon)} \]

\[ \leq \frac{1}{\varepsilon^2} m_{2,k}(\sigma + \varepsilon). \]

Hence the result.

**Lemma 3.2** There exists a monotonic increasing function \( S(\sigma) \) associated with \( m_{2,k}(\sigma) \) such that

\[ \log m_{2,k}(\sigma) = \log m_{2,k}(\sigma_0) + \int_{\sigma_0}^{\sigma} S(x) dx, \quad \ldots(3.3.4) \]

\[ \rho_0 \leq \lim_{\sigma \to \infty} \sup \inf \frac{\log S(\sigma)}{\sigma} \leq \frac{\rho}{\lambda}. \quad \ldots(3.3.5) \]

**Proof.** First assertion follows from the fact that \( \log m_{2,k}(\sigma) \) is a convex function of \( \sigma \).

To prove (3.3.5), we get from (3.3.4) successively

\[ S(\sigma) \leq \int_{\sigma_0}^{\sigma+1} S(x) dx = \log m_{2,k}(\sigma + 1) - \log m_{2,k}(\sigma), \]

\[ \leq \log m_{2,k}(\sigma + 1). \]

This leads to

\[ \lim_{\sigma \to \infty} \sup \inf \frac{\log S(\sigma)}{\sigma} \leq \lim_{\sigma \to \infty} \sup \frac{\loglog m_{2,k}(\sigma + 1)}{\sigma + 1} \cdot \frac{\sigma + 1}{\sigma} \leq \frac{\rho}{\lambda}. \quad \ldots(3.3.6) \]
In virtue of right half of the inequality (3.2.1). We also get from (3.3.4)

\[ \log m_2, k(\sigma) = \log m_2, k(\sigma_0) + \int_{\sigma_0}^{\sigma} S(x)dx \]

\[ \leq \log m_2, k(\sigma_0) + (\sigma - \sigma_0)S(\sigma) \]

\[ \sim \sigma S(\sigma), \sigma \to \infty. \]

Taking the logarithm of both the sides and proceeding to limits we get

\[ \lim_{\sigma \to \infty} \sup \frac{\log \log m_2, k(\sigma)}{\sigma} \leq \lim_{\sigma \to \infty} \inf \frac{\log S(\sigma)}{\sigma}, \]

Combining this with (3.2.1), we have

\[ r^*_n \leq \lim_{\sigma \to \infty} \sup \frac{\log S(\sigma)}{\sigma}. \quad \ldots(3.3.7) \]

(3.3.6) and (3.3.7) together give us (3.3.5).

**Proof of Theorem 3.3.** We have

\[ m_2, k(\sigma, f^{(1)}) = \lim_{T \to \infty} \frac{1}{2T \sigma^{kr}} \int_{-\infty}^{\infty} \int_{-T}^{T} |f(1)(x+it)|^2 e^{kx}dx dt, \]

\[ = \lim_{T \to \infty} \frac{1}{2T \sigma^{kr}} \int_{-\infty}^{\infty} \int_{-T}^{T} \lim_{\epsilon \to 0} |f(x+it) - f(x-\epsilon x + it)|^2 e^{kx}dx dt, \]

\[ \geq \lim_{T \to \infty} \frac{1}{2T \sigma^{kr}} \int_{-\infty}^{\infty} \int_{-T}^{T} \lim_{\epsilon \to 0} \frac{|f(x+it)| - |f(x-\epsilon x + it)|}{\epsilon x} |e^{kx}dx dt. \]

Now, by Minkowski's inequality ([103], p.384).
\[
\left\{ \int_{-T}^{T} \left( |f(x+it)| - |f(x-xe^{it})| \right)^{2} \frac{1}{2} \, dt \right\}^{\frac{1}{2}}
\]
\[
\geq \left\{ \left( \int_{-T}^{T} |f(x+it)| \, dt \right)^{2} - \left( \int_{-T}^{T} |f(x-xe^{it})| \, dt \right)^{2} \right\}^{\frac{1}{2}}.
\]

Hence,
\[
m_{2,k}(\sigma, f(1)) \geq \lim_{T \to \infty} \lim_{\xi \to 0} \frac{1}{e^{\xi \sigma}} \frac{1}{2 \pi c^{2}} \left\{ \int_{-\infty}^{T} \left( \int_{-T}^{T} e^{\xi \sigma} \left( \int_{-T}^{T} e^{\xi \sigma} \left( \frac{1}{2} \right) \right) \, dx \, dt \right) \right\}^{\frac{1}{2}}
\]
\[
\geq \lim_{T \to \infty} \lim_{\xi \to 0} \frac{1}{e^{\xi \sigma}} \frac{1}{2 \pi c^{2}} \left\{ \int_{-\infty}^{T} e^{\xi \sigma} \left( \int_{-T}^{T} e^{\xi \sigma} \left( \frac{1}{2} \right) \right) \, dx \, dt \right\}^{\frac{1}{2}}
\]

Again using Minkowski's inequality,
\[
m_{2,k}(\sigma, f(1)) \geq \lim_{T \to \infty} \lim_{\xi \to 0} \frac{1}{e^{\xi \sigma}} \frac{1}{2 \pi c^{2}} \left\{ \int_{-\infty}^{T} \left( \int_{-T}^{T} e^{\xi \sigma} \left( \int_{-T}^{T} e^{\xi \sigma} \left( \frac{1}{2} \right) \right) \, dx \, dt \right) \right\}^{\frac{1}{2}}
\]
\[
\leq \lim_{\xi \to 0} \frac{1}{e^{\xi \sigma}} \frac{1}{2 \pi c^{2}} \left\{ \frac{m_{2,k}(\sigma)}{2} - \frac{m_{2,k}(\sigma - \xi)}{2} \right\}^{\frac{1}{2}}
\]

Now let,
\[
H(\sigma) = \frac{\log m_{2,k}(\sigma)}{\sigma},
\]
then since \( \log m_{2,k}(\sigma) \) is a steadily increasing convex function of \( \sigma \) for \( \sigma > \sigma_{0} \), it follows that \( H(\sigma) \) is a positive increasing function of \( \sigma \) and therefore
\[
m_{2,k}(\sigma, f(1)) \geq \lim_{\xi \to 0} \frac{e^{\frac{\sigma H(\sigma)}{2}} - e^{\frac{\sigma - \xi H(\sigma - \xi)}{2}}}{\xi \sigma} \left( \frac{H(\sigma)}{2} \right)^{2}
\]
\[
\geq e^{\frac{\sigma H(\sigma)}{2}} \left( \frac{H(\sigma)}{2} \right)^{2}
\]
From this we have, for all large \( \sigma \)

\[
\log \left\{ \frac{m_{2,k}(\sigma,f(1)))^{1/2}}{m_{2,k}(\sigma,f)} \right\} \geq \frac{\log \log m_{2,k}(\sigma)}{\sigma} - \frac{\log 2\sigma}{\sigma},
\]

in virtue of left half of inequality of (3.2.1).

Next, we consider the inverse of above inequality. Lemma 3.1 leads to

\[
\log m_{2,k}(\sigma,f(1)) \leq \log m_{2,k}(\sigma + \epsilon) + \log \frac{1}{\epsilon^2}.
\]

From this and (3.3.4), we get

\[
\log m_{2,k}(\sigma,f(1)) \leq \log m_{2,k}(\sigma) + \frac{\sigma + \epsilon}{\sigma} \int_{\sigma}^{\sigma+\epsilon} S(x)dx + \log \frac{1}{\epsilon^2}
\]

\[
\leq \log m_{2,k}(\sigma) + \epsilon S(\sigma + \epsilon) + 2\log \frac{1}{\epsilon^2}. \quad \ldots(3.3.10)
\]

In (3.3.10) we choose \( \epsilon \) in terms of \( \sigma \) as follows. Let the expression in (3.3.10) be considered as a function of \( \epsilon \) in first instance (with \( \sigma \) fixed for the time being). This expression, for varying \( \epsilon \), is least when

\[
S(\sigma + \epsilon) - \frac{2}{\epsilon} = 0. \quad \ldots(3.3.11)
\]

Let \( \epsilon \) be chosen to satisfy (3.3.11). Then, \( S(\sigma) \) being a monotonic increasing function of \( \sigma \), \( \epsilon \) satisfies further

\[
\frac{2}{\epsilon} \geq S(\sigma) \rightarrow \infty \quad (\sigma \rightarrow \infty),
\]
as also in view of (3.3.10),

\[
\lim_{\sigma \to \infty} \sup_{\infty} \log \left\{ \frac{m_2, k(\sigma, f(1))}{m_2, k(\sigma, f)} \right\}^{\frac{1}{2}} \leq \lim_{\sigma \to \infty} \sup_{\infty} \log S(\sigma + \epsilon) \frac{\sigma + \epsilon}{\sigma} \cdot \frac{\sigma + \epsilon}{\sigma}
\]

\[
\leq \lim_{\sigma \to \infty} \sup_{\infty} \log S(\sigma) \cdot \frac{\sigma + \epsilon}{\sigma}
\]

\[\ldots(3.3.12)\]

Combining (3.3.5) and (3.3.12), we have

\[
\lim_{\sigma \to \infty} \sup_{\infty} \log \left\{ \frac{m_2, k(\sigma, f(1))}{m_2, k(\sigma, f)} \right\}^{\frac{1}{2}} \leq \rho
\]

\[\ldots(3.3.13)\]

Finally, combining (3.3.9) and (3.3.13), we obtain the following

\[
\frac{\rho}{\lambda^*} \leq \lim_{\sigma \to \infty} \inf_{\infty} \log \left\{ \frac{m_2, k(\sigma, f(1))}{m_2, k(\sigma, f)} \right\}^{\frac{1}{2}} \leq \frac{\rho}{\lambda^*}
\]

The particular case of Theorem 3.3, now follows immediately on lines of Theorem 3.1.

Next we deduce from Theorem 3.3, the following

**Theorem 3.4** If \( 0 \leq \rho < \infty \), then \( \log m_2, k(\sigma) \sim \log m_2, k(\sigma, f(1)) \).

**Proof.** From (3.3.4) we have

\[
\frac{\sigma}{2} S\left(\frac{\sigma}{2}\right) \leq \int_{\frac{\sigma}{2}}^{\sigma} S(x)dx = \log m_2, k(\sigma) - \log m_2, k\left(\frac{\sigma}{2}\right) < \log m_2, k(\sigma),
\]
or

\[
\frac{\log m_2, k(\sigma)}{\sigma} > \frac{1}{2} s(\frac{\sigma}{2}) \to \infty, \quad \ldots (3.3.14)
\]

since, in case \(s(\frac{\sigma}{2})\), which is monotonic increasing has finite limit as \(\sigma \to \infty\), the series for \(m_2, k(\sigma)\) must have a finite number of terms, as also the series for \(f(s)\).

To prove the theorem, we have only to note that \((3.3.1)\) gives us, for any small \(\epsilon > 0\) and sufficiently large \(\sigma\),

\[
\frac{2(\lambda - \epsilon)\sigma}{\log m_2, k(\sigma)} < \frac{\log m_2, k(\sigma, f^{(1)})}{\log m_2, k(\sigma)} - 1 < \frac{2(\rho + \epsilon)\sigma}{\log m_2, k(\sigma)}
\]

since, by \((3.3.14)\), the extreme members tend to 0 as \(\sigma \to \infty\), the middle member also tends to 0 as we wished to prove.

**Remark.** Theorem 3.4 is not only more general than Theorem 2 of Juneja and Awasthi [32], but has a proof different from that as well as shorter and more widely applicable.

Towards the close of this section, we give a result concerning \(m_2, k(\sigma)\) and \(m_2, k(\sigma, f^{(m)})\).

**Theorem 3.5** If \(f(s)\) is an entire function and \(f^{(m)}(s)\) is its \(m^{th}\) derivative, then

\[
\rho_* \leq \lim \sup_{\sigma \to \infty} \frac{\log \left\{ \frac{m_2, k(\sigma, f^{(m)})}{m_2, k(\sigma, f)} \right\}}{\sigma} \leq \rho
\]
Here, if \( \{\lambda_n\} \) satisfies the condition

\[
\lim_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0
\]

then

\[
\limsup_{\sigma \to \infty} \frac{\log \left( \frac{m_2, k(\sigma, f(m))}{m_2, k(\sigma, f)} \right)^{\frac{1}{2m}}}{\sigma} = \rho
\]

and if (3.3.2) holds, then

\[
\liminf_{\sigma \to \infty} \frac{\log \left( \frac{m_2, k(\sigma, f(m))}{m_2, k(\sigma, f)} \right)^{\frac{1}{2m}}}{\sigma} = \lambda
\]

The proof of this theorem is quite similar to the proof of Theorem 2.9 proved earlier.

3.4 Further Results on General Mean Values of \( f(s) \).

In this section, we have considered the more general case involving the mean values. Let us define the following mean values of \( f(s) \):

\[
\left\{ I_0(\sigma) \right\}^{\delta} = A_0(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta \, dt, \quad 0 < \delta < \infty
\]

\[
N_0, k(\sigma) = \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} I_0(x) \, e^{kx} \, dx, \quad \ldots (3.4.2)
\]

and

\[
J_0, k(\sigma) = \exp \left\{ \frac{1}{e^{k\sigma}} \int_{0}^{\sigma} \log I_0(x) \, e^{kx} \, dx \right\} \ldots (3.4.3)
\]
Now, we prove the following:

**Theorem 3.6** (i) For $0 < k < \infty$, $\delta \geq 1$

$$\lambda_* \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log N_{\delta}, k(\sigma)}{\delta} \leq \lambda. \quad \ldots(3.4.4)$$

Under the additional condition on $\{\lambda_n\}$ in (3.2.2')

$$\limsup_{\sigma \to \infty} \frac{\log \log N_{\delta}, k(\sigma)}{\sigma} = \rho. \quad \ldots(3.4.5)$$

Further, if $\{\lambda_n\}$ satisfies the condition (3.2.2), then

$$\liminf_{\sigma \to \infty} \frac{\log \log N_{\delta}, k(\sigma)}{\sigma} = \lambda. \quad \ldots(3.4.6)$$

(ii) For $0 < k < \infty$, $\delta > 0$

$$\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log N_{\delta}, k(\sigma)}{\sigma} \leq \rho. \quad \ldots(3.4.7)$$

**Proof.** For fixed $\sigma$, 

$$f(\sigma + it) = \sum_{n=1}^{\infty} (a_n e^{\lambda_n t}) e^{i \lambda_n t} \quad (-\infty < t < \infty)$$

is an absolutely and uniformly convergent function of $t$ and hence ([7], p.6) a function of $t$ which is uniformly almost periodic (briefly u.a.p.). If $|f(\sigma + it)|^\delta$, $\delta > 0$, is also a function of $t$ which is u.a.p., as shown by familiar considerations (e.g. as in [7], p.3) involving the following well known inequalities for $a \geq 0$, $b > 0$
By the result ([7], p. 12), the mean value of $|f(\sigma + it)|^\delta$, $\delta > 0$ defined by $A_\delta(\sigma)$ exists.

For $\delta > 0$, it is obvious that

$$I_\delta(\sigma) \leq M(\sigma).$$

This with (3.4.2) will give us

$$N_{\delta, k}(\sigma) \leq \frac{M(\sigma)}{k}, \quad \text{...(3.4.8)}$$

From which it follows that

$$\lim_{\sigma \to \infty} \sup_{\delta > 0} \frac{\log \log N_{\delta, k}(\sigma)}{\sigma} \leq \frac{p}{\gamma}, \quad 0 < k < \infty, \delta > 0.$$

...(3.4.9)

This completes the proof of (3.4.7).

If $\delta > 1$, we start with Hadamard's formula:

$$a_n = \lim_{n \to \infty} \frac{1}{T} \int_{-T}^{T} e^{-(\sigma + it)\lambda n} f(\sigma + it) dt, (n \geq 1). \quad \text{...(3.4.10)}$$

For fixed $n$ and $\sigma$, this formula gives us

$$a_{n\lambda} = \lim_{n \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-it\lambda n} f(\sigma + it) dt + \epsilon(T, n, \sigma)$$

where $\epsilon(T, n, \sigma) \to 0$ as $T \to \infty$. Hence, for fixed $n$ and $\sigma$, we have

$$|a_n| e^{\sigma \lambda n} \leq \lim_{T \to \infty} \int_{-T}^{T} |f(\sigma + it)| dt + |\epsilon(T, n, \sigma)|.$$

Here we choose $n = \mu(\sigma)$ (rank of maximum term), so that the left hand member has its maximum value $\mu(\sigma)$. With this
choice of \( n \), letting \( T \to \infty \), we get

\[
\mu(\sigma) \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)| \, dt = I_1(\sigma). \tag{3.4.11}
\]

If \( \delta > 1 \), we also get, by Hölder's integral inequality

\[
\mu(\sigma) \leq \lim_{T \to \infty} \left[ \frac{1}{2T} \int |f(\sigma + it)| \, dt \right] \delta \left[ \frac{1}{2T} \int dt \right] \delta' \tag{3.4.12}
\]

where \( \frac{1}{\delta} + \frac{1}{\delta'} = 1 \).

(3.4.11) and (3.4.12) give us

\[
\mu(\sigma) \leq I_0(\sigma) \text{ for } \delta \geq 1.
\]

From (3.4.2), we have for \( h > 0 \),

\[
N_{\delta, k}(\sigma + h) \geq \frac{\mu(\sigma)}{k} (1 - e^{-kh}). \tag{3.4.13}
\]

This leads to

\[
\frac{\log \log N_{\delta, k}(\sigma + h)}{(\sigma + h)} \geq \frac{\log \log \mu(\sigma)}{\sigma + h} + o(1).
\]

Proceeding to limits, we get

\[
\lim_{\sigma \to \infty} \sup_{\sigma \to \infty} \inf \frac{\log \log N_{\delta, k}(\sigma)}{\sigma} \geq \rho^*, \tag{3.4.14}
\]

by (3.1.3).

Finally, combining (3.4.9) and (3.4.14), we get

\[
\rho^* \leq \lim_{\sigma \to \infty} \sup_{\sigma \to \infty} \inf \frac{\log \log N_{\delta, k}(\sigma)}{\sigma} \leq \lambda^*.
\]
The particular cases (3.4.5) and (3.4.6), now follow readily since condition (3.2.2') ensures \( \rho = \rho_* \) and (3.2.2) ensures \( \lambda = \lambda_* \).

**Theorem 3.7**

(i) For \( a > 0, \ a < k < \infty \)

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log N_{\sigma,k}(\sigma)}{e^{\rho \sigma}} \leq \frac{T}{t} \quad (0 < \rho < \infty).
\]

...(3.4.15)

(ii) For \( 0 > 1, \ a < k < \infty \) and under the condition (3.2.2)

\[
T_* \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log N_{\sigma,k}(\sigma)}{e^{\rho \sigma}} \leq \frac{T}{t} \leq \frac{T_* e^{\rho D}}{t_* e^{\rho D}}.
\]

...(3.4.16)

In particular case if \( D = 0 \),

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log N_{\sigma,k}(\sigma)}{e^{\rho \sigma}} = \frac{T_+}{t_*} = \frac{T}{t}.
\]

...(3.4.17)

**Proof.** From (3.4.8), we get

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log N_{\sigma,k}(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log M(\sigma)}{e^{\rho \sigma}}.
\]

Combining this with (3.1.4), we get (3.4.15).

To prove (3.4.16), we use (3.4.8), (3.4.13) and the known result

\[ M(\sigma) < K\mu(\sigma + D + \varepsilon) \quad ([111], p.68) \]

where \( \varepsilon \) is an arbitrary small and positive number \( K \) is a.
constant depending on \( D \), and \( \epsilon \). Finally, we have

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \mu(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \inf_{\sigma} \frac{\log M(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \mu(\sigma + D + \epsilon)}{e^{\rho \sigma}}
\]

and

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \mu(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \inf_{\sigma} \frac{\log N_{\delta}(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log M(\sigma)}{e^{\rho \sigma}}.
\]

Combining these two, we get desired conclusion (3.4.16).

The particular case (3.4.17) is obvious.

**Theorem 3.8** (i) For \( \delta > \delta \),

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log I_0(\sigma)}{e^{\rho \sigma}} \leq \frac{T}{t} \quad \ldots (3.4.18)
\]

(ii) For \( \delta \geq 1 \),

\[
\sup_{\sigma \to \infty} \frac{\log I_0(\sigma)}{e^{\rho \sigma}} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log I_0(\sigma)}{e^{\rho \sigma}} \cdot \quad \ldots (3.4.19)
\]

Further, under the condition (3.2.2)

\[
\frac{T_0}{t_0} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log I_0(\sigma)}{e^{\rho \sigma}} \leq \frac{T}{t} \leq \frac{T_0 e^{\rho D}}{t_0 e^{\rho D}} \cdot \quad \ldots (3.4.20)
\]

In particular case if \( D = 0 \)

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log I_0(\sigma)}{e^{\rho \sigma}} = \frac{T_0}{t_0} = \frac{T}{t} \cdot \quad \ldots (3.4.21)
\]
We omit the proof, since this easily follows from the fact

\[ I_0(\sigma) \leq M(\sigma), \quad \delta > 0 \]
\[ \mu(\sigma) \leq I_0(\sigma), \quad \delta \geq 1 \]

and the known result

\[ M(\sigma) \leq K \mu(\sigma + D + \epsilon). \]

**Remark.** Theorem 3.7 and Theorem 3.8 include the Theorem of Jain [27], which in turn includes the theorem of Juneja [31] and also a theorem of Gupta [21]. The method of proofs of our results is still different technically from that of Jain [ibid]. Jain uses in his proof the following result of Kamthan ([39] p.222):

\[ A_0(\sigma) \leq \left\{ M(\sigma) \right\}^\delta \leq O(1) A_0(\sigma + \eta), \eta > 0, \]

which is proved under the conditions

\[ \liminf_{n \to \infty} (\lambda_n - \lambda_{n-1}) > 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{n}{\lambda_n} = h > 0. \]

The conditions used by Kamthan [ibid] are more restrictive than condition (3.2.2).

### 3.5 SOME RELATIONS BETWEEN TWO OR MORE ENTIRE FUNCTIONS.

In this section, we shall obtain the relations between two or more entire functions and study the mean values of entire functions and their orders. Let
THEOREM 3.9. If

\[ \log \log N_{0,k}(\sigma,f) \sim \log \{ \log N_{0,k}(\sigma,f_1) \cdot \log N_{0,k}(\sigma,f_2) \} \]

then

(i) \( \bar{\rho} \leq \rho_1 + \rho_2 \), \( \sigma > 0 \),

(ii) \( \lambda_1 + \lambda_2 \leq \bar{\lambda} \), \( \sigma \geq 1 \).

Further, if

\[ \log \log N_{0,k}(\sigma,f) \sim \left\{ \log \log N_{0,k}(\sigma,f_1) \cdot \log \log N_{0,k}(\sigma,f_2) \right\}^{\frac{1}{2}} \]

then

(iii) \( \bar{\rho} \leq (\rho_1, \rho_2)^{\frac{1}{2}} \), \( \sigma > 0 \),

(iv) \( \bar{\lambda} \geq (\lambda_1, \lambda_2)^{\frac{1}{2}} \), \( \sigma \geq 1 \).

Where \( \rho_j \) and \( \lambda_j \) (\( j = 1,2 \)) are finite non-zero and correspond to \( f_j (j = 1,2) \).

PROOF. From (3.4.4), we have

\[ \limsup_{\sigma \to \infty} \frac{\log \log N_0,k(\sigma,f_1)}{\sigma} \leq \rho_1 \]

and

\[ \limsup_{\sigma \to \infty} \frac{\log \log N_0,k(\sigma,f_2)}{\sigma} \leq \rho_2 \cdot \]

Hence, for an arbitrary number \( \epsilon > 0 \) and \( \sigma > \sigma_0 \),

\[ \log \log N_{0,k}(\sigma,f_1) \leq \left( \rho_1 + \frac{\epsilon}{2} \right) \ldots (3.5.1) \]
and
\[
\frac{\log \log N_{0,k}(\sigma, f_1)}{\sigma} < \left( \rho_1^2 + \frac{\epsilon}{2} \right).
\] ...(3.5.2)

Adding these two inequalities, we get
\[
\frac{\log \left\{ \log N_{0,k}(\sigma, f_1) \cdot \log N_{0,k}(\sigma, f_2) \right\}}{\sigma} < \left( \rho_1^2 + \rho_2^2 + \epsilon \right), \delta > 0.
\]

Similarly, we get
\[
\frac{\log \left\{ \log N_{0,k}(\sigma, f_1) \cdot \log N_{0,k}(\sigma, f_2) \right\}}{\sigma} > \left( \lambda_{*1} + \lambda_{*2} - \epsilon \right), \delta > 1.
\]

Therefore, if
\[
\log \log N_{0,k}(\sigma, f) \sim \log \left\{ \log N_{0,k}(\sigma, f_1) \log N_{0,k}(\sigma, f_2) \right\},
\]
we get
\[
\bar{\rho} \leq \rho_1 + \rho_2, \quad \delta > 0
\]
and
\[
\bar{\lambda} \geq \lambda_{*1} + \lambda_{*2}, \quad \delta > 1.
\]

Again, let us multiply (3.5.1) and (3.5.2), we have
\[
\frac{\log \log N_{0,k}(\sigma, f_1)}{\sigma^2} \cdot \frac{\log \log N_{0,k}(\sigma, f_2)}{\sigma^2} < \left( \rho_1^2 + \frac{\epsilon}{2} \right) \left( \rho_2^2 + \frac{\epsilon}{2} \right),
\]
which leads to
\[
\bar{\rho} \leq \left( \rho_1 \rho_2 \right), \quad \delta > 0.
\]

Similarly, we can obtain the rest part of the Theorem e.g.
\[
\bar{\lambda} \geq \left( \lambda_{*1} \lambda_{*2} \right), \quad \delta > 1.
\]

Remark. We omit the numerous obvious corollaries.
3.6 GROWTH RELATIONS IN TERMS OF $N_{\delta,k}(\sigma)$ AND $N_{\delta,k}'(\sigma)$.

In this section, we shall study a few properties of $N_{\delta,k}(\sigma)$ and its derivative $N_{\delta,k}'(\sigma)$.

**THEOREM 3.10**

(i) For all $\delta > 0$, $0 < k < \infty$,

\[
\lim_{\sigma \to \infty} \sup_{\tau} \inf_{\lambda} \log \left\{ \frac{N_{\delta,k}^{\prime}(\sigma)}{N_{\delta,k}(\sigma)} \right\} / \sigma \leq \rho. \quad \ldots(3.6.1)
\]

(ii) For all $\delta > 1$, $0 < k < \infty$,

\[
\lim_{\sigma \to \infty} \sup_{\tau} \inf_{\lambda} \log \left\{ \frac{N_{\delta,k}^{\prime}(\sigma)}{N_{\delta,k}(\sigma)} \right\} / \sigma \leq \rho. \quad \ldots(3.6.2)
\]

(iii) For all $\delta > 1$, $0 < k < \infty$, under the additional condition (3.2.2) on $\{\lambda_n\}$,

\[
\lim_{\sigma \to \infty} \sup_{\tau} \inf_{\lambda} \log \left\{ \frac{N_{\delta,k}^{\prime}(\sigma)}{N_{\delta,k}(\sigma)} \right\} / \sigma = \rho = \rho_{*}. \quad \ldots(3.6.3)
\]

**PROOF.** We know ([26], p.114) that $\log N_{\delta,k}(\sigma)$ is an increasing convex function of $\sigma$. Therefore, $\log N_{\delta,k}(\sigma)$ is differentiable almost everywhere with an increasing derivative, the set of points where the left hand derivative is less than the right-hand derivative is of measure zero. This enables us to express $\log N_{\delta,k}(\sigma)$ in the following form:

\[
\log N_{\delta,k}(\sigma) = \log N_{\delta,k}(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{N_{\delta,k}'(x)}{N_{\delta,k}(x)} \, dx
\]

for an arbitrary $\sigma_0$. 
Thus we have
\[
\log N_{\sigma, k}(\sigma) \leq \log N_{\sigma_0, k}(\sigma_0) + \frac{N'_{\sigma, k}(\sigma)}{N_{\sigma, k}(\sigma)} (\sigma - \sigma_0)
\]
or
\[
\lim_{\sigma \to \infty} \sup \inf \frac{\log \log N_{\sigma, k}(\sigma)}{\sigma} \leq \lim_{\sigma \to \infty} \sup \inf \left\{ \frac{1}{\sigma} \left\{ \log \left( \frac{N'_{\sigma, k}(\sigma)}{N_{\sigma, k}(\sigma)} \right) \right\} \right\}.
\]

Again, for an arbitrary fixed \( h > 0 \)
\[
\log N_{\sigma, k}(\sigma + h) = \log N_{\sigma_0, k}(\sigma) + \int_{\sigma}^{\sigma+h} \frac{N'_{\sigma, k}(x)}{N_{\sigma, k}(x)} \, dx
\]
\[
\geq h \left\{ \frac{N'_{\sigma_0, k}(\sigma)}{N_{\sigma_0, k}(\sigma)} \right\},
\]
and therefore,
\[
\lim_{\sigma \to \infty} \sup \inf \frac{\log \log N_{\sigma, k}(\sigma)}{\sigma} \geq \lim_{\sigma \to \infty} \sup \inf \log \left( \frac{N'_{\sigma_0, k}(\sigma)}{N_{\sigma_0, k}(\sigma)} \right).
\]

Thus,
\[
\lim_{\sigma \to \infty} \sup \inf \frac{\log \log N_{\sigma, k}(\sigma)}{\sigma} = \lim_{\sigma \to \infty} \sup \inf \log \left( \frac{N'_{\sigma_0, k}(\sigma)}{N_{\sigma_0, k}(\sigma)} \right).
\]

Combining (3.4.7) and (3.6.4), we have
\[
\lim_{\sigma \to \infty} \sup \inf \frac{\log \left( \frac{N'_{\sigma_0, k}(\sigma)}{N_{\sigma_0, k}(\sigma)} \right)}{\sigma} \leq \rho,
\]
which is (3.6.1).

(3.6.4) in conjunction with (3.4.4) gives us
The result (3.6.3) is an immediate deduction from (3.6.5) combined with the fact $\rho_* = \rho$, $\lambda_* = \lambda$ under the condition (3.2.2).

**Remark.** 'limsup' part of (3.6.3) is also true under (3.2.2).

### 3.7 A Further Result on Comparative Growth of $\log I_0(\sigma)$ and $\log J_{0,k}(\sigma)$.

Let us define

$$\lim_{\sigma \to \infty} \sup_{\sigma} \inf \log \log I_0(\sigma) = \frac{\sigma}{\bar{\lambda}}.$$ 

Following Shah ([76], p. 31) there exists a lower proximate order $\bar{\lambda}(\sigma)$ relative to $\log I_0(\sigma)$, satisfying the following conditions:

(i) $\lim_{\sigma \to \infty} \bar{\lambda}(\sigma) = \bar{\lambda}^-$,

(ii) $\lim_{\sigma \to \infty} \sigma \bar{\lambda}(\sigma) = 0$,

(iii) $\log I_0(\sigma) \geq e^{\sigma \bar{\lambda}(\sigma)}$ for all large $\sigma$,

(iv) $\liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{e^{\sigma \bar{\lambda}(\sigma)}} = 1$.

**Lemma 3.3** For $h > 0$,

$$\lim_{\sigma \to \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp(h \bar{\lambda})$$

where $S(\sigma) = \exp(\sigma \bar{\lambda}(\sigma))$. 

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PROOF. By a simple calculation, we have

\[
\frac{S'(\sigma)}{S(\sigma)} = \sigma \chi(\sigma) + \overline{\chi}(\sigma).
\]

Therefore, using the properties (i) and (ii), we see that for any \( \varepsilon > 0 \), there is a \( \sigma_0 \) such that for every \( \sigma \geq \sigma_0 \)

\[
(\overline{\chi} - \varepsilon) \frac{S'(\sigma)}{S(\sigma)} < \frac{S'(\sigma)}{S(\sigma)} < (\overline{\chi} + \varepsilon).
\]

Integrating the above inequality from \( \sigma \) to \( (\sigma + h) \), we have

\[
(\overline{\chi} - \varepsilon) h < \log \frac{S(\sigma + h)}{S(\sigma)} < (\overline{\chi} + \varepsilon) h.
\]

So that,

\[
\lim_{\sigma \to \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp (h \overline{\chi}).
\]

Now, we are in position to prove the following theorem.

**THEOREM 3.11** Let \( f(s) \) be an entire function represented by Dirichlet series, then

\[
\liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{\log J_{\overline{\chi}, k}(\sigma)} \leq \overline{\chi} \left\{ (1 + \frac{k}{\overline{\chi}}) \right\} \ ... \ (3.7.1)
\]

**PROOF.** We have

\[
\log J_{\overline{\chi}, k}(\sigma + h) = \frac{1}{e^{k(\sigma + h)} - 1} \int_{\sigma}^{\sigma + h} \log I_0(x) \ e^{kx} \ dx,
\]

\[
\geq \frac{1}{e^{k(\sigma + h)} - 1} \int_{\sigma}^{\sigma + h} \log I_0(x) e^{kx} \ dx,
\]

\[
\geq \frac{\log I_0(\sigma)}{k e^{k h - 1}} \left\{ e^{k h} - 1 \right\}.
\]
Dividing each side by \( S(\sigma) \) and taking the limit inferior, we have

\[
\liminf_{\sigma \to \infty} \frac{\log J_{0,k}(\sigma+h)}{S(\sigma)} \geq \liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{S(\sigma)} \cdot \left\{ \frac{e^{kh-1}}{k e^{kh}} \right\}.
\]

\[
\geq \left\{ \frac{e^{kh-1}}{k e^{kh}} \right\} \cdot \ldots \quad (3.7.2)
\]

Put

\[
\frac{\log J_{0,k}(\sigma+h)}{S(\sigma)} = \frac{\log J_{0,k}(\sigma+h)}{S(\sigma+h)} \cdot \frac{S(\sigma+h)}{S(\sigma)}.
\]

Here \( \log J_{0,k}(\sigma+h) / S(\sigma+h) \) and \( S(\sigma+h) / S(\sigma) \) are non-negative, so we have

\[
\liminf_{\sigma \to \infty} \frac{\log J_{0,k}(\sigma+h)}{S(\sigma)} \leq \liminf_{\sigma \to \infty} \frac{\log J_{0,k}(\sigma+h)}{S(\sigma+h)} \cdot \exp(h \lambda)
\]

by lemma 3.3. This inequality with (3.7.2) will give us

\[
\liminf_{\sigma \to \infty} \frac{\log J_{0,k}(\sigma)}{S(\sigma)} \geq \frac{e^{kh-1}}{k \exp\{(k+\lambda)h\}}
\]

Using this inequality and from the equality

\[
\frac{\log I_0(\sigma)}{\log J_{0,k}(\sigma)} = \frac{\log I_0(\sigma)}{S(\sigma)} \cdot \frac{S(\sigma)}{\log J_{0,k}(\sigma)},
\]

we get

\[
\liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{\log J_{0,k}(\sigma)} \leq \liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{S(\sigma)} \cdot \limsup_{\sigma \to \infty} \frac{S(\sigma)}{\log J_{0,k}(\sigma)},
\]

\[
= 1 \cdot \frac{1}{\liminf_{\sigma \to \infty} \frac{\log J_{0,k}(\sigma)}{S(\sigma)}} \leq \frac{k \exp\{(k+\lambda)h\}}{e^{kh-1}} \cdot \ldots \quad (3.7.3)
\]
Now by usual method of calculus we minimize the right hand side of (3.7.3). We find that its minima is attained for that value of $h$ which satisfies the relation

$$e^{kh} = \left( \frac{k + \lambda}{\lambda} \right), \quad \lambda > 0.$$ 

Substituting this value of $h$ in (3.7.3), we get

$$\liminf_{\sigma \to \infty} \frac{\log I_0(\sigma)}{\log J_{0,k}(\sigma)} \leq \lambda \left(1 + \frac{k}{\lambda} \right) \left(1 + \frac{\lambda}{k} \right).$$

The case $\lambda = 0$, is obvious.

This completes the proof of the theorem.

REFERENCES

AZPEITIA [5]; BESICOVITCH [7]; GUPTA [21]; JAIN [26],[27]; JUNEJA [31]; JUNEJA AND AWASTHI [32]; KAMTHAN [37], [39]; REDDY [66]; SHAH [76]; SUGIMURA [100]; TANAKA [102]; TITCHMARSH [103]; YUNG [111].

***
4.1 Introduction. In the second chapter, we have discussed some growth properties of the mean values of an entire function of one complex variable and in the third those of an entire function represented by Dirichlet Series. It appears natural to investigate the growth aspects of the mean values of entire function of several complex variables.

Let
\[ f(z_1, z_2) = \sum_{m_1, m_2 \geq 0} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \quad \ldots \ (4.1.1) \]

be an entire function of two complex variables \( z_1 \) and \( z_2 \), holomorphic in the closed polydisc \( D \{ |z_n| \leq r_n, \ n = 1, 2 \} \). The maximum modulus of \( f(z_1, z_2) \) is denoted by
\[ M(r_1, r_2; f) = M(r_1, r_2) = \max_{|z_n| \leq r_n} |f(z_1, z_2)|, \ (n = 1, 2). \]

The finite order \( \rho \) of an entire function \( f(z_1, z_2) \) is denoted as
\[ \limsup_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \rho. \quad [11] \quad \ldots (4.1.2) \]

Similarly, we can define the lower order \( \lambda \) of \( f(z_1, z_2) \) as
\[ \liminf_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \lambda. \quad \ldots (4.1.3) \]
Let us write:

\[ I_\theta(r_1, r_2, f) = I_\theta(r_1, r_2) = \left( \frac{2\pi}{2\pi} \right) \int_0^\theta \int_0^\theta f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) d\theta_1 d\theta_2 \]

where \( \theta \geq 1 \). ... (4.1.4)

\[ I_\theta(r_1, r_2) \] is an increasing function of

(i) \( r_1 \) for a given \( r_2 \),
(ii) \( r_2 \) for a given \( r_1 \),
(iii) \( r_1 \) and \( r_2 \) (both increasing).

For a given \( r_2 \), let \( 0 < r'_1 < r''_1 \) and define \( k(\theta_1, \theta_2) \) as

\[ k(\theta_1, \theta_2) \{ f(r'_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \}^{\theta} = | f(r'_1 e^{i\theta_1}, r_2 e^{i\theta_2}) |^{\theta}, \]

where \( \theta \geq 1, 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi \) and

\[ F(z_1, r_2) = \frac{1}{2\pi} \int_0^{2\pi} G(z_1, \theta_1, r_2) d\theta_1, \]

where

\[ G(z_1, \theta_1, r_2) = \frac{1}{2\pi} \int_0^{2\pi} k(\theta_1, \theta_2) \{ f(z_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \}^{\theta} d\theta_2. \]

Then \( F(z_1, r_2) \) is regular for \( |z_1| < r''_1 \) and it attains maximum on the boundary of \( z_1 = r''_1 e^{i\alpha_1} \). Therefore,

\[ I_\theta(r'_1, r_2) = F(r'_1, r_2) \leq |F(r''_1 e^{i\alpha_1}, r_2)| \leq I_\theta(r''_1, r_2). \]

Hence, \( I_\theta(r_1, r_2; f) \) is an increasing function of \( r_1 \) for given \( r_2 \). Similarly, it can be proved that \( I_\theta(r_1, r_2; f) \) is an increasing function of \( r_2 \) for a given \( r_1 \). Therefore, \( I_\theta(r_1, r_2; f) \) is an increasing function of \( r_1 \) and \( r_2 \) when one remains fixed and other increases or both increase.
We define

\[ J_{\delta,k_1,k_2}(r_1,r_2;f) = J_{\delta,k_1,k_2}(r_1,r_2) \]

\[ = \exp \left\{ \frac{1}{k_1+1} \int_0^{r_1} \int_0^{r_2} x_1^{k_1} x_2^{k_2} \log I_0(x_1,x_2;f) dx_1 dx_2 \right\} \]

where \( \delta \geq 1, \ 0 < k_1,k_2 < \infty \).

4.2 COMPARATIVE GROWTH OF \( I_0(r_1,r_2) \) AND \( J_{\delta,k_1,k_2}(r_1,r_2) \)

**THEOREM 4.1** Let \( f(z_1,z_2) \) be holomorphic in the closed polydisc \( D \{ |z_n| \leq R_n, \ n = 1,2 \} \) then for \( 0 < r_n < R_n (n = 1,2, \) )

\[ \log J_{\delta,k_1,k_2}(r_1,r_2;f) \]

\[ \leq \frac{\log I_0(r_1,r_2;f)}{(k_1+1)(k_2+1)} \leq \frac{R_1}{k_1+1} \frac{R_2}{k_2+1} \log J_{\delta,k_1,k_2}(R_1,R_2) \]

\[ \ldots (4.2.1) \]

**PROOF.** Since \( I_0(x_1,x_2;f) \) is an increasing function of :

(i) \( x_1 \) for a given \( x_2 \), (ii) \( x_2 \) for given \( x_1 \), (iii) \( x_1 \) and \( x_2 \) (both increasing), from (4.1.5) we have

\[ \log J_{\delta,k_1,k_2}(r_1,r_2) = \frac{1}{k_1+1} \int_0^{r_1} \int_0^{r_2} x_1^{k_1} x_2^{k_2} \log I_0(x_1,x_2;f) dx_1 dx_2 \]

\[ \leq \frac{\log I_0(r_1,r_2;f)}{(k_1+1)(k_2+1)} . \]

\[ \ldots (4.2.2) \]

Further,
\[
\log J_{\delta, k_1, k_2}(R_1, R_2; f) = \frac{1}{R_1 R_2} \int_{\Omega} \int_{\Omega} x_1 x_2 \log I_{\delta}(x_1, x_2; f) dx_1 dx_2,
\]
\[
\geq \frac{1}{R_1 R_2} \int_{\Omega} \int_{\Omega} x_1 x_2 \log I_{\delta}(x_1, x_2; f) dx_1 dx_2,
\]
\[
\geq \frac{k_1 + 1}{R_1 R_2 + 1} \cdot \frac{k_2 + 1}{R_2 + 1} \cdot \frac{\log I_{\delta}(r_1, r_2; f)}{(k_1 + 1)(k_2 + 1)}.
\]

From (4.2.2) and (4.2.3) the theorem follows:

**COROLLARY 4.1** Let \( f(z_1, z_2) \) be an entire function of finite order \( \rho \) and lower order \( \lambda \), then

\[
\lim \sup_{r_1, r_2 \to \infty} \frac{\log J_{\delta, k_1, k_2}(r_1, r_2)}{\log (r_1 r_2)} = \lim \sup_{r_1, r_2 \to \infty} \frac{\log I_{\delta}(r_1, r_2; f)}{\log (r_1 r_2)} = \rho.
\]

**4.3 GROWTH RELATIVE TO \((r_1 r_2)^{\rho}\).**

**THEOREM 4.2** Let \( f(z_1, z_2) \) be an entire function of finite non-zero order \( \rho \) and

(1) if

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log I_{\delta}(r_1, r_2; f)}{(r_1 r_2)^{\rho}} = s, \quad \ldots (4.3.1)
\]

then

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log J_{\delta, k_1, k_2}(r_1, r_2; f)}{(r_1 r_2)^{\rho}} \geq \frac{s}{(k_1^{\rho} + 1)(k_2^{\rho} + 1)}, \quad \ldots (4.3.2)
\]
(ii) if
\[ \limsup_{r_1, r_2 \to \infty} \frac{\log I_0(r_1, r_2; f)}{(r_1 r_2)^\rho} = S \quad \ldots (4.3.3) \]
then
\[ \limsup_{r_1, r_2 \to \infty} \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{(r_1 r_2)^\rho} \leq \frac{S}{(k_1 + \rho + 1)(k_2 + \rho + 1)} \quad \ldots (4.3.4) \]

**Proof.** From (4.3.1), for \( r_1 > r^o = r^o_1(e_1), r_2 > r^o_2 = r^o_2(e_2) \)
and taking \( \epsilon = \max(e_1, e_2) \), we have
\[ \log I_0(r_1, r_2; f) > (S - \epsilon)(r_1 r_2)^\rho \quad \ldots (4.3.5) \]
and from (4.3.3), we have
\[ \log I_0(r_1, r_2; f) < (S + \epsilon)(r_1 r_2)^\rho \quad \ldots (4.3.6) \]

Now from (4.1.5), we have
\[ \log J_0, k_1, k_2(r_1, r_2; f) \]
\[ = \frac{1}{k_1 + 1} \frac{1}{k_2 + 1} \int_0^\infty \int_0^\infty \frac{r_1 r_2}{x_1 x_2} k_1 k_2 \log I_0(x_1, x_2; f) \, dx_1 \, dx_2 , \]
\[ = \frac{1}{k_1 + 1} \frac{1}{k_2 + 1} \left[ \int_0^\infty \int_0^\infty \frac{r_1 r_2}{x_1 x_2} k_1 k_2 \log I_0(x_1, x_2; f) \, dx_1 \, dx_2 \right] , \ldots (4.3.7) \]
\[ > \frac{1}{k_1 + 1} \frac{1}{k_2 + 1} \int_0^\infty \int_0^\infty \frac{r_1 r_2}{x_1 x_2} k_1 k_2 \log I_0(x_1, x_2; f) \, dx_1 \, dx_2 , \]
Proceeding to limit inferior, we get

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log J_{o, k_1, k_2,(r_1, r_2); f}}{(r_1 r_2)^\gamma} \geq \frac{1}{(k_1 + \gamma + 1)(k_2 + \gamma + 1)}
\]

Further, from (4.3.6) and (4.3.7) for \( r_1 > r_0^o \) and \( r_2 > r_2^o \), we have

\[
\log J_{o, k_1, k_2,(r_1, r_2); f} \leq \frac{K}{r_1^{k_1+1} r_2^{k_2+1}} + \frac{(r_1^o)^{k_1+1} (r_2^o)^{k_2+1}}{(k_1+1)(k_2+1)} \log I_{o}(r_1, r_2; f) + \frac{(r_2^o)^{k_1+1} (r_1^o)^{k_2+1}}{(k_1+1)(k_2+1)} \log I_{o}(r_1, r_2^o; f) + \frac{1}{(k_1 + \gamma + 1)(k_2 + \gamma + 1)} (s+c)(r_1 r_2)^\gamma,
\]

where \( K \) is some positive constant.

Taking limit superior, we get

\[
\limsup_{r_1, r_2 \to \infty} \frac{\log J_{o, k_1, k_2,(r_1, r_2); f}}{(r_1 r_2)^\gamma} \leq \frac{1}{(k_1 + \gamma + 1)(k_2 + \gamma + 1)}
\]

This completes the proof of the theorem.
4.4 **GROWTH RELATIVE TO** \((r_1 r_2) L(r_1, r_2)\).

In this section, we confine our interest to the limits of \(\log I_0(r_1, r_2; f)\) and \(\log J_0, k_1, k_2(r_1, r_2; f)\) when they are compared with the function like \((r_1 r_2) L(r_1, r_2)\), where \(L(r_1, r_2)\) is a "slowly changing" function, with the following properties:

(i) \(L(r_1, r_2)\) is continuous for \(r_1 > r_1^0\) and \(r_2 > r_2^0\),

(ii) \(L(r_1, r_2) > 0\) for \(r_1 > r_1^0\) and \(r_2 > r_2^0\),

(iii) for every constant \(\alpha, \beta > 0\), \(L(\alpha r_1, \beta r_2) \sim L(r_1, r_2)\) as \(r_1\) or \(r_2\) or \(r_1\) and \(r_2\) tend to infinity.

Next, for \(0 < \rho < \infty\) let us define

\[
\lim_{r_1, r_2 \to \infty} \inf \sup \log I_0(r_1, r_2; f) = S, \quad (0 < H \leq S < \infty)
\]

\[
= \frac{(r_1 r_2)^\rho L(r_1, r_2)}{H}, \quad (0 < H \leq S < \infty)
\]

and

\[
\lim_{r_1, r_2 \to \infty} \sup \log J_0, k_1, k_2(r_1, r_2; f) = \frac{A}{B}, \quad (0 < B \leq A < \infty).
\]

\[
= \frac{(r_1 r_2)^\rho L(r_1, r_2)}{B}, \quad (0 < B \leq A < \infty)
\]

**THEOREM 4.3** Let \(f(z_1, z_2)\) be an entire function of finite non-zero order \(\rho\). Then

\[
\frac{1}{(k_1 + \rho + 1)(k_2 + \rho + 1)} \leq \lim_{r_1, r_2 \to \infty} \inf \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} \leq \frac{1}{(k_1 + \rho + 1)(k_2 + \rho + 1)} \leq \frac{1}{H} \leq S.
\]

\[
\leq \frac{1}{(k_1 + \rho + 1)(k_2 + \rho + 1)} \leq \frac{1}{H} \leq S.
\]

\[
\ldots (4.4.3)
\]
PROOF. For $0 < t < 1$, from (4.1.5), we have

$$
\log J_{o,k_1,k_2}(r_1+tr_1,r_2+tr_2;f)
= \frac{1}{k_1+1} \frac{k_2}{r_2+1} \frac{(1+t)^{k_1+k_2+2}}{r_1} \log I_o(x_1,x_2;f)dx_1dx_2,
$$

$$
= \frac{1}{k_1+1} \frac{k_2}{r_2+1} \frac{(1+t)^{k_1+k_2+2}}{r_1} \left\{ \log I_o(x_1,x_2;f)dx_1dx_2 \right\},
$$

$$
= P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8 + P_9 \quad \text{say}.
$$

Now,

$$
P_1 < \frac{C}{k_1+1} \frac{k_2}{r_2+1}, \quad \text{where } C \text{ is some positive constant.}
$$

$$
P_2 < \frac{r_1^0}{r_1} \frac{k_1+1}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \log I_o(r_1,r_2;f)
$$

$$
P_3 < \frac{r_1^0}{r_1} \frac{k_1+1}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \left\{ \frac{k_2}{r_2+1} \frac{(1+t)^{k_1+k_2+2}}{r_1} \log I_o(x_1,x_2;f)dx_1dx_2 \right\}.
$$

$$
P_4 < \frac{r_2^0}{r_2} \frac{k_2}{r_2+1} \frac{(1+t)^{k_1+k_2+2}}{r_1} \log I_o(r_1,r_2;f)
$$

$$
= \frac{r_1^0}{r_1} \frac{k_1+1}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \log I_o(x_1,x_2;f)dx_1dx_2,
$$

$$
= \frac{r_2^0}{r_2} \frac{k_2}{r_2+1} \frac{(1+t)^{k_1+k_2+2}}{r_1} \log I_o(r_1,r_2;f)
$$
\[
P_5 < \frac{(s+\varepsilon)}{(1+t)^{k_1+k_2+2} k_1+1 k_2+1} \int_0^{r_1} \int_0^{r_2} \frac{k_1+\rho}{k_2+\rho} x_1 x_2 L(x_1, x_2) dx_1 dx_2,
\]

\[
(1+t) \log I_0(x_1, r_2+tr_2; f) dx_1,
\]

by using (4.4.1) and by repeated application of Lemma 5, [24].

\[
P_6 < \frac{\frac{k_2+1}{k_1+1} - 1}{(1+t)^{k_1+1} k_1+k_2+2} \int_0^{r_1} x_1^k \log I_0(x_1, r_2+tr_2; f) dx_1,
\]

\[
(1+t)^{k_1+1} \int_0^{r_2} x_1^{k_1+\rho} L(x_1, r_2+tr_2) dx_1,
\]

\[
\left\{ \frac{(s+\varepsilon)}{(k_1+\rho+1)(k_2+1)} \right\} \frac{(1+t)^{k_2+1} - 1)(r_1 r_2)^f L(r_1, r_2+tr_2)}{(1+t)^{k_1+k_2+2}},
\]

by using (4.4.1) and applying Lemma 5, [24].

\[
P_7 < \frac{\frac{r^o}{r_2}}{2} \frac{k_2+1}{k_1+1} \frac{(1+t)^{k_1+1} - 1)}{(1+t)^{k_1+1} k_2+2} \log I_0(r_1+tr_1, r_2^o; f),
\]

\[
P_8 < \frac{(s+\varepsilon)}{(k_1+1)(k_2+1)} \frac{(1+t)^{k_1+1} - 1)}{(1+t)^{k_1+1} k_2+2} \log I_0(r_1+tr_1, r_2),
\]

and

\[
P_9 < \frac{(1+t)^{k_1+1} - 1)}{(1+t)^{k_2+1} - 1)} \frac{(s+\varepsilon)}{(k_1+1)(k_2+1)} \frac{(1+t)^{k_1+1} - 1)}{(1+t)^{k_1+1} k_2+2} \log I_0(r_1+tr_1, r_2+tr_2; f),
\]

\[
(1+t)^{k_1+1} \int_0^{r_2} x_1^{k_1+\rho} L(x_1, r_2+tr_2) dx_1,
\]
Hence, (4.4.4) becomes

\[ \log J_{\theta, k_1, k_2}(r_{1+tr_1}, r_{2+tr_2}; f) \]

\[ < O\left( r_1 \frac{(k_1+1) - (k_2+1)}{r_2} \right) + \left( \frac{r_1}{r_1} \right)^{k_1+1} \log I_{\theta}(r_{1}, r_{2}; f) \]

\[ \frac{(k_1+1) - (k_2+1)}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \]

\[ + \left( \frac{r_1}{r_1} \right)^{k_1+1} \left[ (1+t)^{k_2+1} - 1 \right] \log I_{\theta}(r_{1}, r_{2}; f) \]

\[ \frac{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \]

\[ + \frac{r_2^{k_2+1}}{r_2} \log I_{\theta}(r_{1}, r_{2}; f) \]

\[ \frac{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}}{(k_1+1)(k_2+1)(1+t)^{k_1+k_2+2}} \]

\[ + \frac{(S+\varepsilon) L(r_1, r_2)}{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}} \]

\[ \left( k_1+l + 1 \right) \left( k_2+l + 1 \right) \log I_{\theta}(r_{1+tr_1}, r_{2+tr_2}; f) \]

\[ \frac{(S+\varepsilon)(r_1 r_2)^l L(r_1, r_2; tr_2)}{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}} \]

\[ + \frac{r_2^{k_2+1}}{r_2} \left( k_1+l - 1 \right) \log I_{\theta}(r_{1+tr_1}, r_{2}; f) \]

\[ \frac{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}}{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}} \]

\[ + \frac{(S+\varepsilon)(r_1 r_2)^l L(r_{1+tr_1}, r_2; tr_2)}{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}} \]

\[ + \frac{r_2^{k_2+1}}{r_2} \left( k_1+l - 1 \right) \log I_{\theta}(r_{1+tr_1}, r_{2+tr_2}; f) \]

\[ \frac{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}}{(k_1+l)(k_2+l)(1+t)^{k_1+k_2+2}} \]
Now dividing both the sides by 

\[(r_1 + tr_1)^p (r_2 + tr_2)^p L(r_1 + tr_1, r_2 + tr_2)\]

and taking limit superior, we get

\[A \leq S \left[ \frac{1}{(k_1 + l + 1)(k_2 + l + 1)(1 + t)^{k_1 + k_2 + 2}} \frac{((1 + t)^{k_1 + l} - 1)}{((1 + t)^{k_2 + l} - 1)} \frac{((1 + t)^{k_2 + l} - 1)}{((1 + t)^{k_1 + 1} - 1)} \frac{((1 + t)^{k_1 + l} - 1)}{((1 + t)^{k_2 + 1} - 1)} \right].\]

Since \(t\) is arbitrary, we get

\[A \leq \frac{S}{(k_1 + l + 1)(k_2 + l + 1)} \cdot \] \[\ldots (4.4.5)\]

Next, from (4.4.4), we have

\[\log J_0, k_1, k_2(r_1 + tr_1, r_2 + tr_2; f)\]

\[> \frac{1}{(r_1)^{k_1 + l + 1}(r_2)^{k_2 + l + 1}(1 + t)^{k_1 + k_2 + 2}} \left\{ \int_1^{r_1} \int_1^{r_2} \int_1^{r_2} + \int_1^{r_2} \int_1^{r_1} \int_1^{r_1} + \int_1^{r_2} \int_1^{r_2} \right\} x_1 x_2 \log I_0(x_1, x_2; f) dx_1 dx_2.\]

\[= M_1 + M_2 + M_3 + M_4, \text{ say}. \] \[\ldots (4.4.6)\]

Now,

\[M_1 > \frac{(H-\xi)}{k_1 + l + 1 k_2 + l + 1 k_1 + k_2 + 2} \frac{r_1}{r_1} \frac{r_2}{r_2} \frac{k_1 + l}{k_1 + l} \frac{k_2 + l}{k_2 + l} \frac{L(x_1, x_2)}{L(x_1, x_2)} dx_1 dx_2.\]
\[ \sim \frac{(H-\varepsilon) r_1^{\rho}}{(k_1 + \rho + 1) r_1^2 (1+t) k_1 + k_2 + 2} \int_0^{r_1} x_1^{k_1 + \rho} L(x_1, r_2) dx_1, \]

\[ \sim \frac{(H-\varepsilon) (r_1 r_2)^{\rho} L(r_1, r_2)}{(k_1 + \rho + 1)(k_2 + \rho + 1)(1+t)^{k_1 + k_2 + 2}}, \]

using (4.4.1) and repeated application of Lemma 5 [24].

Similarly,

\[ M_2 > \frac{(H-\varepsilon) r_2^{\rho} ((1+t)^{k_2 + 1} - 1)}{(k_2 + 1) r_1^2 (1+t) k_1 + k_2 + 2} \int_0^{r_2} x_1^{k_1 + \rho} L(x_1, r_1) dx_1, \]

\[ \sim \frac{(k_2 + 1) (1+t) - 1) (r_1 r_2)^{\rho} L(r_1, r_2)}{(k_1 + \rho + 1)(k_2 + 1)(1+t)^{k_1 + k_2 + 2}}, \]

using (4.4.1) and applying Lemma 5, [24].

Further,

\[ M_3 > \frac{k_1 + 1}{(k_2 + 1)(k_2 + 1)(1+t) L(r_1, r_2)} \]

\[ \sim \frac{k_1 + 1}{(k_1 + 1)(k_2 + 1)(1+t)} \log I_0(r_1, r_2; \ell). \]

Hence (4.4.6) becomes
\[
\log J_0, k_1, k_2 (r_1 + t r_1, r_2 + t r_2; f)
\]
\[
> \frac{1}{(1+t)^{k_1+k_2+2}} \left[ (H-c)(r_1 r_2)^\rho L(r_1, r_2) \left\{ \frac{1}{k_1+\rho+1} \left( \frac{1}{k_2+\rho+1} \right) \right. \right.
\]
\[
+ \frac{k_2+1}{(k_1+\rho+1) (k_2+\rho+1)} \left. \left. \right. \right. \right.
\]
\[
+ \frac{(1+t)^{k_1+1}}{(k_1+1) (k_2+\rho+1)} \left. \left. \right. \right. \right.
\]
\[
+ \frac{(1+t)^{k_2+1}}{(k_1+1) (k_2+\rho+1)} \right] \log L_0 (r_1, r_2; f). \right.
\]

Now, dividing both the sides of this inequality by
\[
(r_1 + t r_1)^\rho (r_2 + t r_2)^\rho L(r_1 + t r_2, r_2 + t r_2)
\]
and taking limit inferior, we get
\[
B \geq \frac{H}{(1+t)^{k_1+k_2+2}} \left[ \frac{1}{k_1+\rho+1} \left( \frac{1}{k_2+\rho+1} \right) + \frac{(1+t)^{k_2+1}}{k_2+\rho+1} \right.
\]
\[
+ \frac{k_2+1}{(k_1+1) (k_2+\rho+1)} \left. \left. \right. \right. \right.
\]
\[
+ \frac{(1+t)^{k_1+1}}{(k_1+1) (k_2+\rho+1)} + \frac{(1+t)^{k_2+1}}{(k_1+1) (k_2+\rho+1)} \right].
\]

For the best possible value of \( t \), we get
\[
B \geq \frac{H}{(k_1+\rho+1) (k_2+\rho+1)}. \right.
\]

From (4.4.1) and (4.4.2), we obtain
\[
\begin{align*}
\frac{B-\varepsilon}{S+\varepsilon} & < \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} < \frac{A+\varepsilon}{H-\varepsilon}.
\end{align*}
\]

Taking limits and using (4.4.5) and (4.4.8), we get the result.

**Corollary 4.2** If \( S = H \), then

\[
\log J_0, k_1, k_2(r_1, r_2; f) \sim \frac{1}{(k_1+\rho+1)(k_2+\rho+1)} \log I_0(r_1, r_2; f).
\]

**Theorem 4.4** Let \( f(z_1, z_2) \) be an entire function of order \( \rho \) \((0 < \rho < \infty)\), then

\( (i) \) \( S \leq A \sum \left( 1 + \frac{k_1+1}{\rho} \right) \left( 1 + \frac{k_2+1}{\rho} \right) \left( 1 - \frac{\rho}{\rho+1} \right) \left( 1 + \frac{\rho}{\rho+1} \right) \).

\( (ii) \) \( S + H \left\{ \frac{\rho^2}{(k_1+\rho+1)(k_2+\rho+1)} + \frac{\rho}{(k_1+\rho+1)} + \frac{\rho}{(k_2+\rho+1)} \right\} \leq A (\rho + k_1+1) (\rho + k_2+1) \left( 1 + \frac{k_1+1}{\rho} \right) \left( 1 + \frac{k_2+1}{\rho} \right) \left( 1 + \frac{\rho}{\rho+1} \right) \).

**Proof.** From (4.4.7), we have

\[
\log J_0, k_1, k_2(r_1+\tau_1, r_2+\tau_2; f)
\]

\[
> \frac{1}{(1+t)^{k_1+k_2+2}} \left[ (H-\varepsilon)(r_1 r_2) L(r_1, r_2) \left\{ \frac{1}{k_1+\rho+1} \left( \frac{1}{k_2+\rho+1} \right) \right. \right.
\]

\[
+ \frac{k_2+1}{(k_2+1)(k_1+\rho+1)} + \frac{k_1+1}{(k_1+1)(k_2+\rho+1)} \right\} +
\]

\[
+ \frac{(1+t)^{-1}-1}{(k_1+1)(k_2+1)} L(r_1, r_2) \left\{ \left( \frac{1}{k_1+\rho+1} \right) \left( \frac{1}{k_2+\rho+1} \right) \right.
\]

\[
+ \frac{k_1+1}{(k_1+1)(k_2+1)} \left( \frac{1}{k_1+\rho+1} \right) \left( \frac{1}{k_2+\rho+1} \right) \right\}.
\]
Now, dividing both the sides of this inequality by
\[ \{(r_1 + tr_1)^\ell (r_2 + tr_2)^\ell \} \]
and taking limit superior, we get
\[
A \geq \frac{1}{(l+t)^{k_1 + k_2 + 2p + 2}} \left[ H \left\{ \frac{1}{k_1 + p + 1} \frac{1}{k_2 + p + 1} \right\} + \frac{(1+t)^k - 1}{(k_1 + 1)(k_2 + 1)} \left\{ \frac{k_1 + 1}{(1+t)^l - 1} \right\} \right] 
\]
\[ \text{or} \]
\[
S \leq \frac{(k_1 + 1)(k_2 + 1)}{(l+t)^{k_1 + k_2 + 2p + 2}} \left[ A(l+t)^{k_1 + k_2 + 2p + 2} - H \left\{ \frac{1}{k_1 + p + 1} \frac{1}{k_2 + p + 1} \right\} + \frac{(1+t)^{k_2 + 1} - 1}{(k_1 + 1)(k_2 + 1)} \right] 
\]
\[ \text{or} \]
\[
S \leq \frac{(k_1 + 1)(k_2 + 1)}{(l+t)^{k_1 + k_2 + 2p + 2}} A(l+t)^{k_1 + k_2 + 2p + 2} \]
\[ \text{Minimizing the right hand side of this expression, we get} \]
\[
S \leq \frac{k_1 + 1}{(l+t)^k} \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{k_2 + 1}{k_1 + 1} \right) \left( 1 + \frac{k_1 + 1}{k_2 + 1} \right),
\]
which proves first part of the theorem.

Also, from (4.4.9), we have
This proves the last part of the theorem. Thus, the theorem is completely proved.

**Theorem 4.5** Let \( f(z_1, z_2) \) be an entire function of finite non-zero order \( \rho \) and if

\[
\lim_{r_1, r_2 \to \infty} \sup_{k_1, k_2} \inf \frac{\log J_{k_1, k_2}(r_1, r_2; f)}{(r_1 r_2)^\rho L(r_1, r_2)} = \frac{A}{B}, \quad (0 < B \leq A < \infty)
\]

and

\[
\log I_0(r_1, r_2; f) \sim \alpha (r_1 r_2)^\rho L(r_1, r_2), \quad (0 < \alpha < \infty)
\]

where \( L(r_1, r_2) = O(\log r_1 r_2) \) for large values of \( r_1 \) and \( r_2 \), then

(i) \( f(z_1, z_2) \) is of regular growth.

(ii) \( A = B = \frac{\alpha}{(k + \rho + 1)(k_2 + \rho + 1)} \).

(iii) \( \lim_{r_1, r_2 \to \infty} \frac{\log J_{k_1, k_2}(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} = \frac{1}{(k + \rho + 1)(k_2 + \rho + 1)} \).

**Proof.** (i) Since

\[
\log I_0(r_1, r_2; f) \sim \alpha (r_1 r_2)^\rho L(r_1, r_2)
\]

as \( r_1 \) and \( r_2 \) tend to infinity.
Taking logarithm, we have
\[ \log \log I_0(r_1, r_2; f) \sim \log \alpha + \rho \log r_1 r_2 + \log L(r_1 r_2), \]
as \( r_1 \) and \( r_2 \) tend to infinity.

Hence, \( f(z_1, z_2) \) is of regular growth, if we note that
\[ \frac{\log L(r_1 r_2)}{\log r_1 r_2} \to 0 \] as \( r_1 \) and \( r_2 \) tend to infinity.

(ii) From (4.1.5), we have
\[
\log J_{\alpha, k_1, k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \left\{ \frac{r_1^0}{r_1} \int_{r_1}^{r_2^0} + \frac{r_1^0}{r_1} \int_{r_1}^{r_2} + \frac{r_1^0}{r_1} \int_{r_1}^{r_2^0} + \frac{r_1^0}{r_1} \int_{r_1}^{r_2^0} \right\} x_1^{k_1} x_2^{k_2} \log I_0(x_1, x_2; f) dx_1 dx_2.
\]

\[ = P_1 + P_2 + P_3 + P_4, \] say. \( \ldots (4.4.1) \)

Now,
\[ P_1 < \frac{C}{r_1^{k_1+1} r_2^{k_2+1}}, \] where \( C \) is positive constant.
\[ P_2 < \left( \frac{r_1^0}{r_1} \right)^{k_1+1} \frac{1}{(k_1+1)(k_2+1)} \log I_0(r_1^0, r_2^0; f), \]
\[ P_3 < \left( \frac{r_2^0}{r_2} \right)^{k_2+1} \frac{1}{(k_1+1)(k_2+1)} \log I_0(r_1, r_2^0; f). \]
and
\begin{align*}
P_4 &= \frac{1}{k_1+1} \int_0^r \int_0^s x_1 x_2 \log I_\circ(x_1, x_2; f) \, dx_1 \, dx_2 \\
&\quad + \frac{r_1 r_2}{r_1} x_1 x_2 \log I_\circ(x_1, x_2; f) \, dx_1 \, dx_2 \\
&\quad + \frac{r_1 r_2}{r_2} \frac{k_1+1}{r_1} \frac{k_2+1}{r_2} \frac{L(x_1, x_2)}{(k_1+1)(k_2+1)}
\end{align*}

by repeated application of Lemma 5 [24].

Putting the values of \( P_1, P_2, P_3 \) and \( P_4 \) in (4.4.11), we have

\begin{align*}
\log J_{\circ, k_1, k_2}(r_1, r_2; f) \\
&< \frac{c}{k_1+1} + \left( \frac{r_1}{r_1} \right) \frac{k_1+1}{r_1} \log I_\circ(r_1, r_2; f) + \\
&\quad + \left( \frac{r_2}{r_2} \right) \frac{k_2+1}{r_2} \log I_\circ(r_1, r_2; f) + \frac{\alpha(\gamma, \beta)}{(k_1+1)(k_2+1)} L(r_1, r_2) \\
&\quad + \frac{\alpha(\gamma, \beta)}{(k_1+1)(k_2+1)} L(r_1, r_2)
\end{align*}

...(4.4.12)

Now, dividing both the sides by \( (r_1 r_2)^\rho L(r_1, r_2) \), a positive increasing function, the inequalities (4.4.12) gives us

\begin{align*}
\lim_{r_1, r_2 \to \infty} \log J_{\circ, k_1, k_2}(r_1, r_2; f) &\leq \frac{\alpha}{(k_1+1)(k_2+1)} \\
&\leq \frac{\alpha}{(k_1+1)(k_2+1)} L(r_1, r_2)
\end{align*}

(4.4.13)
From (4.4.11), we have

\[
\log \mathcal{J}_{0,k_1,k_2}(r_1, r_2|f) > \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_0^1 \int_0^1 \frac{r_1^{k_1} r_2^{k_2} \log I_0(x_1, x_2|f) \, dx_1 \, dx_2}{r_1^{k_1+1} r_2^{k_2+1}} \cdot \alpha (r_1 r_2)^{\ell} L(r_1, r_2)
\]

\[
\sim \frac{\alpha (r_1 r_2)^{\ell} L(r_1, r_2)}{(k_1 + \ell + 1)(k_2 + \ell + 1)}
\]

Hence,

\[
\lim_{r_1, r_2 \to \infty} \frac{\log \mathcal{J}_{0,k_1,k_2}(r_1, r_2|f)}{(r_1 r_2)^{\ell} L(r_1, r_2)} \geq \frac{\alpha}{(k_1 + \ell + 1)(k_2 + \ell + 1)}
\]

\[
\ldots \ldots (4.4.14)
\]

Finally, the inequalities (4.4.13) and (4.4.14) lead to

\[
\lim_{r_1, r_2 \to \infty} \frac{\log \mathcal{J}_{0,k_1,k_2}(r_1, r_2|f)}{(r_1 r_2)^{\ell} L(r_1, r_2)} = \frac{\alpha}{(k_1 + \ell + 1)(k_2 + \ell + 1)}
\]

i.e.

\[
A = B = \frac{\alpha}{(k_1 + \ell + 1)(k_2 + \ell + 1)}
\]

(iii) If we divide both the sides of inequality (4.4.12) by \(\log I_0(r_1, r_2|f)\), a positive increasing function, we have

\[
\frac{\log \mathcal{J}_{0,k_1,k_2}(r_1, r_2|f)}{\log I_0(r_1, r_2|f)} < \frac{1}{\log I_0(r_1, r_2|f)} \cdot \frac{(k_1+1) - (k_2+1)}{O(r_1^{k_1+1} r_2^{k_2+1})}
\]

\[
\ldots \ldots (4.4.15)
\]
\[
\frac{r_1^0}{r_1} k_1 + 1 \log I_0(r_1, r_2; f) + \frac{r_2^0}{r_2} k_2 + 1 \log I_0(r_1, r_2; f) + \\
\frac{\alpha}{r_2} (r_1, r_2) L(r_1, r_2) \left( \frac{(k_1 + 1)(k_2 + 1)}{(k_1 + 1)(k_2 + 1)} \right) \\
+ \frac{\alpha}{r_1} (r_1, r_2) L(r_1, r_2) \left( \frac{(k_1 + 1)(k_2 + 1)}{(k_1 + 1)(k_2 + 1)} \right)
\]

Or

\[
\lim_{r_1, r_2 \to \infty} \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} \leq \frac{1}{(k_1 + 1)(k_2 + 1)}.
\] ...

(4.4.15)

Also from (4.4.15), we have

\[
\frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} > \frac{1}{(1, 1)} \frac{r_1}{r_1} \frac{r_2}{r_2} k_1 k_2 \log I_0(x_1, x_2; f) dx_1 dx_2 \\
\alpha (r_1, r_2) L(r_1, r_2) \left( \frac{(k_1 + 1)(k_2 + 1)}{(k_1 + 1)(k_2 + 1)} \right)
\]

\[
\lim_{r_1, r_2 \to \infty} \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} \geq \frac{1}{(k_1 + 1)(k_2 + 1)}.
\] ...

(4.4.16)

Combining (4.4.15) and (4.4.16), we get

\[
\lim_{r_1, r_2 \to \infty} \frac{\log J_0, k_1, k_2(r_1, r_2; f)}{\log I_0(r_1, r_2; f)} = \frac{1}{(k_1 + 1)(k_2 + 1)}.
\]

Thus the theorem is completely established.

REFERENCES

BOSE AND SHARMA [11]; HARDY AND ROGOINSKI [24].

***
5.1 INTRODUCTION. In section 5.2 and section 5.3 of this chapter, some results on entire functions of infinite order have been obtained. While in section 5.4, an application of proximate order \( B \) of an entire function has been shown. In the last section, we have proved a theorem on entire function represented by a Dirichlet series.

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of order \( \rho \) and lower order \( \lambda \). It is known that

\[
\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \rho = \limsup_{n \to \infty} \frac{n \log n}{\log |\frac{1}{a_n}|}, \quad (0 \leq \rho \leq \infty)
\]

Further, Shah [75] has proved that

\[
\lambda \geq \liminf_{n \to \infty} \frac{n \log n}{\log |\frac{1}{a_n}|} = \liminf_{n \to \infty} \frac{\log n}{\log |\frac{a_n}{a_{n+1}}|} \quad \ldots (5.1.2)
\]

If an entire function is either of order \( \rho = 0 \) or of lower order \( \lambda = \infty \), then we cannot expect satisfactory results as in (5.1.1) and (5.1.2). For that purpose, in section 5.2, we assume that \( \rho \) is infinite, but there exists a positive integer \( k \geq 2 \) for which,

\[
\lim_{r \to \infty} \sup_{k+1} \frac{\min(r)}{\log r} = \frac{\rho}{\lambda(k)} \quad (0 \leq \lambda(k) \leq \rho(k) < \infty)
\]

\[ k = 2, 3, 4, \ldots \quad \ldots (5.1.3) \]
is finite, by using the familiar notation

$$l_k(x) = \log \log \ldots (k \text{ times})x \quad (k = 1, 2, 3, \ldots),$$

and observing that $l_k(x) > 0$ for a real $x$ after a stage.

An entire function $f(z)$ with $\rho(k-1) = \infty$ and $\rho(k) < \infty$ is called an entire function of index $k$. If $\rho(k) = \gamma(k)$, then the function $f(z)$ is called the function of $k$th regular growth. Thus $\rho(k)$ and $\gamma(k)$ naturally extend the definitions of $\rho$ and $\gamma$ which correspond to $k = 1$.

If $\rho(k)$ is strictly positive and finite, we can as usual defined $T(k)$ and $t(k)$ as

$$\lim_{r \to \infty} \sup_{0 < r < \rho(k)} \frac{l_k(M(r))}{r \rho(k)} = T(k), \quad 0 < \rho(k) < \infty$$

$$\lim_{r \to \infty} \sup_{0 < r < \rho(k)} \frac{l_k(M(r))}{r \rho(k)} = t(k), \quad 0 \leq t(k) \leq T(k) \leq \infty.$$

If $T(k) = t(k)$, the function $f(z)$ is called the function of $k$th perfectly regular growth.

5.2 THEOREMS ON FUNCTIONS OF REGULAR GROWTH.

**THEOREM 5.1** The functions

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$f_2(z) = \sum_{n=0}^{\infty} b_n z^n,$$

are of $k$th regular growth of same finite non-zero order $\rho(k)$ with index $k$, if and only if

$$\log |\frac{b_n}{a_n}| = o(n \log n).$$
for large n, provided \( |\frac{a_{n-1}}{a_n}| \) and \( |\frac{b_{n-1}}{b_n}| \) are non-decreasing functions of n for \( n > n_0 \).

**Proof**  For an entire function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) of kth order \( \rho(k) \), with index k it is necessary and sufficient that

\[
\liminf_{n \to \infty} \frac{\log |a_n|}{n \ln_k n} = \frac{1}{\rho(k)} .
\]

([66], by choosing \( e^s = z \), and \( \gamma_n = n \))

Also, if \( \lambda(k) \) denotes the kth lower order of \( f(z) \) and \( |\frac{a_{n-1}}{a_n}| \) be a non-decreasing function of n for \( n > n_0 \), then

\[
\limsup_{n \to \infty} \frac{\log |a_n|}{n \ln_k n} = \frac{1}{\lambda(k)} .
\]

Hence, if \( f(z) \) is of kth regular growth, \( \rho(k) = \lambda(k) \), we have

\[
\lim_{n \to \infty} \frac{\log |a_n|}{n \ln_k n} = \frac{1}{\rho(k)} .
\]

...(5.2.1)

Now, let \( f_1(z) \) and \( f_2(z) \) be of each of finite kth order \( \rho(k) \), then

\[
\lim_{n \to \infty} \frac{\log |a_n|}{n \ln_k n} = \frac{1}{\rho(k)} = \lim_{n \to \infty} \frac{\log |b_n|}{n \ln_k n} .
\]

Hence,

\[
\log |\frac{b_n}{a_n}| = o(n \ln_k n) .
\]

...(5.2.2)

Further, if \( \rho_1(k) \) and \( \rho_2(k) \) denote the kth orders of \( f_1(z) \) and \( f_2(z) \), then using (5.2.1), we have
If (5.2.2) holds, then

\[
\frac{\log |b_n|}{a_n} = o(n) \quad \text{for large } n, \text{ provided }
\]

\[
\frac{1}{\rho_1(k)} - \frac{1}{\rho_2(k)} = o \quad n \to \infty
\]

Hence, \( \rho_1(k) = \rho_2(k) \).

**Theorem 5.2** The entire functions \( f_1(z) \) and \( f_2(z) \) are of the same finite non-zero \( k \)th perfectly regular growth if, and only if,

\[
\log \left| \frac{b_n}{a_n} \right| = o(n) \quad \text{for large } n, \text{ provided }
\]

(1) \( f_1(z) \) and \( f_2(z) \) are of the same finite non-zero \( k \)th order \( \rho(k) \) of index \( k \).

(2) \( \frac{a_{n-1}}{a_n} \) and \( \frac{b_{n-1}}{b_n} \) are non-decreasing functions of \( n \) for large \( n \).

**Proof.** A necessary and sufficient condition that \( f_1(z) \) should be an entire function of order \( \rho(k) \) \((0 < \rho(k) < \infty)\) and \( k \)th type \( T(k) \), is that

\[
\limsup_{n \to \infty} \left( \frac{\rho(k)}{n} \right) (1_k - 1) |a_n|^n = T(k). \quad [66] \quad (5.2.3)
\]

It is also known that

\[
\liminf_{n \to \infty} \left( \frac{\rho(k)}{n} \right) |a_n|^n = t(k), \quad [67]
\]
provided \( \left| \frac{a_{n-1}}{a_n} \right| \) is a non-decreasing function for large \( n \).

Hence, if \( f_1(z) \) is of \( k \)th perfectly regular growth \( T(k) = t(k) \), we have
\[
\lim_{n \to \infty} \left( l_{k-1,n} \right) \frac{a_n}{a_n} = T(k). \quad \ldots(5.2.4)
\]

Now, let \( f_1(z) \) and \( f_2(z) \) be each of type \( T(k) \), then,
\[
\lim_{n \to \infty} \left( l_{k-1,n} \right) \frac{a_n}{a_n} = T(k) = \lim_{n \to \infty} \left( l_{k-1,n} \right) \frac{b_n}{b_n}. \quad \ldots(5.2.5)
\]

Therefore,
\[
\lim_{n \to \infty} \left\{ \frac{f(k)}{n} \log |a_n| - \frac{f(k)}{n} \log |b_n| \right\} = 0.
\]

Hence,
\[
\log \left| \frac{b_n}{a_n} \right| = o(n). \quad \ldots(5.2.5)
\]

Further, if \( f_1(z) \) and \( f_2(z) \) be of finite \( k \)th perfectly regular \( T_1(k) \) and \( T_2(k) \) respectively, using (5.2.4), we get
\[
\log T_1(k) - \log T_2(k) = \lim_{n \to \infty} \left\{ \frac{f(k)}{n} \left( \log |a_n| - \log |b_n| \right) \right\} = 0,
\]
if (5.2.5) holds. Hence, \( T_1(k) = T_2(k) \).

5.3 GROWTH OF DERIVATIVES OF AN ENTIRE FUNCTION OF INFINITE ORDER.

It is known that
\[
\liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} = 0. \quad (\nu = \infty). \quad [72] \quad \ldots(5.3.1)
\]
A result better than (5.3.1) viz.

\[ \liminf_{r \to \infty} \frac{\log M(r)}{\nu(r)} = 0, \quad \text{... (5.3.2)} \]

for every entire function of infinite order has been proved by Shah [78]. Later on Shah and Khanna [85] proved that for an entire function of infinite order

\[ \liminf_{r \to \infty} \frac{\log \{ r M(1)(r) \}}{\nu(r)} = 0, \quad \text{... (5.3.3)} \]

a result better than (5.3.2), since

\[ r M(1)(r) > \frac{M(r) \log M(r)}{\log r}, \quad r \geq r_0 = r_0(f) \cdot [108] \]

Clunie [16], has gone still further to prove that if \( s \) is any function of \( \nu \) such that \( s(\nu) = o\left(\frac{\nu}{\log \nu}\right) \), then

\[ \liminf_{r \to \infty} \frac{\log \{ r M(r,f(s)) \}}{\nu(r,f)} = 0 \quad \text{... (5.3.4)} \]

The object of this section is to prove a theorem which is more precise than (5.3.4). In what follows, we shall prove the following:

**Theorem 5.3** Let \( f(z) \) be an entire function of infinite order, then

\[
\log \left[ \left( r + \frac{\lambda r \log \mu(r)}{\nu^2(r) J(r)} \right)^s \cdot M(r + \frac{\nu^2(r) J(r)}{\lambda r \log \mu(r)} ; f(s)) \right]
\]

\[ \liminf_{r \to \infty} \frac{\log \{ r M(r,f(s)) \}}{\nu(r,f)} = 0, \quad \text{where}
\]

(1) \( s \) is any function of \( \nu \) such that \( s(\nu) = o\left(\frac{\nu}{\log \nu}\right) \)
(ii) $J(r)$ is any positive function such that \[ \sum_{m=1}^{\infty} \frac{1}{\nu(R_m) J(R_m)} \]
is convergent and $J(r) = o(\nu(r))$, $\lambda$ being any fixed positive number.

PROOF. We have

\[ r^s M(r, f^{(s)}) \leq \sum_{n=s}^{\infty} n(n-1) \cdots (n-s+1) |a_n| r^n. \]

Also, in the notations of Valiron ([106], p. 30) for $n > p$, we have

\[ n(n-1) \cdots (n-s+1) |a_n| r^n \leq n(n-1) \cdots (n-s+1) e^{-G_n} r^n. \]

Therefore, we have

\[ r^s M(r, f^{(s)}) \leq \sum_{n=s}^{p-1} n(n-1) \cdots (n-s+1) \mu(r) + \sum_{n=p}^{\infty} n(n-1) \cdots (n-s+1) \mu(r) \left( \frac{r}{R_p} \right)^{n-p+1}, \]

\[ \leq \mu(r) p^s + \mu(r) p^2 s \left[ \frac{r}{R_p - r} + \frac{r^2}{(R_p - r)^2} + \cdots + \frac{r^{s+1}}{(R_p - r)^{s+1}} \right]. \]

Now, if we take

\[ p = \nu \left( r + \frac{1}{\nu^2(r)} \right) + 1, \]

so that

\[ R_p - r \geq \frac{1}{\nu^2(r)}. \]

Therefore, we have
\[ r^s M(r, f(s)) \leq \mu(r)^p + \mu(r)^p \left[ r^2 \nu^2(r) + \ldots + \left\{ r \nu(r)^{2s+2} \right\} \right], \]
\[ < \mu(r) \left\{ \nu \left( r + \frac{1}{r \nu^2(r)} \right) r \nu(r)^{2s+2} \right\}. \]
\[ \log \left( r^s M(r, f(s)) \right) < \log \mu(r) + (2s+2) \{ \log \nu \left( r + \frac{1}{r \nu^2(r)} \right) + \log r + \log \nu(r) \}, \]
\[ < (1+o(1)) \log \mu(r) + (2s+2) \log \nu \left( r + \frac{1}{r \nu^2(r)} \right). \]

Further, \( \nu(r) \) is constant in the positive intervals \( R_n < r < R_{n+1} \) \((n = 1, 2, 3, \ldots)\), so that \( R_n \) tends to infinity with \( n \) or \( r \). Then by (5.3.1) and our hypothesis \( f = \infty \) there is a subsequence of positive integers

\[ N : N_1 < N_2 < \ldots < N_g < \ldots, N_j \to \infty, (\frac{j}{N}) \to \infty, \ldots (5.3.6) \]
such that, given any small \( \varepsilon > 0 \) we have

\[ \frac{\log \mu(R_{N_j})}{\nu(R_{N_j})} < \varepsilon. \]

...(5.2.7)

There are now two possibilities or cases.

**CASE A.** There is an infinite subsequence of \( N \), say

\[ N_{j_1} < N_{j_2} < N_{j_3} < \ldots < N_{j_k} \to \infty \quad (n_k \to \infty), \ldots (5.3.8) \]

which we call (for convenience)
\( \mathbf{M} : M_1 < M_2 < \ldots < M_k < \ldots, M_k \to \infty, \ (k \to \infty) \ \ \ \ (5.2.9) \)

and which satisfies the following:

\[
R_{M_k} + 1 > R_{M_k} + \frac{\lambda^* R_{M_k} \log \mu(R_{M_k})}{2 (R_{M_k}) J(R_{M_k})}, \ (\lambda^* > \lambda), \ \ \ \ (5.3.10)
\]

in which case

\[
\nu(R_{M_k} + \frac{\lambda^* R_{M_k} \log \mu(R_{M_k})}{2 (R_{M_k}) J(R_{M_k})}) = \nu(R_{M_k}). \ \ \ \ (5.3.11)
\]

**CASE B.** There is no infinite subsequence of \( \mathbb{N} \) such as \( \mathbf{M} \)

i.e. for all large \( j \) say \( j \geq j_0 \)

\[
R_{N_j} + 1 \leq R_{N_j} (1 + \frac{\lambda^* \log \mu(R_{N_j})}{2 (R_{N_j}) J(R_{N_j})}),
\]

in which case either \( R_{N_j+1} = R_{N_j} \) and then \( N_{j+1} \in \mathbb{N} \)

or \( R_{N_j+1} > R_{N_j} \).

Our proof consists in showing that, in case A, Theorem 5.3

is established while, in case B, there is contradiction which

automatically rules out this case.

**CASE A** We take the sequence \( \mathbf{M} = \{M_k\} \) of (5.3.9) and use

(5.3.5) with \( J(r) = o(\nu(r)) \), we have
\[ \log \left\{ (R_{mk} + \lambda R_{mk}^s)^S \cdot M(R_{mk} + \lambda R_{mk}^s \cdot f(s)) \right\} \]
\[ \leq (1 + o(1)) \left[ \left\{ \log \mu(R_{mk}) + \int_{R_{mk}}^\infty \frac{\nu(x)}{x} \, dx \right\} + (2s+2) \log \nu(R_{mk}) \right], \]

(\text{where } \beta = \frac{\log \mu(R_{mk})}{\nu(R_{mk})} J(R_{mk})^2).

\[ \leq (1 + o(1)) \left[ \left\{ \log \mu(R_{mk}) + \nu(R_{mk}) \log(1 + \beta) \right\} + (2s+2) \log \nu(R_{mk}) \right], \]

\[ \leq (1 + o(1)) \left\{ \log \mu(R_{mk}) + \nu(R_{mk}) \log(1 + \beta) + (2s+2) \log \nu(R_{mk}) \right\}, \]

\[ \leq (1 + o(1)) \left\{ \log \mu(R_{mk}) + \frac{\nu(R_{mk})}{\nu(R_{mk}) J(R_{mk})} \log(1 + \beta) + (2s+2) \log \nu(R_{mk}) \right\}. \]

Now, we divide both the sides of this inequality by \( \nu(R_{mk}) \) and use the hypothesis
\[ s = o\left( \frac{\nu(R_{mk})}{\log \nu(R_{mk})} \right), \quad (k \to \infty), \]

Thus, we get
\[ \frac{\log \left[ (R_{mk} + \lambda R_{mk}^s)^S \cdot M(R_{mk} + \lambda R_{mk}^s \cdot f(s)) \right]}{\nu(R_{mk})} \]
\[ \leq (1 + o(1)) \left\{ \frac{\log \mu(R_{mk})}{\nu(R_{mk})} + \frac{\nu(R_{mk}) \log \mu(R_{mk})}{\nu(R_{mk}) J(R_{mk})} \right\}, \]
i.e.,

\[ \liminf_{k \to \infty} \frac{\gamma(R_k) \log \mu(R_k)}{\nu^2(R_k)J(R_k)} = 0. \]

This leads to the desired conclusion as explained at the outset of the proof.

**Case B** The proof depends on the inference that now the subsequence of integers \( N = \{ N_j \} \) defined in (5.3.8), beginning with (say) a certain \( M_0 = N_q \geq N_j \) consists of all integers without exception or that \( \{ N_j \} \) for \( j > q > j_0 \), consists of \( M_0 = N_q, M_0+1, M_0+2, \ldots, M_0+k \ldots \). It is known that

\[ \log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x} \, dx. \]

Hence, \( \log \mu(R_{M_0+1}) \)

\[ \leq \frac{1}{M_0+1} \left\{ \log \mu(R_m) + \int_{R_m}^{R_{M_0+1}} \frac{\nu(x)}{x} \, dx \right\}, \]

\[ \leq \frac{1}{M_0+1} \left\{ \log \mu(R_m) + M_0 \log (1 + \frac{\gamma \log \mu(R_m)}{\nu^2(R_m)J(R_m)}) \right\}, \]

\[ < \frac{1}{M_0+1} \left\{ \log \mu(R_m) + \frac{\gamma \log \mu(R_m)}{M_0 J(R_m)} \right\}, \]

\[ < \frac{\log \mu(R_m)}{M_0}, \]

and so \( M_0+1 \in N \).
Similarly,

\[ M_{o} + 2, M_{o} + 3, M_{o} + 4 \cdots \in \mathbb{N} \]

Let \( M_{o} \in \mathbb{N} \) and \( M_{o} > N_{o} \). Then

\[
\begin{align*}
R_{M_{o} + 1} & \leq R_{M_{o}} \prod_{n=M_{o}}^{M_{o} + j - 1} \left(1 + \frac{\lambda \log \mu(R_{n})}{\nu^{2}(R_{n})J(R_{n})}\right), \\
R_{M_{o} + 1} & \leq R_{M_{o}} \prod_{n=M_{o}}^{M_{o} + j - 1} \left(1 + \frac{\lambda \in \nu(R_{n})}{\nu^{2}(R_{n})J(R_{n})}\right), \\
\end{align*}
\]

\(< \) a constant,

which leads to a contradiction since \( R_{M_{o} + j} \) tends to infinity with \( j \).

Hence the alternative (B) is not possible. Thus, the Theorem is proved.

**REMARKS**

1. It is adaptation of Shah's argument [73] combined with (5.3.5) that forms the basis of the proof of Theorem 5.3.

2. It is known ([59], p.21) for any entire function of finite or infinite order

\[ \nu(r,f) \leq \nu(r,f^{(1)}) \leq \ldots \leq \nu(r,f^{(s)}). \]

Hence, it follows that \( \nu(r,f) \) can be replaced by \( \nu(r,f^{(s)}) \) where \( s \) may have any integer value.

### 5.4 PROXIMATE ORDER B.

When the order of the function is not an integer, we have Bourtroux's proximate order [13], having the following properties:
(i) $\rho(r)$ is real continuous piecewise, differentiable for $r > r_0$ and $r \log r \rho'(r) \to 0$ as $r \to \infty$.

(ii) $\liminf_{r \to \infty} \rho(r) > p$, where $p$ is the genus of the function.

(iii) In this proximate order, we can always find a number $\alpha$ such that $r^{\rho(r)-p-\alpha}$ and $r^{p+1-\rho(r)-\alpha}$ are increasing functions of $r$.

(iv) $n(r,a) < C r^{\rho(r)}$, $r > r_0$, $C$ is any constant $> 0$.

(v) $n(r,a) < \frac{1}{C} r^{\rho(r)}$ for a sequence of values of $r$ proportional to the values for which

$$\log M(r,f) = r^{\rho(r)}.$$

Now, we prove the following:

**THEOREM 5.4** Let $P(z)$ be the canonical product of genus $p$ and order $\rho (\rho > p)$, for some finite $\beta > 0$

$$n(r) \sim \beta r^{\rho(r)}, \text{ as } r \to \infty.$$

Then

$$|\sin \pi(\rho - p)\frac{P'(re^{i\theta})}{P(re^{i\theta})}| = \pi(\rho - p) \beta \cdot r^{\rho(r)-p-1} |< \varepsilon(r^{\rho(r)-1})$$

for $\varepsilon > 0$, $r > r_0 = r(\varepsilon)$.

The following lemmas will be used in the proof of this theorem.

**LEMMA 5.1**

$$\int_0^{\infty} \frac{x^{\rho - p}}{(x+r)^2} \, dx = \frac{\pi(\rho - p)}{\sin \pi(\rho - p)} r^{\rho - p - 1}.$$

This can be easily established, since $0 < \rho - p < 1$. 
LEMMA 5.2A If \( \mu > 1 \), \( n(x) \sim \beta x^p \), then

\[
\int_{r_0}^{r/\mu} \frac{n(x)^p}{x^p(x+r)^2} \, dx < \varepsilon r^\varphi(r) - 1.
\]

PROOF.

\[
\int_{r_0}^{r/\mu} \frac{n(x)^p}{x^p(x+r)^2} \, dx < \int_{r_0}^{r/\mu} \frac{2\beta x^\varphi(x)}{x^p(x+r)^2} \, dx,
\]

\[
< \varepsilon 2\beta x^\varphi(x) - p - 2 < r^\varphi(r) - 2 - \alpha \int_{r_0}^{r/\mu} x^\varphi(x) \, dx,
\]

\[
< \varepsilon r^\varphi(r) - 1. \quad *
\]

Since \( x^\varphi(x) - p - \alpha \) is an increasing function in proximate order \( B \).

LEMMA 5.2B

\[
\int_{r_0}^{r/\mu} \frac{x^\varphi(x)}{x^p(x+r)^2} \, dx < \varepsilon r^\varphi(r) - 1.
\]

This can be easily established, since \( p - p > 0 \).

LEMMA 5.3A If \( \mu > 1 \) and \( n(x) \sim \beta x^p \), then

\[
\int_{r}^{\infty} \frac{n(x)^p}{\mu x^p(x+r)^2} \, dx < \varepsilon r^\varphi(r) - 1.
\]

PROOF.

\[
\int_{r}^{\infty} \frac{n(x)^p}{\mu x^p(x+r)^2} \, dx < r^p \int_{\mu r}^{\infty} \frac{2\beta x^\varphi(x)}{x^p + 2} \, dx,
\]

* \( \varepsilon \) need not be same at each occurrence.
\[ 2^r \int_{t^r}^{\infty} x^{-\alpha} \, dx, \]

\[ < 2 \int_{t^r}^{x} x^{-\alpha} \, dx, \]

\[ \rho(\alpha) = \frac{1}{\alpha}, \quad r^r \alpha, \]

\[ \rho(\alpha) = 1, \quad r^r \alpha. \]

Since \( x^{\rho(x)} + \alpha - 1 \) is a decreasing function in the proximate order \( B \).

**Lemma 5.3B**

\[ \int_{t^r}^{\infty} \frac{x^\rho}{x^{\rho(x)+\alpha}} \, dx < \epsilon. \]

**Proof.**

\[ \int_{t^r}^{\infty} \frac{x^\rho}{x^{\rho(x)+\alpha}} \, dx < \epsilon. \]

Since, \( \rho - p - 1 < 0 \).

**Lemma 5.4** If \( \mu > 1 \) and \( n(x) \sim \beta x^\rho(x) \), then

\[ E = \left| \int_{t^r}^{\infty} \frac{n(x)x^\rho}{x^{\rho(x)+\alpha}} \, dx - \beta \int_{t^r}^{\infty} x^\rho \, dx \right| < \epsilon. \]

**Proof.** Now,

\[ E = \left| \int_{t^r}^{\infty} \frac{n(x)-\beta x^\rho}{x^{\rho(x)+\alpha}} \, dx \right|,

\[ < \beta \int_{t^r}^{\infty} x^\rho \, dx, \]

\[ < \epsilon. \]
PROOF OF THE THEOREM 5.4  P(z) be a canonical product of

genus p and order f (f > p). Then for z outside the circles

with centre \(a_n\) and radius \(|a_n|^{-q}\) (q > f), we have

\[
\left| \frac{P'(z)}{P(z)} \right| < K \int_{0}^{\infty} \frac{n(x)r^p}{x^{p(x+r)^2}} \, dx , \quad [36] \quad \ldots (5.4.1),
\]

where \(K\) is a constant.

For every \(\epsilon > 0\),

\[
J = \left| \frac{P'(r e^{i\theta})}{P(r e^{i\theta})} \right| = \frac{\beta \pi (f - p)}{\sin \pi (f - p)} r^{f(r) - p - 1} .
\]

From (5.4.1) and Lemma 5.1, we have

\[
J < \int_{0}^{r_0} \frac{K n(x)r^p}{x^{p(x+r)^2}} \, dx + \int_{r_0}^{r / \mu} \frac{K n(x)r^p}{x^{p(x+r)^2}} \, dx + \int_{r / \mu}^{\infty} \frac{K n(x)r^p}{\mu r x^{p(x+r)^2}} \, dx
\]

\[
+ \int_{0}^{r_0} \frac{\beta r^{f(r) - p} x^{f - p}}{(x+r)^2} \, dx + \int_{r_0}^{r / \mu} \frac{\beta r^{f(r) - p} x^{f - p}}{(x+r)^2} \, dx
\]

\[
+ \int_{r / \mu}^{r} \frac{K n(x)r^p}{\mu r x^{p(x+r)^2}} \, dx + \beta r^{f(r) - p} \int_{r / \mu}^{r} \frac{x^{f - p}}{(x+r)^2} \, dx
\]

\[
< \epsilon r^{f(r) - 1},
\]

by Lemmas 5.2A, 5.2B, 5.3A, 5.3B and 5.4. This completes the

proof of Theorem 5.4.
5.5 Entire Function Represented by Dirichlet Series.

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad s = \sigma + it, \quad \ldots (5.5.1)$$

be a function represented by a Dirichlet series convergent in the whole plane, where \( \{\lambda_n\}_1^\infty \) is a sequence of positive, non-decreasing numbers with

$$\liminf_{n \to \infty} \{\lambda_{n+1} - \lambda_n\} = h > 0.$$

For convenience, we state certain terminology related to the densities of the sequence \( \{\lambda_n\} \). The number \( \nu(x), x > 0 \), of quantities \( \lambda_n \) smaller than \( x \), is called the distribution function (in short, the distribution) of the sequence \( \{\lambda_n\} \). The function \( D(x) = \frac{\nu(x)}{x} \) is called the density function and

$$D^* = \limsup_{x \to \infty} D(x)$$

is referred to as the upper density of \( \{\lambda_n\} \).

For finer purposes, we have the logarithmic density of sequence \( \{\lambda_n\} \). If we associate with \( \{\lambda_n\} \) the characteristic function \( \gamma(x) = \sum_{\lambda_n \leq x} \frac{1}{2 \lambda_n} \) and denote \( \Delta(x) = \frac{\lambda(x)}{2 \log x} \), then the quantities

$$\Delta^0 = \limsup_{x \to \infty} \Delta(x) \quad \ldots (5.5.2)$$

is known the upper logarithmic density of \( \{\lambda_n\} \). The sequence
is said to be logarithmically measurable if "limsup" can be replaced by "limit" in (5.5.2).

We shall denote by $\Delta^*$ the maximum logarithmic density of $\{\lambda_n\}$, the greatest lower bound of densities of logarithmically measurable sequences containing the sequence $\{\lambda_n\}$.

As usual we use the following notations:

For a fixed $t_0$, let $S(t_0, R)$ denotes the horizontal strip $|t - t_0| < R$. For any $\sigma$ real and finite

$$M_s(\sigma) = \max_{|t-t_0| \leq R} |f(\sigma + it)|$$

and let

$$\lim_{\sigma \to \infty} \sup \log \log M_s(\sigma) = \rho; \lim_{\sigma \to \infty} \inf \frac{\log \log M_s(\sigma)}{-\sigma} = \lambda_s.$$

Further, suppose

$$\overline{M}_s(\sigma_0) = \max_{|t-t_0| \leq R, \sigma \geq \sigma_0} |f(\sigma + it)|$$

and let

$$\lim_{\sigma_0 \to \infty} \sup \log \log \overline{M}_s(\sigma_0) = \overline{\rho}_s; \lim_{\sigma_0 \to \infty} \inf \frac{\log \log \overline{M}_s(\sigma_0)}{-\sigma_0} = \overline{\lambda}_s.$$

5.6 ORDER AND LOWER ORDER IN A HORIZONTAL STRIP.

Mandelbrojt and Gergen ([50], pp. 219 - 220) have proved that order $\rho_s$ of $f(s)$ in each horizontal strip $S(\pi a)$, with $a > \Delta^*$ is equal to the order $\rho$ of $f(s)$ in the whole plane.
This result has been extended to the lower order by Rahman [60]. But, the proof of this theorem is not complete.

In this section, we prove a theorem which is better and more precise than the theorem of Rahman ([60],[61]) and the theorem of Srivastava [95]. In the proof we use the Malliavin's version ([49],p.232) of Mandelbrojt fundamental inequality. This gives, a sharper result, since Malliavin's inequality involves a logarithmic density that is finer than the arithmetical density of Mandelbrojt. Our method of proof is entirely different from that of Rahman ([60],[61]), Srivastava [95] and Sunyer i Balaguer [101].

**THEOREM 5.5** The lower order $\lambda_\sigma$ of $f(s)$ in the each horizontal strip $S(\pi a)$ with $a > \Delta^*$, is equal to the lower order $\lambda$ of $f(s)$.

**PROOF.** It is known that there exists an increasing subsequence \( \{n_j\} \) of \( n \) for which

$$\limsup_{j \to \infty} \frac{\log |\frac{1}{\lambda_{n_j}}|}{\lambda_{n_j} \log n_{n_j}} = \frac{1}{\lambda} < \infty. \quad [69]$$

Now, by the Malliavin's version ([49],p.232) of Mandelbrojt fundamental inequality, we get

$$\log M_{a}(\sigma_0) \geq - \left( \frac{1}{\lambda} + 2 \varepsilon \right) \lambda_{n_j} \log n_{n_j} - \sigma_{n_j} \lambda_{n_j} \left[ k(\lambda_{n_j}) - k(\lambda_{n_j}) \right].$$

...(5.6.1)

Since for sufficiently large \( x \),

$$2(\Delta_0 - \varepsilon) \log x < \lambda(x) < 2(\Delta_0 + \varepsilon) \log x, \quad **$$

** \( \Delta_0 = \liminf_{x \to \infty} \Delta(x) \).
and
\[ k(x) = 2a \log x - \lambda(x), \]
hence
\[ 2(a-\Delta^0 - \epsilon) \log x < k(x) < 2(a-\Delta^0 + \epsilon) \log x \]
\[ k_s(x) > 2(a-\Delta^0 - \epsilon) \log x. \]

Further, we have
\[ A = \limsup_{x} \frac{k(x) - k(x)}{\log x} \leq 2(\Delta^0 - \Delta^0). \]

Under the hypothesis of the theorem \( \Delta^0 = \Delta^0 \), therefore, we get from (5.6.1)
\[ \log \bar{m}_s(\sigma_0) \geq \{ \left( \frac{1}{\lambda} + 2 \epsilon \right) \log \lambda_n - \sigma_0 + o(1) \} \lambda_n. \]

Choose
\[ \sigma_{j+1} = - \left( \frac{1}{\lambda} + 3 \epsilon \right) \log \lambda_n. \]

For any \( \sigma_0 \) satisfying the inequalities
\[ \sigma_{j+1} < \sigma_0 \leq \sigma_j, \]
\( \bar{m}_s(\sigma_0) \) is decreasing function of increasing \( \sigma_0 \). Hence, we have
\[ -\bar{\lambda}_s = \liminf_{\sigma_0 \to -\infty} \frac{\log \log \bar{m}_s(\sigma_0)}{-\sigma_0} \geq \liminf_{j \to \infty} \frac{(1+o(1)) \log \lambda_n}{\left( \frac{1}{\lambda} + 3 \epsilon \right) \log \lambda_n}, \]
\[ = \frac{1}{\left( \frac{1}{\lambda} + 3 \epsilon \right)}. \]

Since \( \epsilon \) is arbitrary, \( -\bar{\lambda}_s \geq \lambda \).

But \( -\bar{\lambda}_s \leq \lambda \) always. This leads to the desired conclusion.

REFERENCES
BOUTROUX [13]; CLUNIE [16]; KAMTHAN [36]; MALLIAVIN [49]; MANDELBROJT [50]; RAHMAN [58],[60],[61]; REDDY [66],[67]; ROUX [69]; SHAH [72],[75],[78]; SHAH AND KHANNA [85]; SRIVASTAVA[95]; SUNYER I BALAGUER [101]; VALIYON [106]; VIJAYARAGHAVAN [108].

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CHAPTER VI

ON THE APPROXIMATION OF AN ENTIRE FUNCTION

6.1 INTRODUCTION. The object of this chapter is to study the relationship of growth of an entire function $f(z)$, with the rate of growth of $E_n^{1/n}(f)$.

Let $f(x)$ be a real valued continuous function on $[-1,1]$, as usual, let

$$E_n(f) = \inf_{p \in \pi_n} \| f - p \|, \quad \text{for } n = 0, 1, 2, \ldots,$$

(6.1.1)

where the norm is the maximum norm on $[-1,1]$ and $\pi_n$ denotes the set of all polynomials with real coefficients of degree at most $n$.

Further, let us denote $f(z)$ as an entire function, and

$$M(r) = \max_{|z|=r} |f(z)|, \quad \rho = \lambda \quad (0 \leq \lambda \leq \rho \leq \infty)$$

$$\lim_{r \to \infty} \sup_{r \to \infty} \inf \frac{\log \log M(r)}{\log r} = \lambda$$

$$\lim_{r \to \infty} \sup_{r \to \infty} \inf \frac{\log M(r)}{r^\rho} = T \quad (0 < \rho < \infty).$$

The following results are due to S.N. Bernstein.

THEOREM A ([6], p.118). Let $f(x)$ be a real valued continuous function on $[-1,1]$. Then

$$\lim_{n \to \infty} E_n^{1/n}(f) = 0, \quad \ldots \quad (6.1.2)$$

if and only if, $f(x)$ is the restriction to $[-1,1]$ of an entire function of order $\rho$.
THEOREM B ([6], p.114). Let \( f(x) \) be a real valued continuous function on \([-1,1]\). Then there exist constants \( \rho \) (positive), \( \alpha, T \) (non-negative) such that

\[
\limsup_{n \to \infty} \left\{ \frac{1}{\rho} \frac{1}{n} E_n(f) \right\} = \alpha, \quad \ldots \ (6.1.3)
\]

if and only if, \( f(x) \) has an analytic extension \( f(z) \) such that \( f(z) \) is an entire function of order \( \rho \) and type \( T \).

By considering only the order of an entire function Varga has shown:

THEOREM C [107]. Let \( f(x) \) be a real valued continuous function on \([-1,1]\). Then

\[
\limsup_{n \to \infty} \left\{ \frac{n \log n}{-\log E_n(f)} \right\} = \rho, \quad \ldots \ (6.1.4)
\]

where \( \rho \) is a non-negative real number if and only if \( f(x) \) has an analytic extension \( f(z) \) such that \( f(z) \) is an entire function of order \( \rho \).

This result of Varga has been extended here to the lower order \( \lambda \), in Theorem 6.1 and Theorem 6.2.

By these results, we can expect that rate at which \( E_n(f) \) tends to zero is dependent on the order and type of an entire function. If an entire function is either of order \( \rho = 0 \) or of order \( \rho = \infty \), then we cannot expect the satisfactory results. For that purpose, a finer distinction has been introduced among such class of all entire functions whose order \( \rho = 0 \), by
means of the logarithmic order \( \rho(l) \) and corresponding lower order \( \gamma(l) \), defined as

\[
\lim_{r \to \infty} \sup_{r} \frac{\log \log M(r)}{\log \log r} = \frac{\rho(l)}{\gamma(l)}, \quad (1 \leq \gamma(l) \leq \rho(l) \leq \infty),
\]

\( \ldots (6.1.5) \)

having implications analogous to those of \( \rho \).

The theorems concerning \( \rho(l) \) and \( \gamma(l) \), have been considered in section 6.5.

Further, we assume that \( \rho \) is infinite, but there exists a positive integer \( k > 2 \), for which,

\[
\lim_{r \to \infty} \sup_{r} \frac{l_{k+1} M(r)}{l_{k} r} = \frac{\rho(k)}{\gamma(k)} = \frac{\rho_k}{\gamma_k}, \quad (k=2,3,4,\ldots)
\]

\( \ldots (6.1.6) \)

is finite, where

\[
l_{k} x = \log \log \ldots (k \text{ times}) x \quad (k=1,2,3,\ldots),
\]

\( l_{k} x > 0 \) for a real \( x \) after a certain stage. An entire function with \( \rho(k-1) = \infty \) and \( \rho(k) < \infty \) is called an entire function of index \( k \).

If \( 0 < \rho_k < \infty \), then there exists a proximate function \( \rho_k^*(r) \) (may be named as \( k \)th proximate order of \( f(z) \)) satisfying the following conditions:

(i) \( \lim_{r \to \infty} \rho_k^*(r) = \rho_k \),

(ii) \( \lim_{r \to \infty} r \rho_k^*(r) \log r \to \infty \), where \( \rho_k^*(r) \) is either the right hand or the left hand derivative at points where they are different,
(iii) \[ \limsup_{r \to \infty} \frac{1_k M(r)}{r^\rho_k(r)} = T_k. \]

\( T_k \) will be called the \( k \)th proximate type. In the last section we study the relationship of order \( \rho_k(r) \) and its corresponding \( T_k \) with the rate of growth of \( \mathbb{E}_n(f) \).

To state and prove the theorem very precisely, we introduce a function \( \mathcal{F}(x) \) which is defined as a single solution (when \( x > x_0 \)) of the equation,

\[ x = r^{\rho_k(r)}. \] \((6.1.6)\)

6.2 EXTENSION OF A RESULT OF VARGA.

We prove in this section the following theorems.

**THEOREM 6.1** Let \( f(x) \) be a real valued continuous function on \([-1,1]\). Then

\[ \limsup_{n \to \infty} \frac{\log \frac{1}{\mathbb{E}_n(f)}}{n \log(n-1)} \geq \frac{1}{\lambda}, \]

where \( \lambda \) is non-negative real number, if \( f(x) \) is the restriction to \([-1,1]\) of an entire function \( f(z) \) of lower order \( \lambda \).

**THEOREM 6.2** Let \( f(x) \) be a real valued continuous function on \([-1,1]\). Then

\[ \frac{1}{\lambda} = \min_{\{n_h\}} \limsup_{h \to \infty} \frac{\log \frac{1}{\mathbb{E}_{n_h}(f)}}{n_h \log n_{h-1}}, \]

where \( \{n_h\} \) is subsequence of non-negative integers such that \( n_0 < n_1 < n_2 < \ldots \) and \( \lambda \) is a non-negative real number, if \( f(x) \) is the restriction to \([-1,1]\) of an entire function \( f(z) \)
Here we prove only Theorem 6.2, since Theorem 6.1 follows from Theorem 6.2. We need the following lemma for our purpose.

**Lemma 6.1** [69] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of lower order \( \lambda \), then

\[
\frac{1}{\lambda} = \min_{\{n_h\}} \limsup_{h \to \infty} \frac{\log\{1/a_{n_h}\}}{n_h \log n_{h-1}}
\]

where \( \{n_h\} \) is a subsequence of non-negative integers such that \( n_0 < n_1 < n_2 < \ldots \).

**Proof of Theorem 6.2** First, assume that \( f(x) \) has an analytic extension \( f(z) \), which is an entire function of lower order \( \lambda \). Following Bernstein's original proof, we have ([48], p.73) for each \( n \geq 0 \),

\[
E_n(f) \leq \frac{2 B(\sigma)}{\sigma^n (\sigma - 1)} \quad \text{for any } \sigma > 1.
\]

(6.2.1)

Where \( B(\sigma) \) is the maximum of the absolute value of \( f(z) \) on \( E_\sigma \), and \( E_\sigma \) with \( \sigma > 1 \) denotes the closed interior of an ellipse with foci at \( \pm 1 \), and semi-major axis \( \frac{\sigma^2 + 1}{2\sigma} \) and semi-minor axis \( \frac{\sigma^2 - 1}{2\sigma} \). The closed discs \( D_1(\sigma) \) and \( D_2(\sigma) \) bound the ellipse \( E_\sigma \) in the sense that

\[
D_1(\sigma) = \{ z \mid |z| \leq \frac{\sigma^2 - 1}{2\sigma} \} \subseteq E_\sigma \subseteq D_2(\sigma) = \{ z \mid |z| \leq \frac{\sigma^2 + 1}{2\sigma} \}.
\]

From this inclusion, it follows by definition that
\[ M_f \left( \frac{\sigma^2 - 1}{2\sigma} \right) \leq B(\sigma) \leq M_f \left( \frac{\sigma^2 + 1}{2\sigma} \right) \text{ for all } \sigma > 1. \quad \ldots(6.2.2) \]

From this, it can be verified that
\[ f = \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\log \sigma} = \lim_{\sigma \to \infty} \inf \frac{\log \log B(\sigma)}{\log \sigma}. \quad \ldots(6.2.3) \]

From (6.2.1), we have
\[ E_n(f) \leq \frac{KB(\sigma)}{\sigma^n}, \text{ where } K \text{ is some positive constant } \ldots(6.2.4) \]
since \( \sigma > 1, \frac{2}{\sigma^2 - 1} < K. \)

From (6.2.4), we can have for any \( \varepsilon > 0, \)
\[ \sum_{n=0}^{\infty} E_n(f) \sigma^n \leq \sum_{n=0}^{\infty} \frac{KB(\sigma + \varepsilon)}{(\sigma + \varepsilon)^n} \sigma^n, \]
\[ = K B(\sigma + \varepsilon) \sum_{n=0}^{\infty} \left( \frac{\sigma}{\sigma + \varepsilon} \right)^n, \]
\[ = K B(\sigma + \varepsilon) \left( \sigma + \varepsilon \right)^{-\varepsilon}. \quad \ldots(6.2.5) \]

We note [107] that
\[ B(\sigma) \leq |P_0(z)| + 2\sigma \sum_{j=0}^{\infty} E_j \sigma^j, \quad \ldots(6.2.6) \]
where \( |P_0(z)| \) is a constant. \( E_j \) is a decreasing sequence of real numbers along with \( j. \) Let us write
\[ J(\sigma) = \sum_{j=0}^{\infty} E_j \sigma^j, \quad \ldots(6.2.7) \]
which represents an entire function, then we have from (6.2.5) and (6.2.6),
\[ B(\sigma) \leq K' \sigma J(\sigma) \leq K'' \sigma (\sigma + \varepsilon) B(\sigma + \varepsilon) \quad \ldots(6.2.8) \]
where $K'$ and $K''$ are constants. From (6.2.8) and (6.2.3), we can have

$$
\lambda = \lim_{\sigma \to \infty} \sup \frac{\log \log B(\sigma)}{\log \sigma} = \lim_{\sigma \to \infty} \inf \frac{\log \log J(\sigma)}{\log \sigma}.
$$

Now, by applying Lemma 6.1 to $J(\sigma)$, we have

$$
\frac{1}{\lambda} = \min_{\{n\}} \limsup_{n \to \infty} \frac{\log 1/E_n(f)}{n \log \log n-1}.
$$

Hence the result.

6.3 ENTIRE FUNCTION OF IRREGULAR GROWTH AND $B_n(f)$.

In this section, we consider the case when $f(x)$ has an analytic extension $f(z)$ such that $f(z)$ is an entire function of irregular growth.

**Theorem 6.3** Let $f(x)$ be a real valued continuous function on $[-1,1]$. If $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of order $\rho$ and lower order $\lambda$ ($0 < \lambda < \rho < \infty$), then

$$
\liminf_{n \to \infty} (n E_n^H(f)) = 0 \quad \ldots (6.3.1)
$$

**Proof.** From (6.2.2) and (6.2.8), we have

$$
2^{-\rho} \lim_{\sigma \to \infty} \sup_{\sigma^f} \frac{\log M(\sigma)}{\sigma^f} = \lim_{\sigma \to \infty} \inf_{\sigma^f} \frac{\log B(\sigma)}{\sigma^f} = \lim_{\sigma \to \infty} \inf_{\sigma^f} \frac{\log J(\sigma)}{\sigma^f}.
$$

Now, by applying a result of Shah [80] to $J(\sigma)$, we get
We note ([4], p.187) that the lower type of an entire function of irregular growth of finite order is zero. Hence we have

\[
\liminf_{n \to \infty} \left( \frac{n}{E_n(f)} \right) = 0.
\]

This proves the theorem.

6.4 \( t_\lambda \) AND THE RATE OF GROWTH OF \( E_n(f) \).

The result of Theorem 6.3 suggests that the rate at which \( E_n(f) \) tends to zero depends on the rate of growth of entire function \( f(z) \). If an entire function is of irregular growth, then we cannot expect satisfactory result. For that purpose, in this section type \( T_\lambda \) and lower type \( t_\lambda \) of \( f(z) \) are defined by

\[
\lim_{r \to \infty} \sup \inf \frac{\log M(r)}{r^\lambda} = T_\lambda \quad (0 < \lambda < \infty),
\]

\[
\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \leq \lambda
\]

Let \( T_\lambda \) be finite, then

\[
\log M(r) < (T_\lambda + \varepsilon)r^\lambda.
\]

Therefore,

\[
\log \log M(r) < \log (T_\lambda + \varepsilon) + \lambda \log r
\]

or

\[
\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \leq \lambda
\]

i.e.

\[
\rho \leq \lambda.
\]
But $f > \gamma$ always.

Hence $f = \gamma$, function is of regular growth, $T_\gamma$ is same as $T$.

The object of this section is to investigate the relationship of $t_\gamma$ with the asymptotic behaviour of $\frac{1}{n} \gamma_n(f)$.

**Theorem 6.4** Let $f(x)$ be a real valued continuous function on $[-1,1]$. Then there exist constants $\gamma$ (positive), $\beta, t_\gamma$ (non-negative) such that

$$\liminf_{n \to \infty} \left\{ n \frac{\gamma}{n} \left( \frac{f}{f_n} \right) \right\} = (2 e \gamma t_\gamma)$$

provided that $E_n / E_{n+1}$ is a non-decreasing function of $n$, for $n > n_0$, if $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of lower order $\gamma$ and lower $\gamma$-type $t_\gamma$.

To prove this theorem we require the following lemma.

**Lemma 6.2** Let $f(z)$ be an entire function of lower order $\gamma$ ($0 < \gamma < \infty$), lower $\gamma$-type $t_\gamma$, such that $|\frac{a_n}{a_{n+1}}|$ is a non-decreasing function of $n$ for all large $n$, then

$$\liminf_{n \to \infty} \left\{ n \frac{\gamma}{n} \left( \frac{f}{f_n} \right) \right\} = (e \gamma t_\gamma).$$

We omit the proof since it is based on the same lines as given by S.M. Shah [80], for the lower type $t$.

**Proof of Theorem 6.4** From (6.2.2) and (6.2.8),

$$\liminf_{\sigma \to \infty} \frac{\log J(\sigma)}{\sigma^{-\gamma}} = t_\gamma 2^{-\gamma}.$$
Now, by applying Lemma 6.2 to $J(\sigma)$, we have the required result i.e.

$$\liminf_{n \to \infty} \left\{ n \frac{\lambda/n}{E_n(f)} \right\} = (2 e^{\lambda t_\lambda})^\beta = 1/n.$$

6.5 **GROWTH OF** $E_n(f)$ **WHEN $\beta$ IS ZERO.**

In this section, we consider the case when $\beta = 0$, here we prove the following:

**THEOREM 6.5** If $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of $f(z)$ of logarithmic order $\rho(\ell)$ and lower logarithmic order $\gamma(\ell)$ and if

$$\limsup_{n \to \infty} \inf \left\{ \frac{E_{n-1}(f)}{E_n(f)} \right\}^{\frac{1}{n}} = \frac{S}{H}, \quad (1 < H \leq S < \infty), \quad \cdots (6.5.1)$$

then

$$\frac{1}{2 \log S} \leq \liminf_{\sigma \to \infty} \frac{\log M(\sigma)}{(\log \sigma)^2} \leq \limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{(\log \sigma)^2} \leq \frac{1}{2 \log H} \cdots (6.5.2)$$

and

$$\rho(\ell) = \gamma(\ell) = 2. \quad \cdots (6.5.3)$$

**PROOF.** We may take

$$\frac{E_{n-1}(f)}{E_n(f)} < (S + \epsilon)^n, \quad \text{for all } n > 0.$$

So,

$$E_n(f) > (S + \epsilon)^{\frac{n+1}{2}}.$$

Combining this with (6.2.4), we get
\[
\log K + \log B(\sigma) > n \log \sigma - \frac{n(n+1)}{2} \log (s+\epsilon).
\] .... (6.5.4)

The right hand side of this inequality considered as a function of \(n\) for a fixed \(\sigma\), will attain its maximum value for a value of \(n\) at \(n = \frac{\log \sigma}{\log (s+\epsilon)} - \frac{1}{2}\).

For this value of \(n\), (6.5.4) will become

\[
\log K + \log B(\sigma) \geq \left\{ \frac{1}{2} \frac{1}{\log (s+\epsilon)} \left\{ \log \sigma - \frac{1}{2} \log (s+\epsilon) \right\}^2 \right\}.
\]

This leads to

\[
\liminf_{\sigma \to \infty} \frac{\log B(\sigma)}{(\log \sigma)^2} \geq \frac{1}{2 \log s}, \quad \ldots (6.5.5)
\]

and

\[
\liminf_{\sigma \to \infty} \frac{\log \log B(\sigma)}{\log \log \sigma} \geq 2. \quad \ldots (6.5.6)
\]

On the other hand, we also have for \(n > 0\) (say)

\[
E_n(f) < (H - \epsilon)^2.
\]

Hence,

\[
J(\sigma) < \sum_{n=0}^{\infty} \left\{ \frac{n(n+1)}{(H - \epsilon)^2} \right\} \sigma^{-n}.
\]

We split the right hand side sum into two parts \(S_1\) and \(S_2\), where \(S_1\) contains the terms for which
\[ n < \frac{2 \log \sigma}{\log (H-\epsilon)} \] and \( S_2 \) contains the rest. Then
\[ S_1 < \frac{\left( \frac{2 \log \sigma}{\log (H-\epsilon)} \right)}{\sigma^2} \Sigma (H-\epsilon) \Sigma (H-\epsilon), \]
\[ = K_1 \sigma. \]

Since the series involved in \( S_1 \) is convergent. Now in \( S_2 \) we have \( \sigma \leq (H - \epsilon) \). Therefore,
\[ S_2 < \Sigma (H-\epsilon) \Sigma (H-\epsilon), \]
\[ = \Sigma (H-\epsilon)^2, \]
which is also convergent since \( H > 1 \).

So,
\[ \log J(\sigma) < \frac{2(\log \sigma)^2}{\log (H-\epsilon)} + \log K_1 + \log (1 + o(1)) \]
giving
\[ \limsup_{\sigma \to \infty} \frac{\log J(\sigma)}{(\log \sigma)^2} \leq \frac{2}{\log H} \quad \ldots (6.5.7) \]
and
\[ \limsup_{\sigma \to \infty} \frac{\log \log J(\sigma)}{\log \log \sigma} \leq 2. \quad \ldots (6.5.8) \]

From (6.2.2) and (6.2.8), we can verify that
\[ \lim_{\sigma \to \infty} \sup_{\sigma \to \infty} \inf \frac{\log M(\sigma)}{(\log \sigma)^2} = \lim_{\sigma \to \infty} \sup_{\sigma \to \infty} \inf \frac{\log B(\sigma)}{(\log \sigma)^2} \]
\[ = \lim_{\sigma \to \infty} \sup_{\sigma \to \infty} \frac{\log J(\sigma)}{(\log \sigma)^2} \quad \ldots (6.5.9) \]
\begin{align*}
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log M(\sigma)}{\log \log \sigma} &= \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log \mathcal{B}(\sigma)}{\log \log \sigma} = \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log \mathcal{J}(\sigma)}{\log \log \sigma}.
\end{align*}

Finally, combining (6.5.9) with (6.5.5), (6.5.7), we obtain (6.5.2), (6.5.3) follows from (6.5.6), (6.5.8) and (6.5.10).

**Theorem 6.6** Let $f(x)$ be a real valued continuous function on $[-1,1]$. Then

\[ \liminf_{n \to \infty} \frac{\log n}{\log \left( \frac{1}{n} \log \frac{1}{E_n(f)} \right)} \leq \lambda(\ell) - 1, \]

where $\lambda(\ell)$ is a real number necessarily $> 1$, if $f(x)$ is the restriction to $[-1,1]$ of an entire function $f(z)$ of logarithmic lower order $\lambda(\ell)$.

Following result of Shah and Ishaq, is needed for the proof of the theorem.

**Lemma 6.3** ([84], Theorem 1). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of logarithmic order $\lambda(\ell) \geq 1$, then

\[ \liminf_{n \to \infty} \frac{\log n}{\log \left( \frac{1}{n} \log \frac{1}{|a_n|} \right)} \leq \lambda(\ell) - 1. \]
PROOF OF THEOREM 6.6 From (6.2.2) and (6.2.8), we get

$$\gamma(l) = \liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\log \log \sigma} = \liminf_{\sigma \to \infty} \frac{\log \log B(\sigma)}{\log \log \sigma}$$

$$= \liminf_{\sigma \to \infty} \frac{\log \log J(\sigma)}{\log \log \sigma}.$$

Now, by applying Lemma 6.3 to $J(\sigma)$, we get

$$\liminf_{n \to \infty} \frac{\log n}{\log \left\{ \frac{1}{n} \log \frac{1}{E_n(f)} \right\}} \leq \gamma(l) - 1.$$ 

This completes the proof of the theorem.

6.6 PROXIMATE TYPE $T_k$ AND THE RATE OF GROWTH OF $E_n(f)$. 

To conclude this chapter, we show how the $k$th proximate type $T_k$ is related with the rate of growth of $E_n(f)$.

THEOREM 6.7 Let $f(x)$ be a real valued continuous function on $[-1,1]$, which is the restriction to $[-1,1]$ of an entire function $f(z)$ of $k$th order $\rho_k(0 < \rho_k < \infty)$ and proximate order $\rho_k(r)$, then

$$\limsup_{n \to \infty} \left\{ \frac{1}{n} \log E_n(f) \right\} \leq \frac{1}{n} (f).$$  ...(6.6.1)

and

$$\limsup_{n \to \infty} \left\{ \frac{1}{n} \log E_n(f) \right\} \leq \frac{1}{n} (f), \quad k = 2, 3, \ldots.$$ ...(6.6.2)
Before we start with the actual proof, we first consider the following lemmas which will be needed in the proof of the theorem.

**Lemma 6.4** ([45], Theorem 2, p.42) The type $T^*$ (proximate type) of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with the proximate order $\rho(r)$ is given by equation,

$$\limsup_{n \to \infty} \Phi(n) |a_n|^\frac{1}{n} = \left( T^* \rho \right)^\frac{1}{\rho}, \quad (0 < T^* < \infty)$$

... (6.6.3)

where $\Phi(x)$ is the inverse of $x = \rho(r)$.

**Lemma 6.5** Let $f(z)$ be an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of proximate order $\rho_k(r)$ and proximate type $T_k$. Then necessary and sufficient condition for $T_k$ to be a $k$th proximate type is that

$$\limsup_{n \to \infty} \Phi(1_{k-1}(n)) |a_n|^\frac{1}{n} = T_k,$$

... (6.6.4)

where $\Phi(x)$ is the inverse of $x = \rho_k(r)$.

We omit the proof since it can be proved on the same lines as given by Jain [28] for entire Dirichlet Series.

**Proof of Theorem 6.7** (6.2.2) and (6.2.8) lead to

$$-\frac{\rho_k(\sigma)}{2} \limsup_{\sigma \to \infty} \frac{\log_k M(\sigma)}{\rho_k(\sigma)} = \limsup_{\sigma \to \infty} \frac{\log_k M(\sigma)}{\rho_k(\sigma)} = \limsup_{\sigma \to \infty} \frac{\log_k B(\sigma)}{\rho_k(\sigma)} = \limsup_{\sigma \to \infty} \frac{\log_k J(\sigma)}{\rho_k(\sigma)}.$$

By applying Lemma 6.4 to $J(\sigma)$, we obtain (6.6.1). Similarly, on applying Lemma 6.5 to $J(\sigma)$, we get (6.6.1). This completes the proof of the theorem.

**References**

Anderson [4]; Bernstein [6]; Jain [28]; Levin [45]; Lorentz [48]; Roux [69]; Shah [80]; Shah and Ishaq [84]; Varga [107].
CHAPTER VII

ON THE ORDER AND TYPE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

7.1 INTRODUCTION. In this chapter, we have obtained the relations between entire functions of finite non-zero orders and types and also studied the relations between the coefficients in the Taylor expansion of entire functions and their orders and types. For simplicity, we confine ourselves to the case of functions of two complex variables. The case of an arbitrary finite number of variables can be examined in the same way.

Let

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n \quad \ldots \quad (7.1.1)$$

be a function of two complex variables $z_1$ and $z_2$, where the coefficients $a_{m,n}$ are complex numbers. The series $(7.1.1)$ represents an entire function of two complex variables $z_1, z_2$, if it converges absolutely for all values of $|z_1| < \infty$ and $|z_2| < \infty$.

M.M. Džrbašyan ([18], p.1) has shown that the necessary and sufficient condition for the series $(7.1.1)$ to represent an entire function of variables $z_1$ and $z_2$, is

$$\limsup_{m+n \to \infty} \left| \frac{a_{m,n}}{m+n} \right| = 0. \quad \ldots \quad (7.1.2)$$

Let $\bar{G}_r$ be a family of closed polycircular domains in space $(z_1, z_2)$ dependent on parameter $r > 0$ and possess the
property that \((z_1, z_2) \in \overline{G_r}\) if and only if \((\frac{z_1}{r}, \frac{z_2}{r}) \in \overline{G_1}\).

The maximum modulus of the entire function \(f(z_1, z_2)\) is denoted by

\[
M_G(r, f) = \max_{(z_1, z_2) \in \overline{G_r}} |f(z_1, z_2)|,
\]

and the G-order and G-type of the function are defined respectively, by

\[
\rho_G = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}, \quad \ldots \ (7.1.3)
\]

\[
T_G = \limsup_{r \to \infty} \frac{\log M_G(r, f)}{r^\rho_G}. \quad \ldots \ (7.1.4)
\]

Set

\[
\Phi = \Phi_G(m, n) = \max_{(z_1, z_2) \in \overline{G_r}} |z_1|^m |z_2|^n. \quad \ldots \ (7.1.5)
\]

A.A. Goldberg ([20], p.146) has proved the following theorems:

**THEOREM A.** All order \(\rho_G\) are equal and

\[
\rho = \rho_G = \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log(1/a_{m,n})}. \quad \ldots \ (7.1.6)
\]

**THEOREM B.** G-type \(T_G\) satisfies the condition

\[
(e \rho T_G)^{1/\rho} = \limsup_{m+n \to \infty} \frac{1}{(m+n)} \left\{ \frac{\Phi_G(m, n)}{|a_{m,n}|} \right\}^{1/(m+n)}], \quad \ldots \ (7.1.7)
\]

or, in short
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\[ (e^p T) = \limsup_{m+n \to \infty} \left\{ \frac{1}{p} \Phi \left( a_{m,n} \right) \right\}. \quad ... (7.1.8) \]

7.2 THEOREMS ON THE COEFFICIENTS AND ORDERS.

THEOREM 7.1 Let

\[ f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{a_{m,n}}{k} z_1^m z_2^n \quad (k=1, 2, \ldots, p) \]

be p entire functions of finite non-zero orders \( \rho_1, \rho_2, \ldots, \rho_p \) respectively. Then the function

\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n, \]

where

\[ \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\} \sim \sum_{k=1}^{p} \alpha_k \left\{ \log \left( \frac{1}{|a_{m,n}|_k} \right) \right\} \]

\( (0 < \alpha_k < 1, \alpha_1 + \alpha_2 + \ldots + \alpha_k = 1), \]

is an entire function such that

\[ \rho \leq \sum_{k=1}^{p} \alpha_k \rho_k \quad [\rho \text{ is the order of } f(z_1, z_2)]. \]

PROOF. Since \( f_k(z_1, z_2) \) is an entire function. Therefore, using (7.1.2), we have, for an arbitrary \( \epsilon \) and enough large \( R \),

\[ \frac{1}{|a_{m,n}|_k} > (R-\epsilon)^{m+n}, \quad \text{for } m+n > k_k, \]

or

\[ \log \left\{ \frac{1}{|a_{m,n}|_k} \right\} > (m+n) \log (R-\epsilon), \quad \text{for } m+n > k_k, \]
Putting \( k = 1, 2, \ldots, p \) and adding the \( p \) inequalities thus obtained, we get, for large \( m + n \),

\[
\sum_{k=1}^{p} \alpha_k \left\{ \log \left( \frac{1}{|a_{m,n}^k|} \right) \right\}^{-1} < \frac{\alpha_1 + \alpha_2 + \ldots + \alpha_p}{(m+n) \log (R - \varepsilon)} \\
\text{for } m + n > k_k.
\]

Thus, if

\[
\left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{-1} \sim \sum_{k=1}^{p} \alpha_k \left\{ \log \left( \frac{1}{|a_{m,n}^k|} \right) \right\}^{-1}, \sum_{k=1}^{p} \alpha_k = 1.
\]

Then, for large \( m + n \),

\[
\left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{-1} < \frac{1}{(m+n) \log (R - \varepsilon)} \quad \text{for } m + n > k,
\]

or

\[
\log \left( \frac{1}{|a_{m,n}|} \right) > (m+n) \log (R - \varepsilon), \quad \text{for } m + n > k,
\]

or

\[
\limsup_{m+n \to \infty} \frac{1}{|a_{m,n}|} = 0.
\]

Hence, \( f(z_1, z_2) \) is an entire function.

Now using (7.1.6) for the function \( f_k(z_1, z_2) \), we have

\[
\limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log \left( \frac{1}{|a_{m,n}^k|} \right)} = f_k.
\]
Therefore, for an arbitrary $\epsilon$ we get

$$\alpha_k \left\{ \log\left(\frac{1}{|(a_{m,n})_k|}\right) \right\} < \alpha_k (\rho_k + \epsilon) \left\{ (m+n)\log (m+n) \right\}^{-1}$$

for $m+n > k_k$.

Putting $k = 1, 2, \ldots, p$ and adding the $p$ inequalities thus obtained, we get

$$\sum_{k=1}^{p} \alpha_k \left\{ \log\left(\frac{1}{|(a_{m,n})_k|}\right) \right\} < \sum_{k=1}^{p} \alpha_k (\rho_k + \epsilon) \left\{ (m+n)\log (m+n) \right\}^{-1}$$

$$[ m+n > k = \max (k_1, k_2, \ldots, k_p) ].$$

Since,

$$\left\{ \log\left(\frac{1}{|(a_{m,n})_k|}\right) \right\} \sim \sum_{k=1}^{p} \alpha_k \left\{ \log\left(\frac{1}{|(a_{m,n})_k|}\right) \right\},$$

and we have

$$\limsup_{m+n \to \infty} \frac{(m+n)\log (m+n)}{\log \left(\frac{1}{|(a_{m,n})_k|}\right)} \leq \sum_{k=1}^{p} \alpha_k \rho_k.$$

Hence we get

$$\rho \leq \sum_{k=1}^{p} \alpha_k \rho_k,$$

where $\rho$ is the order of $f(z_1, z_2)$.

**Theorem 7.2** Let

$$f_k(z_1, z_2) = \sum_{m, n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n (k=1, 2, \ldots, p)$$

be $p$ entire functions of finite non-zero orders $\rho_1, \rho_2, \ldots, \rho_p$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{m,n} z_1^m z_2^n,$$
where
\[ \log \left( \frac{1}{|a_{m,n}|} \right) \sim \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|(a_{m,n})_k|} \right) \right\}^{\alpha_k} \]

\[ (0 < \alpha_k < 1, \sum_{k=1}^{p} \alpha_k = 1), \]
is an entire function such that
\[ \rho \leq \prod_{k=1}^{p} \rho_k^{\alpha_k}, \]
where \( \rho \) is the order of \( f(z_1, z_2) \).

**Proof.** Since \( f_k(z_1, z_2) \) is an entire function, therefore, using (7.1.2), we have for arbitrary \( \varepsilon \) and enough large \( R \)
\[ \frac{1}{|a_{m,n}|} > (R - \varepsilon)^{m+n}, \text{ for } m+n > k_k, \]
or
\[ \log \left( \frac{1}{|a_{m,n}|} \right) > (m+n) \log (R - \varepsilon), \text{ for } m+n > k_k, \]
or
\[ \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{\alpha_k} > (m+n)^{\alpha_k} \log(R-\varepsilon), \text{ for } m+n > k_k. \]

Putting \( k = 1, 2, \ldots, p \) and multiplying the \( p \) inequalities thus obtained, we get, for large \( m+n \),
\[ \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{\alpha_k} > (m+n) \log (R - \varepsilon). \]

Therefore, if
\[ \log(1/|a_{m,n}|) \sim \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|(a_{m,n})_k|} \right) \right\}^{\alpha_k}, \]
then, for large \( m+n \),
log(1/|a_{m,n}|) > (m+n) \log (R-\varepsilon).

Hence

$$\limsup_{m+n \to \infty} \frac{1}{(m+n)} |a_{m,n}| = 0,$$

and \(f(z_1, z_2)\) is an entire function.

Now, using (7.1.6) for the function \(f_k(z_1, z_2)\), we have

$$\limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log \{1/|a_{m,n}| \}} = \rho_k.$$ 

Therefore, for an arbitrary \(\varepsilon\) we get

$$\frac{1}{\{\log(1/|a_{m,n}|)\}} < (\rho_k + \varepsilon) \{ (m+n) \log (m+n) \}^{-1}$$

for \(m+n > k_k\),

or

$$\{ \log(1/|a_{m,n}|) \}^{-\alpha_k} < (\rho_k + \varepsilon)^{\alpha_k} \{ (m+n) \log (m+n) \}^{-\alpha_k}$$

for \(m+n > k_k\).

Putting \(k = 1, 2, \ldots, p\) and multiplying the \(p\) inequalities thus obtained, we get, for large \(m+n\),

$$\prod_{k=1}^{p} \{ \log(1/|a_{m,n}|) \}^{-\alpha_k} < \prod_{k=1}^{p} (\rho_k + \varepsilon)^{\alpha_k} \{ (m+n) \log (m+n) \}^{-1},$$

or

$$\limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\prod_{k=1}^{p} \{ \log \{1/|a_{m,n}| \} \}^{\alpha_k}} \leq \prod_{k=1}^{p} \rho_k^{\alpha_k}.$$ 

Thus, if

$$\log (1/|a_{m,n}|) \sim \prod_{k=1}^{p} \{ \log (1/|a_{m,n}|) \}^{\alpha_k},$$
then
\[ \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log (1/|a_{m,n}|)} \leq \prod_{k=1}^{p} \rho_k^{\alpha_k}. \]

Hence
\[ \rho \leq \prod_{k=1}^{p} \rho_k^{\alpha_k}. \]

### 7.3 Theorems on the Coefficients and Types

**Theorem 7.3** Let
\[ f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n, \quad (k = 1, 2, \ldots, p) \]
be \( p \) entire functions of finite non-zero orders \( \rho_1, \rho_2, \ldots, \rho_p \) and finite non-zero types *\( T_1, T_2, \ldots, T_p \)* respectively. Then the function
\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n, \]
where
\[ \log(1/|f| a_{m,n}) \sim \prod_{k=1}^{p} (\log(1/|f(a_{m,n})|))^{\alpha_k}, \quad (0 < \alpha_k < 1, \sum_{k=1}^{p} \alpha_k = 1), \]
is an entire function such that
\[ T \leq \prod_{k=1}^{p} (T_k)^{\alpha_k}, \]
where \( \rho \) and \( T \) are the order and type of \( f(z_1, z_2) \) respectively provided
\[ \rho = \prod_{k=1}^{p} \rho_k^{\alpha_k}. \]

* The types \( T_1, T_2, \ldots, T_p \) correspond the same family of polycircular domains \( G_r \).
PROOF. It is easy to prove that \( f(z_1, z_2) \) is an entire function. Further, using (7.1.8), for the function \( f_k(z_1, z_2) \) we have

\[
\limsup_{m+n \to \infty} \left[ (m+n) \left\{ \Phi \left( \left\{ a_{m,n} \right\} \right) \right\} \right] = \left( e \rho_k T_k \right) \left( \frac{1}{T_k} \right).
\]

For arbitrary \( \varepsilon > 0 \) and sufficiently large \( m+n \), we have

\[
\frac{1}{\rho_k} \frac{1}{m+n} \left\{ \Phi \left( \left\{ a_{m,n} \right\} \right) \right\} < \left( e \rho_k \left( T_k + \varepsilon \right) \right), \text{ for } m+n > k_k.
\]

Put \( e \rho_k (T_k + \varepsilon) = A_k \),

\[
\frac{1}{\rho_k} \frac{1}{m+n} \left\{ \Phi \left( \left\{ a_{m,n} \right\} \right) \right\} < \left( A_k \right), \text{ for } m+n > k_k,
\]

or

\[
\log \left( \frac{1}{\Phi \left( \left\{ a_{m,n} \right\} \right)} \right) > \frac{m+n}{\rho_k} \frac{1}{A_k}, \text{ for } m+n > k_k,
\]

or

\[
\{ \log \left( \frac{1}{\Phi \left( \left\{ a_{m,n} \right\} \right)} \right) \} > \frac{(m+n)}{\rho_k} \frac{1}{A_k} \left\{ \log \frac{m+n}{A_k} \right\}^{\alpha_k},
\]

for sufficiently large \( m+n \).

Putting \( k = 1, 2, \ldots, p \) and multiplying \( p \) inequalities thus obtained, we have

\[
\prod_{k=1}^{p} \{ \log (1/\Phi \left( \left\{ a_{m,n} \right\} \right)) \}^{\alpha_k} > \prod_{k=1}^{p} \frac{m+n}{\rho_k} \left\{ \log \frac{m+n}{A_k} \right\}^{\alpha_k}, \text{ for } m+n > k_k,
\]

or

\[
\prod_{k=1}^{p} \{ \log (1/\Phi \left( \left\{ a_{m,n} \right\} \right)) \}^{\alpha_k} > \frac{m+n}{\rho} \prod_{k=1}^{p} \left\{ \log \frac{m+n}{A_k} \right\}^{\alpha_k}, \text{ for } m+n > k_k,
\]
Thus, if
\[ \log(1/\Phi | a_m, n|) \sim \prod_{k=1}^{p} \left\{ \log(1/\Phi | (a_m, n)_k|) \right\}^{\alpha_k} \]
\[ (0 < \alpha_k < 1, \quad \sum_{k=1}^{p} \alpha_k = 1), \]
we obtain, for sufficiently large \((m+n)\),
\[ \log(1/\Phi | a_m, n| > \frac{m+n}{p} \left[ 1 - \sum_{k=1}^{p} \frac{\log A_k^{\alpha_k}}{\log (m+n)} + O(\log(m+n)) \right] \log(m+n), \]
or
\[ (m+n) \{ F | a_m, n| \}^{\beta} \]
with
\[ \beta = \sum_{k=1}^{p} \frac{\log A_k^{\alpha_k}}{\log (m+n)} + O(\log(m+n)) . \]
Since,
\[ \lim_{m+n \to \infty} (m+n)^{\beta} = \prod_{k=1}^{p} A_k^{\alpha_k} , \]
where
\[ A_k^{\alpha_k} = e^{\beta_k T_k} , \]
we obtain,
\[ \limsup_{m+n \to \infty} (m+n) \{ F | a_m, n| \}^{\beta/(m+n)} \leq \prod_{k=1}^{p} (e^{\beta_k T_k})^{\alpha_k} . \]
\[
(e \rho \Sigma) \leq \prod_{k=1}^{p} (e \rho_k \Sigma_k)^{\alpha_k},
\]
or
\[
\rho \Sigma \leq \prod_{k=1}^{p} (\alpha_k).\]

**Theorem 7.4** Let

\[
f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_k^{m} z_1^{m} z_2^{n}, \quad (k = 1, 2, \ldots, p)
\]

be \(p\) entire functions of finite non-zero orders \(\rho_1, \rho_2, \ldots, \rho_p\) and finite non-zero types \(\Sigma_1, \Sigma_2, \ldots, \Sigma_p\) respectively. Then the function

\[
f(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_1^{\alpha_1} (a_{m,n})_2^{\alpha_2} \cdots (a_{m,n})_p^{\alpha_p} z_1^{m} z_2^{n}
\]

\((0 < \alpha_k < 1, \sum_{k=1}^{p} \alpha_k = 1)\)

is an entire function such that

\[
\frac{1}{\rho} \leq \prod_{k=1}^{p} \left( \frac{\rho_k}{\Sigma_k} \right)^{\alpha_k},
\]

where \(\rho\) and \(\Sigma\) are the order and type of \(f(z_1, z_2)\) respectively, provided

\[
\frac{1}{\rho} = \sum_{k=1}^{p} \frac{\alpha_k}{\rho_k}.
\]

**Proof.** It is easy to prove that \(f(z_1, z_2)\) is an entire function. Using (7.1.9) for the function \(f_k(z_1, z_2)\), we have

\[
\limsup_{m+n \to \infty} \left[ (m+n)^{k} \left( e \rho_k \Sigma_k \right) \frac{1}{m+n} \right] = (e \rho_k \Sigma_k)^{k}.
\]
or
\[
\limsup_{m+n \to \infty} \left( \frac{1}{\rho_k} \left( \frac{1}{m+n} \right)^{\alpha_k} \right) = (e \rho_k T_k)^{\frac{\alpha_k}{\rho_k}}.
\]

Putting \( k = 1, 2, \ldots, p \) and multiplying \( p \) inequalities thus obtained, we get
\[
\prod_{k=1}^{p} \limsup_{m+n \to \infty} \left( \frac{1}{\rho_k} \left( \frac{1}{m+n} \right)^{\alpha_k} \right) = \prod_{k=1}^{p} \left( e \rho_k T_k \right)^{\frac{\alpha_k}{\rho_k}},
\]

or
\[
\limsup_{m+n \to \infty} \left( \frac{1}{\rho_k} \left( \frac{1}{m+n} \right)^{\alpha_k} \right) \leq \prod_{k=1}^{p} \left( e \rho_k T_k \right)^{\frac{\alpha_k}{\rho_k}},
\]

or
\[
\limsup_{m+n \to \infty} \left( \frac{1}{\rho_k} \right) \left( \frac{1}{m+n} \right)^{\alpha_k} \leq \prod_{k=1}^{p} \left( e \rho_k T_k \right)^{\frac{\alpha_k}{\rho_k}}.
\]

Hence again using (7.1.8) for \( f(z_1, z_2) \), we obtain
\[
(e \rho T) \leq \prod_{k=1}^{p} \left( e \rho_k T_k \right)^{\frac{\alpha_k}{\rho_k}},
\]

or
\[
(\rho T) \leq \prod_{k=1}^{p} \left( \rho_k T_k \right)^{\frac{\alpha_k}{\rho_k}}.
\]

**COROLLARY 7.1** If \( f_k(z_1, z_2) \) are of some finite non-zero order, then
\[
T \leq \prod_{k=1}^{p} (T_k)^{\alpha_k}.
\]

**7.4 EXTENSION OF A RESULT OF S.K. SINGH.**

Here, we prove a theorem of entirely different kind. S.K. Singh has proved that
\[
\frac{M(kr^{''})}{M(r^{''})} \leq \frac{M(kr^{'})}{M(r^{'})}, \text{ for } r^{'} < kr^{''} \text{ and } r^{'} < r^{''} \quad \text{(39)}
\]

where \( k \) is a number lying between zero and one.

We extend this result to two complex variables.

Let \( \mathbb{H}(r) \) be the set of all points

\[
r = ||z|| = \left[ |z_1|^2 + |z_2|^2 \right]^{\frac{1}{2}} = \left[ r_1^2 + r_2^2 \right]^{\frac{1}{2}},
\]

and the maximum modulus of \( f(z_1, z_2) \) be

\[
M(r) = \max_{||z|| = r} |f(z_1, z_2)|.
\]

Further, it is known that if

\[
f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{m, n} z_1^m z_2^n,
\]

is an analytic function in the closed unit hypersphere

\[
|z_1|^2 + |z_2|^2 \leq 1,
\]

then

\[
\{M(R_2)\}^{\alpha_2} \leq \{M(R_1)\}^{\alpha_1} \{M(R_3)\}^{\alpha_3} \quad ([9], p. 60) \quad \ldots (7.4.1)
\]

where

\[
M(R_j) = \max_{||z|| = R_j} |f(z_1, z_2)| \quad (j = 1, 2, 3), \quad (0 < R_1 < R_2 < R_3 < 1)
\]

and

\[
\alpha_1 = \log \frac{R_3}{R_2}, \quad \alpha_2 = \log \frac{R_3}{R_1}, \quad \alpha_3 = \log \frac{R_2}{R_1}.
\]

Now we prove,
**Theorem 7.5** \[ \frac{M(kr^{**})}{M(r^{**})} \leq \frac{M(kr')}{M(r')} \, , \text{ for } r' < kr^{**} , 0 < k < 1. \]

**Proof.** Take four hyperspheres \( \| z \| = kr' \), \( \| z \| = r' \), \( \| z \| = kr^{**} \), \( \| z \| = r^{**} \) such that
\[ 0 < kr' < r' < kr^{**} < r^{**}, 0 < k < 1. \]

Applying (7.4.1), we get
\[
\left\{ \frac{M(r')}{M(kr')} \right\} \log \left( \frac{r^{**}}{r'} \right) \leq \left\{ \frac{M(kr^{**})}{M(kr')} \right\} \log \left( \frac{1}{k} \right).
\]
and
\[
\left\{ \frac{M(kr^{**})}{M(r')} \right\} \log \left( \frac{r'}{r^{**}} \right) \leq \left\{ \frac{M(r^{**})}{M(r')} \right\} \log \left( \frac{kr^{**}}{r'} \right).
\]

Multiplying these two and simplifying, we get
\[
\left\{ \frac{M(kr^{**})}{M(kr')} \right\} \log \left( \frac{r^{**}}{r'} \right) = \log \left( \frac{1}{k} \right) \leq \left\{ \frac{M(r^{**})}{M(r')} \right\} \log \left( \frac{kr^{**}}{r'} \right)
\]
or
\[
\frac{M(kr^{**})}{M(r^{**})} \leq \frac{M(kr')}{M(r')}.
\]

This proves the theorem.

**References**

Bochner and Martin [9]; Džrbašyan [18]; Goldberg [20]; Singh [89].

***
CHAPTER VIII

ON THE CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION

8.1 INTRODUCTION. This chapter is devoted to the study of the rate of growth of Ahlfors - Shimizu characteristic function $T_0(r)$ of a meromorphic function $f(z)$ and the area function $S(r)$ of the image of the disc $|z| < r$ on the Riemann sphere under the mapping $f(z)$. In section 8.2, we have obtained some results by using a general kind of comparison function which on specialization gives the results of S.K. Singh ([90], p.10 part 2). In section 8.3, we have studied the comparative growth of $T_0(r)$ and $S(r)$ functions. Towards the end of this chapter we have established a result on simultaneous convergence or divergence of two integrals involving $T_0(r)$ and $S(r)$, respectively.

Let $f(z)$ be meromorphic and non-constant in the open complex plane. Following Hayman [25], p.11], we write

$$m_0(r,a) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{k(f(re^{i\theta}),a)} d\theta , \quad z = re^{i\theta}$$

where $k(w,a)$ denotes the chordal distance on the Riemann sphere of the two points which project to $w$ and $a$, and

$$S(r,f) = S(r) = \left(\frac{1}{\pi}\right) \int_0^{2\pi} \int_0^r \frac{|f'(te^{i\theta})|^2}{(1+|f(re^{i\theta})|^2)^2} t dt d\theta .$$

$\int S(r)$ gives, with due regard to the multiplicity, the area of image of $|z| < r$ onto the Riemann sphere under the mapping $f(z)$. $S(r) \equiv 0$, if and only if $f(z)$ is constant, otherwise $S(r)$ is non-negative, non-decreasing and continuous.
function of \( r \) \([88]\). This being so, \( S(r) \) represents the behaviour of the growth of \( f(z) \) on \( |z| = r \). In fact, if \( n(r,a) \) denotes the number of \( a \)-points of \( f(z) \) in \( |z| \leq r \), then \( S(r,f) \) is the average value of \( n(r,a) \) as \( a \) moves over the Riemann sphere.

Below is the statement of the first fundamental theorem of Nevanlinna in the form of Ahlfors \([2]\) and Shimizu \([88]\).

"Suppose that \( f(z) \) is meromorphic in the open disc \( |z| < R, 0 < R < \infty \). Then for every \( a \), finite or infinite and \( 0 < r < R \), we have

\[
\int_0^r \frac{S(t)}{t} \, dt = N(r,a) + m_0(r,a) - m_0(0,a) \quad \text{[25]} \quad \ldots (8.1.1)
\]

provided \( f(0) \neq a \) ".

The integral in the left side of (8.1.1) is called the Ahlfors-Shimizu characteristic function of \( f(z) \) and subsequently will be denoted as \( T_0(r,f) = T_0(r) \). Nevanlinna ([51], p.167) calls \( T_0(r,f) " \text{spherical normal} " \) form of his characteristic function \( T(r,f) \). For non-constant meromorphic functions, it is clear that \( T_0(r) \) is a positive, strictly increasing and convex function of \( \log r \). Also it can easily be shown ([25], p.13) that \( T(r,f) \) and \( T_0(r,f) \) differ by a bounded function of \( r \).

By fairly elementary methods, it can be shown that for a meromorphic function of order \( \rho \) and lower order \( \gamma \), we have

\[
\lim_{r \to \infty} \sup_{\rho(r)} \frac{\log T_0(r)}{\log r} = \rho = \lim_{r \to \infty} \sup_{\gamma(r)} \frac{\log S(r)}{\log r}
\]

\([87],[88]\) \ldots (8.1.2)
Now, as $0 < \rho < \infty$, following Shah [74], we define $f(t)$ to be proximate order of $f(z)$, having the following properties:

(i) $\phi(r)$ is real continuous and piecewise differentiable for $r > r_0$.

(ii) $\phi(r) = r$ as $r \to \infty$.

(iii) $r \phi(r) \log r \to 0$ as $r \to \infty$.

8.2 GENERALISATION OF SOME RESULTS OF S.K. SINGH.

If $0 < \rho < \infty$,

\[
\alpha = \lim_{r \to \infty} \sup \frac{T_0(r)}{r^\rho}, \quad \nu = \lim_{r \to \infty} \inf \frac{S(r)}{r^\rho},
\]

S.K. Singh ([90], p. 10 Part 2) has established the following results:

(i) $\delta \leq \rho \beta \leq \delta(1 + \log \frac{\nu}{\delta})$.

(ii) $0 \leq \frac{\nu}{\delta} \leq \rho \alpha \leq \nu$.

(iii) $\nu + \delta \leq e^{\rho \alpha}$.

Here we establish the results which include the conclusions of S.K. Singh [ibid] as special case. We set

\[
A(r) = \exp \left( \int_0^r \frac{P(x)}{x} \, dx \right), \quad B(r) = \int_0^r \frac{f(x)S(x)}{x} \, dx, \quad (\delta > 0)
\]

\[
a = \lim_{r \to \infty} \sup \frac{B(r)}{A(r)}, \quad b = \lim_{r \to \infty} \inf \frac{a}{A(r)}, \quad c = \lim_{r \to \infty} \sup \frac{S(r)}{A(r)}, \quad d = \lim_{r \to \infty} \inf \frac{S(r)}{A(r)}
\]

and prove.
THEOREM 8.1 If $0 < f < \infty$, then

\[
\begin{align*}
    a &\leq c, & \cdots & \text{(8.2.1)} \\
    b &\leq d(1 + \log \frac{c}{d}), & \cdots & \text{(8.2.2)} \\
    b &> d, & \cdots & \text{(8.2.3)} \\
    a &> \frac{d}{e} e^{c}, & \cdots & \text{(8.2.4)}
\end{align*}
\]

To prove the theorem the following immediate lemma is required.

LEMMA 8.1 For every finite $k > 0$

\[
\frac{A(r+kr)}{A(r)} \to (1+k)^{f}, \text{ as } r \to \infty.
\]

PROOF. We have

\[
\log \left\{ \frac{A(r+kr)}{A(r)} \right\} = \int_{r}^{r+kr} \frac{f(x) \log x}{x} dx,
\]

But

\[
\left| \int_{r}^{r+kr} f(x) \log x \ dx \right| < \varepsilon \log(1+k), \quad r > r_0(\varepsilon).
\]

Hence for sufficiently large $r$

\[
\log \left\{ \frac{A(r+kr)}{A(r)} \right\} = \log \left[ \frac{(1+k)^{r}}{r^{f(r)}} \right] + o(1).
\]

Therefore,

\[
\frac{A(r+kr)}{A(r)} \to (1+k)^{f}, \text{ as } r \to \infty \text{ by (1.6) [19]}.
\]
**Proof of Theorem 8.1.** For every finite \( k \geq 0 \) and \( d < \infty \)

\[
B(r+kr) = O(1) + \int_0^r \frac{S(x) f(x)}{x} dx + \int_r^\infty \frac{S(x) f(x)}{x} dx,
\]

\[\cdots \text{(8.2.5)}\]

\[
> O(1) + \int_0^r \frac{S(x)}{A(x)} A'(x) dx + \int_r^\infty \frac{S(x)}{A(x)} A'(x) dx,
\]

\[\cdots \text{(8.2.5)}\]

\[
> O(1) + (d-c) \int_0^r A'(x) dx + S(r) \int_r^\infty \frac{A'(x)}{A(x)} dx.
\]

Therefore,

\[
\frac{B(r+kr)}{A(r+kr)} > O(1) + (d-c) \frac{A(r)}{A(r+kr)} + S(r) \frac{A(r)}{A(r+kr)} \log \left\{ \frac{A(r+kr)}{A(r)} \right\}.
\]

Hence, we get

\[
a > \frac{d}{(1+k)^\rho} + \frac{c \rho}{(1+k)^\rho} \log (1+k), \quad \cdots \text{(8.2.6)}
\]

\[
b > \frac{d}{(1+k)^\rho} + \frac{d \rho}{(1+k)^\rho} \log (1+k), \quad \cdots \text{(8.2.7)}
\]

On the other hand (8.2.5) gives for \( c < \infty \),

\[
\frac{B(r+kr)}{A(r+kr)} < O(1) + \frac{(c+\epsilon)A(r)}{A(r+kr)} + \frac{S(r+kr)}{A(r+kr)} \log \left\{ \frac{A(r+kr)}{A(r)} \right\}.
\]

Hence,

\[
a \leq \frac{c}{(1+k)^\rho} + \int c \log (k+1) \quad \cdots \text{(8.2.8)}
\]

and

\[
b \leq \frac{c}{(1+k)^\rho} + \int d \log (1+k) \quad \cdots \text{(8.2.9)}
\]

which also hold when \( c = \infty \).
It can be seen that the minima of the right hand expressions in (8.2.8) and (8.2.9) occur at \( k = 0 \) and \( k = (\frac{c}{d})^{\frac{1}{k}} - 1 \) respectively. Substituting \( k = 0 \) in (8.2.8) and \( k = (\frac{c}{d})^{\frac{1}{k}} - 1 \) in (8.2.9), we get (8.2.1) and (8.2.2) respectively.

The right hand sides of (8.2.6) and (8.2.7) are maximum when \( k = \exp\left\{\left(\frac{c-d}{p_c}\right)\right\} - 1 \) and \( k = 0 \) respectively. On substitution, these values give us (8.2.4) and (8.2.3). Thus, we have completed the proof of the theorem.

### 8.3 COMPARATIVE GROWTH OF \( T_0(r) \) AND \( S(r) \).

**THEOREM 8.2** For a meromorphic function \( f(z) \) of order \( \rho \) and lower order \( \lambda \), we have

\[
\limsup_{r \to \infty} \frac{T_0(r)}{S(r) \log r} \leq (1 - \frac{\lambda}{\rho}).
\]

... (8.3.1)

**PROOF.** When \( \lambda = 0 \) or \( \rho = \infty \) (i.e. \( \rho^{-1} = 0 \)), the result, to be proved is obvious. Hence we suppose that \( \lambda > 0 \), \( \rho < \infty \) and deduce from (8.1.1),

\[
\frac{T_0(r)}{S(r) \log r} = 0(1) + 1 - L(r). M(r) \quad (r \to \infty),
\]

... (8.2.2)

where

\[
L(r) = \int_{r_0}^{r} \log x \, ds(x) / \int_{r_0}^{r} (\log S(x)) \, ds(x)
\]

\[
M(r) = \int_{r_0}^{r} (\log S(x)) \, ds(x) / S(r) \log r,
\]
\[
\frac{S(r) \log s(r) - S(r) + \text{a constant}}{S(r) \log r}
\]

Now,
\[
\liminf L(r) \geq \liminf \frac{\log r}{\log S(r)} = \frac{1}{\beta},
\]
and
\[
\liminf M(r) = \liminf \frac{\log S(r)}{\log r} = \gamma.
\]

Substituting these values in (8.3.2), we get (8.3.1).

Thus we have completed the proof of the Theorem.

8.4 A THEOREM ON CONVERGENCE AND DIVERGENCE.

Towards the end of this chapter, we wish to prove the following theorem.

**THEOREM 8.3** The integrals
\[
I_1 = \int_{r_0}^{\infty} \frac{T_0(x)}{x (\log x)^{j+1}} \, dx
\]
and
\[
I_2 = \int_{r_0}^{\infty} \frac{S(x)}{x (\log x)^{j}} \, dx \quad (j > 0)
\]
converge or diverge together.

**PROOF.** We have
\[
\int_{r_0}^{r} \frac{S(x)}{x} \, dx = T_0(r) - T_0(r_0), \quad r > r_0.
\]

Hence,
\[
\int_{\frac{t}{r_0}}^{r} \frac{\mathrm{S}(x)}{x} \, dx = \int_{\frac{t}{r_0}}^{r} \left[ \frac{T_o(r) - T_o(r_0)}{r \log r} \right] \frac{dr}{r \log r} \frac{1}{j+1},
\]

\[
= \int_{\frac{t}{r_0}}^{r} \frac{T_o(r) - T_o(r_0)}{-j \log r} \frac{dr}{r \log r} \frac{1}{j+1} + \frac{1}{j} \int_{\frac{t}{r_0}}^{r} \frac{T_o(r)}{r \log r} \frac{dr}{r \log r} \frac{1}{j+1},
\]

\[
= \frac{T_o(t) - T_o(r_0)}{-j \log t} + \frac{1}{j} \int_{\frac{t}{r_0}}^{r} \frac{T_o(r)}{r \log r} \frac{dr}{r \log r} \frac{1}{j+1}.
\]

Also,

\[
\int_{\frac{t}{r_0}}^{r} \left\{ T_o(x) - T_o(r_0) \right\} \frac{1}{r \log r} \frac{dr}{r \log r} \frac{1}{j+1} = \int_{\frac{t}{r_0}}^{r} \frac{T_o(r)}{r \log r} \frac{dr}{r \log r} \frac{1}{j+1} - \int_{\frac{t}{r_0}}^{r} \frac{T_o(r_0)}{r \log r} \frac{dr}{r \log r} \frac{1}{j+1},
\]

\[
= \int_{\frac{t}{r_0}}^{r} \frac{T_o(r)}{r \log r} \frac{dr}{r \log r} + \int_{\frac{t}{r_0}}^{r} \frac{T_o(r_0)}{r \log r} \frac{dr}{r \log r} \frac{1}{j} \left[ \frac{1}{(\log t)^j} - \frac{1}{(\log r_0)^j} \right].
\]

Therefore,

\[
\int_{\frac{t}{r_0}}^{r} \frac{T_o(x)}{x \log x} \frac{dx}{x \log x} + \int_{\frac{t}{r_0}}^{r} \frac{T_o(r_0)}{j} \left[ \frac{1}{(\log r)^j} - \frac{1}{(\log r_0)^j} \right] = \frac{T_o(r) - T_o(r_0)}{-j \log r} + \frac{1}{j} \int_{\frac{t}{r_0}}^{r} \frac{T_o(r_0)}{x \log x} \frac{dx}{x \log x}.
\]

and so

\[
\int_{\frac{t}{r_0}}^{r} \frac{T_o(x)}{x \log x} \frac{dx}{x \log x} + \int_{\frac{t}{r_0}}^{r} \frac{T_o(r)}{x \log x} \frac{dr}{x \log x} \frac{1}{j} \left[ \frac{1}{(\log r)^j} - \frac{1}{(\log r_0)^j} \right] = \int_{\frac{t}{r_0}}^{r} \frac{T_o(r_0)}{x \log x} \frac{dx}{x \log x}.
\]

\[\text{(8.4.1)}\]
Let us now suppose that $I_1$ is convergent, then

$$\epsilon > \int_{r}^{r_0} \frac{T_0(x)}{x(\log x)^{j+1}} dx > \frac{T_0(r)}{(\log r)^{j}} \{1 - 2^{-j}\}$$

$\epsilon$ is an arbitrary chosen positive number, however small and so we have

$$\frac{T_0(r)}{(\log r)^{j}} \to 0 \text{ as } r \to \infty.$$ 

Therefore, we have

$$j I_1 + H = I_2,$$  

$H$ being a constant.

From this the convergence of $I_2$ follows.

On the other hand if $I_2$ is convergent then, from (8.4.1) we get

$$r \int_{r_0}^{r} \frac{T_0(x)}{x(\log x)^{j+1}} dx + \frac{T_0(x)}{(\log r)^{j}} < K, \quad ...(8.4.2)$$

$K$ being some constant and since,

$$r \int_{r_0}^{r} \frac{T_0(x)}{x(\log x)^{j+1}} dx > T_0(r_0) \frac{1}{j} \left\{ (\log r_o)^{-j} - (\log r)^{-j}\right\} > 0,$$

both the terms of the left hand side of (8.4.2) are positive and this ensures the convergence of $I_1$.

**PROOF FOR DIVERGENCE.** Suppose $I_1$ is divergent, choose $\beta, 0 < \beta < 1$ then for large $r$ and arbitrary large constant $K$, we have

$$K < \int_{r_0}^{r} \frac{T_0(x)}{x(\log x)^{j+1}} dx < \frac{T_0(r)}{(\log r)^{j}} \left\{ \frac{1}{\beta^j} - 1 \right\}, \quad j > 0.$$
and so,

\[ \frac{T_o(r)}{(\log r)^j} > K, \quad r > r_0. \]

Therefore, from (8.4.1) we see that \( I_2 > K \). This ensures the divergence of \( I_2 \). Now, suppose that \( I_2 \) is divergence. Then for large \( r \)

\[ \int_{r_0}^{r} \frac{T_o(x)}{x(\log x)^{j+1}} \, dx + \frac{T_o(r)}{(\log r)^j} > K. \quad \ldots (8.4.3) \]

We now, say that \( I_1 \) is divergent if \( I_2 \) is so, for \( I_1 \) is convergent then

\[ \int_{r_0}^{r} \frac{T_o(x)}{x(\log x)^{j+1}} \, dx < \infty \]

and

\[ \frac{T_o(x)}{(\log x)^{j+1}} \]

would have become arbitrary small for large \( r \).

This leads to a contradiction in view of (8.4.3). Hence the theorem is completely established.

REFERENCES

AHLFORS [2]; DWIVEDI [19]; HAYMAN [25]; NEVANLINNA [51]; SHAH [74]; SHANKAR [87]; SHIMIZU [88]; SINGH [90].

***
BIBLIOGRAPHY


[74] SHAH, S.M. : On proximate orders of Integral functions,
    Bull. Amer. Math. Soc. 52(1946), 326-328.

[75] _________ : On the lower order of Integral functions,

[76] _________ : A note on lower proximate orders, J. Indian

[77] _________ : The maximum term of an entire series III,

[78] _________ : The maximum term of entire series IV, Quart.

[79] _________ : The maximum term of an entire series V,
    J. Indian Math. Soc. (N.S) 13(1949), 60-64.

[80] _________ : On the coefficients of an entire series of


[82] _________ : A note on entire functions of perfectly regular
    growth, Math. Z. 56(1952), 254-257.

[83] _________ : On entire functions of infinite order, Arch.

    of an entire series, J. Indian Math. Soc. 16(1952), 177-182.


[90] __________: Exceptional values of entire and meromorphic functions and maximum term of an entire function, Thesis for Ph.D., Aligarh Muslim University (1953).


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APPENDIX
Rivista di Matematica
della Università di Parma
On the Order and Type of Entire Functions of Several Complex Variables. (**)

1. - Introduction.

In this paper we shall obtain the relations among entire functions of finite non-zero orders and types and study the relations among the coefficients in the Taylor expansion of entire functions and their orders and types. For simplicity we confine our selves to the case of two complex variables. The case of an arbitrary finite number of variables is examined in the same way.

Let

\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n \]

be a function of two complex variables \( z_1 \) and \( z_2 \), where the coefficients \( a_{m,n} \) are complex numbers. The series (1.1) represents an entire function of two complex variables \( z_1, z_2 \) if it converges absolutely for all values of \( |z_1| < \infty \) and \( |z_2| < \infty \).

M. M. DŽERBAŞYAN ([1], p. 1) has shown that the necessary and sufficient condition for the series (1.1) to represent an entire function of variables \( z_1 \) and \( z_2 \) is

\[ \limsup_{m+n \to \infty} |a_{m,n}|^{1/(m+n)} = 0. \]

Let \( \mathcal{G} \) be a family of closed polycircular domains in space \( (z_1, z_2) \) depen-
dent on parameter \( r > 0 \) and possess the property that \((z_1, z_2) \in G\), if and only if \((z_1/r, z_2/r) \in \overline{G}\). The maximum modulus of the entire function \(f(z_1, z_2)\) is denoted by

\[ M_\phi(r, f) = \max_{(a_1, a_2) \in \phi} |f(z_1, z_2)|, \]

and the function will be called \(G\)-order and \(G\)-type respectively, if

\[ \alpha_o := \limsup_{r \to \infty} \frac{\log \log M_\phi(r, f)}{\log r}, \]

\[ \gamma_0 = \limsup_{r \to \infty} \frac{\log M_\phi(r, f)}{r^\phi}. \]

Set

\[ \Phi = \Phi_\phi(m, n) = \max \{ |z_1|^m |z_2|^n \}, \]

A. A. Goldberg ([2], p. 146) has proved the following theorems:

**Theorem A.** All orders \( \gamma_o \) be equal and

\[ \alpha = \gamma_o = \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log (1/|a_{m+n}|)}. \]

**Theorem B.** \( G\)-type \( T_\phi \) satisfies the correlation

\[ (e \varrho T_\phi)^{1/\phi} = \limsup_{m+n \to \infty} [(m+n)^{1/\phi} \{\Phi_\phi(m, n) | a_{m+n}|^{1/(m+n)}\}], \]

or, by (1.5),

\[ (e \varrho T_\phi)^{1/\phi} = \limsup_{m+n \to \infty} [(m+n)^{1/\phi} \{\Phi_\phi | a_{m+n}|^{1/(m+n)}\}]. \]

2. **Theorem 1.** Let

\[ f_s(z_1, z_2) = \sum_{m,n=0}^\infty (a_{m,n})_k \overline{z_1}^m \overline{z_2}^n \]

be \( p \) entire functions of finite non-zero orders \( \theta_1, \theta_2, \ldots, \theta_p \) respectively. Then
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the function

\[ f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{m,n} z_1^m z_2^n, \]

where

\[ \{\log (1/|a_{m,n}|)\}^{-1} \sim \sum_{k=1}^{p} \alpha_k \{\log (1/|a_{m,n}|)\}^{-1} \]

\(0 < \alpha_k < 1, \quad \alpha_1 + \alpha_2 + \ldots + \alpha_p = 1\),

is an entire function such that

\[ q < \sum_{k=1}^{p} \alpha_k q_k \quad [q is the order of f(z_1, z_2)].\]

Proof. Since \(f(z_1, z_2)\) is an entire function. Therefore, using (1.2), we have, for an arbitrary \(\varepsilon\) and enough large \(R\),

\[ 1/|a_{m,n}| > (R - \varepsilon)^{m+n} \quad for \ m + n > k, \]

or

\[ \log \{|a_{m,n}|\} > (m + n) \log(R - \varepsilon) \quad for \ m + n > k, \]

or

\[ \alpha_k \{\log (1/|a_{m,n}|)\}^{-1} \leq \alpha_k \{(m + n) \log(R - \varepsilon)\} \quad for \ m + n > k. \]

Putting \(k = 1, 2, \ldots, p\) and adding the \(p\) inequalities thus obtained, we get, for large \(m + n, \)

\[ \sum_{k=1}^{p} \alpha_k \{\log (1/|a_{m,n}|)\}^{-1} \leq (\alpha_1 + \alpha_2 + \ldots + \alpha_p) / \{(m + n) \log(R - \varepsilon)\} \]

\[ [m + n > k = \max (k_1, k_2, \ldots, k_p)]. \]

Thus, if

\[ \{\log (1/|a_{m,n}|)\}^{-1} \sim \sum_{k=1}^{p} \alpha_k \{\log (1/|a_{m,n}|)\}^{-1}, \quad \sum_{k=1}^{p} \alpha_k = 1. \]
Then, for large $m + n$, 

\[ \{ \log \left( 1/|a_{m,n}| \right) \}^{-1} < 1/\{(m + n) \log(R - \varepsilon)\} \quad \text{for } m + n > k, \]

or

\[ \log \left( 1/|a_{m,n}| \right) > (m + n) \log(R - \varepsilon) \quad \text{for } m + n > k, \]

or

\[ \limsup_{m+n \to \infty} |a_{m,n}|^{1/(m+n)} = 0. \]

Hence $f(z_1, z_2)$ is an entire function.

Now using (1.6) for the function $j_k(z_1, z_2)$, we have

\[ \limsup_{m+n \to \infty} \frac{(m + n) \log(m + n)}{\log \left( 1/|a_{m,n}| \right)} = \varrho_k. \]

Therefore, for an arbitrary $\varepsilon$ we get

\[ \alpha_x/\{ \log \left( 1/|a_{m,n}| \right) \} < \alpha_x (\varrho_k + \varepsilon) \{(m + n) \log(m + n)\}^{-1} \quad \text{for } m + n > k_x, \]

or

\[ \alpha_x \{ \log \left( 1/|a_{m,n}| \right) \}^{-1} < \alpha_x (\varrho_k + \varepsilon) \{(m + n) \log(m + n)\}^{-1} \quad \text{for } m + n > k_x. \]

Putting $k = 1, 2, \ldots, p$ and adding the $p$ inequalities thus obtained, we get

\[ \sum_{k=1}^{p} \alpha_x \{ \log \left( 1/|a_{m,n}| \right) \}^{-1} < \sum_{k=1}^{p} \alpha_x (\varrho_k + \varepsilon) \{(m + n) \log(m + n)\}^{-1}, \]

\[ [m + n > k = \max (k_1, k_2, \ldots, k_p)]. \]

Since

\[ \{ \log \left( 1/|a_{m,n}| \right) \}^{-1} \sim \sum_{k=1}^{p} \alpha_k \{ \log \left( 1/|a_{m,n}| \right) \}^{-1}, \]

and we have

\[ \limsup_{m+n \to \infty} \frac{(m + n) \log(m + n)}{\log \left( 1/|a_{m,n}| \right)} \leq \sum_{k=1}^{p} \alpha_k \varrho_k. \]
Hence we get

\[ q \leq \sum_{k=1}^{p} x_k q_k , \]

where \( q \) is the order of \( f(z_1, z_2) \).

3. - Theorem 2. Let

\[ f_k(z_1, z_2) = \sum_{m, n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n \quad (k = 1, 2, \ldots, p) \]

be \( p \) entire functions of finite non-zero orders \( q_1, q_2, \ldots, q_p \) respectively. Then the function

\[ f(z_1, z_2) = \sum_{m, n=0}^{\infty} \alpha_{m,n} z_1^m z_2^n , \]

where

\[ \log (1/|a_{m,n}|) \sim \prod_{k=1}^{p} \left\{ \log (1/|a_{m,n}|) \right\}^{x_k} \quad (0 < x_k < 1, \sum_{k=1}^{p} x_k = 1) , \]

is an entire function such that

\[ q \leq \prod_{k=1}^{p} q_k^{x_k} , \]

where \( q \) is the order of \( f(z_1, z_2) \).

Proof. Since \( f_k(z_1, z_2) \) is an entire function, therefore, using (1.2), we have for arbitrary \( \varepsilon \) and enough large \( R \)

\[ 1/|a_{m,n}| < (R - \varepsilon)^{m+n} \quad \text{for} \ m + n > k_k , \]

or

\[ \log (1/|a_{m,n}|) > (m + n) \log(R - \varepsilon) \quad \text{for} \ m + n > k_k , \]

or

\[ \left\{ \log (1/|a_{m,n}|) \right\}^{x_k} > (m + n)^{x_k} (\log(R - \varepsilon))^{x_k} \quad \text{for} \ m + n > k_k . \]
Putting \( k = 1, 2, \ldots, p \) and multiplying the \( p \) inequalities thus obtained, we get, for large \( m + n \),

\[
\prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{\alpha_k} > (m + n) \log(R - \varepsilon).
\]

Therefore, if

\[
\log \left( \frac{1}{|a_{m,n}|} \right) \sim \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{\alpha_k},
\]

then, for large \( m + n \),

\[
\log \left( \frac{1}{|a_{m,n}|} \right) > (m + n) \log(R - \varepsilon).
\]

Hence

\[
\limsup_{m+n \to \infty} \log \left( \frac{1}{|a_{m,n}|} \right) = 0,
\]

and \( f(z_1, z_2) \) is an entire function.

Now using (1.6) for the function \( f_{(z_1, z_2)} \), we have

\[
\limsup_{m+n \to \infty} \frac{(m + n) \log(m + n)}{\log \left( \frac{1}{|a_{m,n}|} \right)} = q_k.
\]

Therefore, for an arbitrary \( \varepsilon \) we get

\[
\frac{1}{\left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}} < (q_k + \varepsilon) \left\{ (m + n) \log(m + n) \right\}^{-1} \quad \text{for } m + n > k,
\]

or

\[
\left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{-\alpha_k} < (q_k + \varepsilon)^{\alpha_k} \left\{ (m + n) \right\}^{\alpha_k} \left\{ \log (m + n) \right\}^{-\alpha_k} \quad \text{for } m + n > k.
\]

Putting \( k = 1, 2, \ldots, p \) and multiplying the \( p \) inequalities thus obtained, we get, for large \( m + n \),

\[
\prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|a_{m,n}|} \right) \right\}^{-\alpha_k} < \prod_{k=1}^{p} (q_k + \varepsilon)^{\alpha_k} \left\{ (m + n) \log(m + n) \right\}^{-1},
\]

or

\[
\limsup_{m+n \to \infty} \left( \prod_{k=1}^{p} \log \left( \frac{1}{|a_{m,n}|} \right) \right)^{\alpha_k} < \prod_{k=1}^{p} q_k^{\alpha_k}.
\]
Thus, if
\[ \log \left( \frac{1}{|a_{m,n}|} \right) \sim \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|(a_{m,n})_k|} \right) \right\}^{\alpha_k}, \]
then
\[ \limsup_{m+n \to \infty} \frac{(m + n) \log(m + n)}{-\log |a_{m,n}|} \leq \prod_{k=1}^{p} \varrho_k^n. \]
Hence
\[ \varrho \leq \prod_{k=1}^{p} \varrho_k^n, \]
where \( \varrho \) is the order of \( f(z_1, z_2) \).

4. - Theorem 3. Let
\[ f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n \]
be \( p \) entire functions of finite non-zero orders \( \varrho_1, \varrho_2, \ldots, \varrho_p \) and finite non-zero types \( (\nu_1), (\nu_2), \ldots, (\nu_p) \) respectively. Then the function
\[ f(z_1, z_2) = \sum_{m,n=0}^{\infty} \varrho_{m,n} z_1^m z_2^n, \]
where
\[ \log \left( \frac{1}{|\varphi a_{m,n}|} \right) \sim \prod_{k=1}^{p} \left\{ \log \left( \frac{1}{|\varphi \cdot (a_{m,n})_k|} \right) \right\}^{\alpha_k} \]
\[ (0 < \alpha_k < 1, \quad \sum_{k=1}^{p} \alpha_k = 1), \]
is an entire function such that
\[ T \leq \prod_{k=1}^{p} (T_k)^{\alpha_k}, \]
where \( \varrho \) and \( T \) are the order and type of \( f(z_1, z_2) \) respectively provided
\[ \varrho = \prod_{k=1}^{p} \varrho_k^{\alpha_k}. \]

\( (\nu) \) The types \( T_1, T_2, \ldots, T_p \) correspond the same family of polycircular domains \( \mathcal{G}_\nu \).
Proof. It is easy to prove that \( f(z_1, z_2) \) is an entire function. Further, using (1.8), for the function \( f(\varepsilon_1, \varepsilon_2) \) we have

\[
\limsup_{m+n \to \infty} (m+n)^{1/\varepsilon_k} \left\{ \Phi \left| (a_{m,n}) \right|^{1/(m+n)} \right\} = (e \varrho_k T_k)^{1/\varepsilon_k}.
\]

For arbitrary \( \varepsilon_k > 0 \) and sufficiently large \( m + n \), we have

\[
(m+n)^{1/\varepsilon_k} \left\{ \Phi \left| (a_{m,n}) \right|^{1/(m+n)} \right\} < (e \varrho_k (T_k + \varepsilon_k))^{1/\varepsilon_k}
\]

for \( m + n > k \).

Put \( e \varrho_k (T_k + \varepsilon_k) = A_k \),

\[
(m+n)^{1/\varepsilon_k} \left\{ \Phi \left| (a_{m,n}) \right|^{1/(m+n)} \right\} < (A_k)^{1/\varepsilon_k}
\]

or

\[
\log \frac{1}{\Phi \left| (a_{m,n}) \right|} > \frac{m+n}{\varrho_k} \log \frac{m+n}{A_k}
\]

for \( m + n > k \), or

\[
\left\{ \log \frac{1}{\Phi \left| (a_{m,n}) \right|} \right\}^{\varepsilon_k} > \frac{(m+n)^{\varepsilon_k}}{\varrho_k^{\varepsilon_k}} \left( \log \frac{m+n}{A_k} \right)^{\varepsilon_k}
\]

for sufficiently large \( m + n \).

Putting \( k = 1, 2, ..., p \) and multiplying \( p \) inequalities thus obtained, we have

\[
\prod_{k=1}^{p} \left\{ \log \frac{1}{\Phi \left| (a_{m,n}) \right|} \right\}^{\varepsilon_k} > \prod_{k=1}^{p} \frac{m+n}{\varrho_k} \left( \log \frac{m+n}{A_k} \right)^{\varepsilon_k}
\]

for sufficiently large \( m + n > k \), or

\[
\prod_{k=1}^{p} \left\{ \log \frac{1}{\Phi \left| (a_{m,n}) \right|} \right\}^{\varepsilon_k} > \frac{m+n}{\varrho} \prod_{k=1}^{p} \left\{ \log \frac{m+n}{A_k} \right\}^{\varepsilon_k}
\]

for \( m + n > k \)

\[
> \frac{m+n}{\varrho} \prod_{k=1}^{p} \left( 1 - \frac{\log A_k}{\log (m+n)} \right)^{\varepsilon_k} \log(m+n)
\]

\[
> \frac{m+n}{\varrho} \left[ 1 - \frac{\log A_k^{\varepsilon_k}}{\log (m+n)} + O((\log (m+n))^{-1}) \right] \log(m+n)
\]

for \( m + n > k \).
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Thus, if
\[
\log \frac{1}{|\phi|_{a_{m,n}}^n} \sim \prod_{k=1}^{p} \left\{ \log \frac{1}{|\phi|_{(a_{m,n})_k}^n} \right\}^{\sigma_k} \quad (0 < \sigma_k < 1, \sum_{k=1}^{p} \sigma_k = 1),
\]

we obtain, for sufficiently large \(m + n\),
\[
\log \frac{1}{|\phi|_{a_{m,n}}^n} > \frac{m + n}{\theta} \left[ 1 - \sum_{k=1}^{p} \frac{\log A_k}{\log (m + n)} + O\left(\log (m + n)^{-2}\right) \right] \log (m + n),
\]
or
\[
(m + n) \left\{ \phi|_{a_{m,n}}^n \right\}^{\sigma/(m+n)} < (m + n)^\beta
\]
with
\[
\beta = \sum_{k=1}^{p} \frac{\log A_k^{\sigma_k}}{\log (m + n)} + O\left(\log (m + n)^{-2}\right).
\]

Since
\[
\lim_{m+n \to \infty} (m + n)^\beta = \prod_{k=1}^{p} A_k^{\sigma_k}, \quad \text{where} \quad A_k = e^{\theta_k} T_k,
\]
we obtain
\[
\lim \sup_{m+n \to \infty} \left\{ (m + n) \left\{ \phi|_{a_{m,n}}^n \right\}^{\sigma/(m+n)} \right\} \leq \prod_{k=1}^{p} \left( e^{\theta_k} T_k \right)^{\sigma_k}.
\]

Hence
\[
e^{\theta} T \leq \prod_{k=1}^{p} \left( e^{\theta_k} T_k \right)^{\sigma_k},
\]
or
\[
T \leq \prod_{k=1}^{p} (T_k)^{\sigma_k}, \quad \text{where} \ T \text{ is type of } f(z_1, z_2).
\]

5. - Theorem 4. Let
\[
f_k(z_1, z_2) = \sum_{a_{m,n} \geq 0} (a_{m,n})_k z_1^m z_2^n \quad (k = 1, 2, \ldots, p)
\]
be \( p \) entire functions of finite non-zero orders \( q_1, q_2, \ldots, q_p \), and finite non-zero types \( T_1, T_2, \ldots, T_p \) respectively. Then the function

\[
f(z_1, z_2) = \sum_{m, n=0}^{\infty} (a_{m,n})_1^{\alpha_1} (a_{m,n})_2^{\alpha_2} \cdots (a_{m,n})_p^{\alpha_p} z_1^m z_2^n
\]

\((0 < \alpha_1 < 1, \alpha_1 + \alpha_2 + \cdots + \alpha_p = 1)\)

is an entire function such that \((e T)^{1/q} < \prod_{k=1}^{p} (e q_k T_k)^{\gamma_k/q_k}\), where \( q \) and \( T \) are the order and type of \( f(z_1, z_2) \) respectively provided \( \frac{1}{q} = \sum_{k=1}^{p} \alpha_k/q_k \).

**Proof.** It is easy to prove that \( f(z_1, z_2) \) is an entire function. Using (1.8) for the function \( f(z_1, z_2) \), we have

\[
\lim_{m + n \to \infty} \sup \left[ (m + n)^{1/q_k} \left\{ \Phi \left( a_{m,n} \right) \right\}^{1/(m+n)} \right] = (e q_k T_k)^{1/q_k},
\]

or

\[
\lim_{m + n \to \infty} \sup \left[ (m + n)^{1/q_k} \left\{ \Phi \left( a_{m,n} \right) \right\}^{1/(m+n)} \right]^{\gamma_k/q_k} = (e q_k T_k)^{\gamma_k/q_k}.
\]

Putting \( k = 1, 2, \ldots, p \) and multiplying \( p \) inequalities thus obtained, we get

\[
\prod_{k=1}^{p} \lim_{m + n \to \infty} \sup \left[ (m + n)^{1/q_k} \left\{ \Phi \left( a_{m,n} \right) \right\}^{1/(m+n)} \right]^{\gamma_k/q_k} = \prod_{k=1}^{p} (e q_k T_k)^{\gamma_k/q_k},
\]

or

\[
\lim_{m + n \to \infty} \prod_{k=1}^{p} \left[ (m + n)^{1/q_k} \left\{ \Phi \left( a_{m,n} \right) \right\}^{1/(m+n)} \right]^{\gamma_k/q_k} < \prod_{k=1}^{p} (e q_k T_k)^{\gamma_k/q_k},
\]

or

\[
\lim_{m + n \to \infty} \left[ (m + n)^{1/q_k} \left\{ \Phi \left( a_{m,n} \right) \right\}^{1/(m+n)} \right]^{\gamma_k/q_k} < \prod_{k=1}^{p} (e q_k T_k)^{\gamma_k/q_k}.
\]

Hence again using (1.8) for \( f(z_1, z_2) \), we obtain

\[
(e q T)^{1/q} < \prod_{k=1}^{p} (e q_k T_k)^{\gamma_k/q_k},
\]
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or

$$(q \cdot T)^{\lambda_0} < \prod_{k=1}^{p} (q_k \cdot T_k)^{\alpha_k / \gamma_k},$$

where $q$ and $T$ are order and type of $f(z_1, z_2)$.

**Corollary.** If all $f_k(z_1, z_2)$ are of some finite non-zero order, then

$$T \leq \prod_{k=1}^{p} (T_k)^{\gamma_k}.$$  

I take the opportunity to express my thanks to Dr. S. H. Dwivedi for his helpful suggestions and guidance in the preparation of this paper.

**References.**


**Summary.**

In this paper we obtain certain relationships among entire functions of finite non-zero orders and types. Further, we study the relations among the coefficients in the Taylor expansion of entire functions and their orders and types.

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ON THE MEANS OF AN ENTIRE FUNCTION AND ITS DERIVATIVES

J. P. Singh

Some properties of the lower and upper limits of particular means of an entire function and its derivatives are obtained. These properties are extensions of some previously proved, under less general conditions, by R. P. Srivastava, P. K. Kamthan and O. P. Jervia.

1. Let \( f(z) \) be an entire function of order \( \theta \) and lower order \( \lambda \). For \( 0 < \delta < \infty \) and \( z = re^{i\theta} \), let

\[
M_\delta(r) = \sup_{0 < \theta < \pi} \left| \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \right|
\]

and

\[
\beta_\delta(r, f^{(m)}) = \sup_{0 < \theta < \pi} \left| \frac{1}{2\pi} \int_0^{2\pi} |f^{(m)}(re^{i\theta})|^\delta \, d\theta \right|
\]

where \( f^{(m)}(z) \) denotes the \( m \)-th derivative of \( f(z) \). Then the following results are known:

**Theorem 1.** For every entire function \( f(z) \), other than a polynomial,

\[
\lim_{r \to \infty} \sup_{0 < \theta < \pi} \left\{ \frac{M_\delta(r, f^{(m)})}{M_\delta(r, f)} \right\}^{\frac{1}{m}} = \delta
\]

**Theorem 2.** For \( 1 \leq \delta < \infty \)

\[
\lim_{r \to \infty} \sup_{0 < \theta < \pi} \left\{ \frac{M_\delta(r, f^{(m)})}{M_\delta(r, f^{(s)})} \right\}^{\frac{1}{s}} = \delta
\]

(here \( r \to \infty \) excluding a set of values of \( r \) having measure zero).

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JUNEJA [1] proved the following result as a corollary:

If \( f(z) \) is an entire function of lower order \( \lambda > 1 + \frac{1}{m} \) and order \( \varrho \) \((< \infty)\), then

\[
\lim_{r \to \infty} \sup \frac{1}{\log r} \left( \frac{\mu_{\delta}(r, f)}{\mu_{\delta}(r, f^{(m)})} \right) \frac{1}{m^{\frac{1}{\lambda}}} = \ln \frac{1}{\lambda}, \quad (\delta \geq 1)
\]

(1.5)

Theorem [1]. For every entire function \( f(z) \), other than a polynomial,

\[
\lim_{r \to \infty} \sup \frac{1}{\log r} \left( \frac{\mu_{\delta}(r, f)}{\mu_{\delta}(r, f^{(m)})} \right) \frac{1}{m^{\frac{1}{\lambda}}} = \frac{\varrho}{\lambda} \quad (0 < \delta < 1)
\]

(1.6)

where \( r \) tends to infinity through values excluding an exceptional set of at most finite measure.

For \( \delta = 1 \), R. P. SRIVASTAVA [5] has given a proof of (1.3), when the upper limit is only considered. For \( 1 \leq \delta < \infty \), P. K. KAMTHAN [2] has given a proof of (1.4) for the upper limit only and under the condition that \( r \to \infty \) excluding a set of values of \( r \) having measure zero. Their methods, in fact, fail to give the lower limit. The result (1.5) has been obtained by O. P. JUNEJA, imposing the condition on the lower order. In the case \( 0 < \delta < 1 \) O. P. JUNEJA has given a proof of the result (1.6) (see [1]), when the upper limit is only considered. His method, in fact, fails to give the lower limit of the left-hand expression in (1.6).

Our aim in this paper is to prove some results on the means of an entire function and its derivatives. We prove the following:

2. Theorem 1. If \( f(z) \) is an entire function of lower order \( \lambda \) and order \( \varrho \) \((< \infty)\), then

\[
\lim_{r \to \infty} \sup \frac{1}{\log r} \left( \frac{M_{\delta}(r, f^{(m)})}{M_{\delta}(r, f)} \right) \frac{1}{m^{\frac{1}{\lambda}}} = \frac{\varrho}{\lambda}.
\]

(2.1)

where \( \delta \geq 1 \) and \( m = 1, 2, \ldots, m \).

Remark. This result is stronger than that of JUNEJA in the sense that it does not impose the additional condition \( \lambda > 1 + \frac{1}{m} \).

To prove this theorem we require the following lemmas:

Lemma 1. For an entire function \( f(z) \)

\[
\lim_{r \to \infty} \sup \frac{1}{\log r} \frac{\log M_{\delta}(r, f)}{\log r} = \frac{\varrho}{\lambda}.
\]

(2.2)

This lemma has been proved by RAHMAN [6].

Lemma 2. If \( f(z) \) is an entire function of order \( \varrho \) and lower order \( \lambda \) and \( \delta \geq 1 \), then
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\[ (2.3) \quad \frac{M_b(r, f^{(m)})}{M_b(r,f)} > \left[ \log \frac{M_b(r, f)}{r \log r} \right]^m (1 - \epsilon)^{m-1} \cdot \alpha, \quad \alpha < \epsilon < 1 \]

**Proof.** It is known that [1]

\[ \frac{M_b(r, f')}{M_b(r,f)} > \frac{\log M_b(r, f) - \log M_b(r_0, f)}{r \log r} \]

Replacing \( f(z) \) by \( f^{(k-1)} (z) \), we have

\[ \frac{M_b(r, f^{(k)})}{M_b(r,f)} > \frac{\log M_b(r, f^{(k-1)}) - \log M_b(r_0, f^{(k-1)})}{r \log r} \]

\[ > \frac{\log M_b(r, f^{(k-1)})}{r \log r} \cdot \alpha_k, \quad 0 < \alpha_{k-1} < 1. \]

Putting \( k = 1, 2, \ldots, m \) and multiplying the \( m \) inequalities thus obtained, we get

\[ \frac{M_b(r, f^{(m)})}{M_b(r,f)} > \prod_{k=1}^{m} \frac{\log M_b(r, f^{(k-1)})}{(r \log r)^m} \cdot \alpha, \quad \alpha = \prod_{k=1}^{m} \alpha_k. \]

Now by Lemma [1], we have

\[ \log M_b(r, f^{(k)}) \geq (1 - \epsilon) \log M_b(r, f). \]

Hence for all \( r > r_0 > 1 \), we get

\[ \frac{M_b(r, f^{(m)})}{M_b(r,f)} > \left\{ \frac{\log M_b(r, f)}{r \log r} \right\}^m (1 - \epsilon)^{m-1} \cdot \alpha \]

which is (2.3).

**Lemma 3.** If \( f(z) \) is of finite order \( \varrho \) and lower order \( \lambda \), then for \( r > r_0 \)

\[ (2.4) \quad M_b(r, f^{(m)}) < r^{(\varrho-1+\epsilon) \cdot m} M_b(r, f) \]

and for an infinite sequence of values of \( r \) tending to infinity

\[ (2.5) \quad M_b(r, f^{(m)}) < r^{(\lambda-1+\epsilon) \cdot m} M_b(r, f) \]

where \( \delta \geq 1 \) and \( \epsilon \) is positive.

**Proof.** It is enough to prove (2.4) since the proof of (2.5) is similar.

We know [1] that for every \( \epsilon > 0 \) and large \( r \)

\[ M_b(r, f^{(k)}) < r^{(\varrho-1+\epsilon) \cdot m} M_b(r, f^{(k-1)}). \]

Giving \( k \) the values \( 1, 2, 3, \ldots, m \) and multiplying the \( m \) inequalities thus obtained, we get for large \( r \)
Proof of Theorem 1. Lemma 2 leads to
\[
\lim_{r \to \infty} \sup_{f \in F} \inf_{m \geq 0} \frac{\log r \left( \frac{M_b(r, f^{(m)})}{M_b(r, f)} \right) \frac{1}{m}}{\log r} = 0.
\]

From Lemma 3, we have
\[
\lim_{r \to \infty} \sup_{f \in F} \inf_{m \geq 0} \frac{\log r \left( \frac{M_b(r, f^{(m)})}{M_b(r, f)} \right) \frac{1}{m}}{\log r} = 0.
\]

Comparing (2.6) and (2.7), we get (2.1).

3. Theorem 2. For every entire function \( f(z) \), other than a polynomial,
\[
\lim_{r \to \infty} \sup_{f \in F} \inf_{m \geq 0} \frac{\log r \left( \frac{M_b(r, f^{(m)})}{M_b(r, f)} \right) \frac{1}{m}}{\log r} = 0, \quad 0 < \delta < 1,
\]
where \( r \) tends to infinity trough values excluding an exceptional set of at most finite measure.

The proof of this theorem requires the following lemmas:

Lemma 4. For every entire function \( f(z) \), other than a polynomial, outside an exceptional set of at most finite measure,
\[
M_b(r, f^{(m)}) \leq M_b(r, f) \left( \frac{v(r)}{r} \right)^m \left\{ 1 - k v(r) \frac{1}{16} \right\}
\]
where \( v(r) \) denotes the rank of maximum term in \( f(z) \) for \( |z| = r \), \( k \) is positive constant and \( 0 < \delta < 1 \).

Lemma 5. If \( f(z) \) is of finite order \( \delta \) and finite lower order \( \lambda \), then for \( r > r_0 \)
\[
M_b(r, f^{(m)}) \leq r^{(\alpha - 1 + \varepsilon) m} M_b(r, f), \quad 0 < \delta < 1, \quad r \geq r_0 \varepsilon
\]
and for an infinite sequence of values of \( r \) tending to infinity
\[
M_b(r, f^{(m)}) \leq r^{(\alpha - 1 + \varepsilon) m} M_b(r, f), \quad 0 < \delta < 1.
\]
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Proof. It is known [1] that

\[ M_b (r,f) \leq r^{b-1+\epsilon} M_b (r,f), \quad 0 < \delta < 1. \]

Then the proof of the Lemma follows on the lines of that of Lemma 3.

Proof of Theorem 2. Lemma 4 leads to

\[
\lim_{r \to \infty} \sup \inf \log r \left\{ \frac{M_b (r,f^{(n)})}{M_b (r,f)} \right\}^{1/m} \leq \lim_{r \to \infty} \sup \inf -\log r = \frac{\varrho}{\lambda}
\]

while from (3.3) and (3.4) we get

\[
\lim_{r \to \infty} \sup \inf \log r \left\{ \frac{M_b (r,f^{(n)})}{M_b (r,f)} \right\}^{1/m} \leq \frac{\varrho}{\lambda}.
\]

Comparing (3.5) and (3.6), we get (3.1).

REFERENCES


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ALIGARH, U.P. (INDIA)

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ÖZET