ORDERED RANDOM VARIATES

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

in

Statistics

By

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ALIGARH MUSLIM UNIVERSITY
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2011
Dedicated
To My
Loving Parents
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Acknowledgement

I bow in reverence to the Almighty Allah, the most Gracious and most Merciful. I express from the depth of my humility and humbleness before the Almighty Allah without whose endless and the blessing this tedious task could not have been accomplished.

I owe a deep sense of gratitude to my supervisor Dr. Haseeb Athar for introducing the topic, constant supervision and encouragement.

I would like to thank Prof. M. Z. Khan, former Dean, Faculty of Science and Prof. A. H. Khan, Ex-Chairman, Department of Statistics & O.R., for their encouragement and interest.

I am grateful to Prof. I.A. Khan, Chairman, Department of Statistics & O.R., for his encouragement and co-operation during this work.

I would like to express my appreciation for the encouragement and co-operation I received from all other teachers of the Department, especially Dr. R.U. Khan, Prof. M. Yaqub and Dr. Mohd. Faizan.

My special thanks are due to Mr. Ziaul Haque and Mr. Devendar Kumar for his valuable guidance, cooperation, great involvement and sympathetic behavior at all stages of this work.

I must express my thanks to staff members of the Department, for their cooperation, assistance and good wishes.

I also take this opportunity to acknowledge the support extended by all my seniors and colleagues.
My heart goes out in reverence to my parents for their tremendous patience, forbearance, endurance and affection. All my appreciation for their support will not be enough to match their good wishes.

I am also thankful to my sister and brothers for their support and love.

Date: 31.12.2010

(Nayabuddin)
Since Kamps (1995a) had introduced the unifying concept of generalized order statistics (gos), the use of such concept has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been separately treated in statistical literature. Examples of such concepts are the order statistics (os), sequential order statistics (sos), progressive type II censored order statistics, record values, $k$–th record values and pfeifer’s records. Application is multifarious in a variety of disciplines and particularly in reliability.

Order statistics have been extensively used in literature and were discussed, among other topics, in several textbooks, books that were devoted to order statistics are Sarhan and Greenberg (1962), Harter (1970), David and Nagaraja (2003), Balakrishnan and Cohen (1991), Arnold, Balakrishnan and Nagaraja (1992) and Balakrishnan and Rao (1997,1998).

Records have been discussed in books by Ahsanullah (1995) and Arnold, Balakrishnan and Nagaraja (1992, 1998).

Sequential order statistics have been presented by Kamps (1995a) and Cramer and Kamps (1996, 2001) among others.

The present dissertation entitled “Ordered Random Variates” is a brief collection of work done so far on the subject. I have tried my best to include sufficient and relevant materials in a systematic way, which is spread over seven chapters.

Chapter I is introductory in nature and consists of basic concepts and results about order statistics, records, generalized order statistics, lower (dual) generalized order statistics, which may be needed in subsequent chapters.
Chapter II consists of recurrence relation for moments of order statistics for some specific continuous distributions namely, normal, standard exponential, gamma, Weibull, power function, doubly truncated power function, doubly truncated pareto, doubly truncated Cauchy, symmetric truncated logistic, beta, doubly truncated Burr, Burr, standard logistic and some general form of distributions.

Chapter III deals with characterization of probability distributions through properties of order statistics for some specific distributions as well as for general class of distributions.

Chapter IV is based on recurrence relations for single and product moments of record values. Here recurrence relations for some specific distributions and general class of distributions are given.

Chapter V deals with results based on characterization of probability distributions through record statistics.

Chapter VI consists of recurrence relations for moments of generalized order statistics (gos) and lower generalized order statistics (Igos) for some specific continuous distributions like, Burr, doubly truncated Weibull, power function, erlang truncated exponential and some general form of distributions.

Chapter VII is related to the characterization of probability distributions through properties of generalized and lower generalized order statistics.
In this chapter we have introduced those concepts/results which are needed to grasp the idea in subsequent chapters.

1 ORDER STATISTICS

1.1 Definition

Let $X_1,X_2,...,X_n$ be a random sample of size $n$ from a continuous population having probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. Let they be arranged in ascending order of magnitude as

$$X_{1:n} \leq X_{2:n} \leq ... \leq X_{r:n} \leq ... \leq X_{n:n}$$

then $X_{1:n}, X_{2:n},...,X_{n:n}$ are collectively called the order statistics of the sample and $X_{r:n} \,(r=1,2,...,n)$ is called the $r-th$ order statistic of the sample. $X_{1:n} = \text{min}(X_1,X_2,...,X_n)$ and $X_{n:n} = \text{max}(X_1,X_2,...,X_n)$ are called extreme order statistics or the smallest and the largest order statistics.

David and Nagaraja (2003) is the basic book on order statistics dealing in detail with its different aspects. Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). Also, references may be made to Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991), Arnold et al. (1992) and the references therein.

1.2 Distribution of order statistics

Here in this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous.
Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a continuous population having probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics.

The pdf of $X_{r:n}$, the $r$-th order statistic is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1}[1-F(x)]^{n-r} f(x), \quad -\infty < x < \infty \quad (1.1)$$

The pdf's of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x) \quad ; \quad -\infty < x < \infty \quad (1.2)$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) \quad ; \quad -\infty < x < \infty \quad (1.3)$$

The df of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \leq x)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} \quad ; \quad -\infty < x < \infty \quad (1.4)$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_0^1 u^{r-1} (1-u)^{n-r} du \quad (1.5)$$

$$= B_r(n-r+1) \quad (1.6)$$

RHS is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991)
Preliminaries and basic concepts

\[ F_{r,n}(x) = \sum_{i=0}^{n-r} \binom{n-1}{r-1-i} [F(x)]^r [1-F(x)]^{n-r-i}; \quad -\infty < x < \infty \quad (1.7) \]

For continuous case the pdf of \( X_{r,n} \) may also be obtained by differentiating (1.5) w.r.t. \( x \).

From the density function given in (1.1), we may obtain the \( k-th \) moment of \( X_{r,n} \) as below:

\[ \mu_{r,n}^{(k)} = E[X_{r,n}^k] = \int_{-\infty}^{\infty} x^k f_{r,n}(x) \, dx \quad (1.8) \]

The joint pdf of \( X_{r,n}, X_{s,n}, 1 \leq r < s \leq n \) is given by

\[ f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x)f(y); \quad -\infty < x < y < \infty \quad (1.9) \]

The joint df of \( X_{r,n} \) and \( X_{s,n}, (1 \leq r < s \leq n) \) can be obtained as follows:

\[ F_{r,s:n}(x,y) = P(X_{r,n} \leq x, X_{s:n} \leq y) \]

\[ = P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x \]

\[ \quad \text{and at least } s \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \]

\[ = \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^{i} [F(y) - F(x)]^{j-i} [1-F(y)]^{n-j} \quad (1.10) \]

We can write the joint df of \( X_{r,n} \) and \( X_{s:n} \) in (1.10) equivalently as:

\[ F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{0}^{F(x)} \int_{0}^{F(y)} u^{r-1} (v-u)^{s-r-1} \]

\[ \times (1-v)^{n-s} \, du \, dv \]

\[ = I_{F(x),F(y)}(r,s-r,n-s+1); \quad -\infty < x < y < \infty \quad (1.11) \]

which is incomplete bivariate beta function.
It may be noted that for $x \geq y$

$$F_{r,s:n}(x,y) = F_{s:n}(y)$$  \hfill (1.12)

The product moments of the $j-th$ and $k-th$ order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1 \leq r < s \leq n)$ is given by:

$$\mu_{r,s:n}^{(j,k)} = E[X_r^j X_s^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{r,s:n}(x,y) \, dx \, dy$$  \hfill (1.13)

In general, the joint pdf of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n},x_{i_2:n},\ldots,x_{i_k:n}) = n! \left[ \prod_{j=1}^{k} f(x_{i_j}) \right] \prod_{j=0}^{k-1} \left[ \frac{[F(x_{i_{j+1}}) - F(x_{i_j})]^{j+1-i_j-1}}{(i_{j+1} - i_j - 1)!} \right]$$

$$-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty$$  \hfill (1.14)

where $x_0 = -\infty, x_{k+1} = +\infty, i_0 = 0, i_{k+1} = n + 1$

Remarks:

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.

2. Regarding the probability integral transformation, if $X_{r:n}, 1 \leq r \leq n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the $r-th$ order statistic from a uniform distribution on $U(0,1)$.

3. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.
4. Let $X_1, X_2, \ldots, X_n$ be iid random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

5. Let $X$ be a continuous random variable with $E[X_{r:n}] = \alpha_{r:n}$,

a) If $\alpha = E(X)$ exists then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of $r$, even though $\alpha$ does not exist.

b) $\alpha_{r:n}$ for all $n$ determine the distribution completely.

1.3 Truncated and conditional distribution of order statistics

Let $X$ be a continuous random variable having pdf $f(x)$ and df $F(x)$ in the interval $[-\infty, \infty]$.

Let $\int_{-\infty}^{Q_1} f(x)dx = Q$ and $\int_{-\infty}^{P_1} f(x)dx = P$ (1.15)

where $Q_1$ and $P_1$ are known constants. Then doubly truncated pdf of $X$ is given by:

$$\frac{f(x)}{P-Q}; \; x \in (Q_1, P_1)$$ (1.16)

and the corresponding cdf is given by

$$\frac{F(x) - Q}{P-Q}; \; x \in (Q_1, P_1)$$ (1.17)

The lower and upper truncation points are $Q_1, P_1$ respectively; the degrees of truncation are $Q$ (from below) and $1-P$ (from above). If we put $Q = 0$, the distribution will be truncated to the right. Similarly, for $P = 1$, the distribution will be truncated to the left. Whereas for $Q = 0, P = 1$, we get the non truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.
1.4 Some important results

**Result 1 (David and Nagaraja, 2003):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with the df \( F(x) \) and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order statistics obtained from this sample. Then the conditional distribution of \( X_{r:n} \), given that \( X_{s:n} = y \) for \( s > r \), is the same as the distribution of the \( r-th \) order statistic obtained from a sample of size \((s - 1)\) from a population whose distribution is truncated on the right at \( y \).

**Result 2 (David and Nagaraja, 2003):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with the df \( F(x) \) and pdf \( f(x) \), and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order statistics obtained from this sample. Then the conditional distribution of \( X_{s:n} \), given that \( X_{r:n} = x \) for \( r < s \), is the same as the distribution of the \((s - r)-th\) order statistic obtained from a sample of size \((n - r)\) from a population whose distribution is truncated on the left at \( x \).

**Result 3 (David and Nagaraja, 2003):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with df \( F(x) \) and pdf \( f(x) \), and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order statistics obtained from this sample. Then the conditional distribution of \( X_{s:n} \) given that \( X_{r:n} = x \) and \( X_{k:n} = z \) for \( 1 \leq r < s < k \leq n \), is the same as the distribution of the \((s - r)-th\) order statistic obtained from a sample of size \((k - r - 1)\) from a population whose distribution is truncated on the left at \( x \) and on the right at \( z \).

**Result 4:** Order statistics in a sample from a continuous distribution form a Markov chain, that is

\[
 f(X_{k:n} | X_{1:n} = x_1, \ldots, X_{r:n} = x_r, \ldots, X_{s:n} = x_s, \ldots, X_{n:n} = x_n) = f(X_{k:n} | X_{r:n} = x_r, X_{s:n} = x_s)
\]

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.
Result 5 (Ali and Khan, 1997): Let \( g(x) \) be a Borel measurable function of \( x \) in the interval \([\alpha, \beta]\) then, for \( 1 \leq r \leq n, \ n = 1, 2, \ldots \)

(i) \[ E[g(X_{r:n})] - E[g(X_{r-1:n-1})] = \binom{n-1}{r-1} \int_{\mathcal{D}_1}^\beta \int_{\mathcal{D}_1}^\beta g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \] (1.18)

(ii) \[ E[g(X_{r:n})] - E[g(X_{r-1:n-1})] = \binom{n}{r-1} \int_{\mathcal{D}_1}^\beta \int_{\mathcal{D}_1}^\beta g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \] (1.19)

(iii) \[ E[g(X_{r-1:n-1})] - E[g(X_{r-1:n-1})] = \binom{n-1}{r-2} \int_{\mathcal{D}_1}^\beta \int_{\mathcal{D}_1}^\beta g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \] (1.20)

In view of (1.18), (1.19) and (1.20), we have

\[ (n-r+1)E[g(X_{r-1:n})] + (r-1)E[g(X_{r:n})] = nE[g(X_{r-1:n-1})]. \] (1.21)

At \( g(x) = x \) in (1.21), we get the well known relation established by (David and Nagaraja, 2003).

Result 6 (Ali and Khan, 1998): If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)

\[ E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] = \frac{C_{r,s,n}}{(n-s+1)} \int_{\mathcal{D}_1}^\beta \int_{\mathcal{D}_1}^\beta \frac{\partial}{\partial y} g(x, y)[F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} f(y) \, dy \, dx. \] (1.22)

Result 7 (Khan et al., 2001): If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)

\[ E[g(X_{r:n}, X_{s:n})] - E[g(X_{r-1:n}, X_{s:n})] = \frac{C_{r,s:n}}{(s-r)} \int_{\mathcal{D}_1}^\beta \int_{\mathcal{D}_1}^\beta \frac{\partial}{\partial x} g(x, y)[F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) \, dy \, dx. \] (1.23)
2. RECORD VALUES AND RECORD TIMES

2.1 Definition

Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables with df $F(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say $X_j$ is an upper (lower) record values of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 1$. By definition $X_1$ is an upper as well as lower record values. One can transform the upper record by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or if $P(X_i > 0) = 1$ for all $i$ by $\left\{\frac{1}{X_i}, i \geq 1\right\}$, the lower record value of this sequence will correspond to the upper record values of the original sequence (Ahsanullah, 1995).

The indices at which upper record values occur are given by the record times $\{U(n)\}, n > 0$. That is $X_{U(n)}$ is the $n$-th upper record, where $U(n) = \min\{j | j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(n) = 1$. The distribution of $U(n), n \geq 1$ does not depend on $F$. Further, we will denote $L(n)$ as the indices where the lower record values occur. By assumption $U(1) = L(1) = 1$.

The distribution of $L(n)$ also does not depend on $F$.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recoding them: e.g. Olympic records or world records in sports.

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.
To be precise, record values are defined by means of record times. That is, those times have to be described at which successively largest values appear.

Chandler (1952) shows several properties of record values and notes their Markovian structure. Two recent books on records by Ahsanullah (1995) and Arnold et al. (1998) are worth mentioning.

### 2.2 Distribution of record values

Let $R(x)$ be a continuous function of $x$ with $R(x) = -\ln \bar{F}(x)$ and $0 < \bar{F}(x) = 1 - F(x)$, where 'ln' is the natural logarithm.

If we define $F_n(x)$ as the df of $X_{U(n)}$ for $n \geq 1$, then we have (Ahsanullah, 1995)

$$F_n(x) = P(X_{U(n)} \leq x)$$

$$= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty$$

(2.1)

and the pdf $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty$$

(2.2)

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i,j}(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j)$$

$$-\infty < x_i < x_j < \infty$$

(2.3)

The joint pdf of the $n$ record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \ldots r(x_{n-1}) f(x_n),$$

$$-\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty$$

(2.4)

where $r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{1 - F(x)}$, $0 < F(x) < 1$

is known as hazard rate.
In particular at \( i = 1, \ j = n \) we have

\[
f_{1,n}(x_1, x_n) = r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} \frac{f(x_n)}{f(x_1)}, \quad -\infty < x_1 < x_2 < \infty.
\]

The conditional distribution of \( X_{U(j)} \mid X_{U(i)} = x_i \) is

\[
f(X_{U(j)} \mid X_{U(i)} = x_i) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)}
\]

\[
= \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)}, \quad -\infty < x_i < x_j < \infty \tag{2.5}
\]

and for \( X_{U(i)} \mid X_{U(j)} = x_j \) is

\[
f(X_{U(i)} \mid X_{U(j)} = x_j)
\]

\[
= \frac{(j-1)!}{(i-1)!(j-i-1)!} \left[ \frac{R(x_i)}{R(x_j)} \right]^{i-1} \left[ 1 - \frac{R(x_i)}{R(x_j)} \right]^{j-i-1} \frac{r(x_i)}{R(x_j)} \quad -\infty < x_i < x_{i+1} < \infty \tag{2.6}
\]

### 2.3 \( k \)-Records

In some situations record values themselves are viewed as 'outlier' and hence second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example.

Let \( X_1, X_2, \ldots, X_n \) be an identically and independent sequence of random variables with a continuous distribution function \( F(x) \) and let \( k \) be a positive integer.

Then the random variables \( L^{(k)}(n) \) is given by (Kamps, 1995b)

\[
L^{(k)}(n) = 1
\]

\[
L^{(k)}(n+1) = \min\{ j \in N; X_j, X_{j+k-1} > X_{L^{(k)}(n), L^{(k)}(n)+k-1} \}, n \in N,
\]

are called \( k - th \) record times and the quantities \( X_{L^{(k)}(n)}, n \in N \) are called \( k - th \) record values or \( k \) - records.
We can obtain ordinary record values at \( k = 1 \).

The joint density of the \( k \)-records \( X_{L(k)}^{(1)}, \ldots, X_{L(k)}^{(r)} \) is given as

\[
fx_{L(k)}^{(1)} \cdots X_{L(k)}^{(r)}(x_1, \ldots, x_r) = k^r \left( \prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)} \right) [1 - F(x_r)]^{k-1} f(x_r)
\]  

(2.7)

and the marginal densities and marginal distribution functions are given by:

\[
fX_{L(k)}^{(r)}(x) = \frac{k^r}{(r-1)!} \left[ R(x) \right]^{r-1} [1 - F(x)]^{k-1} f(x)
\]  

(2.8)

and

\[
FX_{L(k)}^{(r)}(x) = 1 - [1 - F(x)]^k \sum_{j=0}^{r-1} \frac{1}{j!} [kR(x)]^j
\]  

(2.9)

3 GENERALIZED ORDER STATISTICS

The concept of generalized order statistics (gos) have been introduced and extensively studied by Kamps (1995). A variety of ordered models of random variables is contained in this concept.

3.1 Definition

Let \( n \geq 2 \) be a given integer and \( \bar{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \), \( k \geq 1 \) be the parameters such that

\[
\gamma_i = k + n - i + \sum_{j=i}^{n-i} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n - 1.
\]

Then \( X(1,n,\bar{m},k), X(2,n,\bar{m},k), \ldots, X(n,n,\bar{m},k) \) are called generalized order statistics if their joint probability density function (pdf) has the form

\[
k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} [1 - F(x_i)] \right)^{m_i} f(x_i) [1 - F(x_n)]^{k-1} f(x_n)
\]

(3.1)

on the cone \( F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1) \) of \( \mathbb{R}^n \).
with absolutely continuous distribution function \( df \) \( F() \) with probability density function \( pdf \) \( f() \). The model of generalized order statistics contains as special cases such as ordinary order statistics \( \gamma_i = n - i + 1; i = 1, 2, ..., n \) i.e. \( m_1 = m_2 = ... = m_{n-1} = 0, k = 1 \), \( k \)-th record values \( \gamma_i = k \) i.e. \( m_1 = m_2 = ... = m_{n-1} = -1, k \in N \), sequential order statistics \( \gamma_i = (n - i + 1)\alpha_i; \alpha_1, \alpha_2, ..., \alpha_n > 0 \), order statistics with non-integral sample size \( \gamma_i = \alpha - i + 1; \alpha > 0 \), Pfeifer’s record values \( \gamma_i = \beta_i; \beta_1, \beta_2, ..., \beta_n > 0 \) and progressive type II censored order statistics \( m_i \in N_0, k \in N \) are obtained [Kamps (1995), Kamps and Cramer (2001)].

3.2 Distribution of generalized order statistics

**Case I:** \( m_1 = m_2 = ... = m_{n-1} = m \)

The marginal density of the \( r \)-th generalized order statistic \( (gos) \) is given by [Kamps, 1995b]

\[
f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [1 - F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \quad (3.2)
\]

and the joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k), 1 \leq r < s \leq n \) is

\[
f_{X(r,n,m,k), X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s - r - 1)!} [1 - F(x)]^m g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{s-1} f(x) f(y) \quad (3.3)
\]

where \( C_{r-1} = \prod_{i=1}^{r} \gamma_i, \gamma_i = k + (n - i)(m+1) \)

\[
h_m(x) = \begin{cases} 
\frac{-1}{m+1} (1-x)^{m+1}, & m \neq -1 \\
-\log(1-x), & m = -1 
\end{cases}
\]

\[
g_m(x) = \int_0^x (1-t)^m \, dt = h_m(x) - h_m(0), x \in [0,1]
\]
The conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \leq r < s \leq n$ is given by

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y | x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[1 - F(y)]^{y-1}f(y)}{[1 - F(x)]^{y-1}}, \ x < y$$

(3.4)

and the conditional pdf of $X(r,n,m,k)$ given $X(s,n,m,k) = y$, $1 \leq r < s \leq n$ is

$$f_{X(r,n,m,k)|X(s,n,m,k)}(x | y) = \frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}$$

$$\times \left[ \frac{F(x)}{1-(F(x))^{m+1}} \right]^{y-1} \left[ (F(x))^{m+1} - (F(y))^{m+1} \right]^{s-r-1} \frac{f(x)}{1-(F(y))^{m+1}}^{s-1}f(x), \ x < y$$

(3.5)

**Case II**: $\gamma_i \neq \gamma_j$ ; $i, j = 1, 2, \ldots, n - 1$

The pdf of $X(r,n,m,k)$ is [Kamps and Cramer, 2001]

$$f_{X(r,n,m,k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i-1}$$

(3.6)

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$ is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i}$$

$$\times \left[ \sum_{i=r+1}^{s} a_i(r)(1 - F(x))^{\gamma_i} \right] \frac{f(x)}{1 - F(x)} \frac{f(y)}{1 - F(y)}$$

(3.7)

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \ \gamma_i = k + n - i + M_i$$

$$a_i(r) = \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}^{1}, 1 \leq i \leq r \leq n$$

and

$$a_i^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}^{1}, r + 1 \leq i \leq s \leq n$$
Thus, the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \leq r < s \leq n$ is given by

$$f_{X(s,n,m,k) | X(r,n,m,k)}(y | x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{1 - F(y)}{1 - F(x)} \right]^{\gamma_i} \frac{f(y)}{[1 - F(y)]}, \quad x \leq y$$

(3.8)

and the conditional pdf of $X(r,n,m,k)$ given $X(s,n,m,k) = y$, $1 \leq r < s \leq n$ is given by

$$f_{X(r,n,m,k) | X(s,n,m,k)}(x | y) = \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ \sum_{i=1}^{r} a_i^{(r)}(F(x))^{\gamma_i} \right\} \frac{f(x)}{F(x)}$$

(3.9)

3.3 Some important results

**Result 1:** (Athar and Islam, 2004)

Let $\xi(x)$ is a measurable function of $x$ which is differentiable, then for any arbitrary distribution function $F$ and $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$, following relations hold:

**Case I:** $m_1 = m_2 = \ldots = m_{n-1} = m$

$$E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}]$$

$$= \frac{C_{r-2}}{(r-1)!} \left[ \frac{\xi'(x)}{\alpha} \right] \left[ 1 - F(x) \right]^{\gamma_r} g_m^{-1}(F(x)) \, dx$$

(3.10)

$$E[\xi\{X(r-1,n,m,k)\}] - E[\xi\{X(r-1,n-1,m,k)\}]$$

$$= - \frac{(m+1) C_r^{(n)}}{\gamma_1} \left[ \frac{\xi'(x)}{\alpha} \right] \left[ 1 - F(x) \right]^{\gamma_r} g_m^{-1}(F(x)) \, dx$$

(3.11)
\[ E[\xi \{ X(r, n, m, k) \}] - E[\xi \{ X(r - 1, n - 1, m, k) \}] \]

\[
= \frac{C_{r-2}^{(n-1)}}{(r-1)!} \frac{\beta}{\alpha} \int_0^\alpha \xi'(x) [1 - F(x)]^{\gamma_r} g_m^{-1}(F(x)) \, dx
\]

(3.12)

**Case II:** \( m_i \neq m_j (\gamma_i \neq \gamma_j); i \neq j = 1, 2, \ldots n - 1 \)

\[
E[\xi \{ X(r, n, \bar{m}, k) \}] - E[\xi \{ X(r - 1, n, \bar{m}, k) \}] \]

\[
= C_{r-2}^{(n-1)} \frac{\beta}{\gamma_1} \int_0^\alpha \xi'(x) \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i} \, dx
\]

(3.13)

\[
E[\xi \{ X(r - 1, n, \bar{m}, k) \}] - E[\xi \{ X(r - 1, n - 1, \bar{m}^*, k) \}] \]

\[
= \frac{(r - 1) + \sum_{j=1}^{r-1} m_j}{\gamma_1} C_{r-2}^{(n-1)} \frac{\beta}{\gamma_1} \int_0^\alpha \xi'(x) \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i} \, dx
\]

(3.14)

\[
E[\xi \{ X(r, n, \bar{m}, k) \}] - E[\xi \{ X(r - 1, n - 1, \bar{m}^*, k) \}] \]

\[
= \frac{\gamma_r}{\gamma_1} C_{r-2}^{(n-1)} \frac{\beta}{\alpha} \int_0^\alpha \xi'(x) \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i} \, dx
\]

(3.15)

where \( \bar{m}^* = (m_2, m_3, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \).

**Result 2:** (Athar and Islam, 2004)

For \( 1 \leq r < s \leq n - 1 \), \( n \geq 2 \) and \( k = 1, 2, \ldots \)

**Case I:** \( m_1 = m_2 = \ldots = m_{n-1} = m \)

\[
E[\xi \{ X(r, n, m, k), X(s, n, m, k) \}] - E[\xi \{ X(r, n, m, k), X(s - 1, n, m, k) \}] \]

\[
= \frac{C_{s-2}}{(r-1)! (s-r-1)!} \frac{\beta}{\alpha} \int_0^\alpha \int_0^\alpha \xi(x, y) [1 - F(x)]^m f(x) g_m^{-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} \, [1 - F(y)]^r dy \, dx
\]

(3.16)
Preliminaries and basic concepts

Case II: $m_i \neq m_j (\gamma_i \neq \gamma_j); i \neq j = 1, 2, \ldots n - 1$

$$E[\xi(X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))] = E[\xi(X(r, n, \tilde{m}, k), X(s-1, n, \tilde{m}, k))]$$

$$= \frac{1}{(s-1)!} \int_{a \leq x < y \leq b} \frac{\partial}{\partial y} \xi(x, y) \left( \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1-F(y)}{1-F(x)} \right)^{\gamma_i} \right)$$
$$\times \left( \sum_{i=1}^{r} a_i(r) \left( 1 - F(x) \right)^{\gamma_i} \right) \frac{f(x)}{1-F(x)} \, dy \, dx \quad (3.17)$$

where, $\xi(x, y) = \xi_1(x) \xi_2(y)$

4 LOWER (DUAL) GENERALIZED ORDER STATISTICS

Generalized order statistics can be easily applicable in practice problems except that when $F(\cdot)$ is so called inverse distribution function. So the concept of lower generalized order statistics is needed. Pawlas and Szynal (2001) introduced the concept of lower generalized order statistics (Igos) to enable a common approach to descending ordered rv's like reversed order statistics and lower record values. The work of Burkschat et al. (2003) may also be seen for dual (lower) generalized order statistics.

4.1 Definition

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}, k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-i} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n - 1.$$ 

By the lower generalized order statistics from an absolutely continuous distribution function $F(\cdot)$ with the density function $f(\cdot)$ we mean random variables $X'(1, n, \tilde{m}, k), \ldots, X'(n, n, \tilde{m}, k)$ having joint density function of the form
Preliminaries and basic concepts

\[
k\left(\prod_{i=1}^{n-1} \gamma_i \right) \prod_{i=1}^{n-1} \left[ F(x_i) \right]^{m_i} f(x_i) \left[ F(x_n) \right]^{k-1} f(x_n) \tag{4.1}
\]

for \( F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0) \).

Here it may be noted that the joint density (4.1) is obtained by replacing \( 1-F(x) \) with \( F(x) \) in (3.1).

4.2 Distribution of lower (dual) generalized order statistics

Case I: \( m_1 = m_2 = \ldots = m_{n-1} = m \).

The density function of \( r \)-th lower generalized order statistic is given by

\[
f_{X^r(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} \left[ F(x) \right]^{r-1} f(x) g_{m}^{r-1}(F(x)) \tag{4.2}
\]

The joint density function of \( r \)-th and \( s \)-th lower generalized order statistics is

\[
f_{X^r(r,n,m,k),X^s(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(s-1)!} \left[ F(x) \right]^{m} f(x) g_{m}^{r-1}(F(x))
\times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [F(y)]^{r-1-s} f(y), \ \alpha \leq y < x \leq \beta ,
\]  

(4.3)

where,

\[
h_m(x) = \begin{cases} 
\frac{-1}{m+1} x^{m+1}, & m \neq -1 \\
- \log x, & m = -1
\end{cases}
\]

and

\[g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1) .\]

Case II: \( \gamma_i \neq \gamma_j, \ i, j = 1, 2, \ldots, n-1 \).

The pdf of \( r \)-th lower generalized order statistic is given by

\[
f_{X^r(r,n,m,k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i-1} \tag{4.4}
\]
and the joint pdf of $r$-th and $s$-th lower generalized order statistics is

$$f_{X'(r,n,ar{m},k),X'(s,n,ar{m},k)}(x,y) = C_{r-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{F(y)}{F(x)} \right]^\gamma_i \times \sum_{i=1}^{r} a_i(r)[F(x)]^\gamma_i \frac{f(x)f(y)}{F(x)F(y)}, \quad \alpha \leq y < x \leq \beta,$$

(4.5)

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + M_j$$

$$a_i(r) = \prod_{j=1}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n$$

and

$$a_i^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad r + 1 \leq i \leq s \leq n.$$

5 SOME CONTINUOUS DISTRIBUTIONS

5.1 Pareto distribution

A random variable $X$ is said to have the Pareto distribution if its probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$ are of the form given below:

$$f(x) = p \lambda^p x^{-(p+1)}; \quad \lambda < x < \infty; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^p x^{-p}; \quad \lambda < x < \infty; \quad \lambda, p > 0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

5.2 Power function distribution

A random variable $X$ is said to have a power function distribution if its pdf and df are of the form given below:

$$f(x) = p \lambda^{-p} x^{p-1}; \quad 0 \leq x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p} x^p; \quad 0 \leq x < \lambda; \quad \lambda, p > 0$$
The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if $X$ has a power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

**5.3 Beta distribution**

**i) Beta distribution of first kind**

A random variable $X$ is said to have the beta distribution of first kind if its pdf is of the form

$$f(x) = \frac{1}{B(p, q)} x^{p-1} (1-x)^{q-1}; \quad 0 \leq x \leq 1, \quad p, q > 0$$

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents $p$ and $q$ equal to 1. If $q=1$, the distribution reduces to power function distribution.

**ii) Beta distribution of second kind**

The continuous random variable $X$ which is distributed according to probability law:

$$f(x) = \frac{1}{B(p, q)} \frac{x^{p-1}}{(1+x)^{p+q}}; \quad p, q > 0, \quad 0 \leq x < \infty$$

is known as a beta variate of the second kind with parameters $p$ and $q$.

**Remark 5.3.1:** Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $\frac{1}{y}$. 

Usage: The Beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

5.4 Weibull distribution

A random variable $X$ is said to have a Weibull distribution if its pdf is given by:

$$f(x) = \theta \alpha^{p-1} x^{p-1} e^{-\theta x^p}; \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

and the df is given by

$$F(x) = 1 - e^{-\theta x^p}; \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

Remark 5.4.1: If we put $p = 1$ in Weibull distribution, we get the pdf of exponential distribution.

Remark 5.4.2: If we put $p = 2$, it gives pdf of Rayleigh distribution.

Remark 5.4.3: If $X$ has a Weibull distribution, then the pdf of

$$Y = -p \log \left( \frac{X}{\alpha} \right)$$

is

$$f(y) = e^{-y} e^{-e^{-y}}$$

which is a form of an Extreme Value distribution.

Remark 5.4.4: The pdf and the cdf of inverse Weibull distribution is given by

$$f(x) = \theta \alpha^{-(p+1)} x^{-(p+1)} e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

$$F(x) = e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict
randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

5.5 Exponential distribution

A random variable $X$ is said to have an exponential distribution if its pdf is given by

$$f(x) = \theta e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0$$

and the df is given by

$$F(x) = 1 - e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0$$

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable $X$ assuming non-negative values satisfies the assumption,

$$P(X > s + t | X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

then $X$ will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

5.6 Rectangular distribution

A random variable $X$ is said to have a rectangular distribution if its pdf is given by

$$f(x) = \frac{1}{\lambda - \beta}; \quad \beta \leq x \leq \lambda$$

and the df is given by

$$F(x) = \frac{x - \beta}{\lambda - \beta}; \quad \beta \leq x \leq \lambda.$$
The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(x)$ follows rectangular distribution $R(0,1)$. This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

5.7 Burr distribution

Let $X$ be a continuous random variable, then different forms of cumulative distribution function of $X$ are listed below (Johnson and Kotz, 1970):

i) $F(x) = x$, $0 < x < 1$

ii) $F(x) = (1 + e^{-x})^{-k}$, $-\infty < x < \infty$

iii) $F(x) = (1 + x^{-c})^{-k}$, $0 \leq x < \infty$

iv) $F(x) = \left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$, $0 \leq k \leq c$

v) $F(x) = [1 + c e^{-\tan x}]^{-k}$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

vi) $F(x) = [1 + c e^{-k \sinh x}]^{-k}$, $-\infty < x < \infty$

vii) $F(x) = 2^{-k} (1 + \tanh x)^{K}$, $-\infty < x < \infty$

viii) $F(x) = \left(\frac{2}{\pi} \tan^{-1} e^{x}\right)$, $-\infty < x < \infty$

ix) $F(x) = 1 - \frac{2}{c[(1 + e^{x})^{k} - 1] + 2}$, $-\infty < x < \infty$

x) $F(x) = (1 + e^{-x^{2}})^{k}$, $0 \leq x < \infty$

xi) $F(x) = \left(x - \frac{1}{2\pi} \sin 2\pi x\right)^{k}$, $0 \leq x \leq 1$

xii) $F(x) = 1 - (1 + x^{c})^{-k}$, $0 \leq x < \infty$

where $k$ and $c$ are positive parameters.
Special attention is given to type XII, whose pdf is given as:

\[ f(x) = kcx^{c-1}(1+x^c)^{-(k+1)}; \quad 0 \leq x < \infty; \quad k, c > 0 \]

This distribution is frequently used for the purpose of graduation and in reliability theory. At \( c = 1 \), it is called Lomax distribution whereas at \( k = 1 \), it is known as Log-logistic distribution.

### 5.8 Cauchy distribution

The special form of the Pearson type VII distribution, with pdf

\[ f(x) = \frac{1}{\pi \lambda} \frac{1}{\left[1 + \left((x-\theta)/\lambda\right)^2\right]} \quad -\infty < x < \infty; \quad \lambda > 0; \quad -\infty < \theta < \infty \]

is called the Cauchy distribution.

The cdf is given by

\[ F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\theta}{\lambda}\right) \quad -\infty < x < \infty; \quad \lambda > 0; \quad -\infty < \theta < \infty \]

The distribution is symmetrical about \( x = \theta \). The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, \( \theta \) and \( \lambda \) are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting \( \theta = 0, \lambda = 1 \). The standard probability density function is given by

\[ f(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \quad -\infty < x < \infty \]

and the standard cumulative distribution function is

\[ F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x \quad -\infty < x < \infty. \]
CHAPTER II

ORDER STATISTICS: MOMENTS AND RECURRENCE RELATIONS

1 INTRODUCTION

Order statistics and their moments have been of great interest from the turn of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of two successive order statistics. The moments of order statistics assume considerable and fetching importance in the statistical literature and have been numerically tabulated extensively for several distributions. Many authors have investigated and derived several recurrence relations and identities satisfied by single as well as product moments of order statistics primarily to reduce the amount of direct computations. However, one could list the following three main reasons why these recurrence relations and identities for the moments of order statistics are important:

i) They reduce the amount of direct computations considerably,

ii) They usefully express the higher order moments of order statistics in terms of the lower order moments and hence make the evaluation of higher order moments easy,

iii) They are very useful in checking the computation of the moments of order statistics.

Shah (1966) obtained moments of order statistics from logistic distribution.

Krishnaiah and Rizvi (1967) extended the results of Gupta (1960) for gamma distribution with any positive shape parameter. Young (1971) established a very simple relation between moments of order statistics of the symmetrical inverse multinomial distribution and the moments of order statistics of independent standardized gamma variables with integer parameter $\lambda$.

Joshi (1978) obtained some recurrence relations between the moments of order statistics from exponential and right truncated exponential distributions and later on Joshi (1979a, b) obtained similar recurrence relations for the moments
of order statistics from doubly truncated exponential distribution. Joshi and Balakrishnan (1981) obtained an identity for the moments of normal order statistics and showed their applications.

Balakrishnan and Joshi (1982) obtain the recurrence relations for doubly truncated Pareto distribution. Joshi (1982) also obtained some recurrence relations for mixed moments of order statistics from exponential and truncated exponential distributions.

Khan et al. (1983 a) obtained general results for finding the k-th moments of order statistics for an arbitrary distribution. These results were utilized to obtain recurrence relations for doubly truncated and non truncated distributions. The examples considered were Weibull, exponential, Pareto, power function, Cauchy, logistic, gamma and beta distribution. Further, Khan et al. (1983 b) also established general results for obtaining the product moments from an arbitrary continuous distribution. Then they utilized these results to determine the recurrence relations between product moments of some doubly truncated and non truncated distributions, viz. Weibull, exponential, Pareto, power function and Cauchy.

Balakrishnan and Joshi (1983a, b, 1984) obtained recurrence relations for single and product moments of order statistics from symmetrically truncated logistic distribution and doubly truncated exponential distributions.

Khan et al. (1984) obtain the inverse moments of order statistics from Weibull distribution.

Balakrishnan (1985) established some recurrence relations for the single and product moments from half logistic distribution.


Balakrishnan and Malik (1987a, b) obtained relations for moments of order statistics from truncated log-logistic distribution.
Ali and Khan (1987) obtained the recurrence relations between moments of order statistics for log-logistic distribution.


Ali and Khan (1995) have obtained ratio and product moment of two order statistics of different order from Burr distribution. Further they have deduced the moments and inverse moments of single order statistics from the product moments.

Balakrishana and Aggarwala (1996) obtain the relationship for moments of order statistics from right truncated generalized half logistic distribution.

Ali and Khan (1996) obtain the ratio and product moments of order statistics from Weibull and exponential distribution.


2 Single Moments

Normal Distribution

Theorem 2.1: (Govindrajulu, 1963)

For any arbitrary absolutely continuous distribution for which \( f'(x) = -x f(x) \), (that is for standard normal, half normal or generalized truncated normal densities) one has, for \( 1 \leq r \leq n \),

\[
\mu_{r,n}^{(2)} = 1 + r \sum_{m=0}^{n-r} (-1)^m \frac{1}{m+r} \binom{n-r}{m} \mu_{m+r-1,m+r:m+r}.
\]  

(2.1)
Proof: we have,

\[
\mu_{r;n}^{(2)} = \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^2 [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) \, dx \tag{2.2}
\]

Integrating (2.2) by parts and noting that \(x f(x) \, dx = d[-f(x)]\), we get

\[
\frac{(r-1)!(n-r)!}{n!} \mu_{r;n}^{(2)} = -x f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} \bigg|_{-\infty}^{\infty}
\]

\[
+ \int_{-\infty}^{\infty} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) \, dx
\]

\[
+ (r - 1) \int_{-\infty}^{\infty} x f^2(x) [F(x)]^{r-2} [1 - F(x)]^{n-r} \, dx
\]

\[-(n - r) \int_{-\infty}^{\infty} x f^2(x) [F(x)]^{r-1} [1 - F(x)]^{n-r-1} \, dx
\]

\[
= \frac{(r-1)!(n-r)!}{n!} + (r - 1) \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} 
\]

\[
\times \int_{-\infty}^{\infty} x f^2(x) [F(x)]^{r+m-2} \, dx + (n - r) \sum_{l=0}^{n-r-1} (-1)^{l+1} \binom{n-r-1}{l} 
\]

\[
\times \int_{-\infty}^{\infty} x f^2(x) [F(x)]^{r+l-2} \, dx. \tag{2.3}
\]

Since \(y f(y) \, dy = d[-f(y)]\) and \(\int_{-\infty}^{\infty} y f(y) d(y) = f(x)\),

therefore,

\[
\int_{-\infty}^{\infty} x f^2(x) [F(x)]^k \, dx = \int_{-\infty}^{\infty} x f(x) [F(x)]^k \{ \int_{-\infty}^{\infty} y f(x) \, dy \} \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{y} x f(x) f(y) [F(x)]^k \, dx \, dy
\]

\[
= [(k + 1)(k + 2)]^{-1} \mu_{k+1,k+2,k+2}, \quad k = 1, 2, \ldots \tag{2.4}
\]

Thus, in view of (2.3) and (2.4), we get
\[
\frac{(r-1)!(n-r)!}{n!} \mu_{r:n}^{(2)} = \frac{(r-1)!(n-r)!}{n!} + (r-1) \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \frac{\mu_{r+m-1,r+m:r+m}}{(r+m)(r+m-1)} \\
+ \sum_{l=0}^{n-r-1} (-1)^{l+1} \binom{n-r-1}{l} \frac{\mu_{r+l+1,r+l+1:r+l+1}}{(r+l+1)(r+l)}
\]

\[
= \frac{(r-1)!(n-r)!}{n!} + \frac{1}{r} \mu_{r-1,r} \sum_{m=1}^{n-r} (-1)^m \frac{\mu_{r+m-1,r+m:r+m}}{(r+m)(r+m-1)} \binom{n-r}{m}(r-1+m).
\]

Hence,

\[
\mu_{r:n}^{(2)} = 1 + \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m}(r-1+m)^{-1} \mu_{r+m-1,r+m:r+m}.
\]

**Standard exponential distribution**

The pdf of standard exponential distribution is given as

\[
f(x) = e^{-x}, x \geq 0 \quad (2.5)
\]

and corresponding df is

\[
F(x) = 1 - e^{-x}, x \geq 0 \quad (2.6)
\]

In view of (2.5) and (2.6), we have

\[
f(x) = 1 - F(x) \quad (2.7)
\]

**Theorem 2.2: (Joshi, 1978)**

For \(1 \leq r \leq n\) and \(k = 1, 2, \ldots\)

\[
\mu_{k:n}^{(k)} = \frac{k}{n} \mu_{1:n}^{(k-1)} \quad (2.8)
\]

and

\[
\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + \frac{k}{n} \mu_{r:n}^{(k-1)} \quad (2.9)
\]

**Proof:** In view of (2.7), we have

\[
\mu_{r:n}^{(k-1)} = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty x^{k-1}[F(x)]^{r-1}[1 - F(x)]^{n-r+1} dx \quad (2.10)
\]
Integrating (2.10) by part treating $x^{k-1}$ for integration and the rest of the integrand for differentiation, we get

$$\mu_{r,n}^{(k-1)} = \frac{n!}{(r-1)!(n-r)!k} [(n-r+1) \int_0^\infty x^k [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) \, dx$$

$$- (r-1) \int_0^\infty x^k [F(x)]^{r-2} [1 - F(x)]^{n-r+1} f(x) \, dx]$$

(2.11)

The relation in (2.8) follows immediately from (2.11) upon setting $r = 1$.

Further, by splitting the first integral on the right hand-side of (2.11) into two and combining one of them with the second integral, we get

$$\mu_{r,n}^{(k-1)} = \frac{n!}{(r-1)!(n-r)!k} \left[ n \int_0^\infty x^k [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) \, dx ight.$$  

$$- (r-1) \int_0^\infty x^k [F(x)]^{r-2} [1 - F(x)]^{n-r+1} f(x) \, dx \right]$$

(2.12)

$$\mu_{r:n}^{(k-1)} = \frac{n}{k} \left[ \mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} \right],$$

which proves the relation in (2.9).

**Weibull distribution**

The cdf of two parameter weibull distribution is given as

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - \exp(-x^{p/\theta}), & x > 0; \ p, \theta > 0 \end{cases}$$

(2.13)

**Theorem 2.3: (Balakrishana and Joshi, 1981a)**

For $1 \leq r \leq n$ and $k \geq 1$

$$\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + k \binom{n-1}{r-1} \int_0^\infty x^{k-1} \{\exp(-x^{p/\theta})\}^{n-r+1} \{\exp(-x^{p/\theta})\}^{r-1} \, dx$$

(2.14)

**Proof:** David and Shu (David, 1978) have shown that for an arbitrary distribution with df $F(x)$,
\[ F_{r:n}(x) = F_{r-1:n}(x) - \binom{n}{r-1} [F(x)]^{r-1} [1 - F(x)]^{n-r+1}, \quad (2.15) \]

with \( F_{0:n}(x) = 1 \) for all \( x \), so that (2.15) is true for \( r = 1 \) as well. Further, for any arbitrary distribution, we have (David, 1981)

\[ (r-1)F_{r:n}(x) + (n-r+1)F_{r-1:n}(x) = nF_{r-1:n-1}(x). \quad (2.16) \]

Then on substituting for \( F_{r-1:n}(x) \) from (2.16) in equation (2.15), we get

\[ F_{r:n}(x) = F_{r-1:n-1}(x) - \binom{n-1}{r-1} [F(x)]^{r-1} [1 - F(x)]^{n-r+1}. \quad (2.17) \]

Now with \( F(x) \) as in (2.13) we have (for example, see Parzen, 1960)

\[
\begin{aligned}
\mu_{r:n}^{(k)} &= k \int_0^\infty x^{k-1} [1 - F_{r:n}(x)] dx \\
&= \mu_{r-1:n-1}^{(k)} + k \binom{n-1}{r-1} \int_0^\infty x^{k-1} [\exp(-x^{p/\theta})]^{n-r+1} [\exp(-x^{p/\theta})]^{r-1} dx
\end{aligned}
\]

on using equation (2.17), which establishes (2.14).

In order to evaluate moments of order statistics using the recurrence relation in (2.17) we proceed as follows. For \( m \geq 1 \), consider the integral,

\[
J_k(l, m) = \int_0^\infty x^{k-1} [\exp(-x^{p/\theta})]^{l} [1 - \exp(-x^{p/\theta})]^{m} dx
\]

\[ = J_k(l, m-1) - J_k(l+1, m-1). \quad (2.18) \]

By writing \([1 - \exp(-x^{p/\theta})]^{m}\) as \([1 - \exp(-x^{p/\theta})]^{m-1} [1 - \exp(-x^{p/\theta})]\) and splitting the integral in two. Also

\[
J_k(l, 0) = \int_0^\infty x^{k-1} [\exp(-x^{p/\theta})]^{l} dx
\]

\[ = \left( \frac{\theta}{l} \right)^{k/p} \Gamma\left( \frac{k/p}{p} \right). \]
Thus, $J_k(l,0)$, $l \geq 1$ can be calculated by using gamma function. The function $J_k(l,m)$ for $m \geq 1$ can now be obtained by using equation (2.14) recursively.

In this notation equation (2.18) can be rewritten as

$$
\mu^{(k)}_{r:n} = \mu^{(k)}_{r-1:n-1} + k \binom{n-r}{r-1} J_k(n-r+1,r-1).
$$

Starting with $\mu^{(k)}_{1:1} = k J_k(1,0)$, equation (2.19) enables us to obtain all the single moments of order statistics from a two-parameter Weibull distribution. An advantage of this method is that it avoids the usage of gamma function to a great extent.

**Doubly truncated power function distribution**

The pdf of doubly truncated power function distribution is given as

$$
f(x) = \frac{v a^{-v} x^{-1}}{P - Q}, \quad a Q^{1/v} \leq x \leq a P^{1/v}, \quad a, v \geq 0.
$$

Here, $Q_1 = a Q^{1/v}$, $P_1 = a P^{1/v}$ and $P_2 = P/(P - Q)$, $Q_2 = Q/(P - Q)$.

**Theorem 2.4:** (Balakrishana and Joshi, 1981b)

For $n \geq 1$, and $k = 1, 2, ...$

$$
\mu^{(k)}_{1:n} = (P_2 \mu^{(k)}_{1:n-1} - Q_2 Q_1^k) \frac{n v}{n v + k}
$$

For $n \geq 2$, $2 \leq r < n$ and $k = 1, 2, ...$

$$
\mu^{(k)}_{r:n} = (P_2 \mu^{(k)}_{r:n-1} - Q_2 \mu^{(k)}_{r-1:n-1}) \frac{n v}{n v + k}.
$$

**Proof:** See reference.

**Remark 2.1:** At $P_2 = 1$ and $Q_2 = 0$ in (2.21), we get the result for non-truncated power function distribution, that is
\[ \mu_{r:n}^{(k)} = \mu_{r:n-1}^{(k)} \left( \frac{n\nu}{n\nu + k} \right) \]

as obtained by (Malik, 1967).

**Remark 2.2:** At \( r = n \) in (2.21), we get

\[ \mu_{n:n}^{(k)} = \left( P_2 \nu_k^k - Q_2 \mu_{n-1:n-1}^{(k)} \right) \frac{n\nu}{n\nu + k} \]

as obtained by (Khan et al., 1983a).

**Doubly truncated Pareto distribution**

The pdf of doubly truncated Pareto distribution is given as

\[ f(x) = \frac{\nu a^\nu x^{-\nu-1}}{P - Q}, a(1 - Q)^{-1/\nu} \leq x \leq a(1 - P)^{1/\nu}, a, \nu > 0 \quad (2.22) \]

Here,

\[ P_1 = a(1 - P)^{-1/\nu}, \quad Q_1 = a(1 - Q)^{-1/\nu}, \]

\[ Q_2 = (Q - 1)/(P - Q) \text{ and } P_2 = (P - 1)/(P - Q). \]

Also we have,

\[ 1 - F(x) = \frac{x}{\nu} f(x) + P_2 \quad (2.23) \]

**Theorem 2.5:** (Khan et al., 1983 a)

For \( n > 1 \)

\[ \mu_{1:n-1}^{(k)} = \frac{Q_2}{P_2} Q_1^k \]

For \( 2 \leq r \leq n - 1 \)

\[ \mu_{r:n-1}^{(k)} = \frac{Q_2}{P_2} Q_1^k \]

\[ \mu_{n-1:n-1}^{(k)} = \frac{Q_2}{P_2} P_1^k \]
Proof: In view of (2.23), we have

\[ \mu_{1:n}^{(k)} - Q_1^k = k \int_{Q_1}^{P_1} x^{k-1} \left[ 1 - F(x) \right]^{n-1} \left\{ \frac{x}{y} f(x) + P_2 \right\} dx \]

or

\[ \frac{k}{n} \mu_{1:n}^{(k)} + P_2 [\mu_{1:n-1}^{(k)} - Q_1^k] \]

or,

\[ (nv - k) \mu_{1:n}^{(k)} = [P_2 \mu_{1:n-1}^{(k)} - Q_2 Q_1^k] nv, \ nv \neq k. \] \hspace{1cm} (2.24)

In particular,

\[ (v - k) \mu_{1:1}^{(k)} = [P_2 P_1^k - Q_2 Q_1^k] v, \ v \neq k. \]

For the \( r \)-th order statistic, \( 2 \leq r \leq n - 1 \)

\[ \mu_{r:n}^{(k)} - \mu_{r-1:n}^{(k)} = \frac{k}{nv} \mu_{r:n}^{(k)} + P_2 \left( \frac{n-1}{n-r} \right) [\mu_{r:n-1}^{(k)} - \mu_{r-1:n-2}^{(k)}]. \]

Using the recurrence relation,

\[ r \mu_{r+1:n}^{(k)} = n \mu_{r:n-1}^{(k)} - (n-1) \mu_{r:n}^{(k)} \]

we get

\[ (nv - k) \mu_{r:n}^{(k)} = [P_2 \mu_{r:n-1}^{(k)} - Q_2 \mu_{r-1:n-1}^{(k)}]nv, \ nv \neq k. \] \hspace{1cm} (2.25)

For \( r = n \), it can be seen that

\[ (nv - k) \mu_{n:n}^{(k)} = [P_2 P_1^k - Q_2 \mu_{n-1:n-1}^{(k)}]nv, \ nv \neq k. \] \hspace{1cm} (2.26)

In case of \( nv = k \), from (2.24), (2.25) and (2.26), we get respectively

\[ \mu_{1:n}^{(k)} = \frac{Q_2}{P_2} Q_1^k, \ n > 1, \]

\[ \mu_{r:n}^{(k)} = \frac{Q_2}{P_2} \mu_{r-1:n-1}^{(k)}, \ 2 \leq r < n - 1 \]

\[ \mu_{n-1:n-1}^{(k)} = \frac{P_2}{Q_2} P_1^k. \]
However, this result may not be used to evaluate $\mu_{t_1}^{(k)}$ and $\mu_{n_1}^{(k)}$ when $nv = k$.

For $\mu_{t_1}^{(k)}$, it can be easily seen by direct integration that

$$
\mu_{t_1}^{(k)} = \frac{\log(Q_2/P_2)}{P - Q}, \quad v = k.
$$

Similar results were also obtained by Balakrishnan and Joshi (1982). For non-truncation cases ($P = 1, Q = 0$) one may refer to Malik (1966).

**Burr Distribution**

The pdf of Burr distribution is given as

$$
f(x) = mp\theta x^{p-1}[1 + \theta x^p]^{-(m+1)}, \quad m, p, \theta, x > 0
$$

and corresponding df is given as

$$
F(x) = 1 - [1 + \theta x^p]^{-m}, \quad m, p, \theta, x > 0
$$

(2.27)

**Theorem 2.6: (Ali and Khan, 1995)**

For the Burr distribution as given in (2.25)

$$
\mu_{r:n}^{(k-p)} = mp\theta C_{r:n} J_{k-p} (r-1, n-r+1),
$$

where,

$$
J_k(a,b) = \int_0^\infty \frac{x^{k+p-1}}{(1 + \theta x^p)} (F(x))^a [1 - F(x)]^b \, dx
$$

(2.28)

$$
= \sum_{i=0}^a (-1)^{a-1} \binom{a}{i} J_k(0, a+b-i)
$$

and

$$
J_k(0, b) = \frac{\theta}{p} B\left(1 + \frac{k}{p}, mb - \frac{k}{p}\right),
$$

provided $\left(1 + \frac{k}{p}\right)$ and $\left(mb - \frac{k}{p}\right)$ are neither zero nor negative integers.

**Proof:** See reference.
General form of distributions

(a) \[ F_1(x) = 1 - [a \ h(x) + b]^c \] \[ \text{[Khan and Abu-Salih, 1989]} \]

where \( a \neq 0, \ b, \ c \neq 0 \) are finite constants and \( h(x) \) is continuous, monotonic and differentiable function of \( x \), then the truncated pdf \( f(x) \) is given by

\[ f(x) = -\frac{ca}{P - Q} [ah(x) + b]^{c-1} h'(x), \quad x \in (Q_1, P_1). \] \[ (2.29) \]

The corresponding truncated df \( F(x) \) by

\[ 1 - F(x) = -P_2 - \frac{ah(x) + b}{ca h'(x)} f(x), \] \[ (2.30) \]

where \( P_2 = \frac{1 - P}{P - Q} \).

Theorem 2.7: (Ali and Khan, 1997)

For the distribution given as in (2.29) and \( 1 \leq r \leq n, \ n = 1, 2, \cdots \)

\[ E[g(X_{r:n})] = (1 + P_2)E[g(X_{r-1:n-1})] - P_2 E[g(X_{r-1:n})] \]

\[ - \frac{1}{nca} E[m(X_{r:n})] , \]

where \( g(x) \) be Borel measurable function of \( x \) in the interval \([\alpha, \beta]\) and

\[ m(x) = [ah(x) + b] \frac{g'(x)}{h'(x)}. \]

Proof: Now in view of (1.1.18) and (2.30), we have

\[ E[g(X_{r:n})] - E[g(X_{r-1:n-1})] \]

\[ = - \left( \frac{n - 1}{r - 1} \right)^r \int_{Q_1}^P \! \! \! [g'(x)[F(x)]^{r-1}[1 - F(x)]^{n-r} \left\{ P_2 + \frac{ah(x) + b}{ca h'(x)} f(x) \right\} dx \]

and hence the result.
Theorem 2.8: (Ali and Khan, 1997)

For $1 \leq r \leq n$, $n = 1, 2, \cdots$

$$E[g(X_{r:n})] = E[g(X_{r-1:n-1})] - \frac{(P - Q)(n - r + 1)}{n(n + 1)c} E[z(X_{r:n+1})],$$

where

$$z(x) = [ah(x) + b]^{1-c} \frac{g'(x)}{h'(x)}.$$

Proof: From (2.29), we get

$$1 = -\frac{(P - Q)(ah(x) + b)^{1-c}}{cah'(x)} f(x). \quad (2.31)$$

Now in view of (1.1.18) and (2.31), we have

$$E[g(X_{r:n})] - E[g(X_{r-1:n-1})]$$

$$= \frac{(P - Q)}{ca} \left( n - r \right)^{P_1} \left( r - 1 \right) \int_{Q_1} z(x)[F(x)]^{r-1}[1 - F(x)]^{n-r+1} f(x) dx$$

$$= -\frac{(P - Q)(n - r + 1)}{n(n + 1)c} E[z(X_{r:n+1})]$$

and hence the result.

(b) $F_1(x) = 1 - be^{-ah(x)}$, $x \in (\alpha, \beta)$, \hspace{1cm} [Khan and Abu-Salih, 1989]

where $a \neq 0$, $b \neq 0$ are constants and $h(x)$ is continuous, monotonic and differential function of $x$ in the interval $[\alpha, \beta]$.

The truncated pdf $f(x)$ is given by

$$f(x) = \frac{ab}{P - Q} e^{-ah(x)} h'(x), \hspace{0.5cm} x \in (Q_1, P_1)$$

and

$$1 - F(x) = P_2 - \frac{1}{ah'(x)} f(x) \quad (2.32)$$
Theorem 2.9 (Ali and Khan, 1997)

For $1 \leq r \leq n$, $n = 1, 2, \cdots$

\[
E[g(X_{r:n})] = (1 + P_2)E[g(X_{r-1:n-1})] - P_2E[g(X_{r:n-1})]
+ \frac{1}{na}E[w(X_{r:n})],
\]

where

\[
w(x) = \frac{g'(x)}{h'(x)}
\]

Proof: In view of (1.1.18) and (2.32), we have

\[
E[g(X_{r:n})] - E[g(X_{r-1:n-1})]
= -P_2 \left( \binom{n-1}{r-1} \right) \int_0^1 g'(x)[F(x)]^{r-1}[1-F(x)]^{n-r} dx
+ \frac{1}{a} \left( \binom{n-1}{r-1} \right) \int \left[ w(x) \left[ F(x) \right]^{r-1}[1-F(x)]^{n-r} f(x) \right] dx.
\]

Now on using (1.1.18) and (1.1.19), we can get the required result.

Theorem 2.10: (Ali and Khan, 1997)

For $1 \leq r \leq n$, $n = 1, 2, \cdots$

\[
\left( 1 - \frac{r-1}{n-r+1} P_2 \right) E[g(X_{r:n})]
\]

\[
= (1 + P_2)E[g(X_{r-1:n-1})] - \frac{n P_2}{n-r+1} E[g(X_{r:n-1})]
+ \frac{1}{(n-r+1)a} E[w(X_{r:n})].
\]

Proof: See reference.
3 PRODUCT MOMENTS

Normal distribution

Theorem 3.1: (Govindrajulu, 1963)

For normal and half normal distribution

\[ \mu_{r;n}^{(2)} = 1 + \sum_{s=r}^{n} \mu_{r-1,s;n} - \sum_{s=r+1}^{n} \mu_{r,s;n}, \]

where

\[ \mu_{0,i,n} = 0 \text{ for all } r \geq 1. \]

Proof: In view of (1.1.8), we have

\[ \mu_{r;n}^{(2)} = C_{r;n} \int_{-\infty}^{\infty} x^2 [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \, dx, \]

since \( f'(x) = -xf(x) \), hence we can write

\[ \mu_{r;n}^{(2)} = -C_{r;n} \int_{-\infty}^{\infty} x [F(x)]^{r-1} [1-F(x)]^{n-r} f'(x) \, dx. \] (3.1)

Integrating (3.1) by part, treating \( f'(x) \) for integration and rest of integrand for differentiation, we get

\[ \mu_{r;n}^{(2)} = -C_{r;n} \int_{-\infty}^{\infty} x [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \, dx \]

\[ + C_{r;n} \int_{-\infty}^{\infty} f(x) [F(x)]^{r-1} [1-F(x)]^{n-r} \, dx \]

\[ + x(r-1) [F(x)]^{r-2} [1-F(x)]^{n-r} f(x) \]

\[ - x(n-r) [F(x)]^{r-1} [1-F(x)]^{n-r-1} f(x) \, dx \]

\[ = 0 + 1 + J_1 + J_2, \] (3.2)

where,

\[ J_1 = (r-1)C_{r;n} \int_{-\infty}^{\infty} x [F(x)]^{r-2} [1-F(x)]^{n-r} f^2(x) \, dx, \]

and \[ J_2 = -(n-r)C_{r;n} \int_{-\infty}^{\infty} x [F(x)]^{r-1} [1-F(x)]^{n-r-1} f^2(x) \, dx. \]
The condition \( f'(x) = -xf(x) \) implies that \( \int_{x}^{\infty} y f(y) dy = f(x) \), and hence \( J_1 \) can be written as

\[
J_1 = (r-1)C_{r,n} \int_{\infty}^{\infty} \int_{\infty}^{x} xy[F(x)]^{r-2}[1-F(x)]^{n-r} f(x)f(y) dx dy.
\]

Writing \( 1-F(x) = [F(y)-F(x)] + [1-F(y)] \) and expanding \( [1-F(x)]^{n-r} \) in power of \( [F(y)-F(x)] \) and \( [1-F(y)] \), we get

\[
J_1 = (r-1)C_{r,n} \int_{\infty}^{\infty} \int_{\infty}^{x} xy[F(x)]^{r-2}\sum_{s=r}^{n} \binom{n-r}{s-r} [F(y)-F(x)]^{s-r} \times [1-F(y)]^{n-s} f(x)f(y) dx dy
\]

\[
= \sum_{s=r}^{n} \frac{n!}{(r-2)!(s-r)!(n-s)!} \int_{\infty}^{\infty} \int_{\infty}^{x} xy[F(x)]^{r-2}[F(y)-F(x)]^{s-r} \times [1-F(y)]^{n-s} f(x)f(y) dx dy
\]

\[
J_1 = \sum_{s=r}^{n} \mu_{r-1,s:n}.
\]

Similarly,

\[
J_2 = -\sum_{s=r+1}^{n} \mu_{r,s:n}
\]

Substituting these values of \( J_1 \) and \( J_2 \) in equation (3.2), we get

\[
\mu_{r,n}^{(2)} = 1 + \sum_{s=r}^{n} \mu_{r-1,s:n} - \sum_{s=r+1}^{n} \mu_{r,s:n}.
\]
Standard logistic distribution

The pdf of standard logistic distribution is given as

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty$$  \hfill (3.3)

and corresponding df is

$$F(x) = \frac{1}{(1 + e^{-x})}, \quad -\infty < x < \infty$$  \hfill (3.4)

In view of (3.3) and (3.4), we have

$$f(x) = F(x)[1 - F(x)]$$  \hfill (3.5)

**Theorem 3.2:** (Shah, 1970)

For the standard logistic population, we have

$$\mu_{r, r+b+1} = \frac{n+1}{n-r+1} \left[ \mu_{r, r+b+1} - \frac{r}{n+1} \mu_{r+1, r+1+b+1} - \frac{1}{n-r} \mu_{r, r+b+1} \right], 1 \leq r \leq n-1$$  \hfill (3.6)

and

$$\mu_{r, s, s+b+1} = \frac{n+1}{n-s+2} \left[ \mu_{r, s, s+b+1} - \mu_{r, s+b+1, s+b+1} - \frac{n-s+2}{n+1} \mu_{s, s-b+1, s+b+1} - \frac{1}{n-s+1} \mu_{r, r+b+1} \right],$$

$$1 \leq r < s \leq n-1, s-r \geq 2.$$  \hfill (3.7)

**Prof:** From the joint density function of $X_{r,n}$ and $X_{s,n}$, we may write for $1 \leq r < s \leq n$

$$\mu_{r:n} = E[X_{r:n} X_{s:n}^0]$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{-\infty}^{\infty} x[F(x)]^{r-1}[I(x)]f(x)dx$$  \hfill (3.8)

where,

$$I(x) = \int_{-\infty}^{\infty} [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1} F(y) dy.$$
By writing \( F(y) = [1 - (1 - F(y))] \), \( I(x) \) can be written as

\[
I(x) = \int_{-\infty}^{\infty} [F(y) - F(x)]^{s-r+1} [1 - F(y)]^{n-s} dy
- \int_{-\infty}^{x} [F(y) - F(x)]^{s-r+1} [1 - F(y)]^{n-s+2} F(y) dy.
\]

Now integrating by parts, we obtain for \( s = r + 1 \),

\[
I(x) = (n-r) \int_{-\infty}^{\infty} y [1 - F(y)]^{n-r-1} f(y) dy
- (n-r+1) \int_{-\infty}^{\infty} y [1 - F(y)]^{n-r} f(y) dy - xF(x) [1 - F(x)]^{n-r}
\tag{3.9}
\]

and for \( s - r \geq 2 \)

\[
I(x) = \left\{ (n-s+1) \int_{-\infty}^{\infty} y [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) dy
\right.

\* \left. f(y) dy - (s-r-1) \int_{-\infty}^{\infty} y [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+1} f(y) dy \right\}

- \left\{ (n-s+2) \int_{-\infty}^{\infty} y [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1}
\right.

\* \left. f(y) dy - (s-r-1) \int_{-\infty}^{\infty} y [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+2} f(y) dy \right\}
\tag{3.10}

The recurrence relations in (3.6) and (3.7) follow upon substituting the value of \( I(x) \) from (3.9) and (3.10), into equation (3.8) and then simplifying the resulting equations.

**Standard exponential distribution**

**Theorem 3.3:** (Joshi, 1978)

For the standard exponential distribution as given in (2.5),

\[
\mu_{r,r+1:n} = \mu_{r:n}^{(2)} + \frac{1}{n-r} \mu_{r:n}, \ 1 \leq r \leq n-1
\tag{3.11}
\]
and

\[ \mu_{r,s:n} = \mu_{r,s-1:n} + \frac{1}{n-s+1} \mu_{r:n}, \quad 1 \leq r < s \leq n, \ s-r \geq 2. \]  

(3.12)

**Proof:** To establish the relations, we shall first of all write, for \(1 \leq r < s \leq n\),

\[ \mu_{r:n} = E[X_{r:n} X_{s:n}^0] = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{0 \leq x < y < \infty} x[F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x)f(y) \, dx \, dy \]

(3.13)

where,

\[ I(x) = \int_{x_r}^\infty [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) \, dy. \]  

(3.14)

In view of (2.7) and (3.14), we have

\[ I(x) = \int_{x_r}^\infty [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} f(y) \, dy. \]  

(3.15)

Integrating the right-hand side of (3.15) by parts, treating \(dy\) for integration and the rest of the integrand for differentiation, we obtain, when \(s = r + 1\)

\[ I(x) = (n-r) \int_x^\infty y[1 - F(y)]^{n-r-1} f(y) \, dy - [1 - F(x)]^{n-r} \]  

(3.16)

and, when \(s-r \geq 2\)

\[ I(x) = (n-s+1) \int_x^\infty y[F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) \, dy \]

\[ - (s-r-1) \int_x^\infty y[F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+1} f(y) \, dy. \]  

(3.17)

The relation in (3.11) and (3.12) follow readily when we substitute the expression of \(I(x)\) in equation (3.16) and (3.17) respectively, into equation (3.13) and simplifying the resulting equations.
Doubly truncated Weibull distribution

The pdf of doubly truncated weibull distribution is given as

\[ f(x) = \frac{p x^{p-1} e^{-x^p}}{P - Q}, \quad -\log(1 - Q) \leq x^p \leq -\log(1 - P), \quad P > 0 \]  \hspace{1cm} (3.18)

where,

\[ Q_1^P = -\log(1 - Q), \quad P_1^Q = -\log(1 - P) \]

\[ Q_2 = (1 - Q)/(1 - P), \quad P_2 = (1 - P)/(P - Q) \]

and

\[ [1 - F(x)] = -P_2 + \frac{e^{-x^p}}{P - Q} \]

\[ = -P_2 + \frac{1}{p} x^{1-p} f(x) \]  \hspace{1cm} (3.19)

**Theorem 3.4:** (Khan et al., 1983b)

For \( 1 \leq r < s \leq n, \quad s - r \geq 2 \)

\[ \mu_{r,s:r+l,n}^{(j,k)} = \mu_{r,n}^{(j+k)} - \frac{n P_2}{(n - r)} [\mu_{r,r+l,n-1}^{(j,k)} - \mu_{r,n-1}^{(j+k)}] + \frac{k}{p(n - r)} \mu_{r,r+l,n}^{(j,k,p)} \hspace{1cm} (3.20) \]

**Proof:** In view of (3.19), we get

\[ \mu_{r,s;1:n}^{(j,k)} - \mu_{r,s-1;1:n}^{(j,k)} = C_{r,s-1;1:n} k P_1 \int_{Q_1} x^j y^{k-1} [F(x)]^{r-1}[F(y) - F(x)]^{s-r-1} \]

\[ \times [1 - F(y)]^{n-s} \left\{ -P_2 + \frac{1}{p} y^{1-p} f(y) \right\} f(x) dy \ dx \]

\[ = -P_2 \frac{1}{n - s + 1} C_{r,s;n} k P_1 \int_{Q_1} x^j y^{k-1} [F(x)]^{r-1}[F(y) - F(x)]^{s-r-1} \]

\[ \times [1 - F(y)]^{n-s} f(x) dy \ dx + \frac{k}{p(n - s + 1)} C_{r,s;n} \int_{Q_1} x^j y^{k-p} [F(x)]^{r-1} \]

\[ \times [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(x) f(y) dy \ dx \]
Therefore,

\[
\mu_{r,s:n}^{(j,k)} = \mu_{r,s-1:n}^{(j,k)} - \frac{n P_2}{n-s+1} [\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}] + \frac{k}{p(n-s+1)} \mu_{r,s:n}^{(j,k-p)}
\] (3.21)

At \( s = r+1 \), (3.21) reduces to

\[
\mu_{r,r+1:n}^{(j,k)} = \mu_{r:n}^{(j,k)} - \frac{n P_2}{n-r} [\mu_{r,r+1:n-1}^{(j,k)} - \mu_{r:n-1}^{(j,k)}] + \frac{k}{p(n-r)} \mu_{r,r+1:n}^{(j,k-p)}.
\]

### Doubly truncated Pareto distribution

**Theorem 3.5:** (Khan et al., 1983 b)

For the distribution as given in (2.22) and \( n \geq 2 \),

\[
(v-k)\mu_{n-1:n,n:n}^{(j,k)} = v[\mu_{n-1:n}^{(j+k)} + n P_2 (\mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)})].
\]

**Proof:** In view of (2.23), we have

\[
\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s:n} k \left[ P_1 \int_{\tilde{x}}^{\infty} x j y^{k-1} [F(x)]^{r-1} \times [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} \left\{ \frac{y}{v} f(y) + P_2 \right\} f(x) dy dx \right]
\]

\[
= \frac{k}{v(n-s+1)} \mu_{r,s:n}^{(j,k)} + \frac{n P_2}{n-s+1} [\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}],
\]

or,

\[
[v(n-s+1) - k] \mu_{r,s:n}^{(j,k)} = v[(n-s+1) \mu_{r,s-1:n}^{(j,k)} + n P_2 (\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)})],
\]

\[1 \leq r < s < n - 2, \ s - r \geq 2 \] and \( v(n-s+1) \neq k \).

However, if \( k = v(n-s+1) \), then

\[
(n-s+1) \mu_{r,s-1:n}^{(j,k)} = n P_2 (\mu_{r,s-1:n-1}^{(j,k)} - \mu_{r,s:n-1}^{(j,k)}).
\]

Marginal results for \( k \neq v(n-s+1) \) can easily be seen as
Recurrence relations for $j=k=1$ have been studied by Balakrishnan and Joshi (1982). Malik (1966) has obtained these results for $P=1, Q=0$. To evaluate $\mu_{r,n}^{(j,k)}$, one may require the recurrence relations for $\mu_{r,n}^{(i)}$ for which we refer to Khan et al. (1983a).

**Doubly truncated power function distribution**

**Theorem 3.6:** (Khan et al., 1983b)

For doubly truncated power function distribution as given in (2.20) and $n \geq 2$,

$$
\frac{\mu_{n-1,n,n}^{(j,k)}}{\mu_{n-1,n-1,n}^{(j,k)}} = \frac{v}{v+k}\left[\mu_{n-1:n}^{(j+k)} + nP_2(p_{1:n}^{k} \mu_{n-1:n-1}^{(j)} - \mu_{n-1:n-1}^{(j+k)})\right].
$$

**Proof:** See reference.

**General form of distributions**

(a) $F_1(x) = 1 - [ah(x)+b]^c$, $x \in (\alpha, \beta)$,

where $a \neq 0$, $b$, $c \neq 0$ are finite constants and $h(x)$ is a continuous monotonic and differentiable function of $x$ in the interval $[\alpha, \beta]$.

**Theorem 3.7:** (Ali and Khan, 1998)

For $1 \leq r < s \leq n$, $n = 1, 2, \ldots$

$$
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] = \frac{nP_2}{n-s+1}\{E[g(X_{r:n-1}, X_{s:n-1})] - E[g(X_{r:n-1}, X_{s-1:n-1})]\} - \frac{1}{(n-s+1)ca}E[m(X_{r:n}, X_{s:n})],
$$
where \( m(x, y) = [ah(y) + b]h'(y) \).

**Proof:** From (1.1.22) and (2.28), we have

\[
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] = \frac{C_{r,s,n}}{(n-s+1)} \iint_{Q_{1} \leq x \leq y \leq Q_{n}} \frac{\partial}{\partial y} g(x, y) [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1}
\]

\[
\times [1 - F(y)]^{n-s} \left[ -P_2 - \frac{ah(y) + b}{cah'(y)} f(y) \right] f(x) dx \, dy
\]

and hence the theorem.

**Theorem 3.8:** (Ali and Khan, 1998) For \( 1 \leq r < s \leq n, \, n=1,2,\ldots \)

\[
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] = \frac{(P-Q)}{(n+1)ca} E[z(X_{r:n+1}, X_{s:n+1})],
\]

where \( z(x, y) = [ah(y) + b]^{1-c} \frac{\partial}{\partial y} g(x, y) \).

**Proof:** From (1.1.22) and (2.29), we have

\[
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] = \frac{C_{r,s,n}}{(n-s+1)} \iint_{Q_{1} \leq x \leq y \leq Q_{n}} \frac{\partial}{\partial y} g(x, y) [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1}
\]

\[
\times [1 - F(y)]^{n-s+1} \left[ -\frac{(P-Q)[ah(y) + b]^{1-c}}{cah'(y)} f(y) \right] f(x) dx \, dy
\]

\[
= \frac{(P-Q)}{(n+1)ca} E[z(X_{r:n+1}, X_{s:n+1})].
\]
To obtain results for non-truncation distribution, set \( P = 1 \) and \( Q = 0 \).

(b) \( F_1(x) = 1 - be^{-ah(x)}, \ x \in (\alpha, \beta) \),

where \( a \neq 0, \ b, c \neq 0 \) are constants and \( h(x) \) is continuous, monotonic and differentiable function of \( x \) in the interval \( [\alpha, \beta] \).

**Theorem 3.9:** (Ali and Khan, 1998) For \( 1 < r < s \leq n, \ n = 1, 2, \ldots \)

\[
E[g(X_{r:n}, X_{s:n})] = E[g(X_{r:n}, X_{s-1:n})]
\]

\[
= -\frac{nP_2}{n - s + 1} \{E[g(X_{r:n-1}, X_{s:n-1})] - E[g(X_{r:n-1}, X_{s-1:n-1})]\}
\]

\[
+ \frac{1}{(n - s + 1)a} E[w(X_{r:n}, X_{s:n})],
\]

where \( w(x, y) = \frac{\partial}{\partial y} \frac{g(x, y)}{h'(y)} \).

**Proof:** The result is straightforward in view of (1.1.22) and (2.30).

**Theorem 3.10:** (Ali and Khan, 1998) For \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)

\[
E[g(X_{r:n}, X_{s:n})] = E[g(X_{r:n}, X_{s-1:n})] + \frac{(P - Q)}{(n+1)ab} E[T(X_{r:n+1}, X_{s:n+1})],
\]

where \( T(x, y) = e^{a h(y)} \frac{\partial}{\partial y} \frac{g(x, y)}{h'(y)} \).

**Proof:** The result is obvious.
CHAPTER III
ORDER STATISTICS: CHARACTERIZATION OF DISTRIBUTIONS

1 INTRODUCTION

Characterization is a condition involving certain properties of random variable $X$ which identifies the associated distribution function $F(x)$. The property that uniquely determines $F(x)$ may be based on a function of random variables whose joint distribution is related to that of $X$. A characterization can be of use in the construction of goodness of fit test and in the examinations of the consequences of modeling assumption made by an applied scientist.

Ferguson (1967) introduced the characterization of distribution based on the linearity of regression of adjacent order statistics $E(X_{r+1:n} \mid X_{r:n} = x)$ and its dual $E(X_{r:n} \mid X_{r+1:n} = x)$.

Shanbhag (1970) characterized exponential and geometric distributions in terms of conditional expectations. His result for exponential distributions is further generalized by Hamdan (1972).

Khan and Khan (1987) characterized Burr type XII distribution through linear regression of order statistics for a single order gap, whereas Khan and Abu-Salih (1989) characterized a general class of distribution through conditional expectations of function order statistics by means of the relations

$$E[h(X_{r+1:n}) \mid X_{r:n} = x] = a^* h(x) + b^*$$

and

$$E[h(X_{r:n}) \mid X_{r+1:n} = x] = a_1^* h(x) + b_1^*.$$  

Wesolowski and Ahsanullah (1997) characterized distributions by the regression of non-adjacent order statistics through the relation

$$E[X_{r+2:n} \mid X_{r:n} = x] = ax + b.$$  

Using the result of Rao and Shanbhag (1994) dealing with an extended version of integrated Cauchy functional equation, Dembińska and Wesolowski (1998)
and Athar et al. (2003) characterized distribution by means of regression equation

\[ E[X_{r+1:n} | X_{r:n} = x] = ax + b. \]

Characterization of distribution via linearity of regression of order statistics when gap is higher is also considered by Khan and Ali (1987) and Franco and Ruiz (1997).

Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the general form of distributions for higher order gap. Khan and Athar (2000) also characterized some continuous distributions through linearity of regression when conditioned on a pair of order statistics.

Characterization of continuous distributions by conditional variance of adjacent order statistics is first considered by Beg and Kirmani (1978). They shown that

\[ V[X_{r+1:n} | X_{r:n} = x] = \frac{1}{a^2(n-r)^2}, \]

if and only if \( X \) has exponential distribution.

Khan and Beg (1987) extended the result and proved that the conditional variance of \( X^P_{r+1:n} \) given \( X_{r:n} = x \) does not depend on \( x \) if and only if \( X \) has Weibull distribution.

2 CHARACTERIZATION OF SPECIFIC DISTRIBUTIONS

Uniform distribution

**Theorem 2.1:** (Ahsanullah, 1989)

Let \( X \) be a positive and bound random variable having an absolutely continuous \((w.r.t \text{ Lébesque measure})\) distribution function \( F(x) \). We will assume without any loss of generality that \( F(1) = 1 \), when the following two statements are equivalent,
(a) If $X$ is distributed as $U(0,1)$, then for all $n \geq 2$, $X_{1:n}/X_{2:n}$ is distribution as $U(0,1)$.

(b) If for some fixed $n$, $n \geq 2$, $X_{1:n}/X_{2:n}$ is distributed as $U(0,1)$, then $X$ is distributed as $U(0,1)$.

**Proof:**

Let $U_1 = X_{1:n}/X_{2:n}$, then $U_1$ has the pdf

$$f_{U_1}(u_0) = \int_0^1 n(n-1)[F(u_2)]^{n-2}f(u_0,u_2)du_2, 0 \leq u_0 \leq 1,$$

and corresponding df is

$$F_{U_1}(u_0) = \int_0^1 n(n-1)[F(u_2)]^{n-2}f(u_0,u_2)f(u_2)du_2, 0 \leq u_0 \leq 1$$

where $\bar{F} = 1 - F$, substituted $F(x) = x$, $0 < x < 1$, it follows easily that $F_{U_1}(u_0) = u_0, 0 \leq u_0 \leq 1$.

To prove (b) assume that $U_1$ is defined as $U(0,1)$. Then

$$u_0 = \int_0^1 n(n-1)[\bar{F}(u_2)]^{n-2}\bar{F}(u_0,u_2)f(u_2)du_2$$

(2.1)

for all $u_0$, $0 \leq u_0 \leq 1$. But we know that

$$[n(n-1)]^{-1} = \int_0^1 [\bar{F}(u_2)]^{n-2}\bar{F}(u_2)f(u_2)du_2$$

(2.2)

From (2.1) and (2.2), we have on simplification

$$\int_0^1 [\bar{F}(u_2)]^{n-2}f(u_2)[\bar{F}(u_0,u_2) - u_0\bar{F}(u_2)]du_2 = 0$$

(2.3)

for all $u_0$, $0 \leq u_0 \leq 1$. Hence we have from (2.3)

$$F(u_0,u_2) = u_0F(u_2)$$

(2.4)

for all $u_0, 0 \leq u_0 \leq 1$ and almost all $u_2, 0 \leq u_2 \leq 1$.

The only non zero continuous of (2.4) is $F(x) = x, 0 \leq x \leq 1$. 
It is a characteristic property of uniform distribution (over the unit interval), then $X$ and $X_{1,n}/X_{2,n}$ characterize a family of distributions of which the uniform distribution is a member.

**Weibull distribution**

**Lemma 2.1:** (Khan and Ali, 1987)

Let $X$ be a continuous r.v. having $df\ F(x)$, then $F(x)$ is unique if

$$E[X_{r+i:n}^k | X_{r:n} = x], \ 1 \leq i \leq n-r, \ 1 \leq r \leq n \text{ exists.}$$

**Lemma 2.2:** (Khan and Ali, 1987)

Let $X$ be a continuous r.v. having $df\ F(x)$, then $F(x)$ is unique if

$$E[X_{r:n}^k | X_{r+i:n} = x], \ 1 \leq i \leq n-r, \ 1 \leq r \leq n \text{ exists.}$$

**Theorem 2.2:** (Khan and Ali, 1987)

Let $X$ be a continuous r.v. having $df\ F(x)$ with $F(0) = 0$ and $E(x^k) < \infty$, $k > 0$. If $F(x) < 1$ for all $x < \infty$,

then, $F(x) = 1 - e^{-\theta x^p}$, $x \geq 0, \theta > 0, \ p > 0$, if and only if for $r < n, \ 1 \leq i \leq n-r$,

$$E[X_{r+i:n}^p | X_{r:n} = x] = x^p + \frac{1}{\theta} \sum_{l=0}^{i-1} \frac{1}{n-r-l}.$$ 

**Proof:** In view of Khan et al. (1983a), we have

$$\mu_{r:n}^{(k)} = Q_2 \mu_{r-1:n-1}^{(k)} - P_2 \mu_{r:n-1}^{(k)} + \frac{k}{np\theta} \mu_{r:n}^{(k-p)}$$

In case of left truncation ($P = 1$), $Q_2 = 1, P_2 = 0$

Thus,

$$\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + \frac{k}{np\theta} \mu_{r:n}^{(k-p)}.$$
That is,
\[ \mu_{r:n-r}^{(p)} = \mu_{r-1;n-r-1}^{(p)} + \frac{1}{(n-r)\theta}. \]

Necessary part is proved by noting that
\[ \mu_{0:n-r-i}^{(k)} = Q_1^p = x^p \]
a sufficient part follows from lemma 2.1.

For \( i = 1 \), the theorem was proved by Khan and Beg (1987). Also, if we put \( p = 1 \) and \( i = 1 \), we get the result obtained by Ferguson (1967). At \( p = 2 \), it characterizes Rayleigh distribution.

**Burr distribution**

**Theorem 2.3: (Khan and Ali, 1987)**

Let \( X \) be a r.v. having continuous df \( F(x) \) with \( F(0) = 0 \) and \( E(X^k) < \infty, k > 0 \). If \( F(x) < 1 \) for all \( X < \infty \), then \( F(x) = 1 - (1 + \theta x^p)^{-m} \), \( x \geq 0, \theta > 0, p > 0, m > 0 \) if and only if for \( r < n, 1 \leq i \leq n-r \),

\[
E[x_{r+1:n}^{p} | x_{r:n} = x] =
\begin{cases}
  x^p \frac{m(n-r) \theta^{-1} + m(n-r)}{m(n-r) - 1} & i = 1, \\
  x^p \sum_{i=0}^{i-1} \frac{m(n-r-i)}{m(n-r-i) - 1} + \frac{1}{\theta^{[m(n-r)-1]} \theta^{i}[m(n-r)-i]} \left[ 1 + \sum_{j=0}^{i-2} \prod_{i=0}^{j} \frac{m(n-r-j)}{m(n-r-j) - 1} \right], & i \geq 2
\end{cases}
\]

**Proof:** In view of Khan and Khan (1986),

\[
[1 - \frac{k}{mn \theta}] \mu_{r:n}^{(k)} = Q_2 \mu_{r-1;n-1}^{(k)} - P_2 \mu_{r;n-1}^{(k)} + \frac{k}{mn \theta} \mu_{r;n}^{(k-p)}.
\]

For \( k \neq mn \), \( 1 \leq r \leq n \), where

\[
\theta Q_1^p = [(1 - Q)^{-1} - 1], \theta P_1^p = [(1 - P)^{-1} - 1], Q_2 = \frac{1 - Q}{P - Q}, P_2 = \frac{1 - P}{P - Q}.
\]

Therefore, for the left truncation only, \( P_2 = 0, Q_2 = 1 \).
This gives,

\[ \mu_{i,n-r}^{(p)} = \frac{m(n-r)}{m(n-r)-1} \mu_{i-1,n-r-1}^{(p)} + \frac{1}{\theta(m(n-r)-1)}. \]

Noting that, \( \mu_{0,n}^{(p)} = Q_1^p = x^p \), we prove the necessary part. Sufficient parts follow from lemma 2.1.

If we put \( i = 1 \), we get the result of Khan and Khan (1986 b). For \( m = 1 \), it characterizes log-logistic distribution.

**Pareto distribution**

**Theorem 2.4: (Khan and Ali, 1987)**

Let \( X \) be a continuous r.v. having \( df \ F(x) \) with \( F(a) = 0 \) and \( E(X^k) < \infty, k > 0 \). If \( F(x) < 1 \) for all \( x < \infty \), then \( F(x) = 1 - a^p x^{-p}, x \geq a, \ a > 0, \ p > 0 \), if and only if for \( r < n, 1 \leq i \leq n-r \),

\[ E[x_{r+i:n}^k | x_{r:n} = x] = x^k \prod_{l=0}^{i-1} \frac{p(n-r-l)}{p(n-r-l)-k} \]

**Proof:** See reference.

**Power function distribution**

**Theorem 2.5: (Khan and Ali, 1987)**

Let \( X \) be a continuous r.v. having \( df \ F(x) > 0 \) for \( 0 < x < a \) with \( F(a) = 1 \) and \( E(X^k) < \infty, k > 0 \),

then,

\[ F(x) = a^p x^{-p}, 0 \leq x \leq a, \ a > 0, \ p > 0, \]

if and only if for \( 1 \leq r < n, 1 \leq i \leq n-r \),

\[ E[x_{r:n}^k | x_{r+i:n} = x] = x^p \prod_{l=0}^{i-1} \frac{p(r+l)}{p(r+l)+k}. \]

**Proof:** See reference.
3 CHARACTERIZATION OF SOME GENERAL FORM OF DISTRIBUTIONS

(a) \( F(x) = 1 - [ah(x) + b]^c \), \( x \in (\alpha, \beta) \).

**Theorem 3.1:** (Khan and Abu-Salih, 1989) Let \( X \) be an absolutely continuous random variable with \( df F(x) \) and \( pdf f(x) \). Suppose, \( F(x) < 1 \) for all \( x \in (\alpha, \beta) \), \( F(\alpha) = 0 \) and \( F(\beta) = 1 \).

then,

\[
F(x) = 1 - [ah(x) + b]^c, \quad for \ x \in (\alpha, \beta)
\]

if and only if for \( r < n \),

\[
E[h(X_{r+1:n}) | X_{r:n} = x] = \frac{ac(n-r)h(x)-b}{a[n-r]c+1}
\]

where \( h() \) is a monotonic, continuous and differentiable function on \( (\alpha, \beta) \), \( a \neq 0 \), \( (n-r)+1 \neq 0 \)

**Proof:** Note that

\[
1 - F(x) = [ah(x) + b]^c = \frac{ah(x) + b}{ach'(x)} f(x)
\]

Now in view of (3.1)

\[
E[h(h(X_{r+1:n}) | X_{r:n} = x) = \frac{(n-r)}{(1 - F(x))^{n-r}} \int_{x}^{\beta} h(y)(1 - F(y))^{n-r-1} f(y)dy
\]

Integrating by parts and noting the relation (3.2), it is easy to prove the necessary part. To prove the sufficiency part, we have

\[
(n-r) \int_{x}^{\beta} h(y)(1 - F(y))^{n-r-1} f(y)dy = \frac{[ac(n-r)h(x)-b]}{a[c(n-r)+1]}(1 - F(x))^{n-r}
\]

Differentiating both sides w.r.t. \( x \) and rearranging we get

\[
\frac{f(x)}{1 - F(x)} = \frac{ach'(x)}{ah(x) + b}
\]

which gives

\[
1 - F(x) = [ah(x) + b]^c.
\]
Remark 3.1: At \( r = n - 1 \),
\[
a = \frac{f(k) - 1}{(f(k) - 1)h(\alpha) + g(k)}
\]
\[
b = \frac{g(k)}{(f(k) - 1)h(\alpha) + g(k)}
\]
and
\[
c = \frac{f(k)}{1 - f(k)}
\]

Theorem reduces the result of Talwalkar (1977).

Lemma 3.1: For any continuous and differentiable function \( h() \) and \( 1 \leq r < s \leq n \).
\[
E[h(X_{s:n}) | X_{r:n} = x] - E[h(X_{s-1:n}) | X_{r:n} = x]
= \sum_{n-r}^{n-s} \frac{1}{(s - r - 1)!} \int_{x}^{
fty} h'(y)[F(y) - F(x)]^{s-r-1} \times [1 - F(y)]^{n-s+1} dy.
\] (3.3)

Lemma 3.2: For any continuous and differentiable function \( h() \) and \( 1 \leq r < s \leq n \),
\[
E[h(X_{r:n}) | X_{s:n} = y] - E[h(X_{r-1:n}) | X_{s:n} = y]
= \sum_{r-1}^{s-1} \frac{1}{(r - 1)!} \int_{-\infty}^{y} h'(y)[F(y)]^{r-1}[F(y) - F(x)]^{s-r} dx.
\]

Theorem 3.2: (Khan and Abouammoh, 2000) Let \( X \) be an absolutely continuous random variable with \( df \ F(x) \) and \( pdf \ f(x) \) in the interval \((\alpha, \beta)\), where \( \alpha \) and \( \beta \) may be finite or infinite. Then for continuous and differentiable function \( h(x) \) of \( x \) and for \( 1 \leq r < s \leq n \)
\[ E[h(X_{s:n})|X_{r:n} = x] = h(x) \prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c(n-r-j)+1} \]
\[ -\frac{b}{a} \sum_{j=0}^{s-r-1} \frac{1}{c(n-s+1+j)} \prod_{i=0}^{j} \frac{c(n-s+1+i)}{c(n-s+1+i)+1} \] (3.4)

if and only if,

\[ F(x) = 1 - \left[ ah(x) + b \right]^{c}, \quad \alpha \leq x \leq \beta, \] (3.5)

where,

\[ a \neq 0, c \neq 0, \quad c(n-r-j)+1 \neq 0 \quad \text{for} \quad j = 0, 1, ..., (s-r-1) \quad \text{and} \quad b = -ah(\beta). \]

**Proof:** First we have to prove that (3.5) implies (3.4), we have

\[ E[h(X_{r:n})] - E[h(X_{-1:n})] = \left( \begin{array}{c} n \\ r-1 \end{array} \right) \int_{P_1}^{1} \frac{1}{Q_1 h'(x)[F(x)]^{r-1}} [1 - F(x)]^{n-r+1} dx \] (3.6)

For doubly truncated distribution (3.5), we have

\[ 1 - F(x) = \frac{1-P}{P-Q} \frac{ah(x)+b}{ca h'(x)} f(x). \]

Expressing \( [1 - F(x)]^{n-r+1} \) as \( [1 - F(x)]^{n-r} \left\{ \frac{1-P}{1-Q} \frac{ah(x)+b}{ca h'(x)} f(x) \right\} \)

and putting \( Q_1 = x, \quad P_1 = \beta, \quad Q = F(x), \quad P = 1, \) after noting the relation between truncated and conditional distribution, we get form (3.6),

\[ E[h(X_{s,n})|X_{r:n} = x] - E[h(X_{-1:n})|X_{r:n} = x] \]
\[ = \left( \begin{array}{c} n \\ s-1 \end{array} \right) \frac{1}{ca} \int_{0}^{\beta} (ah(t)+b)[F(t)]^{s-1}[1 - F(t)]^{n-s} f(t) dt \]
\[ - \frac{1}{c(n-s+1)} \left\{ E[h(X_{s,n})|X_{r:n}] + \frac{b}{a} \right\} . \]
Therefore,

\[ E[h(X_{s:n}) \mid X_{r:n} = x] = \frac{c(n-s+1)}{c(n-s+1)+1} E[h(X_{s-1:n}) \mid X_{r:n} = x] \]

\[ - \frac{b}{a} \frac{1}{c(n-s+1)+1}. \]  \hspace{1cm} (3.7)

Using (3.7) recursively and noting that \( E[h(X_{r:n}) \mid X_{r:n} = x] = h(x) \), the relation (3.4) is established.

To prove that (3.4) implies (3.5), we have from (3.3) and (3.7)

\[ c(n-s+1) \left( E[h(X_{s:n}) \mid X_{r:n} = x] - E[h(X_{s-1:n}) \mid X_{r:n} = x] \right) \]

\[ = -E[h(X_{s:n}) \mid X_{r:n} = x] - \frac{b}{a}, \]

or

\[ c(n-s+1) \left( \frac{n-r}{s-r-1} \right) \frac{1}{[1-F(x)]^{n-r}} \int_0^\beta h'(y) \left[ F(y) - F(x) \right]^{s-r-1} [1-F(y)]^{n-s+1} dy \]

\[ = -\frac{(n-r)!}{(s-r-1)!} \frac{1}{(n-s)!} \frac{1}{[1-F(x)]^{n-r}} \int_0^\beta h(y) \left[ F(y) - F(x) \right]^{s-r-1} \]

\[ \times [1-F(y)]^{n-s} f(y) dy - \frac{b}{a}. \]

Differentiating both sides \((s-r)\) times partially w.r.t. \( x \), after noting that

\[ \frac{\partial}{\partial x} \int_{u(x)}^{v(x)} f(x,t) dt = \int_{u(x)}^{v(x)} \left[ \frac{\partial}{\partial x} f(x,t) \right] dt + f(v,t) \frac{\partial v}{\partial x} - f(u,t) \frac{\partial u}{\partial x} \]

we get,

\[ -h'(x) [1-F(x)]^{n-s+1} - h(x)[1-F(x)]^{n-s} f(x) = \frac{b}{a} [1-F(x)]^{n-s} f(x) \]

or

\[ -\frac{f(x)}{1-F(x)} = \frac{cah'(x)}{ah(x)+b} \]
implying

\[ 1 - F(x) = [ah(x) + b]^c \]

and hence the theorem.

**Theorem 3.3:** (Khan and Athar 2002)

For any continuous and differentiable function \( h() \) and \( 1 \leq r < s \leq n \),

\[
E[h(X_{r+1:n}) | X_{r:n} = x, X_{s:n} = y] = (-1)^m \sum_{i=0}^{m-1} \binom{m}{i} (-1)^i P_i \frac{1}{P-Q} \prod_{j=0}^{i-1} \frac{(m-j)c}{(m-j)c+1} 
\]

if and only if

\[
F(x) = 1 - [ah(x) + b]^c, \alpha \leq x \leq \beta \quad (3.9)
\]

where \( \alpha \) and \( \beta \) are such that \( F(\alpha) = 0, F(\beta) = 1 \),

and \( m = s - r - 1, P_2 = \frac{1-P}{P-Q}, Q_2 = \frac{1-Q}{P-Q} \)

\( a \neq 0, (m-i)c \neq 0, (m-i)c+1 \neq 0 \) for \( i = 0, 1, \ldots, m-1 \)

**Proof:** See Reference.

**Theorem 3.4:** (Khan et al., 2009b)

Let \( X \) be an absolutely continuous random variable with \( df F(x) \) and \( pdf f(x) \) in the interval \( (\alpha, \beta) \), where \( \alpha \) and \( \beta \) may be finite or infinite, then for \( 1 \leq m < r < s \leq n \),

\[
E[h(X_{s:n}) - h(X_{r:n}) | X_{m:n} = x] = \frac{1}{a} \sum_{j=r}^{n-1} \frac{1}{(n-j)} \quad (3.10)
\]

if and only if
\[ F(x) = 1 - e^{-ah(x)}, \quad a \neq 0 \tag{3.11} \]

Where \( h(x) \) is a monotonic and differentiable function of \( x \) such that \( h(x) \to 0 \) as \( x \to \alpha \) and \( h(x)\{1 - F(x)\} \to 0 \) as \( x \to \beta \).

**Proof:** First we will prove (3.11) implies (3.10). For \( 1 \leq r < s \leq n \),

\[
E[h(X_{s:n}) - h(X_{r:n}) \mid X_{r:n} = x] = \left( \frac{n-r}{s-r-1} \right) \frac{1}{[1 - F(x)]^{n-r}} \int_x^{\beta} h'(y)[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1} dy
\]

Therefore, for \( 1 \leq m < r < s \leq n \),

\[
E[h(X_{s:n}) - h(X_{r:n}) \mid X_{m:n} = x] = \sum_{i=0}^{s-r-1} \frac{1}{[1 - F(x)]^{n-m}} \int_x^{\beta} h'(y)[F(y) - F(x)]^{j-m}[1 - F(y)]^{n-j} dy
\]

\[
= \frac{1}{a} \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)}, \text{ in view of } 1 - F(x) = \frac{f(x)}{ah'(x)}
\]

This proves the necessary part.

To prove the sufficiency part, let

\[
c = \frac{1}{a} \sum_{i=r}^{s-1} \frac{1}{(n-j)},
\]

then

\[
E[h(X_{s:n}) - h(X_{r:n}) \mid X_{m:n} = x] = c
\]

implies,

\[
\frac{(n-m)!}{(s-m-1)!(n-s)!} \int_x^{\beta} h(y)[F(y) - F(x)]^{s-m-1}[1 - F(y)]^{n-s} f(y) dy
\]

\[
- \frac{(n-m)!}{(r-m-1)!(n-r)!} \int_x^{\beta} h(y)[F(y) - F(x)]^{r-m-1}[1 - F(y)]^{n-r} f(y) dy
\]
Differentiating (3.12) \((r - m)\) times \(w.r.t\ x\), we have,

\[
= \frac{(n-r)!}{(s-r-1)!(n-s)!} \int_x^\beta h(y)[F(x) - F(y)]^{s-r-1} [1 - F(y)]^{n-s} f(y) dy
\]

\[
= \{h(x) + c\}[1 - F(x)]^{n-r} \quad (3.13)
\]

Integrating LHS of (3.13) by parts and simplifying, we get

\[
= \frac{(n-r)!}{(s-r-2)!(n-s+1)!} \int_x^\beta h(y)[F(x) - F(y)]^{s-r-2} [1 - F(y)]^{n-s+1} f(y) dy
\]

\[
+ \frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(x) - F(y)]^{s-r-1} [1 - F(y)]^{n-s+1} dy
\]

\[
= \{h(x) + c\}[1 - F(x)]^{n-r} \quad (3.14)
\]

From (3.13) and (3.14) it follows that

\[
= \frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(x) - F(y)]^{s-r-1} [1 - F(y)]^{n-s+1} dy
\]

\[
+ \{h(x) + c_1\}[1 - F(x)]^{n-r} = \{h(x) + c\}[1 - F(x)]^{n-r}
\]

where \(c_1 = \frac{1}{a} \sum_{j=1}^{s-2} \frac{1}{(n-j)}\).

That is,

\[
\frac{[1 - F(x)]^{n-r}}{a(n-s+1)} = \frac{(n-r)!}{(s-r-1)!(n-s+1)!} \int_x^\beta h'(y)[F(x) - F(y)]^{s-r-1}
\]

\[
\times [1 - F(y)]^{n-s+1} dy.
\]

Differentiating \((s-r)\) times \(w.r.t\ x\), we have

\[
h'(x)[1 - F(x)] = \frac{f(x)}{a}
\]

and hence the Theorem.
CHAPTER IV

RECORD STATISTICS:
MOMENTS AND RECURRENCE RELATIONS

1 INTRODUCTION

The recurrence relation for the moments of record statistics has been investigated extensively in the literature, and there are also many results concerning moments of record values. There are many papers devoted to the study of recurrence relation of record values and $k-th$ record values from several distributions.

Nagaraja (1978) provided the basic result on the topic of moments and their bounds. Grudzien and Szynal (1983) obtained an analogous result for $k-th$ record statistics.

Arnold (1985) discussed the $p$-norm bounds on the moments order statistics.

Kamps (1992) has investigated recurrence relations for the moments of record values for the general class of distributions and some specific distributions, whereas Balakrishana et al. (1993) obtained recurrence relations for the moments of record values from generalized extreme value distribution.

Balakrishana and Ahsanullah (1994 a, b) established recurrence relations for single and product the moments of record values for Lomax distribution. Further, Balakrishana and Ahsanullah (1995) obtained recurrence relations for single and product the moments of record values from exponential distribution.

Pawlas and Szynal (1998) established recurrence relations for single and product the moments $k-th$ record values from exponential distribution and Gumbel distribution, whereas Pawlas and Szynal (1999) established recurrence relations for single and product the moments of $k-th$ record values from Pareto, generalized Pareto and Burr distributions.
Beniek and Szynal (2002) derived recurrence relations for distribution function and moments of \( k - \text{th} \) record values.

Sultan (2007) established some new recurrence relations between the single moments of record values from modified Weibull distribution as well as between double moments.

2 SINGLE MOMENTS

Generalized Pareto distribution:

The pdf of distribution is given as

\[
f(x) = (1 + \beta x)^{-\left(\beta^{-1} + 1\right)}, \quad x \geq 0, \quad \beta > 0.
\]  

(2.1)

Here it may be noted that

\[
1 - F(x) = (1 + \beta x) f(x), \quad x \geq 0, \quad \beta > 0
\]  

(2.2)

Theorem 2.1: (Balakrishnan et al., 1994)

For the existences of the \((r + 1)\)-th moments are derived assuming their existence,

\[
E(X_{U(n)}^{r+1}) = \frac{1}{1 - (r+1)\beta} [(r + 1)E(X_{U(n)}^r) + E(X_{U(n)}^{r+1})]
\]

(2.3)

for \( \beta < (r+1)^{-1} \).

Proof: For \( n \geq 1 \) and \( r = 0,1,2\ldots \), we have

\[
E(X_{U(n)}^r) + \beta E(X_{U(n)}^{r+1}) = \int x^r \left( x^{-\beta} x^{r+1} \right) f_n(x)dx
\]

\[
= \int x^{r+1} \left( \frac{1}{(n-1)!} (R(x))^{n-1} (1 + \beta x) f(x) \right) dx
\]

Now in view of (2.2), we get

\[
= \frac{1}{(n-1)!} \int x^{r} (R(x))^{n-1} (1 - F(x)) dx
\]
Now integrating by parts treating $x^r$ for integration and the rest of the integrand for differentiation, we get

$$E(X_{U(n)}^r) + \beta E(X_{U(n)}^{r+1}) = \frac{1}{(r+1)(n-1)!} \left[ - (n-1) \int x^{r+1} (R(x))^{n-2} \times (f(x)) \, dx + \int x^{r+1} (R(x))^{n-1} f(x) \, dx \right]$$

$$= \frac{1}{(r+1)} \left[ E(X_{U(n)}^{r+1}) - E(X_{U(n-1)}^{r+1}) \right]$$

The relation in (2.3) is derived simply by rewriting the above equation.

**Lomax distribution**

The *pdf* of distribution is given as

$$f(x) = \gamma (1+x)^{-\gamma-1}, \quad x \geq 0, \gamma > 0 \quad (2.4)$$

and corresponding *df*

$$F(x) = 1 - (1+x)^{-\gamma-1}, \quad x \geq 0. \quad (2.5)$$

Now it is easy to see that

$$(1+x)f(x) = \gamma (1-F(x)), \quad x \geq 0, \quad \gamma \geq 0. \quad (2.6)$$

**Theorem 2.2 (Balakrishanan *et al.* 1994)**

For $n \geq 2$, $r = 0,1,2,...$ and $r+1 < \gamma$,

$$E(X_{U(n)}^{r+1}) = \frac{r+1}{(\gamma-r-1)} E(X_{U(n)}^r) + \frac{\gamma}{(\gamma-r-1)} E(X_{U(n-1)}^{r+1}). \quad (2.7)$$

**Proof:** We have,

$$E(X_{U(n)}^r + X_{U(n)}^{r+1}) = \int_0^\infty (x^r + x^{r+1}) f_n(x) \, dx$$

$$= \frac{1}{(n-1)!} \int_0^\infty x^r (1+x) (R(x))^{n-1} f(x) \, dx$$
\[ = \frac{\gamma}{(n-1)!} \int_0^\infty x^r (R(x))^{n-1} (1 - F(x)) \, dx \]

\[ = \frac{\gamma}{\gamma + 1} \left[ \int_0^\infty x^{r+1} \frac{1}{(n-1)!} (R(x))^{n-1} f(x) \, dx \right. \]

\[ - \int_0^\infty x^{r+1} \frac{1}{(n-2)!} (R(x))^{n-2} f(x) \, dx \]

\[ = \frac{\gamma}{\gamma + 1} \left[ E(X_U^{r+1}) + X_U^{r+1} \right] \]

and hence the result.

**Exponential distribution**

The pdf of distribution is given as

\[ f(x) = e^{-x}, \quad x \geq 0 \]

and corresponding df is given as

\[ 1 - F(x) = e^{-x}, \quad x \geq 0. \]

Also we have,

\[ f(x) = 1 - F(x) \]

**Theorem 2.3 (Balakrishnan and Ahsanullah, 1995)**

For \( n \geq 2 \), and \( r = 0,1,2,... \)

\[ E(X_U^{r+1}) = E(X_U^{r+1}) + (r + 1) E(X_U^{r}) \]  \hspace{1cm} (2.8)

and consequently, for \( 0 \leq m \leq n - 1 \) we can write

\[ E(X_U^{r+1}) = E(X_U^{r+1}) + (r + 1) \sum_{p=m+1}^n E(X_U^{r}) \]  \hspace{1cm} (2.9)

where,

\[ E(X_U^{r+1}) = 0 \text{ and } E(X_U^{0}) = 1. \]
Proof: For \( n \geq 1, r = 0,1,2,\ldots \), we have

\[
E(X_{U(n)}^r) = \frac{1}{\Gamma(n)} \int_0^\infty x^r (R(x))^{n-1} f(x) \, dx
\]

\[
E(X_{U(n)}^r) = \frac{1}{\Gamma(n)} \int_0^\infty x^r (R(x))^{n-1} [1 - F(x)] \, dx \quad (2.10)
\]

Integrating (2.10) by parts, treating \( x^r \) for integration and the rest of the integrand for differentiation, we get

\[
E(X_{U(n)}^r) = \frac{1}{(r+1)} \int_0^\infty x^{r+1} (R(x))^{n-1} f(x) \, dx
\]

\[
- (n-1) \int_0^\infty x^{r+1} [R(x)]^{n-2} f(x) \, dx
\]

\[
= \frac{1}{(r+1)} \left[ \frac{1}{\Gamma(n)} \int_0^\infty x^{r+1} (R(x))^{n-1} f(x) \, dx \right.
\]

\[
- \frac{1}{\Gamma(n-1)} \int_0^\infty x^{r+1} (R(x))^{n-2} f(x) \, dx \right]
\]

\[
= \frac{1}{r+1} [E(X_{U(n)}^{r+1} - X_{U(n-1)}^{r+1})],
\]

which on rewriting, gives the recurrence relation in (2.8). Then, by repeatedly applying the recurrence relation (2.8), we simply derive the recurrence relation (2.9).

Power Function Distribution

The pdf of distribution is given as

\[
f(x) = \begin{cases} 
\gamma (1-x)^{\gamma-1}, & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases} \quad (2.11)
\]

and corresponding df

\[
F(x) = 1 - (1-x)^\gamma, \quad 0 < x < 1. \quad (2.12)
\]
It is easy to see that

\[ r[1 - F(x)] = (1 - x)f(x) \]  

(2.13)

**Theorem 2.4 (Ahsanullah, 1995)**

For \( n \geq 2, \) and \( r = 0, 1, 2, \ldots \)

\[
E(X_{U(n)}^{r+1}) = \frac{r + 1}{\gamma + r + 1} E(X_{U(n)}^r) + \frac{\gamma}{\gamma + r + 1} E(X_{U(n-1)}^{r+1})
\]  

(2.14)

**Proof:** For \( n \geq 1, r = 0, 1, 2, \ldots, \) we have

\[
E(X_{U(n)}^r - X_{U(n)}^{r+1}) = \int_0^\infty (x^r - x^{r+1}) f_n(x) \, dx.
\]

On using (2.13), we get

\[
= \frac{\gamma}{(n-1)!} \int_0^\infty x^r (R(x))^{n-1} (1 - F(x)) \, dx
\]

\[
= \frac{\gamma}{r + 1} \left[ \int_0^\infty x^{r+1} \frac{1}{(n-1)!} (R(x))^{n-1} f(x) \, dx
\]

\[ - \int_0^\infty x^{r+1} \frac{1}{(n-2)!} (R(x))^{n-2} f(x) \, dx \right]

\[
= \frac{\gamma}{(r+1)} [E(X_{U(n)}^{r+1} - X_{U(n-1)}^{r+1})],
\]

and hence the result.

**Modified Weibull distribution**

The *pdf* of distribution is given as

\[
f(x) = a(b + \lambda x)^{b-1} e^{\lambda x} \exp(-ax e^{\lambda x}) , \ a, \lambda > 0, b \geq 0, x > 0
\]

(2.15)

and corresponding *df*

\[
F(x) = 1 - \exp(-ax b e^{\lambda x}) , \ a, \lambda > 0, b \geq 0, x > 0.
\]

(2.16)
It is easy to derive the relation between the pdf and df of MDF given in (2.15) and (2.16) respectively as

$$\frac{f(x)}{1-F(x)} = -\log(1-F(x)) \frac{b+\lambda x}{x} \quad (2.17)$$

**Theorem 2.5 (Sultan, 2007)**

For $n \geq 2$, and $r = 0,1,2,...$

$$E(X_{U(n)}^r) = \frac{nb}{r} [E(X_{U(n+1)}^r) - E(X_{U(n)}^r)] + \frac{n\lambda}{r+1} [E(X_{U(n+1)}^{r+1}) - E(X_{U(n)}^{r+1})] \quad (2.18)$$

**Proof:** The single moments of record values from MWD can be written as

$$E(X_{U(n)}^r) = \frac{1}{\Gamma(n)} [b I_{r-1} + \lambda I_r] \ ,$$

where,

$$I_r = \int_0^\infty x^r (R(x))^n (1-F(x))dx. \quad (2.19)$$

Upon integrating (2.19) by parts, treating $x^r$ for integration and the rest of the integrand for differentiation, we get

$$I_r = \frac{\Gamma(n+1)}{r+1} E(X_{U(n+1)}^{r+1}) - \frac{\Gamma(n+1)}{r+1} E(X_{U(n)}^{r+1})$$

where $\Gamma(.)$ is a gamma function.

Similarly, we write

$$I_{r-1} = \frac{\Gamma(n+1)}{r+1} E(X_{U(n+1)}^r) - \frac{\Gamma(n+1)}{r} E(X_{U(n)}^r) \quad (2.20)$$

and hence the result.
3 PRODUCT MOMENTS

Exponential distribution

Theorem 3.1 (Balakrishnan and Ahsanullah, 1995)

For standard exponential distribution and $m \geq 1$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(m)}^r X_{U(m)}^{s+1}) = E(X_{U(m)}^{r+s+1}) + (s + 1)E(X_{U(m)}^r X_{U(m+1)}^s)$$

(3.1)

and for $1 \leq m \leq n - 2$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(m)}^r X_{U(n)}^{s+1}) = [E(X_{U(m)}^r X_{U(n-1)}^{s+1}) + (s + 1)E(X_{U(m)}^r X_{U(n)}^s)].$$

(3.2)

Proof: In view of (1.2.3) and for $1 \leq m \leq n - 2$ and $r, s = 0, 1, 2, \ldots$

$$E(X_{U(m)}^r X_{U(n)}^s) = \frac{1}{\Gamma(m) \Gamma(n - m)} \int_0^\infty x^r (R(x))^{m-1} \frac{f(x)}{1 - F(x)} I(x) \, dx$$

(3.3)

where,

$$I(x) = \int_y^\infty y^s (R(y) - R(x))^{n-m-1} f(y) \, dy .$$

$$= \int_y^\infty y^s (R(y) - R(x))^{n-m-1} [1 - F(y)] \, dy ,$$

upon integrating by parts treating $y^s$ for integration and rest of the integrand

for differentiation, we obtain when $n = m + 1$ that

$$I(x) = \frac{1}{s+1} [ \int_y^\infty y^{s+1} f(y) \, dy - x^{s+1} (1 - F(x))],$$

and when $n \geq m + 2$ that

$$I(x) = \frac{1}{s+1} [ \int_y^\infty y^{s+1} (R(y) - R(x))^{n-m-1} f(y) \, dy$$

$$- (n - m - 1) \int_y^\infty y^{s+1} (R(y) - R(x))^{n-m-2} f(y) \, dy ] .$$

Upon substituting the above expression of $I(x)$ in equation (3.3) and after

simplification, we obtain when $n = m + 1$ that
and when \( n \geq m + 2 \) that

\[
E(X_{U(m)}^r X_{U(n)}^s) = \frac{1}{s+1}[E(X_{U(m)}^r X_{U(n)}^{s+1}) - E(X_{U(m)}^r X_{U(n-1)}^{s+1})].
\]

The recurrence relation in (3.1) and (3.2) follows readily when the above two equations are rewritten.

**Theorem 3.2 (Balakrishnan and Ahsanullah, 1995)**

For standard exponential distribution and \( m \geq 2 \) and \( r, s = 0, 1, 2, ... \)

\[
E(X_{U(m-1)}^{r+1} X_{U(m)}^s) = E(X_{U(m)}^r X_{U(m)}^s) - (r+1)E(X_{U(m)}^s X_{U(m+1)}^{s+1})
\]

for \( 2 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, ... \)

\[
E(X_{U(m-1)}^{r+1} X_{U(n-1)}^s) = [E(X_{U(m)}^{r+1} X_{U(n-1)}^s) - (r+1)E(X_{U(m)}^r X_{U(n+1)}^s)].
\]

**Proof:** See reference.

**Generalized Pareto distribution**

**Theorem 1.3.3 (Balakrishnan and Ahsanullah, 1994)**

For the distribution as given in (2.1) and \( m \geq 1 \) and \( r, s = 0, 1, 2, ... \)

\[
E(X_{U(m)}^r X_{U(n)}^{s+1}) = \frac{1}{1-(s+1)\beta}[(s+1)E(X_{U(m)}^r X_{U(n+1)}^{s+1}) + E(X_{U(m)}^{r+1} X_{U(n+1)}^s)]  \tag{3.4}
\]

and for \( 1 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, ... \)

\[
E(X_{U(m)}^r X_{U(n)}^{s+1}) = \frac{1}{1-(s+1)\beta}[(s+1)E(X_{U(m)}^r X_{U(n)}^{s+1}) + E(X_{U(m)}^{r+1} X_{U(n-1)}^s)].  \tag{3.5}
\]

**Proof:** In view of (1.2.3) and for \( 1 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, ... \)

\[
E(X_{U(m)}^r X_{U(n)}^s) + \beta E(X_{U(m)}^r X_{U(n)}^{s+1})
\]

\[
= \int_{\alpha < y} (x^r y^s + \beta x^r y^{s+1}) f_{m,n}(x,y) \, dy \, dx
\]
\[
\frac{1}{(m-1)!(n-m-1)!} \int_{x} x^r (R(x))^{m-1} \frac{f(x)}{1 - F(x)} I(x) \, dx, \quad (3.6)
\]

where,

\[
I(x) = \int_{y>x} y^s (1 + \beta y) (R(y) - R(x))^{n-m-1} f(y) \, dy
\]

\[
= \int_{y>x} y^s (R(y) - R(x))^{n-m-1} [1 - F(y)] \, dy.
\]

Integrating by parts treating \( y^s \) for integration and rest of the integrand for differentiation, we obtain when \( n = m + 1 \) that

\[
I(x) = \frac{1}{s+1} \left[ -x^{s+1} [1 - F(x)] + \int_{y>x} y^{s+1} f(y) \, dy \right],
\]

and when \( n \geq m + 2 \) that

\[
I(x) = \frac{1}{s+1} \left[ -(n-m-1) \int_{y>x} y^{s+1} [R(y) - R(x)]^{n-m-2} f(y) \, dy \\
+ \int_{y>x} y^{s+1} [R(y) - R(x)]^{n-m-1} f(y) \, dy \right].
\]

Upon substituting the above expression of \( I(x) \) in equation (3.6) and after simplification, we get when \( n = m + 1 \), that

\[
E(X_{U(m)}^r X_{U(m+1)}^s) + \beta E(X_{U(m)}^r X_{U(m+1)}^{s+1})
\]

\[
= \frac{1}{s+1} \left[ -E(X_{U(m+1)}^{s+1}) + E(X_{U(m)}^r X_{U(m+1)}^{s+1}) \right],
\]

and when \( n \geq m + 2 \) that

\[
E(X_{U(m)}^r X_{U(n)}^s) + \beta E(X_{U(m)}^r X_{U(n)}^{s+1})
\]

\[
= \frac{1}{s+1} \left[ -E(X_{U(n)}^r X_{U(n-1)}^{s+1}) + E(X_{U(m)}^r X_{U(n)}^{s+1}) \right].
\]
The relation in (3.4) and (3.5) are derived simply by rewriting the above two equations.

**Theorem 3.4 (Balakrishnan and Ahsanullah, 1994)**

For the distribution as given in (2.1) and \(1 \leq m \leq n - 2\) and \(r, s = 0, 1, 2, \ldots\),

\[
E(X_{U(m)}^{r+1}X_{U(m+1)}^{s}) = \frac{1}{(r + 1)\beta}[E(X_{U(m)}^{r+1}X_{U(m-1)}^{s}) - E(X_{U(m)}^{r+1}X_{U(m)}^{s})] - (r + 1)E(X_{U(m)}^{r}X_{U(m+1)}^{s})]
\]

and for \(1 \leq m \leq n - 2\) and \(r, s = 0, 1, 2, \ldots\),

\[
E(X_{U(m)}^{r+1}X_{U(n)}^{s}) = \frac{1}{(r + 1)\beta}[E(X_{U(m)}^{r+1}X_{U(n-1)}^{s}) - E(X_{U(m)}^{r+1}X_{U(n-1)}^{s})] - (r + 1)E(X_{U(m)}^{r}X_{U(n)}^{s})]
\]

**Proof:** See reference.

**Power Function distribution**

**Theorem 3.5 (Ahsanullah, 1995)**

For power function distribution as given in (2.11) and \(n > 2, r, s = 0, 1, 2, \ldots\)

\[
E(X_{U(m)}^{r}X_{U(m+1)}^{s+1}) = \frac{s + 1}{\gamma + s + 1}E(X_{U(m)}^{r}X_{U(m+1)}^{s}) + \frac{\gamma}{\gamma + s + 1}E(X_{U(m)}^{r+1}X_{U(m+1)}^{s}) \quad (3.7)
\]

and for \(1 \leq m \leq n - 2\) and \(r, s = 0, 1, 2, \ldots\),

\[
E(X_{U(m)}^{r}X_{U(n)}^{s+1}) = \frac{s + 1}{\gamma + s + 1}E(X_{U(m)}^{r}X_{U(n)}^{s}) + \frac{\gamma}{\gamma + s + 1}E(X_{U(m)}^{r+1}X_{U(n)}^{s+1})
\]

**Proof:** For \(1 \leq m \leq n - 2\) and \(r, s = 0, 1, 2, \ldots\), we have

\[
E[X_{U(m)}^{r}X_{U(n)}^{s} - X_{U(m)}^{r}X_{U(n)}^{s+1}]
\]

\[
= \iint_{0 < x, y < \infty} (x^{r}y^{s} - x^{r}y^{s+1})f_{m,n}(x, y) \, dy \, dx
\]
where,

\[ I(x) = \int_0^1 y^s (1 - y) [R(y) - R(x)]^{n-m-1} f(y) dy \]

\[ = \gamma \int_0^1 y^{s+1} f(y) dy - x^{s+1}(1 - F(x)), \text{ for } n = m + 2 \quad (3.9) \]

\[ = \frac{\gamma}{s+1} \int_0^1 y^{s+1} [R(y) - R(x)]^{n-m-1} f(y) dy \]

\[ - (n - m - 1) \int_0^1 y^{s+1} [R(y) - R(x)]^{n-m-2} f(y) dy, \text{ for } n > m + 2 \quad (3.10) \]

In view of (3.9) and (3.10), we get

\[ E(X_{U(m)}^r X_{U(m+1)}^s) - E(X_{U(m)}^r X_{U(m+1)}^{s+1}) = \frac{\gamma}{s+1} [E(X_{U(m)}^r X_{U(m+1)}^{s+1}) - E(X_{U(m)}^r X_{U(m+1)}^s)] \],

\[ \text{for } n = m + 1 \]

and

\[ E(X_{U(m)}^r X_{U(n)}^s) - E(X_{U(m)}^r X_{U(n)}^{s+1}) \]

\[ = \frac{\gamma}{s+1} [E(X_{U(m)}^r X_{U(n)}^{s+1}) - E(X_{U(m)}^r X_{U(n-1)}^{s+1})], \text{ for } n - m \geq 2 \]

The recurrence relation in (3.7) and (3.8) follows by writing above two equations recursively.

**Theorem 3.6 (Ahsanullah, 1995)**

For the distribution as given in (2.11) and \( m \geq 2 \) and \( r, s = 0, 1, 2, \ldots \),

\[ E(X_{U(m)}^{r+1} X_{U(m+1)}^s) = \frac{\gamma}{r+1} [E(X_{U(m)}^{r+1} X_{U(m+1)}^s) - E(X_{U(m)}^{r+1} X_{U(m)}^s) - E(X_{U(m)}^r X_{U(m+1)}^{s+1})], \]

for \( 2 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, \ldots \)

\[ E(X_{U(m)}^{r+1} X_{U(n)}^s) = \frac{\gamma}{r+1} [E(X_{U(m)}^{r+1} X_{U(n)}^s) - E(X_{U(m-1)}^{r+1} X_{U(n)}^s) - E(X_{U(m)}^r X_{U(n)}^{s+1})] \]

**Proof**: See reference.
Modified Weibull distribution

**Theorem 3.7 (Khalaf Sultan, 2007)**

For the distribution as given in (2.15) and for $m \geq 1$, $r = 0,1,2,...$

$$E(X_{U(m,m+1)}^{r,s}) = \frac{mb}{r} [E(X_{U(m+1)}^{r+s}) - E(X_{U(m,m+1)}^{r,s})]$$

$$+ \frac{m\lambda}{r+1} [E(X_{U(m)}^{r+1,s}) - E(X_{U(m,m+1)}^{r+1,s})]$$

and for $1 \leq m \leq n-1$ and $r,s = 0,1,2,...$,

$$E(X_{U(m,n)}^{r,s}) = \frac{mb}{r} [E(X_{U(m+1,m)}^{r+s}) - E(X_{U(m,n)}^{r,s})]$$

$$+ \frac{m\lambda}{r+1} [E(X_{U(m+1,n)}^{r+1,s}) - E(X_{U(m,n)}^{r+1,s})].$$

**Proof:** we have

$$E(X_{U(m,n)}^{r,s}) = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty y^s f(y) I(y) \, dy$$

(3.13)

where,

$$I(y) = \int_0^y x^r (R(x))^{m-1} [R(y) - R(x)]^{n-m-1} \frac{f(x)}{1-F(x)} \, dx$$

(3.14)

For $n = m + 1$ and using (2.17), (3.14) may be written as

$$I(y) = bT_{r-1} + \lambda T_r$$

(3.15)

where,

$$T_r = \int_0^y x^r (R(x))^m \, dx,$$

which upon integrating by parts gives,

$$T_r = \frac{y^{r+1}}{r+1} (R(y))^m - \frac{m}{r+1} \int_0^y x^{r+1} (R(x))^{m+1} \frac{f(x)}{1-F(x)} \, dx$$

(3.16)

and

$$T_{r-1} = \frac{y^r}{r} (R(y))^m - \frac{m}{r} \int_0^y x^r (R(x))^{m-1} \frac{f(x)}{1-F(x)} \, dx.$$ 

(3.17)

Now in view of (3.13), (3.15) and (3.16), (3.11) can be established.
For $1 \leq m \leq n - 1$ and by making use of (3.16), (3.13) can be written as

$$I(y) = bQ_{r-1} + \lambda Q_r,$$  \hspace{1cm} (3.18)

where,

$$Q_r = \frac{n - m - 1}{r + 1} \int_0^y x^{r+1} (R(x))^m [R(y) - R(x)]^{n-m-2} \frac{f(x)}{1 - F(x)} \, dx$$

$$- \frac{m}{r + 1} \int_0^y x^{r+1} (R(x))^{m-1} [R(y) - R(x)]^{n-m-1} \frac{f(x)}{1 - F(x)} \, dx$$  \hspace{1cm} (3.19)

and

$$Q_{r-1} = \frac{n - m - 1}{r} \int_0^y x^r (R(x))^m [R(y) - R(x)]^{n-m-2} \frac{f(x)}{1 - F(x)} \, dx$$

$$- \frac{m}{r} \int_0^y x^r [-\ln(1 - F(x))]^{m+1} [R(y) - R(x)]^{n-m-1} \frac{f(x)}{1 - F(x)} \, dx$$  \hspace{1cm} (3.20)

Now in view of (3.18), (3.19) and (3.20), (3.12) can be proved.
CHAPTER V

RECORD STATISTICS:
CHARACTERIZATION OF DISTRIBUTIONS

1 INTRODUCTION

Characterization of distributions through record values have been considered by many authors in the literature.

Nagaraja (1977) characterized continuous distributions by linearity of regression of record values, through relation

\[ E \left[ X_{u(r+1)} \mid X_{u(r)} = x \right] = ax + b. \]

Further, Nagaraja (1988) also characterized distributions by means of

\[ E \left[ X_{u(r)} \mid X_{u(r+1)} = x \right] = ax + b. \]

Wesolowski and Ahsanullah (1997) extended the result of Nagaraja (1977) and characterized the distributions for double order gap.

Franco and Ruiz (1996) obtained the distribution \( F \) from the conditional expectation of function of record values, i.e.

\[ E h(X_{u(r-1)} \mid X_{u(r)} = x), \]

where \( h() \) is a real, continuous and strictly monotonic function.

Ahsanullah (1979) characterized exponential distribution through conditional distribution of record values, whereas Kirmani and Beg (1984) and Lin (1987) characterized probability distributions based on the moments of record values.


\[ E \left[ X_{u(r+i)} \mid X_{u(r)} = x \right] = ax + b. \]

Gupta and Ahsanullah (2004) characterized distribution through conditional expectation of record values through
\[ E \left[ \xi \left( X_{u(r+2)} \right) \mid X_{u(r)} = x \right] = g(x), \]

where \( g(x) \) may be non linear but differentiable w.r.t. \( x \).


Bairamov et al. (2005) characterized exponential type distribution via regression on pairs of record values, where regression may not be linear, whereas Ahsanullah (2010) characterized exponential distribution based on distribution properties of random variable.

For more results on characterization one may refer to Wu and Lee (2001), Raqab (2002), Wu (2004), Khan et al. (2010a), Khan and Athar (2010) and references therein.

2 CHARACTERIZATION OF SPECIFIC DISTRIBUTIONS

Exponential distribution

Theorem 2.1 (Lee, 2001)

If \( F(x) \) is absolutely continuous with \( F(x) < 1 \) for all \( x \) then

\[ E \left[ X_{u(n+1)} - X_{u(n)} \mid X_{u(m)} = y \right] = y + (n-m)c \]

if and only if \( F(x) = 1 - e^{-x/c}, \quad x > 0 \).

Proof: If \( X_k \in \text{EXP}(c) \), then \( E \left[ X_{u(n)} \mid X_{u(m)} = y \right] = y + (n-m)c \). Hence (2.1) holds.

Conversely, suppose (2.1) holds. Using Ahsanullah formula (1995) it follows the following equation,

\[
\frac{1}{1-F(y)} \int_y^\infty \frac{1}{(n-m)!} \left( \frac{\ln \frac{1-F(y)}{1-F(x)}}{1-F(x)} \right)^{n-m-1} x f(x) dx = c,
\]

where \( c > 0, \quad n \geq m + 1 \).
that is,

\[
\frac{1}{(n-m)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m} x f(x) \, dx
\]

\[
- \frac{1}{(n-m-1)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m-1} x f(x) \, dx = c (1 - F(y)).
\]  

(2.2)

Since \( F(x) \) is absolutely continuous, we can differentiate \( (n-m+1) \) times both sides of (2.2) with respect to \( y \) and simplify, then we obtain the following equation

\[
f(y) + \frac{1}{c} F(y) = \frac{1}{c}.
\]  

(2.3)

When we solve differential equation of (2.3), we get

\[ F(y) = 1 - e^{-y/c}. \]

This completes the proof.

**Theorem 2.2 (Lee, 2001)**

If \( F(x) \) is absolutely continuous with \( F(x) < 1 \) for all \( x \) then

\[ E [X_{u(n+2)} - X_{u(n)} | X_{u(m)} = y] = 2c, \ c > 0, \ n \geq m + 1 \]  

(2.4)

if and only if \( F(x) = 1 - e^{-x/c}, \ x > 0. \)

**Proof:** If \( X_k \in \text{EXP}(c) \), then \( E [X_{u(n)} | X_{u(m)} = y] = y + (n - m)c \). Hence (2.4) holds.

Conversely, suppose (2.4) holds. Using Ahsanullah formula (1995) it follows the following equation, that is

\[
\frac{1}{(n-m+1)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m+1} x f(x) \, dx
\]

\[
- \frac{1}{(n-m-1)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m-1} x f(x) \, dx = 2c (1 - F(y))
\]  

(2.5)
Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+2)$ times both sides of (2.5) with respect to $y$ and simplify, then we obtain the following equation

$$3(1-F(y)) - 2cy + \frac{(1-F(y))^2 f'(y)}{f(y)^2} = 0$$

Let $y = F(y)$ (that is, $y' = f$, $y'' = f'$). Then, the above expression may be expressed in the following form

$$\frac{(1-y)^2 y''}{y^2} + 3(1-y) - 2cy' = 0 \quad \text{i.e.} \quad y'' = f(x, y, y'). \quad (2.6)$$

Therefore there exists a unique solution of the differential equation (2.6), that satisfies the prescribed initial conditions $y(0) = 0$, $y'(0) = \frac{1}{c}$. By the existence and uniqueness theorem, we get $F(x) = 1 - e^{-x/c}$.

This completes the proof.

**Theorem 2.3 (Ahsanullah, 2010)**

Let $X_1, X_2, \ldots$ be a sequence of non-negative i.i.d random variables with cdf $F(x)$ and pdf $f(x)$. We assume $F(0) = 0$ and $F(x) < 1$ for all $x > 0$ and $E(X^k)$ exists for some $k > 0$. Then the following statements are equivalent

a) $F$ is $E(\theta)$

b) $E [Z_{n-1,n} | X_{U(n-1)} = x] = b$, where $b$ is a constant independent of $x$ for all $x \geq 0$ and $F \in C$.

**Proof:** In view of (1.2.3) joint pdf of $X_{U(n-1)}$ and $X_{U(n)}$, $1 \leq n - 1 \leq n$ can be written as

$$f_{n-1,n}(x, y) = \frac{(R(x))^{n-1} \Gamma(n)}{r(x)f(y)}, \quad -\infty < x < y < \infty.$$  

The conditional pdf $f_{c,n-1,n} (z | X_{U(n-1)} = x)$ of $Z_{n-1,n} | X_{U(n-1)} = x$ is

$$f_{c,n-1,n} (z | X_{U(n-1)} = x) = \frac{f(z + x)}{1 - F(x)}, \quad z > 0. \quad (2.7)$$
Thus if $F$ is $E(\theta)$, then

$$\int c_{n-1,n} \, (z | X_{U(n-1)} = x) = \theta e^{-\theta x}, \quad z > 0$$

and hence $E(Z_{n-1,n} | X_{U(n-1)} = x)^k$ is constant independent of $x$.

Suppose $E(Z_{n-1,n} | X_{U(n-1)} = x)^k = b$, where $b$ is constant independent of $x$, then from (2.7)

$$E(Z_{n-1,n} | X_{U(n-1)} = x)^k = \int_0^{\infty} z^k \frac{f(z + x)}{1 - F(x)} \, dz.$$ 

Now $E(Z_{n-1,n} | X_{U(n-1)} = x)^k = b$, implies

$$b[1 - F(x)] = \int_0^{\infty} Z^k f(z + x) \, dz.$$ 

$$= \int_0^{\infty} k Z^{k-1} (1 - F(z + x)) \, dz \quad (2.8)$$

Taking limit as $x \to 0$, we get

$$b = \int_0^{\infty} k Z^{k-1} (1 - F(z)) \, dz \quad (2.9)$$

Substituting (2.8) in (2.9), we get on simplification

$$\int_0^{\infty} k Z^{k-1} [(1 - F(z + x)) - (1 - F(x))(1 - F(z))] \, dz = 0. \quad (2.10)$$

If $F \in C$, then (2.10) is true for all $x$ and almost all $z$, we must have

$$(1 - F(z + x)) = (1 - F(x))(1 - F(z)) = 0 \quad (2.11)$$

The solution of (2.11) for all $x \geq 0$ and almost all $z > 0$ is,

$$F(x) = 1 - e^{-\theta x}, \quad x \geq 0 \quad \text{and} \quad \theta > 0.$$ 

From (2.9) it is easy to see that $b = E(x^k)$. The following theorem gives a Characterization of the exponential distribution using this property.
Theorem 2.4 (Ahsanullah, 2010)

Let \( X_1, X_2, \ldots \) be a sequence of non-negative \( i.i.d \) random variables with cdf \( F(x) \) and pdf \( f(x) \). We assume \( F(0) = 0 \) and \( F(x) < 1 \) for all \( x > 0 \) and \( E(x^k) \) exists for some \( k > 0 \). Then the following statements are equivalent

a) \( F \) is \( E(\theta) \)

b) \( E(Z_{n-1,n} | X_{U(n-1)} = x)^k = E(X^k) \), for all \( x \geq 0 \) and \( F \in C \).

Proof: See reference.

Theorem 2.5 (Ahsanullah, 2010)

Let \( X_1, X_2, \ldots \) be a sequence of non-negative \( i.i.d \) random variables with cdf \( F(x) \) and pdf \( f(x) \). We assume \( F(0) = 0 \) and for \( F(x) < 1 \) for all \( x > 0 \). Then the following two statements are equivalent

a) \( F \) is \( E(\theta) \).

b) \( X_{U(n)} \in G(n, \theta) \).

Proof: It is easy to show that (a) implies (b). To prove (b) implies (a), we have

\[
\int_0^x \frac{x \theta^n t^{n-1}}{\Gamma(n)} e^{-\theta t} dt = \int_0^x \frac{(R(t))^{n}}{\Gamma(n)} f(t) dt
\]

\[
= \int_0^{\ln(1-F(x))} \frac{t^n}{\Gamma(n)} e^{-t} dt.
\]

Now writing

\[
\int_0^x \frac{x \theta^n t^{n-1}}{\Gamma(n)} e^{-\theta t} dt = \int_0^x \frac{\theta x t^{n-1}}{\Gamma(n)} e^{-t} dt,
\]

we have

\[
\int_0^x \frac{\theta x t^{n-1}}{\Gamma(n)} e^{-t} dt = \int_0^{\ln(1-F(x))} \frac{t^n}{\Gamma(n)} e^{-t} dt, \quad \text{for all} \ x
\]

(2.12)

The solution of (2.12) is

\[- \ln(1 - F(x)) = qx,\]
That is,
\[ F(x) = 1 - e^{-\theta x}, x > 0, \ \theta > 0. \]

**Pareto Distribution**

**Theorem 2.2.6 (Lee, 2003)**

\[ F(x) = 1 - x^\theta, x > 0, \ \theta < -1 \text{ if and only if } \]

(i) \( (\theta + 1)E [X_{u(n+1) | X_{u(m)} = y}] = \theta E [X_{u(n) | X_{u(m)} = y}], \ n \geq m + 1 \) \quad (2.13)

(ii) \( (\theta + 1)^2 E [X_{u(n+2) | X_{u(m)} = y}] = \theta^2 E [X_{u(n) | X_{u(m)} = y}], \ n \geq m + 1 \) \quad (2.14)

(iii) \( (\theta + 1)^3 E [X_{u(n+3) | X_{u(m)} = y}] = \theta^3 E [X_{u(n) | X_{u(m)} = y}], \ n \geq m + 1 \) \quad (2.15)

**Proof:** If \( F(x) = 1 - x^\theta \), then \( E[X_{u(n) | X_{u(m)} = y}] = \left(\frac{\theta}{\theta + 1}\right)^{n-m} y \). Hence (2.13) holds. Conversely, suppose (2.13) holds, then from Ahsanullah formula (1995) we can obtain the following equation,

\[
\frac{\theta + 1}{(n-m)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m} x f(x) \, dx 
= \frac{\theta}{(n-m-1)!} \int_y^\infty \left( \ln \frac{1 - F(y)}{1 - F(x)} \right)^{n-m-1} x f(x) \, dx. \quad (2.16)
\]

Since \( F(x) \) is absolutely continuous, we can differentiate \((n-m+1)\) times both side of (2.16) with respect to \( y \) and simplify, then we obtain the following equation,

\[-y f(y) = \theta (1 - F(y)), \]

that is \( \frac{f(y)}{1 - F(y)} = \frac{\theta}{y} \). \quad (2.17)

Integrating both side of (2.2.17) with respect to \( y \), we get \( F(y) = 1 - y^\theta \).

This completes the proof.

(2.14) and (2.15) can be proved on the lines of (2.13).
3 CHARACTERIZATION OF GENERAL FORM OF DISTRIBUTIONS

Theorem 3.1 (Khan et al., 2010a)

Let $X$ be an absolutely continuous random variable with the df $F(x)$ and pdf $f(x)$ on the support $(\alpha, \beta)$, where $\alpha$ and $\beta$ may be finite or infinite. Then for $m \leq r < s$

$$E \left[ h(X_{u(s)}) - h(X_{u(r)}) \mid h(X_{u(m)}) = x \right] = (s - r)c \quad (3.1)$$

if and only if

$$F(x) = e^{-\frac{h(x)}{c}}, c > 0, \quad (3.2)$$

where $h(x)$ is a monotonic and differentiable function of $x$ such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)F(x) \to 0$ as $x \to \beta$.

Proof: we have,

$$E \left[ h(X_{u(s)}) - h(X_{u(r)}) \mid h(X_{u(m)}) = x \right] = \frac{1}{\Gamma(s - m)} \int_{x}^{\beta} h(y)[-\ln F(y) + \ln F(x)]^{s - m - 1} \frac{f(y)}{F(x)} dy - \frac{1}{\Gamma(r - m)} \int_{x}^{\beta} h(y)[-\ln F(y) + \ln F(x)]^{r - m - 1} \frac{f(y)}{F(x)} dy$$

Now, it is easy to see that (3.2) implies (3.1).

For sufficiency part, let $c^* = (s - r)c$, then

$$\frac{1}{\Gamma(s - m)} \int_{x}^{\beta} h(y)[-\ln F(y) + \ln F(x)]^{s - m - 1} f(y) dy = c^* F(x) \quad (3.3)$$

Differentiating $(r - m)$ times both the sides of (3.3) with respect to $x$, we get

$$\frac{1}{\Gamma(s - r)} \int_{x}^{\beta} h(y)[-\ln F(y) + \ln F(x)]^{s - m - 1} \frac{f(y)}{F(x)} dy = h(x) + c^* \quad (3.4)$$

Integrating LHS of (3.4) by parts and simplifying, we have
Record Statistics: Characterization of Distributions

\[
\frac{1}{\Gamma(s-r-1)F(x)} \int_{x}^{\beta} h(y)[-\ln F(y) + \ln F(x)]^{s-r-2} f(y) dy \\
+ \frac{1}{\Gamma(s-r)F(x)} \int_{x}^{\beta} h'(y)[-\ln F(y) + \ln F(x)]^{s-r-1} F(y) dy = h(x) + c^*
\]

This in view of (3.4), reduces to

\[
\frac{1}{\Gamma(s-r)} \int_{x}^{\beta} h'(y)[-\ln F(y) + \ln F(x)]^{s-r-1} F(y) dy = cF(x)
\]

Differentiating (3.5) \((s-r)\) times with respect to \(x\), we obtain

\[h'(x)F(x) = cf(x)\]

and hence the result.

**Remark 3.1:** At \(s=r+1, s=r+2\) and \(h(x) = x\), we get the result as obtained by Lee (2001).

**Remark 3.2:** At \(s=r+3, s=r+4\) and \(h(x) = x\), this reduces to the result as obtained by Lee at al. (2002).

**Remark 3.3:** At \(r = m\), \(E[h(X_{u(s)} | X_{u(r)} = x)] = h(x) + (s-r)c\) as obtained by Athar et al. (2003).

**Theorem 3.2 (Khan et al., 2010a)**

Under the condition as given in Theorem 3.1 and for \(1 \leq r < s\)

\[E \left[ h(X_{u(s)}) - h(X_{u(r)}) \right] + h(x) = E[h(X_{u(s)}) | (X_{u(r)}) = x] \]

if and only if

\[
\frac{h(x)}{F(x)} = e^{-c} , \quad c > 0
\]

**Proof:** See reference.
Lemma 2.3.1 (Khan and Athar, 2010)

If for \(1 \leq r < s \leq n\),
\[
E[\xi(X_{u(s)}) \mid X_{u(l)} = x] = g_{s\mid l}(x), \ l = r, r + 1
\] (3.6)
exists, then
\[
\bar{F}(x) = \exp\left(\int_{x}^{\alpha} A(t)dt\right), \ x \in (\alpha, \beta),
\] (3.7)
where \(A(x) = \frac{g_{s\mid r}'(x)}{[g_{s\mid r+1}(x) - g_{s\mid r}(x)]}, g_{s\mid r}'(x)\) is the first derivative of \(g_{s\mid r}(x)\) w.r.t. \(x\) and \(\bar{F}(\alpha) = 1\).

Proof: We have
\[
E[\xi(X_{u(s)}) \mid X_{u(r)} = x] = \int_{x}^{\beta} \xi(y) \frac{f_{s\mid r}(y \mid x)}{f_{s\mid r}(x)} dy = g_{s\mid r}(x)
\]
Therefore, in view of (1.2.5),
\[
\frac{1}{\Gamma(s-r)} \int_{x}^{\beta} \xi(y)[\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} f(y) dy = g_{s\mid r}(x) \bar{F}(x)
\]
Differentiating both sides with respect to \(x\) and re-arranging the terms, we get
\[
- \frac{1}{\Gamma(s-r-1)} \int_{x}^{\beta} \xi(y)[\ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-2} \frac{f(x)}{F(x)} f(y) dy
\]
\[
= - g_{s\mid r}(x) f(x) + g_{s\mid r}'(x) \bar{F}(x)
\]
or,
\[
- \frac{f(x)}{\bar{F}(x)} = A(x) = \frac{g_{s\mid r}'(x)}{[g_{s\mid r+1}(x) - g_{s\mid r}(x)]}
\]
and hence the result.

Lemma 3.2 (Khan and Athar, 2010)

Let \(q \in (\alpha, \beta)\), such that \(-\ln \bar{F}(q) = 1\), then for \(1 \leq r < s \leq n\), if
\[
E[\xi(X_{u(s)}) \mid X_{u(l)} = y] = g_{r\mid l}(y), \ l = s-1, s
\] (3.8)
exists, then
\[
\bar{F}(x) = \exp(-e^{\int_{x}^{\beta} A(y) dy}), \ x \in (\alpha, \beta),
\] (3.9)
where \( A(y) = \frac{g'_{r|s}(y)}{(s-1)[g_{r|s-1}(y) - g_{r|s}(y)]} \) and \( g'_{r|s}(y) \) is the first derivative of \( g_{r|s}(y) \) w.r.t. \( y \).

**Proof:** We have,
\[
E[\xi(X_{u(r)}) | X_{u(s)} = y] = \int_{\alpha}^{r} \xi(x) f_{r|s}(x | y) dx = g_{r|s}(y)
\]
Therefore, in view of (1.2.6),
\[
\frac{\Gamma(s)}{\Gamma(r) \Gamma(s-r)} \int_{\alpha}^{r} \xi(x)[-\ln \overline{F}(x)]^{r-1}[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \frac{f(x)}{\overline{F}(x)} dx
\]
\[
= g_{r|s}(y)[-\ln \overline{F}(y)]^{s-1}
\]
Differentiating both sides with respect to \( y \) and re-arranging the terms, we get
\[
\frac{(s-1)\Gamma(s-1)}{\Gamma(r) \Gamma(s-r-1)} \int_{\alpha}^{r} \xi(x)[-\ln \overline{F}(x)]^{r-1}
\]
\[
\times[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-2} \frac{f(x)}{\overline{F}(x)\overline{F}(y)} f(y) dx
\]
\[
= [-\ln \overline{F}(y)]^{s-2} \left( (s-1)g_{r|s}(y) \frac{f(y)}{\overline{F}(y)} + g'_{r|s}(y)[-\ln \overline{F}(y)] \right)
\]
or,
\[
\frac{f(y)}{\overline{F}(y)[-\ln \overline{F}(y)]} = \frac{g'_{r|s}(y)}{(s-1)[g_{r|s-1}(y) - g_{r|s}(y)]} = A(y)
\]
Now since \( F(x) \) is a non-decreasing function of \( x \) in \((0,1)\), hence \(-\ln \overline{F}(x)\) is also a non-decreasing function in \((0,\infty)\). Recall that there exits a \( q \), \( \alpha < q < \beta \), such that \(-\ln \overline{F}(q) = 1\). Thus
\[
\ln[-\ln \overline{F}(x)] = C + \int_{\alpha}^{x} A(y) dy,
\]
where \( C \) is constant of integration, implying that
\[
-\ln \overline{F}(x) = \exp(\int_{\alpha}^{x} A(y) dy)
\]
and hence the Lemma.
Theorem 3.3 (Khan and Athar, 2010): Let $X$ be an absolutely continuous rv with df $F(x)$ and pdf $f(x)$ on the support $(\alpha, \beta)$, then for $r < s$

$$g_{s|r}(x) = E[X_{u(s)} | X_{u(r)} = x] = a_{s|r}^* x + b_{s|r}^*$$ (3.10)

if and only if

$$\tilde{F}(x) = [ax + b]^c, \quad \alpha < x < \beta,$$ (3.11)

where

$$a_{s|r}^* = \left(\frac{c}{c+1}\right)^{s-r}, \quad b_{s|r}^* = -\frac{b}{a}(1 - a_{s|r}^*).$$

Proof: For the proof of 'if' part one may refer to Athar et al. (2003) and Khan and Alzaid (2004). To prove 'only if' Athar et al. (2003) and Khan and Alzaid (2004) have used Rao-Shanbhag functional equation [Rao and Shanbhag, 1994] to prove the result. Here we have used Lemma 2.1, which greatly simplifies the steps.

we have

$$g_{s|r}(x) = a_{s|r}^* x + b_{s|r}^*$$

where,

$$b_{s|r}^* = -\frac{b}{a}(1 - a_{s|r}^*), \quad \text{and} \quad a_{s|r+1}^* = \left(\frac{c}{c+1}\right)^{s-r-1} = \left(\frac{c+1}{c}\right)a_{s|r}^*,$$

and

$$g_{s|r+1}(x) - g_{s|r}(x) = (a_{s|r+1}^* - a_{s|r}^*)(x + \frac{b}{a}) = a_{s|r}^*(\frac{ax + b}{ac}).$$

Therefore,

$$\frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{ac}{ax + b}$$

and hence

$$\frac{f(x)}{\tilde{F}(x)} = \frac{ac}{ax + b}$$
implying

\[ F(x) = [ax + b]^c. \]

For examples on Theorem 3.3 and its variants

\[ E[\xi(X_{u(s)}) \mid X_{u(r)} = x] = a_{s\mid r}^* \xi(x) + b_{s\mid r}^*, \]

where \( \xi(x) \) is a monotonic and continuous function of \( x \), refer to Dembińska and Wesolowski (2000), Athar et al. (2003) and Khan and Alzaid (2004).
CHAPTER VI

GENERALIZED AND LOWER GENERALIZED ORDER STATISTICS:
MOMENTS AND RECURRENCE RELATIONS

1 INTRODUCTION

In this chapter a review of moments and recurrence relations of generalized and
dual (lower) generalized order statistics is given.

Moments and recurrence relations of generalized and lower (dual) generalized
order statistics are investigated by many authors in literatures. Kamps (1995)
derived some recurrence relations for moments of generalized order statistics
based on non-identically distributed random variables, which contains order
statistics and record values as special cases. Cramer and Kamps (2000) derived
relations for expectations of functions of generalized order statistics within a
class of distributions including a variety of identities for single and product
moments of ordinary order statistics and record values as particular cases.
Pawlas and Szynal (2001a) established recurrence relations for single and
product moments of generalized order statistics from Pareto, generalized Pareto
and Burr distributions.

Pawlas and Szynal (2001b) defined the concept of lower generalized order
statistics and obtained the recurrence relations for single and product moments
of lower generalized order statistics from the inverse Weibull distribution.

Athar and Islam (2004) established some recurrence relations between
expectation of function of single and joint generalized order statistics from a
general class of distribution. Further, Athar et al. (2007) obtained the ratio and
inverse moments of generalized order statistics from Weibull distribution
whereas Khan et al. (2007) obtained recurrence relations for single and product
moments of generalized order statistics from doubly truncated Weibull
distribution and Athar et al. (2008a) established the relations for the
expectation of function of generalized order statistics for truncated distributions
derived from general class of distributions. For some additional results on
generalized and lower (dual) generalized order statistics, one may refer to
Keseling (1999), Cramer and Kamps (2003), Cramer (2003), Saran and Pandey
(2003), Anwar et al. (2007), Athar et al. (2008b), Faizan and Athar (2008),
Khan et al. (2009a), Khan et al. (2010b), Athar et al. (2010a) and references
therein.

2 SINGLE MOMENTS

Burr Distribution

The pdf of distribution is given as

\[ f(x) = \lambda \tau \beta^\lambda \frac{x^{r-1}}{(\beta + x^r)^{\lambda+1}}, \quad x > 0, \ \beta > 0, \ \lambda > 0, \ \tau > 0 \]  \hspace{1cm} (2.1)

For Burr distribution, we have

\[ (\beta + x^r)f(x) = \lambda \tau (1-F(x)) \; x^{r-1} \]  \hspace{1cm} (2.2)

**Theorem 2.1: (Pawlas and Szynal, 2001a)**

Fix a positive integer \( k \). For \( n \in N, \ m \in Z, \ 1 \leq r \leq n, \) and \( j = 0,1,2,... \), such
that \( \lambda \gamma_r \tau > (j + \tau) \)

\[ E[X^{j+\tau}(r,n,m,k)] = \frac{\beta(j+\tau)}{\lambda \gamma_r \tau - (j + \tau)} E[X^{j}(r,n,m,k)] \]

\[ + \frac{\lambda \gamma_r \tau}{\lambda \gamma_r \tau - (j + \tau)} E[X^{j+\tau}(r-1,n,m,k)] \]  \hspace{1cm} (2.3)

and consequently for \( 0 \leq r \leq n-1 \)

\[ E[X^{j+\tau}(r,n,m,k)] = \prod_{i=s+1}^{r} \frac{\lambda \gamma_i \tau}{\lambda \gamma_i \tau - (j + \tau)} E[X^{j+\tau}(s,n,m,k)] \]

\[ + \sum_{p=s+1}^{r} \left[ \frac{\beta(j+\tau)}{\lambda \gamma_p \tau - (j + \tau)} \right] \prod_{i=p+1}^{r} \left[ \frac{\lambda \gamma_i \tau}{\lambda \gamma_i \tau - (j + \tau)} \right] E[X^{j}(p,n,m,k)] \]
**Proof:** From equation (2.2), we note that for $1 \leq r \leq n$ and $j = 0, 1, 2, \ldots$,

$$E[X^{j+\tau}(r, n, m, k)] + \beta E[X^j(r, n, m, k)]$$

$$= \frac{\lambda \tau c_r-1}{(r-1)!} \int x^{j+\tau-1}(1-F(x))^{\gamma_r} g_{n,m}^{-1}(F(x))dx.$$  

Integrating by parts, treating $X^{j+\tau-1}$ as part for integration, and taking into account that $\lambda \gamma_r > (j + \tau)$, we get

$$E[X^{j+\tau}(r, n, m, k)] + \beta E[X^j(r, n, m, k)]$$

$$= \frac{\lambda \gamma_r}{j+\tau} [E[X^{j+\tau}(r, n, m, k)] - E[X^{j+\tau}(r-1, n, m, k)]],$$

which gives (2.3).

**Remark 2.1:** The recurrence relation for single moments of order statistics from Burr distribution have the form

$$E(X_j^{(n)}) = \frac{\beta(j + \tau)}{\lambda(n-r+1)\tau - (j + \tau)} E(X_{j+\tau}^{(n)}) + \frac{\lambda(n-r+1)\tau}{\lambda(n-r+1)\tau - (j + \tau)} E(X_{j+\tau}^{(n-1)})$$

**Remark 2.2:** The recurrence relation for single moments $k-th$ record values from Burr distribution have the form

$$E(Y_n^{(k)})^{j+\tau} = \frac{\beta(j + \tau)}{\lambda(n-r+1)\tau - (j + \tau)} E(Y_n^{(k)})^j + \frac{\lambda k \tau}{\lambda k \tau - (j + \tau)} E(Y_{n-1}^{(k)})^j.$$  

**Doubly truncated Weibull distribution**

**Theorem 2.2:** (Khan et al., 2007)

For Weibull distribution as given in (2.3.18) and $n \in N$, $m \in \mathbb{R}$, $2 \leq r \leq n$,

(i) $E[X^j(r, n, m, k)] - E[X^j(r-1, n-1, m, k)]$

$$= -P_{2K} E[X^j(r-1, n, m, k+m)]$$

$$- E[X^j(r-1, n-1, m, k+m)] + \frac{j}{p\gamma_1} E[X^{j-p}(r, n, m, k)],$$

(2.4)
(ii) \[ E[X^J(r,n,m,k)] - E[X^J(r-1,n,m,k)] \]
\[ = -P_2K^{**} \{E[X^J(r,n-1,m,k + m)] - E[X^J(r-1,n-1,m,k + m)]\} + \frac{j}{p\gamma_r} E[X^{-p}(r,n,m,k + m)], \tag{2.5} \]

(iii) \[ E[X^J(r-1,n,m,k)] - E[X^J(r-1,n-1,m,k)] \]
\[ = P_2 \frac{(m+1)(r-1)}{\gamma_1} K^{**} \{E[X^J(r,n-1,m,k + m)] - E[X^J(r-1,n-1,m,k + m)]\} \]
\[ - \frac{(m+1)(r-1)}{p\gamma_r\gamma_1} jE[X^{-p}(r,n,m,k)], \tag{2.6} \]

where \( K = \frac{c^{(n-1)}_{r-2}(n-1,k+m)}{c^{(n-1),k+m}_{r-2}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m} \right), \quad \gamma_i^{(n-1)} = k + (n-1-i)(m+1) \)

and \( K^{**} = \frac{c^{(n-1),k+m}_{r-2}}{c^{(n-1),k+m}_{r-2}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i^{(n-1)} + m} \right) = \prod_{i=1}^{r-1} \left( \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m} \right) \).

**Proof:** At \( \xi(x) = x^J \) in (1.3.12), we have
\[ E[X^J(r,n,m,k)] - E[X^J(r-1,n-1,m,k)] \]
\[ = \frac{c_{r-1}}{\gamma_1 (r-1)!} \int_{Q_1} x^{j-1} [1-F(x)]^{\gamma_r-1} d^x \]

Now in view of (2.3.19),
\[ E[X^J(r,n,m,k)] - E[X^J(r-1,n-1,m,k)] \]
\[ = \frac{c_{r-1}}{\gamma_1 (r-1)!} \int_{Q_1} x^{j-1} [1-F(x)]^{\gamma_r-1} \left( -P_2 + \frac{1}{p} x^{1-p} f(x) \right) d^x \]
\[ \begin{align*}
&= -\frac{p_2}{\gamma_1} \frac{c_{r-1}}{(r-1)!} \int_{Q_1}^1 x^{j-1} [1 - F(x)]^{y-1} g_{m-1}^{-1}(F(x)) \, dx \\
&\quad + \frac{j}{p\gamma_1} \frac{c_{r-1}}{(r-1)!} \int_{Q_1}^1 x^{j-p} [1 - F(x)]^{y-1} g_{m-1}^{-1}(F(x)) \, dx \\
&= -\frac{p_2}{\gamma_1} \frac{c_{r-2}}{(r-1)!} \int_{Q_1}^1 x^{j-1} [1 - F(x)]^{y-1} g_{m-1}^{-1}(F(x)) \, dx \\
&\quad + \frac{j}{p\gamma_1} E[X^{j-p}(r,n,m,k)]
\end{align*} \]

as \( y_r-1 = y_{r-1}^{(n-1,k+m)} = (k + m) + (n - r - 1)(m + 1), c_{r-1} = \gamma_1 c_{r-2} \)

and hence the required result.

**Remark 2.3:** If we put \( p = 1 \) in the above expression, we get corresponding result for the exponential distribution. For non-truncated case one has to put \( P = 1, Q = 0 \).

**Remark 2.4:** The recurrence relation for the non-truncated exponential distribution given by Paswlas and Syznal (2001a) are obtained by setting \( P = 1, Q = 0 \) and \( j = j + 1 \).

**Power function distribution**

At \( P = 1 \) and \( Q = 0 \) in (2.2.20), the pdf of non-truncated power function distribution is given as

\[ f(x) = p \nu^{-p} x^{p-1}; \quad 0 < x \leq \nu, \nu > 0 \]  \hspace{1cm} (2.7)

\[ = 0, \text{ otherwise} \]

and the corresponding distribution function (df) if

\[ F(x) = \nu^{-p} x^{p}; \quad 0 < x \leq \nu, \nu > 0 \]  \hspace{1cm} (2.8)

In view of (2.7) and (2.8), we get

\[ F(x) = \frac{x}{P} f(x) \]  \hspace{1cm} (2.9)
Lemma 2.1: (Athar et al., 2010a)

For power function distribution as given in (2.7) and any non-negative finite integers \( a \) and \( b \).

\[
J_\alpha(a,b) = \frac{1}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} J_\alpha[a + (m+1)i, 0] \tag{2.10}
\]

\[
= \frac{\nu^a}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} \frac{1}{t_\alpha[a + (m+1)i]}, \quad m \neq -1 \tag{2.11}
\]

\[
= \frac{b! \nu^a}{[t_\alpha(a)]^{b+1}}, \quad m = -1, \tag{2.12}
\]

where \( J_\alpha(a,b) = \int x^{\alpha-1} [F(x)]^a g_m^b[F(x)] dx \)

\[
J_\alpha(a,0) = \frac{\nu^a}{t_\alpha(a)} \tag{2.14}
\]

and \( t_\alpha(a) = \alpha + a \nu \)

**Proof:** When \( m \neq -1 \)

Result (2.11) can be proved by expanding \( g_m^b[F(x)] = \left[ \frac{1}{m+1} \left( 1 - (F(x))^{m+1} \right) \right]^b \)

binomially in (2.13) and then using the result (2.14).

**When \( m = -1 \)**

At \( m = -1 \) in (2.11)

\[
J_\alpha(a,b) = 0 \quad \text{as} \quad \sum_{i=0}^{b} (-1)^i \binom{b}{i} = 0 .
\]

Since (2.11) is of the form \( 0/0 \) at \( m = -1 \), therefore after applying L'Hospital's rule, we get
\[ \lim_{m \to -1} J_{\alpha}(a, b) = \frac{p^b \nu_\alpha}{(\alpha + a p)^{b+1}} \sum_{i=0}^{b} (-1)^{i+b} \binom{b}{i} i^b, \quad b > 0. \]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n! \]

Therefore,

\[ \sum_{i=0}^{b} (-1)^{i+b} \binom{b}{i} i^b = b!. \]

Hence

\[ \lim_{m \to -1} J_{\alpha}(a, b) = \frac{b! p^b \nu_\alpha}{(\alpha + a p)^{b+1}}. \]

**Theorem 2.3: (Athar et al., 2010a)**

For power function distribution as given in (2.7) and \( \gamma_r \geq 1, k \geq 1, 1 \leq r \leq n, \)

\[ \frac{m-m^l}{m^l} \]

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Identity 2.1: For \( \gamma_r \geq 1, k \geq 1, 1 \leq r \leq n \) and \( m \neq -1 \)

\[
\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}} \frac{(m+1)^{r-1}(r-1)!}{\prod_{j=1}^{r} \gamma_j}.
\] (2.19)

Proof: (2.19) can be proved by putting \( \alpha = 0 \) in (2.16).

Remark 2.5: If we put \( m = 0, k = 1 \) in (2.16), we get the result for order statistics

\[
E(X_{r,n,0,1}) = E(X_{n-r+1:n}) = p \nu^\alpha C_{n-r+1:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha (n+i-r+1)}
\] (2.20)

where \( C_{n-r+1:n} = \frac{n!}{(n-r)!(r-1)!} \)

In view of Identity (2.1), (2.20) may also be expressed as

\[
E(X_{n-r+1:n}) = \frac{\Gamma(n+1)\Gamma[(\alpha/p)+n-r+1] \nu^\alpha}{\Gamma(n-r+1)\Gamma[n+(\alpha/p)+1]}, \quad p > \alpha
\] (2.21)

as obtained by Malik (1967).

Remark 2.6: Moment of \( k-th \) lower record values from the power function distribution may be obtained in view of (2.12) and (2.15) at \( m = -1 \).

\[
E(X_{r,n,-1,k}) = E(R_{n}^{(k)})^\alpha = \frac{(pk)^n \nu^\alpha}{[t_\alpha (k)]^n}
\] (2.22)

by noting \( \gamma_i = k \) and \( C_{r-1} = k^r \).

Erlang truncated exponential distribution

The pdf of distribution is given as

\[
f(x) = \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})}, \quad x \geq 0, \quad \beta, \lambda > 0
\] (2.23)
and the corresponding df is
\[ \overline{F}(x) = e^{-\beta x(1-e^{-\lambda})}. \] (2.24)

Now in view of (2.23) and (2.24), we have
\[ f(x) = \beta (1-e^{-\lambda}) \overline{F}(x) \] (2.25)

**Theorem 2.4: (Khan et al., 2010b)**

For the distribution given in (2.23) and \( n \in N, m \in \mathbb{R}, r = 1 \)
\[ E[X^{j+1}(1,n,m,k)] = \frac{j+1}{\beta(1-e^{-\lambda}) \gamma_1} E[X^j(1,n,m,k)] \] (2.26)

and for \( 2 \leq r \leq n \)
\[ E[X^{j+1}(r,n,m,k)] = \frac{j+1}{\beta(1-e^{-\lambda}) \gamma_r} E[X^j(r,n,m,k)] + E[X^{j+1}(r-1,n,m,k)]. \] (2.27)

**Proof:** In view of (2.25), we have
\[ E[X^j(r,n,m,k)] = \beta(1-e^{-\lambda}) \frac{C_{r-1}}{(r-1)!} I_r(x), \] (2.28)

where \( I_r(x) = \int_0^\infty x^j \overline{F}(x)^{\gamma_r} g_m^{r-1} (F(x)) \, dx \).

Integrating by parts, treating \( x^j \) for integration and the rest of the integrand for differentiation, we get
\[ I_r(x) = \frac{\gamma_r}{j+1} \int_0^\infty x^{j+1} \overline{F}(x)^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) \, dx \]
\[ - \frac{(r-1)}{j+1} \int_0^\infty x^{j+1} \overline{F}(x)^{\gamma_r+m} f(x) g_m^{r-2} (F(x)) \, dx. \]
Now substituting for $I_1(x)$ and $I_r(x)$ in equation (2.28), we drive the relations in (2.26) and (2.27).

**General form of distributions**

(a) $F(x) = 1 - \left[ a h(x) + b \right]^c$

where $a \neq 0$, $b$, $c \neq 0$ are finite constants and $h(x)$ is continuous, monotonic and differentiable function of $x$.

The *pdf* $f(x)$ is given by

$$f(x) = -c a \left[ b h(x) + b \right]^{c-1} h'(x), \; x \in (\alpha, \beta). \tag{2.29}$$

and the corresponding truncated *df* $F(x)$ by

$$1 - F(x) = -\frac{a h(x) + b}{c a h'(x)} f(x). \tag{2.30}$$

**Theorem 2.5: (Athar and Islam 2004)**

For general class of distribution as in (2.29) and $n \in \mathbb{N}$, $m \in \mathbb{N}$, $2 \leq r \leq n$.

$$E[\xi(X(r,n,m,k))] = E[\xi(X(r-1,n,m,k))] - \frac{1}{\gamma_r, ca} E[\varphi(X(r,n,m,k))],$$

where

$$\varphi(x) = \left[ a h(x) + b \right] w(x), w(x) = \frac{\xi'(x)}{h'(x)}.$$

**Proof:** In view of (1.3.10) and (2.30), we have

$$E[\xi(X(r,n,m,k))] - E[\xi(X(r-1,n,m,k))]$$

$$= -\frac{C_{r-2}}{(r-1)!} \int_\alpha^\beta \xi'(x) [1 - F(x)]^{\gamma_r-1} \left\{ -\frac{a h(x) + b}{c a h'(x)} f(x) \right\} g_r^{-1}(F(x)) dx$$
\[ E \{ \xi(X(r,n,m,k)) \} - E \{ \xi(X(r-1,n,m,k)) \} \]

\[ = -\frac{1}{\gamma_r c a (r-1)!} \int \psi(x) [1 - F(x)]^{\gamma_r - 1} g_m^{-1}(F(x)) f(x) dx. \]

Rearranging the terms we find the result.

**Remark 2.7:** Recurrence relation for single moments of order statistics at \((m = 0, k = 1)\) is

\[ E[\xi(X_{rn})] = E[\xi(X_{r-1n})] - \frac{1}{(n-r+1)c a} E[\psi(X_{rn})] \]

as obtain by Ali and Khan (1997).

**Remark 2.8:** Recurrence relation for single moment's \(k\)th record values will be

\[ E[\xi(X_{r,n-1,k})] = E[\xi(X_{r-1,n-1,k})] - \frac{1}{k c a} E[\psi(X_{r,n-1,k})], \]

\[ E[\xi(X_{r}^{(k)})] = E[\xi(X_{r-1}^{(k)})] - \frac{1}{k c a} E[\psi(X_{r}^{(k)})], \]

where \(X_{r}^{(k)}; r = 1,2,...\) is \(r\)th \(k\) records.

**3 PRODUCT MOMENTS**

**Burr Distribution**

**Theorem 3.1:** (Pawlas and Szynal, 2001a)

For distribution as given in (2.1), fix a positive integer \(k \geq 1\). For \(n \in \mathbb{N}, m \in \mathbb{Z}, 1 \leq r \leq n\) and \(i, j = 0,1,2,...\), such that \(\lambda \gamma_r \tau > (j + \tau)\)

\[ E[X^i(r,n,m,k)X^{j+\tau}(r+1,n,m,k)] \]

\[ = \frac{\beta(j + \tau)}{\lambda \gamma_{r+1} \tau - (j + \tau)} E[X^i(r,n,m,k)X^{j}(r+1,n,m,k)] + \frac{\lambda \gamma_{r+1} \tau}{\lambda \gamma_{r+1} \tau - (j + \tau)} E[X^{i+j+\tau}(r,n,m,k)] \quad (3.1) \]
and for $1 \leq r \leq s - 2 \leq n$

$$E[X^i(r,n,m,k)X^j+s(r,n,m,k)] = \frac{\beta(j+s)}{\lambda s^{s-(j+s)}} E[X^i(r,n,m,k)X^j(s,n,m,k)]$$

$$+ \frac{\lambda s^{s-(j+s)}}{\lambda s^{s-(j+s)}} E[X^i(r,n,m,k)X^j+s(r-1,n,m,k)]$$  \hspace{1cm} (3.2)

**Proof:** Note that for $1 \leq r \leq s - 2 \leq n$ and $i, j = 0, 1, 2, \ldots$,

$$E[X^i(r,n,m,k)X^j+s(r,n,m,k)] + \beta E[X^i(r,n,m,k)X^j(s,n,m,k)]$$

$$= \frac{c_{r-1}s^r}{(r-1)!(s-r-1)!} \int x^i(1-F(x))^m f(x)g_{s-r-1}(F(x))I(x)dx$$  \hspace{1cm} (3.3)

where,

$$I(x) = \int y^{i+s-1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}(1-F(y))y^s dy.$$  

Integrating $I(x)$ by parts we obtain

$$I(x) = \int_{j+\tau} \gamma_x \left[ \int y^{i+\tau}[h_m(F(y)) - h_m(F(x))]^{s-r-1}(1-F(y))y^{s-1} f(y)dy \right]$$

$$- \frac{s-r-1}{j+\tau} \left[ \int y^{i+\tau}[h_m(F(y)) - h_m(F(x))]^{s-r-2}(1-F(y))y^{s-1} f(y)dy \right].$$

Substituting this expression into (3.3) and simplifying we obtain (3.2). When $s = r + 1$, we have (3.1).

**Pareto distribution**

The *pdf* of distribution is given as

$$f(x) = \frac{\alpha x^\alpha}{\sigma^{\alpha+1}}, \ x > \sigma, \sigma > 0, \alpha > 0$$  \hspace{1cm} (3.4)

for a simple Pareto distribution, we have

$$xf(x) = \alpha(1-F(x))$$  \hspace{1cm} (3.5)
Theorem 3.2: (Pawlas and Szynal, 2001b)

Fix a positive integer \( k \geq 1 \). For \( n \in \mathbb{N} \), \( m \in \mathbb{Z} \), \( 1 \leq r \leq n \), and \( i, j = 0, 1, 2, \ldots \), such that \( \alpha \gamma_s > s \)

\[
E[X^i(r, n, m, k) X^j(r + 1, n, m, k)] = \frac{\alpha \gamma_{r+1}}{\alpha \gamma_{r+1} - (r + 1)} E[X^{i+j}(r, n, m, k)]
\]

and for \( 1 \leq r \leq s - 2 \leq n \)

\[
E[X^i(r, n, m, k) X^j(s, n, m, k)] = \prod_{j=r}^{s} \left( \frac{\alpha \gamma_j}{\alpha \gamma_j - s} \right) E[X^{i+j}(r, n, m, k)]
\]

Proof: See reference.

Doubly truncated Weibull distribution

Theorem 3.3: (Khan et al., 2007)

For the given Weibull distribution as in (2.3.18) and \( n > 2 \), \( m \in \mathbb{R} \), \( 1 \leq r < s \leq n - 1 \).

\[
E[X^i(r, n, m, k) X^j(s, n, m, k)] - E[X^i(r, n, m, k) X^j(s - 1, n, m, k)]
\]

\[= -P_2 K_1 E[X^i(r, n - 1, m, k + m) X^j(s, n - 1, m, k + m)]
\]

\[+ - E[X^i(r, n - 1, m, k + m) X^j(s - 1, n - 1, m, k + m)]
\]

\[+ \frac{j}{p \gamma_s} E[X^i(r, n, m, k) X^{i+j-p}(s, n, m, k)]
\]

were \( K_1 = \frac{C_{s-2}}{C_{s-2}} = \prod_{i=1}^{s-1} \left( \frac{\gamma_i}{\gamma_i (n - 1) + m} \right) = \frac{\gamma_i^{s-1}}{\gamma_s} \prod_{i=1}^{s-1} \left( \frac{\gamma_i^{(n - 1)}}{\gamma_i (n - 1) + m} \right) \)

Proof: In view of (1.3.13), we have

\[
E[X^i(r, n, m, k) X^j(s, n, m, k)] - E[X^i(r, n, m, k) X^j(s - 1, n, m, k)]
\]

\[= \frac{C_{s-1}}{\gamma_s (r - 1)! (s - r - 1)!} \int_{0}^{x} \int_{0}^{y} x^i y^{j-1} (1 - F(x))^m f(x) g^{r-1}_m(F(x))
\]

\[\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{s-1} dydx \quad (3.6)
\]
Now using (2.3.19) in (3.6), we get

\[
\begin{align*}
    &= \frac{C_s-1}{\gamma_s(r-1)!s(r-1)!} \int_0^1 \int_0^1 x^j y^{j-1} [1-F(x)]^m f(x) g_r^m(F(x)) \\
    &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s} f(y) dy dx \\
    &= -P_2 \frac{C_s-1}{\gamma_s(r-1)!s(r-1)!} \int_0^1 \int_0^1 x^j y^{j-1} [1-F(x)]^m \\
    &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s} f(y) dy dx \\
    &+ \frac{C_s-1}{s(r-1)!s(r-1)!} \int_0^1 \int_0^1 x^j y^{j-1} [1-F(x)]^m f(x) g_r^m(F(x)) \\
    &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s} f(y) dy dx \\
    &= -P_2 \frac{C_s-1}{\gamma_s(r-1)!s(r-1)!} \int_0^1 \int_0^1 x^j y^{j-1} [1-F(x)]^m f(x) g_r^m(F(x)) \\
    &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s} f(y) dy dx \\
    &+ \frac{1}{C_s} E \left[ X^i(r,n,m,k) X^{j-p}(r,n,m,k) \right],
\end{align*}
\]

where, \( \gamma_s = \gamma_s^{(n-1,k+m)} \), \( C_{s-1} = \gamma_s \), \( C_{s-2} \) and hence the result.

**Power function distribution**

**Lemma 3.1 (Athar et al., 2010a)**

For power function distribution as in (2.7) and any non negative distribution \( a,b,c \) with \( m \neq -1 \)

\[
J_{\alpha,\beta}(a,0,c) = \frac{\Gamma^{\alpha+\beta}}{t_\beta(c)t_\alpha(a+c)},
\]

where

\[
J_{\alpha,\beta}(a,b,c) = \int_0^\infty \int_0^\infty x^\alpha y^\beta [F(x)]^a [h_m(F(y)) - h_m(F(x))]^b [F(x)]^c dx dy
\]
Proof: From (2.8) and (3.7), we have

\[ J_{\alpha,\beta}(a,b,c) = \int_0^\infty x^{\alpha+ap-1} \left( \int_0^\infty y^{\beta+cp-1} dy \right) dx \]

\[ = \frac{\nu^{\alpha+\beta}}{\beta + cp} [\alpha + \beta + p(a + c)] \]

and hence the lemma.

Lemma 3.2 (Athar et al., 2010a)

For power function distribution as in (2.7) and any non negative integers \(a, b, c\) with \(m \neq -1\)

\[ J_{\alpha,\beta}(a,b,c) = \frac{\nu^{\alpha+\beta}}{(m+1)^b} \sum_{j=0}^{b} (-1)^j \binom{b}{j} \frac{1}{t \beta_c(m+1) j \alpha+\beta(a+c)+(m+1)b} \; \; m \neq -1 \]  

(3.8)

\[ J_{\alpha,\beta}(a,b,c) = \frac{b! \nu^{\alpha+\beta}}{[t \beta_c(b)]^{b+1} [t \alpha+\beta(a+c)]} \; \; m \neq -1 \]  

(3.9)

Proof: When \(m \neq -1\)

(3.8) can be established by expanding \([h_m(F(y)) - h_m(F(x))]^b\) binomially in (3.7) and there after on application of Lemma 3.1.

When \(m = -1\)

Since at \(m = -1\) (3.8) is of the \(0/0\) form, so after applying L-Hospital’s rule (3.9) can be proved on the lines of (2.12).

Theorem 3.4 (Athar et al., 2010a)

For power function distribution as in (2.7) and \(\gamma_r, \gamma_s \geq 1, k \geq 1, 1 \leq r \leq n, m \neq -1\)

\[ E(X_{r,n,m,k}^{\alpha} X_{s,n,m,k}^{\beta}) = \frac{p^2}{(m+1)^{r-1} (r-1)! (s-r-1)!} \]

\[ \times \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} J_{\alpha,\beta}[(m+1)(j+1),(s-r-1),\gamma_s] \]
\[
\frac{p^{2\alpha + \beta}}{(m + 1)^{s-2} (r-1)! (s-r-1)!} \times \sum_{j=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{t_{\beta(\gamma_{s-1})} t_{\alpha + \beta(\gamma_{r-j})}}
\]
and subsequently for \( s = r+1 \)

\[
E(X_{r,n,m,k}^{\alpha} X_{r+1,n,m,k}^{\beta}) = \frac{p^{2\alpha + \beta}}{(m + 1)^{r-1} (r-1)!} \times \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j} \frac{1}{t_{\beta(\gamma_{r+j})} t_{\alpha + \beta(\gamma_{r-j})}},
\]

where \( X_{r,n,m,k} \) and \( X_{s,n,m,k} \) are \( r-th \) and \( s-th \) lgos respectively.

**Proof:** We have,

\[
E(X_{r,n,m,k}^{\alpha} X_{s,n,m,k}^{\beta}) = \frac{C_{s-1}}{(r-1)! (s-r-1)!} \int_0^x x^\alpha y^{\beta} [F(x)]^m f(x) g_{m-1}^{r-1}(F(x)) \times [h_{m}(F(y)) - F_{m}(F(x))]^{s-r-1} [F(y)]^{y_{s-1}} f(y) dy dx.
\]

Since

\[
g_{m-1}^{r-1}(F(x)) = \left\{ \frac{1}{m+1} \left[ 1 - (F(x))^{m+1} \right] \right\}^{r-1} = \frac{1}{(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j} [F(x)]^{(m+1)j}
\]

Therefore in view of (2.9), we get

\[
E(X_{r,n,m,k}^{\alpha} X_{s,n,m,k}^{\beta}) = \frac{C_{s-1}}{(r-1)! (s-r-1)! (m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j}
\]

\[
\times \int_0^x x^{\alpha-1} y^{\beta-1} [F(x)]^{(m+1)(j+1)} [h_{m}(F(y)) - F_{m}(F(x))]^{s-r-1} [F(y)]^{y_{s}} dy dx.
\]

Thus, theorem can be proved by an application of lemma 3.1 and lemma 3.2.
Erlang–truncated exponential distribution

**Theorem 3.5 (Khan et al., 2010b)**

For the Erlang-truncated exponential distribution as given in (2.23) and \( n \geq 2 \), \( m \in \mathbb{R} \), \( 1 \leq r < r + 1 \leq n \)

\[
E[X^i(r, n, m, k)X^{j+1}(r+1, n, m, k)]
= \frac{j+1}{\beta(1-e^{-\lambda})\gamma_{r+1}}E[X^i(r, n, m, k)X^j(r+1, n, m, k)]
+ E[X^{i+j+1}(r, n, m, k)]
\tag{3.10}
\]

and for \( 1 \leq r < s \leq n \), \( s-r \geq 2 \) and \( i, j \geq 0 \)

\[
E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)]
= \frac{j+1}{\beta(1-e^{-\lambda})\gamma_s}E[X^i(r, n, m, k)X^j(s, n, m, k)]
+ E[X^{i+j+1}(r, n, m, k)]
\tag{3.11}
\]

**Proof:** See reference.

**Remark 3.1:** Putting \( m = 0 \), \( k = 1 \) in (3.10) and (3.11), we obtain recurrence relations for product moments of order statistics of the Erlang-truncated exponential distribution in the form

\[
E(X^{(i,j+1)}_{r,r+1:n}) = \frac{j+1}{\beta(1-e^{-\lambda})(n-r)}E(X^{(i,j)}_{r,r+1:n}) + E(X^{(i,j+1)}_{r:n})
\]

and

\[
E(X^{(i,j+1)}_{r,s:n}) = \frac{j+1}{\beta(1-e^{-\lambda})(n-s+1)}E(X^{(i,j)}_{r,s:n}) + E(X^{(i,j+1)}_{r,s-1:n})
\]

**Remark 3.2:** Putting \( m = -1 \), \( k \geq 1 \) in (3.11), we get the recurrence relations for product moments of upper \( k \)-records of the Erlang-truncated exponential distribution in the form

\[
E[X^i(r,n-1,k)X^{j+1}(s,n-1,k)] = \frac{j+1}{\beta(1-e^{-\lambda})k}E[X^i(r,n-1,k)X^j(s,n-1,k)]
+ E[X^i(r,n-1,k)X^{j+1}(s-1,n-1,k)].
\]
General form of distribution

**Theorem 3.6 (Athar and Islam, 2004)**

For general class of distribution as given in (2.29) and $n \in \mathbb{N}$, $m \in \mathbb{R}$, $1 \leq r < s \leq n-1$,

$$E[\xi\{X(r,n,m,k),X(s,n,m,k)\}] - E[\xi\{X(r,n,m,k),X(s-1,n,m,k)\}]$$

$$= -\frac{1}{cay_s} E[\psi\{X(r,n,m,k),X(s,n,m,k)\}]$$

(3.12)

where, $\psi(x,y) = [ah(y) + b] \frac{\partial}{\partial y} \frac{\xi(x,y)}{h'(y)}$

**Proof:** (3.12) can be established in view of (2.30) and (1.3.16).
CHAPTER VII

GENERALIZED AND LOWER GENERALIZED ORDER STATISTICS:
CHARACTERIZATION OF DISTRIBUTIONS

1 INTRODUCTION

Order statistics and record values are special cases of generalized order statistics (gos), therefore characterization through the generalized order statistics is of special interest. Keseling (1999) gave characterization of exponential distribution under the condition

\[ E[\psi(X(r+1,n,m,k) - X(r,n,m,k)) | X(r,n,m,k) = x] = c, \]

where \( c \) is a constant and \( X(r,n,m,k) \) is the \( r \)-th gos.

Raqab and Lawi (2004) characterized some general continuous distribution based on conditional expectation

\[ E[g(X(r+1,n,m,k)) | X(r,n,m,k) = x] = h(x) + c, \]

where \( h(.) \) and \( g(.) \) are real, continuous and strictly increasing functions.

Khan and Alzaid (2004) characterized a general class of distributions \( \overline{F}(x) = (ax + b)^c \) through linear regression of generalized order statistics using Rao and Shanbhag’s (1994) result. They characterized distributions by using the relation

\[ E[X(s,n,m,k) | X(r,n,m,k) = x] = a^* h(x) + b^*. \]

Further, Khan et al. (2006), Beg and Ahsanullah (2006) have characterized distributions by means of the relation

\[ E[\xi(X(s,n,m,k)) | X(r,n,m,k) = x] = g_{s|r}(x) \]

and its dual

\[ E[\xi(X(r,n,m,k)) | X(s,n,m,k) = x] = g_{r|s}(x). \]
Bienik (2007) characterized continuous distribution based on conditional expectation

\[ E[g\{X(r,n,m,k)\} | X(r+1,n,m,k) = x] = h(x) \]

using Meijer's G-function.

Ahsanullah and Beg (2008) characterized continuous distribution functions conditioned on a pair of adjacent gos through the relation

\[ E[\xi\{X(r+1,n,m,k)\} | X(r,n,m,k) = x, X(r+2,n,m,k) = y] = g_{r+1|r,r+2}(x). \]

Haque and Faizan (2010) characterized Weibull distribution through conditional variance of gos, whereas Khan et al. (2010) characterized a general form of distribution \( \overline{F}(x) = e^{-ah(x)} \) through conditional variance of gos.


2 CHARACTERIZATION OF SOME SPECIFIC DISTRIBUTIONS

Uniform distribution

Lemma 2.1 (Ahsanullah, 2004)

\[ F_{i,r,n,m,k}(x) = I_\alpha(x) \left( r, -\frac{y_r}{m+1} \right) \quad \text{if } m > -1 \]

and

\[ F_{i,r,n,m,k}(x) = \Gamma(x)(r) \quad \text{if } m = -1 \]

where, \( F_{i,r,n,m,k}(x) = F_{X'(r,n,m,k)} \)

\[ \alpha(x) = 1 - (F(x))^{m+1}, \]

\[ \beta(x) = -\ln F(x), \]
Generalized and Lower Generalized Order Statistics: Characterization of...

\[ I_x(p,q) = \frac{1}{B(p,q)} \int_0^x u^{p-1} (1-u)^{q-1} \, du, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \]

\[ \Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} \, du. \]

**Proof:** For \( m > -1 \)

\[ F_{l,r,n,m,k}(x) = \frac{C_{r-1}}{\Gamma(r)} \int_0^x (F(u))^{r-1} (g_m(F(u)))^{r-1} f(u) \, du \]

\[ = \frac{C_r}{\Gamma(r)} \int_0^x (F(u))^{r-1} \left[ \frac{1-(F(u))^{m+1}}{m+1} \right] f(u) \, du \]

\[ = \frac{1}{\Gamma(r)} \int_0^{(F(x))^{m+1}} \frac{r}{m+1} (1-u)^{m+1} - 1(-u)^{r-1} \, du \]

\[ \alpha(x) = 1-(F(x))^{m+1}. \]

For \( m = -1, \gamma_j = k, \quad j = 0,1,\ldots,n \)

\[ F_{l,r,n,m,k}(x) = \frac{k^{r}}{\Gamma(r)} \int_0^x (F(u))^{k-1} (-\ln(F(u)))^{r-1} f(u) \, du \]

\[ = \int_0^{k\ln F(x)} \frac{1}{\Gamma(r)} u^{r-1} e^{-u} \, du \]

\[ \Gamma_{\beta(k)}(r), \beta(x) = -k \ln F(x). \]

**Lemma 2.2 (Ahsanullah, 2004)**

For \( m > -1 \)

\[ \gamma_{r+1}(F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x) \]

and for \( m = -1 \)

\[ \gamma_{r+1}(F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x) \]
Proof: For $m > -1$

\[
F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x) = I_\alpha(x) \left( r, \frac{\gamma_r}{m+1} \right) - I_\alpha(x) \left( r+1, \frac{\gamma_{r+1}}{m+1} \right)
= I_\alpha(x) \left( r, \frac{\gamma_r}{m+1} \right) - I_\alpha(x) \left( r+1, \frac{\gamma_r}{m+1} - 1 \right).
\]

We know that

\[
I_x(a,b) - I_x(a+1,b-1) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^{b-1},
\]

thus,

\[
F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x) = \frac{\Gamma \left( \frac{\gamma_r}{m+1} \right)}{\Gamma(r+1) \Gamma \left( \frac{\gamma_r}{m+1} \right)}
\times (1-(F(x))^{m+1})^r (F(x)^{m+1})^{\gamma_{r+1} - 1}
= \frac{\gamma_1, \ldots, \gamma_r}{\Gamma(r+1)} \left( \frac{1-(F(x))^{m+1}}{m+1} \right)^r (F(x))^\gamma_{r+1}
= \frac{F(x)}{\gamma_{r+1} f(x)} f_{l,r+1,n,m,k}(x).
\]

Therefore,

\[
\gamma_{r+1}(F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x).
\]

For $m = -1$, the proof of the result

\[
k(F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x)
\]

is similar.
Theorem 2.1 (Ahsanullah, 2004)

Let $X$ be an absolutely continuous (with respect to Lebesgue Measure) bounded random variable with strictly increasing cdf $F(x)$ and pdf $f(x)$. Without any loss of generality we will take $F(0) = 0$ and $F(1) = 1$. Then the following two statements are equivalent

(a) $X$ is uniformly distributed random variable in $(0,1)$

(b) $X'(r + 1, n, m, k) \overset{d}{=} X'(r, n, m, k) W_{r+1},$

where $W_{r+1}$ is independent of $X'(r + 1, n, m, k)$ and the pdf of $W_{r+1}$ is

$$f_{r+1}(w) = \lambda_{r+1} w^{\lambda_{r+1}-1}, 0 < w < 1.$$

Proof: The proof of (a) implies (b) can be obtained from the dual representation of dual generalized order statistics in Burkschat et al. (2003). However, here we present a direct proof of result.

Let $m > -1$ and $Y = X'(r, n, m, k) W_{r+1}$

$$F_Y(x) = P(Y \leq x) = P\{X'(r, n, m, k) W_{r+1} \leq x\}$$

$$F_Y(x) = \int_0^x f_{l,r,n,m,k}(u) \, du + \int_x^1 \left(\frac{x}{u}\right)^{\lambda_{r+1}} f_{l,r,n,m,k}(u) \, du$$

$$= F_{l,r,n,m,k}(x) + \int_x^1 \left(\frac{x}{u}\right)^{\lambda_{r+1}} f_{l,r,n,m,k}(u) \, du. \quad (2.3)$$

Differentiating (2.3) w.r.t. $x$, we obtain,

$$f_Y(x) = f_{l,r,n,m,k}(x) - f_{l,r,n,m,k}(x)u$$

$$+ \int_x^1 \frac{Y_{r+1}}{u^{\lambda_{r+1}}} \left(\frac{x}{u}\right)^{\lambda_{r+1}-1} f_{l,r,n,m,k}(u) \, du \quad (2.4)$$

From (2.4), we obtain on simplification
\[
\frac{f_Y(x)}{(y)^{\gamma_{r+1}} - 1} = \int_x^1 \frac{\gamma_{r+1}}{(y)^{\gamma_{r+1}}} f_{i,r,n,m,k}(u) \, du.
\] (2.5)

Differentiating both sides of (2.5) w.r.t. \(x\), we obtain,

\[
\frac{f_Y'(x)}{(y)^{\gamma_{r+1}} - 1} - \frac{f_Y(x)}{(y)^{\gamma_{r+1}}} (\gamma_{r+1} - 1) = \frac{\gamma_{r+1}}{x^{\gamma_{r+1}}} f_{i,r,n,m,k}(x).
\] (2.6)

On simplification from (2.6), we get

\[
f_Y''(x) - \frac{\gamma_{r+1}}{x} f_Y(x) = \frac{c_r x^{\gamma_{r-1}}}{\Gamma(r)x} \left[ \frac{1-x^{m+1}}{m+1} \right]^{r-1}.
\] (2.7)

Multiplying both sides of (2.7) by \(x^{-(\gamma_{r+1})}\), we obtain

\[
\frac{d}{dx} (f_Y(x)x^{-(\gamma_{r+1})}) = \frac{c_r x^m}{\Gamma(r)} \left[ \frac{1-x^{m+1}}{m+1} \right]^{r-1}.
\]

Thus,

\[
f_Y(x)x^{-(\gamma_{r+1})} = c - \int \frac{c_r x^m}{\Gamma(r)} \left[ \frac{1-x^{m+1}}{m+1} \right]^{r-1} \, dx
\]

\[= c + \frac{c_r}{\Gamma(r+1)} \left[ \frac{1-x^{m+1}}{m+1} \right]^r,
\] (2.8)

where \(c\) is a constant.

Therefore,

\[
f_Y(x) = c x^{\gamma_{r+1} - 1} + \frac{c_r x^{\gamma_{r+1} - 1}}{\Gamma(r+1)} \left[ \frac{1-x^{m+1}}{m+1} \right]^r.
\] (2.9)

We have from (2.9), \(F_Y(0) = 0\) and \(F_Y(1) = \frac{c}{\gamma_{r+1}} + 1\). Since \(f_Y(x)\) is a pdf

with boundary condition \(F_Y(0) = 0\) and \(F_Y(1) = 1\), we must have \(c = 0\).
Hence, we obtain
\[ f_Y(x) = \frac{c_r x^\gamma r+1-1}{\Gamma(r+1)} \left[ \frac{1-x^m+1}{m+1} \right]^r. \]  
(2.10)

Thus, \( Y = X'(r+1,n,m,k) \).

To prove (b) implies (a), we have
\[ F_{l,r+1,n,m,k}(x) = P\{X'(r,n,m,k) W_{r+1} \leq x\} \]
\[ = \int_0^1 F_{l,r,n,m,k} \left( \frac{x}{u} \right) u^{\gamma r+1-1} \gamma r+1 \, du \]
\[ = x^{\gamma r+1} + \gamma r+1 \int_0^1 F_{l,r,n,m,k} \left( \frac{x}{u} \right) u^{\gamma r+1-1} \, du. \]

Substituting \( \frac{u}{s} = t \) in the integral, we obtain
\[ F_{l,r+1,n,m,k}(x) = x^{\gamma r+1} + \gamma r+1 \int_0^1 F_{l,r,n,m,k} \left( \frac{1}{t} \right) \frac{1}{t} \gamma r+1 \, dt \]  
(2.11)

Differentiating both sides of \( w.r.t. \ x \), we obtain
\[ f_{l,r+1,n,m,k}(x) = \gamma r+1 x^{\gamma r+1-1} - \gamma r+1 x^{\gamma r+1} F_{l,r,n,m,k}(x) x^{-\gamma r+1-1} \]
\[ + (\gamma r+1)^2 x^{\gamma r+1-1} \int_0^1 F_{l,r,n,m,k}(t) \left( \frac{1}{t} \right) \gamma r+1 \, dt \]

Thus, using (2.11), we obtain
\[ f_{l,r+1,n,m,k}(x) = \gamma r+1 x^{\gamma r+1-1} - \frac{F_{l,r,n,m,k}(x)}{x} \]
\[ + (F_{l,r+1,n,m,k}(x) - x^{\gamma r+1}) \gamma r+1 x^{-1} \]  
(2.12)
On simplification, we obtain from (2.12)

\[ f_{l,r+1,n,m,k}(x) = (-F_{l,r,n,m,k}(x) + F_{l,r+1,n,m,k}(x) \gamma_{r+1}x^{-1}) \]

\[ = \left( \frac{F(x)}{\gamma_{r+1}f(x)} f_{l,r+1,n,m,k}(x) \gamma_{r+1}x^{-1} \right) \] (2.13)

Thus,

\[ \frac{f(x)}{F(x)} = \frac{1}{x} \] (2.14)

The solution of (2.14) with boundary condition \( F(0) = 0 \) and \( F(1) = 1 \) is

\[ F(x) = x, \quad 0 \leq x \leq 1. \]

The proof of Theorem for \( m = -1 \) is similar.

Weibull distribution

**Lemma 2.3 (Haque and Faizan, 2010)**

Let \( F(x) \) be a df such that \( F(0) = 0 \) and has a continuous second order derivative on \((0,\infty)\) with \( F'(x) > 0 \) for all \( x > 0 \) (so that \( F(x) < 1 \) for all \( x \), in particular). If it satisfies the differential equation

\[ \frac{\bar{F}'(x)}{F(x)} + (\gamma_{r+1} - 1) \left[ \frac{\bar{F}'(x)}{F(x)} \right]^2 - \frac{(p-1)}{x} \left[ \frac{\bar{F}'(x)}{F(x)} \right] - \gamma_{r+1} \theta^2 x^{2(p-1)} = 0, \] (2.15)

then \( \bar{F}(x) = e^{-\theta x^p} \) for all \( x > 0 \) where \( \theta, p, \gamma_{r+1} \) are all positive constants.

**Proof:** Let \( \frac{\bar{F}'(x)}{F(x)} = p \gamma_{r+1} x^{p-1} \frac{\gamma_{r+1}}{t} \), then (2.15) reduces to

\[ \frac{dt}{dx} = px^{p-1} [\gamma_{r+1}^2 - \theta^2 t^2] \]
therefore,

\[
\frac{1}{2\gamma_{r+1}} \int \left[ \frac{1}{(\gamma_{r+1} - \theta t)} + \frac{1}{(\gamma_{r+1} + \theta t)} \right] \, dt = p \int x^{p-1} \, dx
\]

implying that

\[
\frac{\gamma_{r+1} + \theta t}{\gamma_{r+1} - \theta t} = A e^{2\gamma_{r+1} \theta x^p},
\]

where \( A \) is the constant of integration.

Thus,

\[
\frac{\tilde{F}'(x)}{\tilde{F}(x)} = \frac{1}{2\gamma_{r+1}} \left[ \frac{2A}{Au - 1} - \frac{1}{u} \right] \frac{du}{dx}, \quad \text{where} \quad u = e^{2\gamma_{r+1} \theta x^p}
\]

and

\[
\tilde{F}(x) = B \left[ A e^{\gamma_{r+1} \theta x^p} - e^{-\gamma_{r+1} \theta x^p} \right]^{1/\gamma_{r+1}},
\]

where \( A \) and \( B \) are constants to be determined. Since \( F(x) \) is bounded, hence \( \tilde{F}(x) = e^{-\theta x^p} \) in view of the initial conditions on \( x \).

**Theorem 2.2 (Haque and Faizan, 2010)**

Let \( X \) be a continuous random variable with the df \( F(x) \) and the pdf \( f(x) \) over the support \((0, \infty)\). Let \( 0 < p < \infty \) and \( F(x) \) has moment of order \( 2p \) then for \( 0 < r < n \),

\[
V[X^P(r+1,n,m,k) \mid X(r,n,m,k) = x] = \frac{1}{\gamma_{r+1}^2 \theta^2}
\]

if and only if

\[
\tilde{F}(x) = e^{-\theta x^p} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad \theta > 0.
\]
Proof: It is easy to see that

\[ E[X^{pk}(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{\gamma_{r+1}}{[\bar{F}(x)]^{\gamma_{r+1}}} \int_0^x y^{pk} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy. \]

Thus, for the Weibull distribution

\[ \bar{F}(x) = e^{-\theta x^p}, x \geq 0, \theta > 0 \]

\[ E[X^{pk}(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{\gamma_{r+1} \theta p}{e^{-\gamma_{r+1} \theta x^p}} \int_0^x y^{pk} y^{p-1} e^{-\gamma_{r+1} \theta y^p} dy \]

\[ = \sum_{m=0}^{k} \frac{k!}{m!} x^{pm} a^{p(k-m)}, \]

where \( a^{-p} = \gamma_{r+1} \theta \).

Therefore,

\[ V[X^{p}(r+1,n,m,k) | X(r,n,m,k) = x] = a^{2p} = \frac{1}{\gamma_{r+1}^2 \theta^2} = c. \]

This proves the necessary part.

For sufficiency part, we have

\[ \frac{\gamma_{r+1}}{[\bar{F}(x)]^{\gamma_{r+1}}} \int_0^x y^{2p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy - \frac{\gamma_{r+1}^2}{[\bar{F}(x)]^{2\gamma_{r+1}}} \]

\[ \times \left[ \int_0^x y^{p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy \right]^2 = c \]

That is,

\[ \gamma_{r+1} [\bar{F}(x)]^{\gamma_{r+1}} \int_0^x y^{2p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy \]

\[ - \left[ \gamma_{r+1} \int_0^x y^{p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy \right]^2 = c [\bar{F}(x)]^{2\gamma_{r+1}} \]  \hspace{1cm} (2.16)

Differentiating (2.16) w.r.t. \( x \) and solving, we get
\[ \gamma_{r+1} P \int_{x}^{\infty} y^p \left( \bar{F}(x) \right)^{\gamma_{r+1}-1} f(y) \, dy \]
\[ = c \gamma_{r+1} x^{1-p} \left( \bar{F}(x) \right)^{\gamma_{r+1}-1} f(x) + px^p \left( \bar{F}(x) \right)^{\gamma_{r+1}} \quad (2.17) \]

Differentiate (2.17) again w.r.t. \( x \), to get
\[ c \gamma_{r+1} x^{1-p} \left( \bar{F}(x) \right)^{\gamma_{r+1}-1} \left[ -\bar{F}^*(x) \left( \frac{\gamma_{r+1}-1}{\bar{F}(x)} \right)^2 + \frac{(p-1)\bar{F}'(x)}{x} \right] \]
\[ + \frac{1}{c \gamma_{r+1}} x^{2(p-1)} \left( \bar{F}(x) \right) p^2 = 0. \]

That is,
\[ \frac{\bar{F}^*(x)}{\bar{F}(x)} + (\gamma_{r+1}-1) \left( \frac{\bar{F}'(x)}{\bar{F}(x)} \right)^2 - \frac{(p-1)}{x} \left[ \frac{\bar{F}'(x)}{\bar{F}(x)} \right] - \gamma_{r+1} \theta^2 p^2 x^{2(p-1)} = 0. \]

Hence \( \bar{F}(x) = e^{-\theta x^p} \) in view of the Lemma 2.3.

At \( p = 1 \), this theorem gives the result for exponential distribution.

**Remark 2.1:** At \( m = 0, k = 1, \gamma_r = n - r + 1 \), Theorem 2.2 reduces for order statistics as obtained by Beg and Kirmani (1978) at \( p = 1 \) and Khan and Beg (1987).

**Remark 2.2:** At \( m = -1 \) and \( \gamma_r = k \), Theorem 2.2 reduces for \( k \)-th record statistics.

**Erlang-truncated exponential**

**Theorem 2.3 (Khan et al. 2010b)**

For the distribution as given in (6.2.23), suppose \( F(x) < 1 \), for all \( x \in (0, \infty) \) with \( F(0) = 0, F(\infty) = 1 \), then
\[ \bar{F}(x) = e^{-\beta x (1-e^{-\lambda x})}, \quad \beta, \lambda > 0 \]
if and only if for $1 < r < s < n$

$$E[X(s, n, m, k) \mid X(r, n, m, k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \sum_{t=0}^{s-r-1} (-1)^t \binom{s-r-1}{t} \frac{1}{\gamma_{s-t}} \left(x + \frac{1}{\beta(1-e^{-\lambda})\gamma_{s-t}}\right)$$

**Proof:** We have

$$E[X(s, n, m, k) \mid X(r, n, m, k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \times \sum_{t=0}^{s-r-1} (-1)^t \binom{s-r-1}{t} \frac{1}{\gamma_{s-t}} \times \left[ \int_{-\infty}^{\infty} y \left[\frac{1}{\beta\gamma_{s-t}}\right] f(y) \, dy \right] (2.18)$$

Integrating by parts and noting the relation (6.2.25), it is easy to prove the necessary part.

To prove sufficient part, we have from (2.18)

$$\sum_{t=0}^{s-r-1} (-1)^t \binom{s-r-1}{t} \gamma_{s-t} \int_{-\infty}^{\infty} y \left[\frac{1}{\beta\gamma_{s-t}}\right] f(y) \, dy$$

$$= \sum_{t=0}^{s-r-1} (-1)^t \binom{s-r-1}{t} \left[x + \frac{1}{\beta(1-e^{-\lambda})\gamma_{s-t}}\right] [\beta(1-e^{-\lambda})\gamma_{s-t}]$$

Differentiating both sides with respect to $x$ and rearranging, we get

$$\frac{f(x)}{F(x)} = \beta(1-e^{-\lambda})$$

which gives

$$F(x) = e^{-\beta x(1-e^{-\lambda})}.$$
3 CHARACTERIZATION OF SOME GENERAL FORM OF DISTRIBUTIONS

(a) \[ F(x) = 1 - [a h(x) + b]^c, \quad x \in (\alpha, \beta), \]

where \(a \neq 0, b, c \neq 0\) are finite constants and \(h(x)\) is continuous, monotonic and differentiable function of \(x\).

**Theorem 3.1 (Khan and Alzaid, 2004)**

Let \(X\) be absolutely continuous random variable with \(df F(x)\) and \(pdf f(x)\) on the support \((\alpha, \beta)\) where \(\alpha\) and \(\beta\) may be finite or infinite. Then for \(r < s\)

\[ E[X(s,n,m,k) \mid X(r,n,m,k)] = a^* x + b^* \]

if and only if

\[ 1 - F(x) = [a x + b]^c, x \in (\alpha, \beta) \]

where

\[ a^* = \prod_{j=1}^{s-r} \frac{c[k + (m+1)(n-r-j)]}{c[k + (m+1)(n-r-j)+1]} = \prod_{j=1}^{s-r} \frac{c \gamma_j}{(1 + c \gamma_j)} \]

\[ b^* = -\frac{b}{a} (1 - a^*) \]

**Proof**: See reference.

**Theorem 3.2 (Khan et al., 2006):**

Let \(\xi(x)\) be a monotonic and continuous function of \(x\). If

\[ E[\xi \{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = g_{s \mid r} (x) \quad (3.1) \]

then

\[ \overline{F}(x) = \exp \left[ -\frac{1}{\gamma_{r+1}} \int_{x}^{\infty} \frac{g'_{s \mid r} (t)}{g_{s \mid r} (t) - g_{s \mid r} (t + 1)} \, dt \right] \quad (3.2) \]
Proof: We have,

\[ E[\xi \{ X(s, n, m, k) \} | X(r, n, m, k) = x] = g_{s|r}(x) \]

that is,

\[
\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\beta \xi(y)[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1} \times [F(y)]^{r-1} f(y) dy = g_{s|r}(x)[F(x)]^{r+1}
\]

Differentiating both sides with respect to \( x \) and re-arranging the terms, we get

\[
\frac{f(x)}{F(x)} = \frac{1}{g_{s|r+1}(x) - g_{s|r}(x)} \frac{g_{s|r+1}'(x)}{r+1}
\]

and hence the Theorem.

The result for \( s = r + 1 \), obtained by Raqab and Abu-Lawi (2004) and Cramer et al. (2004), may be deduced by noting that

\[
g_{r+1|r+1}(x) = E[\xi \{ X(r+1, n, m, k) \} | X(r+1, n, m, k) = x] = \xi(x) .
\]

Remark 3.1: If

\[
E[\{X(s, n, m, k)\} | X(r, n, m, k) = x] = a_{s|r}^* x + b_{s|r}^* = g_{s|r}(x)
\]

then

\[
F(x) = (ax + b)^c,
\]

where

\[
a_{s|r}^* = \prod_{j=r+1}^s \frac{c_j}{1 + c_j}, \quad b_{s|r}^* = -\frac{b}{a}(1 - a_{s|r}^*)
\]

Proof: The proof was given by Khan and Alzaid (2004) using Rao and Shanbhag (1994) result. This can also be obtained from Theorem 3.2 under the assumption \([a x + b]^c = 1\) by noting that

\[
g_{s|r+1}(x) - g_{s|r}(x) = a_{s|r+1}^* x + b_{s|r+1}^* - a_{s|r}^* x - b_{s|r}^*
\]
\[
= (a^\bullet_{s|r+1} - a^\bullet_{s|r}) \left( x + \frac{b}{a} \right)
\]
\[
= \frac{a^\bullet_{s|r}}{ac\gamma_{r+1}} (ax + b) \quad \text{as} \quad a^\bullet_{s|r+1} = \prod_{j=r+2}^{s} \frac{c\gamma_j}{1+c\gamma_j} = \frac{1+c\gamma_{r+1}}{c\gamma_{r+1}} a^\bullet_{s|r}
\]

Therefore,
\[
\frac{g'_{s|r}(x)}{\gamma_{r+1} [g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{ac}{ax + b}
\]

Thus,
\[
\frac{f(x)}{\overline{F}(x)} = -\frac{ac}{ax + b}
\]

implying
\[
\overline{F}(x) = [ax + b]^c
\]

For \( s = r + 1 \), the result was obtained by Keseling (1999).

Further, it has been shown by Bieniek and Szynal (2003), Khan and Alzaid (2004) and Cramer et al. (2004), that for

i) Power function distribution

\[
\overline{F}(x) = \left( \frac{\nu - x}{\nu - \mu} \right)^\theta, \quad \mu < x < \nu
\]

\[
a^\bullet_{s|r} = \prod_{i=r+1}^{s} \frac{\theta \gamma_i}{1+\theta \gamma_i} < 1, \quad b^\bullet_{s|r} = \nu (1 - a^\bullet_{s|r})
\]

ii) Pareto distribution

\[
\overline{F}(x) = \left( \frac{\mu + \delta}{x + \delta} \right)^\theta, \quad \mu < x < \infty
\]

\[
a^\bullet_{s|r} = \prod_{i=r+1}^{s} \frac{\theta \gamma_i}{\theta \gamma_i - 1} > 1, \quad b^\bullet_{s|r} = \delta (a^\bullet_{s|r} - 1)
\]
iii) Exponential distribution

\[ \overline{F}(x) = e^{-\lambda(x - \mu)}, \quad x \geq \mu \quad (3.5) \]

\[ a^*_s|_r = 1, \quad b^*_s|_r = \frac{1}{\lambda} \sum_{i=r+1}^{s} \frac{1}{\gamma_i} \]

Remark 3.2: Let \( \xi(x) \) be a monotonic and continuous function of \( x \), then it has been shown by Khan and Alzaid (2004) that

\[ E[\xi(X(s,n,m,k))|X(r,n,m,k) = x] = a^*_s|_r \xi(x) + b^*_s|_r \]

if and only if \( \overline{F}(x) = [a \xi(x) + b]^c \)

This can be deduced from Theorem 2.1 by considering

\[ g^*_s|_r(x) = a^*_s|_r \xi(x) + b^*_s|_r \]
A number of distributions can be characterized by the proper choice of $a, b, c$ and $\xi(x)$; (Khan and Alzaid, 2004):

**Table 3.1: Specific distributions**

<table>
<thead>
<tr>
<th>Distributions</th>
<th>$\bar{F}(x)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$\xi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>$1 - a^{-p} x^p$</td>
<td>$-a^{-p}$</td>
<td>1</td>
<td>1</td>
<td>$x^p$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0 \leq x \leq a$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Pareto</td>
<td>$a^p x^{-p}$</td>
<td>$a^p$</td>
<td>0</td>
<td>1</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>0</td>
<td>$p$</td>
<td>$x^{-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a^{-p}$</td>
<td>0</td>
<td>$-1$</td>
<td>$x^p$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a^{-1}$</td>
<td>0</td>
<td>$-p$</td>
<td>$x$</td>
</tr>
<tr>
<td>3. Beta of the first kind</td>
<td>$(1 - x)^p$</td>
<td>$-1$</td>
<td>1</td>
<td>$p$</td>
<td>$x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0 \leq x \leq 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Weibull</td>
<td>$\exp[-\theta(x-\mu)^p]$</td>
<td>$-\frac{\theta}{c}$</td>
<td>1</td>
<td>$\infty$</td>
<td>$(x-\mu)^p$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x \geq \mu$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Inverse Weibull</td>
<td>$1 - \exp(-\theta x^{-p})$</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>$e^{-\theta x^{-p}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0 \leq x &lt; \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Burr type XII</td>
<td>$(1 + \theta x^p)^{-m}$</td>
<td>$\theta$</td>
<td>1</td>
<td>$-m$</td>
<td>$x^p$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0 \leq x &lt; \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Cauchy</td>
<td>$\frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$</td>
<td>$-\frac{1}{\pi}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\tan^{-1} x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-\infty &lt; x &lt; \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


b) \(F(x) = e^{-a h(x)}\), \(x \in (\alpha, \beta)\)

Lemma 3.1 (Khan et al. 2010)
Let the df \(F(x)\) be twice differentiable on \((\alpha, \beta)\) and let \(h(x)\) be a non-decreasing and a twice differentiable function of \(x\) such that \(h(x) \to 0\) as \(x \to \alpha\). Then the solution of the differential equation

\[ \frac{\bar{F}(x)}{F(x)} + (\gamma_{r+1} - 1) \left( \frac{\bar{F}(x)}{F(x)} \right)^2 - \frac{h''(x)}{h'(x) \left( \frac{\bar{F}(x)}{F(x)} \right)} - a^2 \gamma_{r+1} \left[ h'(x) \right]^2 = 0 \quad (3.6) \]

is

\[ \bar{F}(x) = e^{-a h(x)} \] for all \(x \in (\alpha, \beta)\),

where \(a > 0\) is a constant.

Proof: Let \(\frac{F'(x)}{F(x)} = \gamma_{r+1} \frac{h'(x)}{t} \).

Then using (3.6), we have

\[ \frac{dt}{dx} = h'(x) \left[ \gamma_{r+1}^2 - a^2 t^2 \right] \]

Therefore,

\[ \frac{1}{2 \gamma_{r+1}} \int \left[ \frac{1}{(\gamma_{r+1} - at)} + \frac{1}{(\gamma_{r+1} + at)} \right] dt = \int h'(x) \, dx \]

implying that

\[ \frac{\gamma_{r+1} + at}{\gamma_{r+1} - at} = A e^{2a \gamma_{r+1} h(x)}, \]

where \(A\) is the constant of integration.
Hence,
\[
\frac{\bar{F}'(x)}{\bar{F}(x)} = \frac{1}{2\gamma_{r+1}} \left[ \frac{2A}{Au-1} - \frac{1}{u} \right] du,
\]
where \( u = e^{2a\gamma_{r+1}h(x)} \), implying that
\[
\bar{F}(x) = B[Ae^{a\gamma_{r+1}h(x)} - e^{-a\gamma_{r+1}h(x)}]^{1/\gamma_{r+1}},
\]
where \( A \) and \( B \) are constants of integration. Since \( F \) is bounded, hence \( \bar{F}(x) = e^{-a\gamma_{r+1}h(x)} \), in view of the initial conditions on \( h(x) \).

**Theorem 3.3 (Khan et al., 2010)**

Let \( X \) be a continuous random variable with the pdf \( f(x) \) and the df \( F(x) \) over the support \((\alpha, \beta)\). Let \( E[h(X)]^2 \) exist, then for some \( 0 < r < n \),
\[
V[h(X(r+1,n,m,k)|X(r,n,m,k) = x] = \frac{1}{a^2 \gamma_{r+1}^2}
\]
if and only if
\[
\bar{F}(x) = e^{-a\gamma_{r+1}h(x)}
\]
where \( a > 0 \), \( h(x) \) is a non-decreasing and a twice differentiable of \( x \) such that \( h(x) \to 0 \) as \( x \to \alpha \) and \( h(x) \bar{F}(x) \to 0 \) and \( x \to \infty \).

**Proof:** It is easy to see that
\[
E[h(X(r+1,n,m,k))|X(r,n,m,k) = x] = \frac{\gamma_{r+1}}{[\bar{F}(x)]^{\gamma_{r+1}}} \int_\alpha^\beta h(y)[\bar{F}(y)]^{\gamma_{r+1}-1} f(y) dy
\]
For \( \bar{F}(x) = e^{-a\gamma_{r+1}h(x)} \)
\[
E[h(X(r+1,n,m,k))|X(r,n,m,k) = x] = h(x) + \frac{1}{a \gamma_{r+1}}
\]
and

\[ E[h^2 \{X(r+1,n,m,k)\} | X(r,n,m,k) = x] \]

\[ = h^2(x) + \frac{2 h(x)}{a \gamma_{r+1}} + \frac{2}{a^2 \gamma_{r+1}^2}. \]

Thus,

\[ \nu [X(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{1}{a^2 \gamma_{r+1}^2}. \]

This proves the necessary part.

For sufficiency part, we have

\[ \frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \int_x^\theta h^2(y)[\bar{F}(y)]^{\gamma_{r+1}-1} f(y) \, dy - \frac{\gamma_{r+1}^2}{[F(x)]^{2\gamma_{r+1}}} \times \left[ \int_x^\theta h(y)[\bar{F}(y)]^{\gamma_{r+1}-1} f(y) \, dy \right]^2 \]

\[ = \frac{1}{a^2 \gamma_{r+1}^2}. \]

That is,

\[ \gamma_{r+1} \frac{h'(x)}{h(x)} \int_x^\theta h(y)[\bar{F}(x)]^{\gamma_{r+1}-1} f(y) \, dy \]

\[ = \frac{1}{a^2 \gamma_{r+1}} [\bar{F}(x)]^{\gamma_{r+1}-1} f(x) + [\bar{F}(x)]^{\gamma_{r+1}} h(x) h'(x) \]

That is,

\[ h'(x) \int_x^\theta h'(y)[\bar{F}(x)]^{\gamma_{r+1}-1} \, dy = \frac{1}{a^2 \gamma_{r+1}} [\bar{F}(x)]^{\gamma_{r+1}-1} f(x) \] (3.8)
Now differentiate (3.8) again w.r.t. x, to get

\[ h^*(x) \int_0^x h'(y)[\overline{F}(x)]^{\gamma_{r+1}} \, dy - [\overline{F}(x)]^{\gamma_{r+1}} [h'(x)]^2 \]

\[ = \frac{1}{a^2 \gamma_{r+1}} \left[ -(\gamma_{r+1} - 1) [\overline{F}(x)]^{\gamma_{r+1} - 2} [f(x)]^2 \right. \]

\[ \left. + [\overline{F}(x)]^{\gamma_{r+1} - 1} f''(x) \right] \]

Therefore,

\[ \frac{\overline{F}''(x)}{\overline{F}(x)} + (\gamma_{r+1} - 1) \left[ \frac{\overline{F}'(x)}{\overline{F}(x)} \right]^2 - \frac{h^*(x)}{h'(x)} \left[ \frac{\overline{F}'(x)}{\overline{F}(x)} \right] - a^2 \gamma_{r+1} [h'(x)]^2 = 0 \]

Hence,

\[ \overline{F}(x) = e^{-ah(x)} \] in view of the Lemma 3.1.


References


Galton, F. (1902): The most suitable proportion between the values of first and second prized. *Biometrika*, 1, 385-390.


References


