ORDER STATISTICS AND CHARACTERIZATIONS OF DISTRIBUTIONS

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FOR THE AWARD OF THE DEGREE OF
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IN
STATISTICS

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TO
MY GRANDPA
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Order statistics has immense role in characterization problems. Characterization results are those which shed light on modeling consequences of certain distributional assumption and those which have potential for development of hypothesis test for model assumption.

In this dissertation an attempt has been made to present the available up to date results on characterization of some specific distributions through functional and moments property of order statistics. The whole dissertation is divided into five chapters.

Chapter - 1 deals with concept of order statistics and some basic results needed in the subsequent chapter.

In Chapter - 2, functions of order statistics have been studied and characterization of some specific distribution have been shown through distributional and independence properties. Characterization through co variance bounds has also been done.

In Chapter - 3, single moments of order statistics have been studied and characterization of uniform, logistic, Pareto and
general class of distribution is shown.

In Chapter - 4, product moments of order statistics are used to characterize the uniform, extended normal, Pareto and general class of distribution.

Chapter - 5 deals with characterization through conditional moments of order statistics. In which characterization of some specific distributions have been shown for single order gap, double order gap and higher order gap. Characterization of general class distribution is also discussed.

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( HASEEB ATHAR )
1.1 ORDER STATISTICS:

If the random variable $X_1$, $X_2$, ..., $X_n$ are arranged in ascending order of magnitude

$$X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{r:n} \leq \ldots \leq X_{n:n}$$

then $X_{r:n}$ (or $X_{r,n}$) is called the rth order statistic. $X_{(r)}$ also be used for rth order statistic in a sample of size $n$. The two terms $X_{1:n} = \min(X_1, X_2, ..., X_n)$ and $X_{n:n} = \max(X_1, X_2, ..., X_n)$ are called extremes.

The subject of order statistics deals with the properties and applications of these ordered random variables and of functions involving them (David, 1981). It is different from the rank order statistic in which the order of the value of observation rather than its magnitude is considered.

It plays an important role both in model building and in statistical inference.

For example: Extreme (largest, smallest) values are important in oceanography (waves and tides), material strength (strength of a chain depends on the weakest link) and meteorology (extremes of
temperature, pressure etc.).

Order statistics have have immense application in life testing and reliability problems. If n similar items are simultaneously placed on life test, the life of the first item to fail is first order statistics, life of the second item to fail is second order statistics and so on. Often the experimenter may wish to terminate the experiment when only m (<n) failures have occurred to save the resources and time. In this case we have only the first m ordered statistics on the basis of which we have to make inferences.

In statistical inference \(X_{(n)} - X_{(1)}\) (range) is widely used to estimate the standard deviation (David, 1981).

It is also used in outliers' detection (Barnett, 1988).

For further applications, one may refer to Malik et al. (1988), Harter (1988), Gumbel (1958), Galambos (1978) and books of Balakrishnan.

1.2 DISTRIBUTION OF ORDER STATISTICS:

Let \(X_1, X_2, \ldots, X_n\) be a random sample of size n from a continuous probability density function (pdf) \(f(x)\) and distribution function (df) \(F(x)\). Then the pdf of \(X_{r:n}\) (1 ≤ r ≤ n), the \(r\)th order statistic, is given by
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Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a continuous probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \). Then the pdf of \( X_{r:n} \) (1 \( \leq r \leq n \)), the \( r \)th order statistic, is given by
\[ f_{r:n}(x) = C_{r:n} \left[ F(x) \right]^{r-1} \left[ 1-F(x) \right]^{n-r} f(x) \quad -\infty < x < \infty \quad \ldots \ldots (1.2.1) \]

where

\[ C_{r:n} = \frac{n!}{(r-1)!(n-r)!} = \left[ B(r, n-r+1) \right]^{-1} = \frac{\Gamma(n+1)}{\Gamma (r) \Gamma (n-r+1)} \quad \ldots \ldots (1.2.2) \]

and df is

\[ F_{r:n}(x) = P[X_{r:n} \leq x] \]

\[ = \sum_{i=r}^{n} \binom{n}{i} \left[ F(x) \right]^i \left[ 1-F(x) \right]^{n-i} \quad \ldots \ldots (1.2.3) \]

\[ F_{r:n}(x) = C_{r:n} \int_{0}^{\frac{F(x)}{1-F(x)}} t^{r-1} (1-t)^{n-r} dt \quad \ldots \ldots (1.2.4) \]

where (1.2.4) is complete beta function for x continuous, (1.2.1) can be obtained from (1.2.4) by differentiating w.r.t. x.

In particular,

\[ F_{1:n}(x) = 1 - (1-F(x))^n \quad \ldots \ldots (1.2.5) \]

and

\[ F_{n:n}(x) = (F(x))^n \quad \ldots \ldots (1.2.6) \]

The joint p.d.f. of \( X_{r:n} \) and \( X_{s:n} \) (1 ≤ r < s ≤ n) is given by

\[ f_{r,s:n}(x,y) = C_{r,s:n} \left[ F(x) \right]^{r-1} \left[ F(y)-F(x) \right]^{s-r-1} \left[ 1-F(y) \right]^{n-s} f(x)f(y) \quad , \quad -\infty < x < y < \infty \quad \ldots \ldots (1.2.7) \]

where

\[ C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} = \frac{1}{B(r,s-r,n-s+1)} \quad \ldots \ldots (1.2.8) \]

\[ F_{r,s:n}(x,y) = P[X_{r:n} \leq x, X_{s:n} \leq y] \]
\[
\sum_{j=s}^{n} \sum_{i=r}^{n} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^i [F(y)-F(x)]^{j-i} [1-F(y)]^{n-j} 
\]

\[\ldots \ldots (1.2.9)\]

1.3 SINGLE AND PRODUCT MOMENTS OF ORDER STATISTICS:

Let \(\mu_{r:n}^{(k)}\) be the kth moment of the rth order statistics and \(\mu_{r,s:n}^{(j,k)}\) be the product moment of the jth power of the rth order statistics and kth power of the sth order statistics.

Then
\[
\mu_{r:n}^{(k)} = E(X_r^{k}) = \int_{-\infty}^{\infty} x^k f_{r:n}(x) \, dx, \quad 1 \leq r \leq n
\]

\[
= C_{r:n} \int_{-\infty}^{\infty} x^k I(F_r; r-1, n-1) \, dF(x) \ldots \ldots (1.3.1)
\]

where
\[I(u; j, k) = u^j (1-u)^k, \quad 0 < u < 1 \ldots \ldots (1.3.2)\]

and
\[
\mu_{r,s:n}^{(j,k)} = E(X_r^{j} X_s^{k}) = \int_{x<y} x^j y^k f_{r,s:n}(x,y) \, dy \, dx
\]

\[
= C_{r,s:n} \int_{x<y} x^j y^k I(F(x), F(y); r-1, s-r-1, n-s) \, dF(x) \, dF(y) \ldots \ldots (1.3.3)
\]

where
\[I(u, v; r, k, n) = u^r (v-u)^k (1-v)^n \text{ for } r, s, k, n \geq 0 \text{ and } 0 < u < v < 1 \]

\[\ldots \ldots (1.3.4)\]

We will use the following conventions in the subsequent chapters.

\[
\sigma_{r,s:n} = Cov(X_r:n, X_s:n) = \mu_{r,s:n} - \mu_{r:n} \mu_{s:n} \text{ for } 1 \leq r \leq s \leq n \ldots \ldots (1.3.5)
\]

with \(\sigma_{r,s:n} = \sigma_{s,r:n} \ldots \ldots (1.3.6)\)
and $\sigma_{r,n} = \text{Var } X_{r;n}$
$$
(1.3.7)
$$
and $\mu_{r;n}^{(0)} = 1$
$$
(1.3.8)
$$
$\mu_{r;n}^{(1)} = \mu_{r;n}$
$$
(1.3.9)
$$
In case of truncated distributions, we have
$$
\mu_{0;n}^{(k)} = q_1^k \quad \text{if } n > 0 \text{ and } k = 1,2, \ldots (1.3.10)
$$
$$
\mu_{n-1;n}^{(k)} = p_1^k, \quad n > 1 \ldots (1.3.11)
$$
$$
\mu_{r,n}^{(j,k)} = E(X_{r;n}^j, x_{r;n}^k) = E(X_{r;n}^{j+k}) = \mu_{r;n}^{(j+k)} \ldots (1.3.12)
$$
$$
\mu_{r,s;n}^{(j,0)} = E(X_{r;n}^j, x_{s;n}^0) = E(X_{r;n}^j) = \mu_{r;n}^{(j)} \ldots (1.3.13)
$$
$$
\mu_{r,s;n}^{(0,k)} = E(x_{r;n}^0, x_{s;n}^k) = E(x_{s;n}^k) = \mu_{s;n}^{(k)} \ldots (1.3.14)
$$
$$
\mu_{n-1;n-1}^{(j,k)} = E(x_{n-1;n-1}^j, x_{n-1;n-1}^k) = \mu_{n-1;n-1} \ldots (1.3.15)
$$

Some relations of frequent use are listed below;

(David, 1981 and Khan et al. 1983a,b)

1. $(n-r) \mu_{r;n}^{(k)} + r \mu_{r+1;n}^{(k)} = n \mu_{r;n-1}^{(k)} \ldots (1.3.16)$

2. $\mu_{r;n}^{(k)} - \mu_{r-1;n-1}^{(k)} = \left[1 - \frac{1}{r-1}\right] k \int_{Q_1}^1 x^{k-1} [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx \ldots (1.3.17)$

3. $\mu_{r;n}^{(k)} - \mu_{r;n-1}^{(k)} = \left[1 - \frac{1}{r-1}\right] k \int_{Q_1}^1 x^{k-1} [F(x)]^{r} [1-F(x)]^{n-r} dx \ldots (1.3.18)$
\( \mu_{r:1:n}^{(k)} - \mu_{r-1:1:n}^{(k)} = \left( \begin{array}{c} n \\ r-1 \end{array} \right) k \int_{-\infty}^{1} x^{k-1} [F(x)]^{r-1} [1-F(x)]^{n-r+1} dx \) 

\( \mu_{r,s:1:n}^{(j,k)} - \mu_{r,s-1:1:n}^{(j,k)} = \frac{C_{r,s:1:n}}{(n-s+1)} k \int_{-\infty}^{1} \int_{-\infty}^{1} x^{j} y^{k-1} [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s+1} f(x)f(y) dy dx \)

\(-\infty \leq x < y < \infty, 1 \leq r < s \leq n \) 

Again let \( F(x), x \in (\alpha, \beta) \) be a absolutely continuous distribution function and \( g(.) \) be a Borel measurable function from \( R \) to \( R \) and \( E\{g(X_{r:1:n})\} \), \( 1 \leq r \leq n \), the expectation of a function of single order statistics with the conventions:

\[ E\{g(c)\} = g(c), \ \text{where} \ c \ \text{is a constant} \]

\[ E\{g(X_{0:1:r})\} = g(\alpha) \]

and

\[ E\{g(X_{r:1:r-1})\} = g(\beta), \ 1 \leq r \leq n. \]

Define the inverse function of \( F \) by

\[ F^{-1}(t) = \inf \left\{ x : F(x) \geq t \right\}; \ t \in (0,1) \]

then it can be verified that (Ali and Khan, 1994c)

\[ E\{g(X_{r:1:n})\} = C_{r:1:n} \int_{0}^{1} g(F^{-1}(u)) u^{r-1} (1-u)^{n-r} du \] 

\[ E\{g(X_{r:1:n})\} - E\{g(X_{r-1:1:n-1})\} = \frac{C_{r:1:n}}{n} \int_{0}^{1} g'(F^{-1}(u)) u^{r-1} (1-u)^{n-r+1} d(F^{-1}(u)) \]
\[ E\{g(X_{r:n})\} - E\{g(X_{r-1:n})\} = \frac{C_{r:n}}{n-r+1} \int_0^1 g'(F^{-1}(u))u^{r-1}(1-u)^{n-r+1}d(F^{-1}(u)) \]

\[ E\{g(X_{r-1:n-1})\} - E\{g(X_{r-1:n})\} = \frac{(r-1)}{n(n-r+1)} C_{r:n} \int_0^1 g'(F^{-1}(u))u^{r-1}(1-u)^{n-r+1}d(F^{-1}(u)) \]

Equations (1.3.22), (1.3.23) and (1.3.24) lead the identity

\[ (r-1)E\{g(X_{r:n})\} + (n-r+1)E\{g(X_{r-1:n})\} = nE\{g(X_{r-1:n-1})\} \]

which was given by David (1981) with \( g(x) = x^k \).

For product moments, consider \( g(.,.) \) is a measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

If \( E\{g(X_{r:n}, X_{s:n})\}, 1 \leq r < s \leq n \) denotes the expectation of a function of two order statistics, then we have

\[ E\{g(X_{r:n}, X_{s:n})\} = C_{r,s:n} \int \int g(F^{-1}(u), F^{-1}(v))u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}dudv \]

and

\[ E\{g(X_{r:n}, X_{s:n})\} - E\{g(X_{r:n}, X_{s-1:n})\} = \frac{C_{r,s:n}}{(n-s+1)} \int \int g\left(F^{-1}(u), F^{-1}(v)\right)u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s+1}dud(F^{-1}(v)) \]
Corollary 1.3.1: (Khan et al., 1988)

For symmetrically truncated logistic distribution with 
\[ 1 \leq r < s \leq n-2, \ s-r > 1 \text{ and } n \geq 4, \]
\[
\mu_{r,s;n} = \mu_{r,s-1;n} + \frac{n}{(n-s+1)} \left[ (\mu_{r,s;n-1} - \mu_{r,s-1;n-1}) ight.
\]
\[
+ \frac{1}{(n-s)(P-Q)} \left\{ \frac{(n-1)PQ}{(P-Q)} (\mu_{r,s;n-2} - \mu_{r,s-1;n-2} - k \mu_{r,s;n-1}) \right\} \]
\[ \quad \ldots \tag{1.3.28} \]

Corollary 1.3.2: (Khan et al., 1988)

For symmetrically truncated logistic distribution with 
\[ 2 \leq r \leq n-1 \]
\[
\mu_{r;n} = \mu_{r-1;n-1} - \frac{(r-1) PQ}{(r-1)(P-Q)} (\mu_{r-1;n-2} - \mu_{r-2;n-2}) 
\]
\[
+ \frac{k}{(r-1)(P-Q)} \mu_{r-1;n-1} \quad \ldots \tag{1.3.29} \]

Corollary 1.3.3: (Khan and Khan, 1987)

For Burr distribution with 
\[ 2 \leq r \leq n-1 \text{ and } k \neq \frac{mnp}{k} \]
\[
\left[ 1 - \frac{k}{mnp} \right] \mu_{r;n} = 2 \mu_{r-1;n-1} - \frac{P_2 \mu_{r-1,n-1}}{2} + \frac{k}{mnp} \mu_{r;n}^{(k-p)} \quad \ldots \tag{1.3.30} \]

The important deductions for \( k \neq \frac{mnp}{k} \), in view of (1.3.15) are:
\[
\left[ 1 - \frac{k}{mnp} \right] \mu_{1;1}^{(k)} = Q_2 \mu_{1;1}^{k} - P_2 \mu_{1;1}^{k} + \frac{k}{mnp} \mu_{1;1}^{(k-p)} \quad \ldots \tag{1.3.31} \]
\[
\left[ 1 - \frac{k}{mnp} \right] \mu_{1;n}^{(k)} = Q_2 \mu_{1;n-1}^{k} - P_2 \mu_{1;n-1}^{k} + \frac{k}{mnp} \mu_{1;n}^{(k-p)} \quad \text{, } n > 1 \ldots \tag{1.3.32} \]
\[
\left[ 1 - \frac{k}{mnp} \right] \mu_{n;1}^{(k)} = Q_2 \mu_{n-1;n-1}^{k} - P_2 \mu_{n-1;n-1}^{k} + \frac{k}{mnp} \mu_{n;1}^{(k-p)} \quad \text{, } n > 1 \ldots \tag{1.3.33} \]
1.4 CENSORING AND TRUNCATION:

Let \( X_1:n \leq X_2:n \leq \ldots \leq X_n:n \) be the order statistics from a population having d.f. \( F(x) \). It may happen that we record first \( r \) or the last \( r \) observations corresponding to these order statistics. This is called Censoring. For example, suppose \( m \) bulbs are put on test and their failure times are recorded. Because of time and cost factors, we may not like to record the failure time of all bulbs and instead observe first \( r \) failure time of bulbs. Also it may be possible that we started recording their failure time after \( r \) failures. This is different from truncation, where the population rather than the sample is curtailed and the number of lost observations is unknown.

If we represent the truncation points by \( Q_1 \) and \( P_1 \) at left and right respectively, then in case of doubly truncation the pdf is given as

\[
\frac{f(x)}{P - Q}, \quad Q_1 \leq x \leq P_1 \quad \ldots (1.4.1)
\]

where

\[
\int_{-\infty}^{Q_1} f(x) \, dx = Q \quad \ldots (1.4.2)
\]

and

\[
\int_{-\infty}^{P_1} f(x) \, dx = P \quad \ldots (1.4.3)
\]

and \( Q \) and \( (1-P) \) are respectively the portion of truncation on the
left and right of the distribution. \( P \) and \( Q \) assumed to be known \((Q < P)\) and \( Q_1 \) and \( P_1 \) are functions of \( Q \) and \( P \).

Distribution of truncated order statistics:

(a) **Left truncated at \( x \)**

Let \( Q = F(x) \), \( P = 1 \), then the truncated distribution has pdf

\[
\frac{f(t)}{1 - F(x)} , \quad x < t < \infty
\]

and the pdf of \( X_{r:n} = Y \), in this case, will be

\[
C_{r:n} \left[ \frac{F(y) - F(x)}{1 - F(x)} \right]^{r-1} \left[ \frac{1 - F(y)}{1 - F(x)} \right]^{n-r} \frac{f(y)}{1 - F(x)} \quad \ldots \ldots \quad (1.4.4)
\]

(b) **Right truncation at \( y \)**

Similarly, at \( Q = 0, P = F(y) \), then the truncated distribution has pdf \( \frac{f(t)}{F(y)} \), \(-\infty < t < y\) and the pdf of \( X_{r:n} = X \) will be

\[
C_{r:n} \left[ \frac{F(x)}{F(y)} \right]^{r-1} \left[ \frac{F(y) - F(x)}{F(y)} \right]^{n-r} \frac{f(x)}{F(y)} \quad \ldots \ldots \quad (1.4.5)
\]

1.5 **CONDITIONAL DISTRIBUTION OF ORDER STATISTICS:**

Let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) be the order statistics from a continuous random variable (r.v.) having probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \). Then the conditional pdf, of \( X_{s:n} = y \) given \( X_{r:n} = x \) \((1 \leq r < s \leq n)\) is
which is just the unconditional pdf of (s-r)th order statistics from a sample of size (n-r) truncated to the left at x, i.e., the pdf of \( X_{s-r:n-r} \) given \( X_{s-r:n-r} \geq x \).

Therefore,

\[
E \left[ x_{s:n}^k \mid X_{r:n} = x \right] = E \left[ x_{s-r:n-r}^k \mid X_{s-r:n-r} \geq x \right] \tag{1.5.2}
\]

Similarly, the conditional pdf of \( X_{r:n} = x \) given \( X_{s:n} = y \) (1 \( \leq r < s \leq n \)) is

\[
\frac{(s-r)!}{(s-1)!} \frac{[F(x)]^{s-1} [F(y)-F(x)]^{s-r-1} f(x)}{[F(y)]^{s-1}}, \quad x \leq y \tag{1.5.3}
\]

which is the unconditional pdf of \( X_{r:s-1} \) truncated at \( y \) on the right.

Therefore,

\[
E \left[ x_{r:n}^k \mid X_{s:n} = y \right] = E \left[ x_{r:s-1}^k \mid X_{r:s-1} \leq y \right] \tag{1.5.4}
\]

**Corollary 1.5.1**: (Khan and Khan, 1987)

For Burr distribution in case of left truncation at \( x \),

\( Q_1 = x, \quad Q = F(x), \quad P_1 = \infty, \quad P = 1, \quad P_2 = 0, \quad Q_2 = 1 \)

and therefore, in view of (1.3.32) and (1.5.1),

\[
E \left[ x_{r+1:n}^p \mid X_{r:n} = x \right] = \mu_{1:n-r}^{(p)} = \frac{m(n-r)}{m(n-r)-1} \left[ x^p + \frac{1}{m(n-r)\theta} \right], \quad m(n-r) \neq 1 \tag{1.5.5}
\]

**Corollary 1.5.2**: (Khan and Khan, 1987)

For Burr distribution in case of right truncation at \( x \),

\[

\]
\( Q_1 = 0, Q = 0, P_1 = x, P = F(x), Q_2 = \frac{1}{F(x)}, P_2 = \frac{1 - F(x)}{F(x)} \)

and therefore, in view of (1.3.33) and (1.5.2)

\[
E\left[X_{1:n}^p | X_{2:n} = x\right] = \mu_{1:1}^{(p)} = \frac{m}{m - 1} \left[ \frac{1}{m\theta} - \frac{1 - F(x)}{F(x)} x^p \right], \ m \neq 1
\]

..........(1.5.6)

Let

\( G(y) \) denotes the conditional d.f. of \( X_{r+2:n} \) given \( X_{r:n} = x \)

\( H_1(y) \) denotes the conditional d.f. of \( X_{r:n} \) given \( X_{r+1:n} = x \)

\( H_2(y) \) denotes the conditional d.f. of \( X_{r:n} \) given \( X_{r+2:n} = x \)

then in view of (1.5.2), (1.5.4) and (1.2.3), it can be seen that

\[
G(y) = 1 - \left[\frac{1-F(y)}{1-F(x)}\right]^{n-r} - (n-r) \left[\frac{F(y)-F(x)}{1-F(x)}\right] \left[\frac{1-F(y)}{1-F(x)}\right]^{n-r-1}
\]

..........(1.5.7)

\[
H_1(y) = \left[\frac{F(y)}{F(x)}\right]^r
\]

..........(1.5.8)

\[
H_2(y) = (r+1) \left[\frac{F(y)}{F(x)}\right]^r - r \left[\frac{F(y)}{F(x)}\right]^{r+1}
\]

..........(1.5.9)

and accordingly,

\[
E\left[X_{r+2:n}^k | X_{r:n} = x\right] = x^k (1-G(x)) + k \int_x^\infty y^{k-1} (1-G(y)) dy
\]

..........(1.5.10)

\[
E\left[X_{r:n}^k | X_{r+2:n} = x\right] = x^k H_2(x) - k \int_0^x y^{k-1} H_2(y) dy
\]

..........(1.5.11)

Also, by Khan and Beg (1987)

\[
E\left[X_{r+1:n} | X_{r:n} \right] = E\left[X | X > y\right]
\]

..........(1.5.12)

and

\[
E\left[X_{r:n} | X_{r+1:n} \right] = E\left[X | X < y\right]
\]

..........(1.5.13)

The conditional p.d.f. of \( X_{s:n} = y \) given \( X_{r:n} = x \) and \( X_{m:n} = z \)

\((1 \leq r \leq s \leq m \leq n)\) is given as
This is the p.d.f. of the \((s-r)\)th order statistics from a sample of size \((m-r-1)\) truncated on the left at \(X_{r:n} = x\) and on the right at \(X_{m:n} = z\). Thus, we have for symmetric truncation of a symmetric logistic distribution (Khan et al., 1987).

\[
Q_1 = x = -z, \quad P_1 = z
\]
\[
Q = F(x) = F(-z), \quad P = F(z)
\]
\[
1 - F(-z) = F(z)
\]

and consequently,

\[
E \left[ X_{r+1:n} \mid X_{r:n} = -z, X_{r+3:n} = z \right] = 2 \int_{-z}^{z} y \frac{[F(z)-F(y)]f(y)}{[2F(z)-1]^2} \, dy = \mu_{1:2}
\]

1.6 BASIC DISTRIBUTION FUNCTIONS:

Definition 1.6.1: (Barlow and Proschan, 1975, p.159)

Let \(F(x)\) be the distribution function of a non-negative random variable and let \(F(x) = 1 - F(x)\), for \(x \geq 0\). We say that \(F(x)\) is "new better than used" (NBUD), if \(F(x+y) \geq F(x) F(y)\), for \(x, y \geq 0\); and \(F(x)\) is "new worse than used" (NWUD), if \(F(x+y) \geq F(x) F(y)\), for \(x, y \geq 0\).

Definition 1.6.2: A distribution function \(F\) will be said to belong to class \(C_1\) if either \(F(x_1 x_2) \geq F(x_1) F(x_2)\) or
\( F(x_1 x_2) \leq F(x_1) F(x_2) \) for all \( x_1 \) and \( x_2 \). We will also say that \( F \) belongs to class \( C_2 \) if the corresponding p.d.f. \( f \) satisfies either \( f(x_1) \geq f(x_2) \) or \( f(x_1) \leq f(x_2) \) for all \( x_1, x_2 \), with \( x_1 \geq x_2 \). We will say that \( F \) is "super additive" if \( F(x+y) \leq F(x) + F(y) \), \( x, y \geq 0 \) and \( F \) is "sub additive" if \( F(x+y) \geq F(x) + F(y) \), \( x, y \geq 0 \). We will say that \( F \) belongs to class \( C_3 \) if \( F \) is either super additive or sub additive.

**H - Notation 1** (Govindrajulu, 1975)

Let \( X_1, X_2, \ldots, X_n \), independent nontrivial random variables each having the distribution function \( F(x) \) with \( F(0)=0 \). Also let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) with \( X_{0:n} = 0 \) and \( X_{n+1:n} = \infty \), denote the ordered values of \( X_1, X_2, \ldots, X_n \). Assume that \( E X_i^2 < \infty \). Let \( U_1, U_2, \ldots, U_n \) be independent, each being uniformly distributed over the open interval \((0,1)\). Define

\[
H(u) = \{x \mid F(x) \geq u\}, \quad 0 < u < 1
\]

where \( F \) is taken to be right continuous. Then, for \( 0 < u < 1 \)

\[
H(u) \leq x \iff u \leq F(x)
\]

and the distribution of \( H(U_1), H(U_2), \ldots, H(U_n) \), is the same as that of \( X_1, X_2, \ldots, X_n \). Also, \( H(U_{n:n}) = \max \{H(U_k) \mid 1 \leq k \leq n\} \) has the same distribution as \( X_{n:n} \), etc. Through, we assume that \( F \) is absolutely continuous. That is, \( H'(u) \) exists almost everywhere for \( 0 < u < 1 \).
2.1 CHARACTERIZATION BY DISTRIBUTIONAL PROPERTIES OF ORDER STATISTICS:

There are several known characterization of the probability distribution based on the identical distribution of suitable functions of order statistics. Puri and Rubin (1970) first treated the case for $n = 2$ and stated that if $X_1$ and $X_2$ are two independent copies of a random variable $X$ with probability density function $f(x)$, then $X$ and $|X_1 - X_2|$ have the same distribution if and only if $f(x) = be^{-bx}$, for $x \geq 0$, with some constant $b > 0$. Desu (1971) showed that if $X$ is a random variable with non degenerate distribution function $F(x)$ and $X_1, X_2, \ldots, X_n$ is a random sample of size $n$ from $F(x)$, then for each positive integer $n$, $n X_1$ and $X$ are identically distributed, if and only if $F(x) = 1 - e^{-bx}$, for $x \geq 0$, with some constant $b > 0$. Ahsanullah (1977) proved that if $X$ is a non negative random variable having an absolutely continuous distribution function $F(x)$, then for some $i$ and $n$, $1 \leq i < n$, the statistics $(n-i)(X_{i+1:n} - X_{i:n})$ and $X$ are identically distributed and $F(x)$ is
either NBU or NWU characterize the exponential distribution. Ahsanullah(1983) characterized the exponential distribution for higher order gap whereas Ouyang(1982) characterized the same distribution for single order gap. Both times they assumed \( F(x) \) is either NBU or NWU. Gather(1989) gave the new proof of exponential distribution by dropping the idea of NBU and NWU. Ahsanullah(1989) characterized the uniform distribution.

**THEOREM 2.1.1:** (Ahsanullah, 1983)

Let \( X \) be a non-negative random variable having a continuous distribution function \( F(x) \) with \( a = \lim \inf_{x \to 0} F(x) > 0 = \inf \) and \( F(x) < 1 \) for all \( x > 0 \). Then the following two properties are equivalent.

(a) \( X \in E(\sigma) \)

(b) \( g_{k:n-k} \) and \( x_{n-k:n-k} \) are identically distributed for some \( k, 1 \leq k < n \) and \( X \in \text{NBU or NWU} \).

where \( g_{k:n-k} = x_{n:n} - x_{k:n}, \quad 1 \leq k < n \).

**PROOF:** We have d.f.

\[
F(x) = \begin{cases} 
1 - \exp\left(-\frac{x}{\sigma}\right); & x > 0, \sigma > 0 \\
0; & x \leq 0 
\end{cases} \quad \text{......(2.1.1)}
\]

(a) \( \Rightarrow \) (b): The d.f. \( F_1 \) of \( g_{k:n-k} \) can be written as
\[ F_1(v_o) = P \left[ g_{k:n-k} \leq v_o \right] \]
\[ = \int_0^\infty \frac{n!}{(k-1)!(n-k)!} (F(u))^{k-1} [F(u+v_o) - F(u)]^{n-k} dF(u) \]
\[ = 0, \text{ otherwise} \] .......(2.1.2)

substituting the value of \( F \) from (2.1.1), it follows that the distribution \( F_1 \) is identical with that of \( X_{n-k:n-k} \).

(b) \( \Rightarrow \) (a): The probability distribution function \( F_2 \) of \( X_{n-k:n-k} \) can be written as
\[ F_2(v_o) = [F(v_o)]^{n-k}, \quad 0 \leq v_o \leq \infty \] .......(2.1.3)
\[ = 0, \text{ otherwise} \]

Since \( g_{k:n-k} \) and \( X_{n-k:n-k} \) are identically distributed we must have from (2.1.2) and (2.1.3)
\[ \int_0^\infty \frac{n!}{(k-1)!(n-k)!} (F(u))^{k-1} [F(u+v_o) - F(u)]^{n-k} dF(u) = \frac{(k-1)!(n-k)!}{n!} [F(v_o)]^{n-k} \]
\[ \text{for all } v_o; \quad 0 \leq v_o \leq \infty \] .......(2.1.4)

But we know
\[ \frac{(k-1)!(n-k)!}{n!} = \int_0^\infty (F(u))^{k-1} [F(u)]^{n-k} dF(u) \] .......(2.1.5)

Substituting (2.1.5) into (2.1.4) and simplifying we get,
\[ \int_0^\infty (F(u))^{k-1} [F(u)]^{n-k} h(u, v_o) dF(u) = 0 \quad \text{for all } v_o, \quad 0 \leq v_o \leq \infty \] .......(2.1.6)
where \( h(u,v_0) \) is given by
\[
 h(u,v_0) = \left\{ 1 - \frac{F(u+v_0)}{F(u)} \right\}^{n-k} - \left[ F(v_0) \right]^{n-k} \quad \ldots \ldots (2.1.7)
\]
Suppose \( F \) is NBU, then \( \frac{F(u+v_0)}{F(u)} \left[ \frac{F(u)}{F(v_0)} \right]^{n-k} \leq 1 - F(v_0) \), and thus
\[
 \left\{ 1 - \frac{F(u+v_0)}{F(u)} \left[ \frac{F(u)}{F(v_0)} \right]^{n-k} \right\}^{n-k} \leq [F(v_0)]^{n-k}.
\]
Therefore, for (2.1.6) to be true we must have
\[
 \frac{F(u+v_0)}{F(u)} \left[ \frac{F(u)}{F(v_0)} \right]^{n-k} = F(v_0) \quad \ldots \ldots (2.1.8)
\]
for all \( v_0, 0 \leq v_0 \leq \infty \) and almost all \( u, 0 \leq u \leq \infty \).
The continuous solution of (2.1.8) with boundary condition \( F(0)=1 \) and \( F(\infty)=0 \) is \( F(x) = \exp(-x/\sigma) \) for all \( x > 0 \) \ldots \ldots (2.1.9)
If \( F \) is NWU, then we similarly obtain (2.1.8) and hence (2.1.9).
If we substitute \( n = k+1 \) in Theorem 2.1.1 then it becomes Ouyang(1982) Theorem 4, which is stated as below,

THEOREM 2.1.2: (Ouyang, 1982)

Let \( F(x) \) be an absolutely continuous distribution function of the non negative random variable \( X \) with \( F(x) < 1 \) for \( x > 0 \) and with probability density function \( f(x) \). Then for some \( k \) and \( n, 1 \leq k < n \), \( X_{k+1:n}, X_{k:n} \) and \( X_{1:n-k} \) are identically distributed and \( F(x) \) is either NBU or NWU if and only if \( F(x) = 1- e^{-bx}, x \geq 0 \), with some constant \( b > 0 \).
A new proof of the characterization of exponential distribution is given by Gather(1989) by dropping the idea of NBU and NWU,
which is stated as below.

**THEOREM 2.1.3:** (Gather, 1989)

A continuous c.d.f $F(x)$, strictly increasing for $x > 0$, is exponential if and only if $X_{j:n} - X_{i:n}$ and $X_{j-1:n-i}$ have identical distribution for some $i, n, j, 1 \leq i < j \leq n, n \geq 3$.

**PROOF:** Please see the given reference.

**THEOREM 2.1.4:** (Ahsanullah, 1989)

Let $X$ be a positive and bounded random variable having an absolutely continuous (w.r.t Lebesgue measure) distribution function $F(x)$. We will assume without any loss of generality that $F(1) = 1$. Then the following two statements are equivalent,

(a) if $X$ is distributed as $U(0,1)$, then for all $n \geq 2$, $X_{1:n}/X_{2:n}$ is distributed as $U(0,1)$.

(b) if for some fixed $n, n \geq 2$, $X_{1:n}/X_{2:n}$ is distributed as $U(0,1)$, then $X$ is distributed as $U(0,1)$.

**THEOREM 2.1.5:** (Ahsanullah, 1989)

Let $X$ be a positive and bounded random variable having an absolutely continuous (w.r.t Lebesgue measure) distribution function $F(x)$.

(i) if $F(x) = x^\alpha, 0 < x < 1$, then $X_{1:n}/X_{2:n}$ are identically distributed.

(ii) if $X$ and $X_{1:n}/X_{2:n}$ are identically distributed and $F$ belongs
to class $C_1$, then $F(x) = x^\alpha$, $0 < x < 1$, for some $\alpha > 0$.

PROOF: Statement (i) is easily verified. To prove (ii), let $Y = -\ln X$ and let $G$ denote the distribution function of $Y$. It can be easily seen that $F \in C_1$ if and only if $G$ is NBU or NWU, (Ahsanullah, 1977). Further, $X$ and $X_{1:n}/X_{2:n}$ will be identically distributed if and only if $Y$ and $-Y_{n-1:n}$ are identically distributed. It follows (Ahsanullah, 1977) that if $X$ and $X_{1:n}/X_{2:n}$ are identically distributed then, for some $\alpha > 0$, $\alpha Y = -\alpha \ln X$ has the exponential distribution with mean 1 so that $X$ has distribution function $F(x) = x^\alpha$, $0 \leq x \leq 1$.

If $X$ is distributed as $U(0,1)$ and $U_{n:n} = 1 - X_{n:n}$ then for all $k$, $0 \leq k \leq n$, $U_{k:n}$ have the same distribution.

Hwang, Arnold and Ghosh (1979) proved that if $F$ belongs to $C_3$ then the identical distribution of $U_{i:n}$ and $U_{0:n}$ characterizes the uniform distribution.

THEOREM 2.1.6: (Ahsanullah, 1989)

Let $X$ be a positive and bounded random variable having an absolutely continuous (w.r.t Lebesgue measure) distribution function $F(x)$. Without any loss of generality we will assume that $\inf\{x/f(x) > 0\} = 0$ and $F(1)$. If $U_{i:n}$ and $U_{i+1:n}$, $0 \leq i < n$, $i \frac{n-1}{2}$ are identically distributed and $F$ belongs to class $C_2$ then $F(x) = x$, $0 < x < 1$. 

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PROOF: Considering the joint probability density of $X_{i+1:n}$ and
$X_i:n$ and using the transformation $V = X_i:n$ and $U_{i+1:n} = X_{i+1:n} - X_i:n$
($0 \leq i < n$), it can show that the probability density function of
$U_{i+1:n}$ is;

$$f_{u_{i+1:n}}(u) = \frac{n!}{(i-1)!(n-i-1)!} \int_0^{1-u} (F(v))^{i-1} (F(u+v))^{n-i-1} f(v)f(u+v)dv,
0 \leq u \leq 1$$

Similarly,

$$f_{u_{i+1:n}}(u) = \frac{n!}{i!(n-i-2)!} \int_0^{1-u} (F(v))^i (F(u_0+v))^{n-i-2} f(v)f(u_0+v)dv$$

Integrating (2.1.11) w.r.t. $u$ from $u_0$ to 1, we get

$$P(U_{i+1:n} > u_0) = \frac{n!}{i!(n-i-1)!} \int_0^{1-u_0} (F(v))^i (F(u_0+v))^{n-i-1} f(v) dv
0 \leq u_0 \leq 1$$

Now integrating (2.1.10) w.r.t. $u$ from $u_0$ to 1, and using integration by parts w.r.t. $v$ and simplifying, we have

$$P(U_{i+1:n} > u_0) = \frac{n!}{i!(n-i-1)!} \int_0^{1-u_0} (F(v))^i (F(u_0+v))^{n-i-1} f(u_0+v) dv
0 \leq u_0 \leq 1$$

if $i = m$, $n = 2m+1$, $m \geq 1$ and assuming that $X$ has any symmetric
distribution about $1/2$, i.e. $F(x) = F(1-x)$, it can be shown that
(2.1.12) and (2.1.13) are identical, we must have
\[
\int_0^{1-u_0} (F(v))^{i} (F(u_0+v))^{n-1} \left[ f(u_0+v) - f(v) \right] dv = 0
\]
for all \( u_0, 0 \leq u_0 \leq 1 \)

Since \( F \in C_2 \), from (2.1.14) we have
\[
f(u_0+v) = f(v)
\]
for all \( u_0, 0 \leq u_0 \leq 1 \) and almost all \( v, 0 \leq v \leq 1-u_0 \)
implies \( X \) is distributed as \( U(0,1) \).

**THEOREM 2.1.7**: (Ahsanullah, 1989)
Let all the assumption is same as in the Theorem 2.1.5. Then the following two properties are equivalent.

(i) \( X \) is distributed as \( U[0,1] \)

(ii) \( X_{n:n} - X_{1:n} \) and \( X_{n-1:n} \) are identically distributed and \( X \in C_3 \), for some \( n > 1 \).

**PROOF**: Please see the given reference.

2.2 CHARACTERIZATION BY INDEPENDENCE OF FUNCTIONS OF ORDER STATISTICS:
It is well known (Epstein and Sobel, 1953 and Renyi, 1953) that
\( X_{1:n}, X_{2:n} - X_{1:n}, \ldots, X_{n:n} - X_{n-1:n} \) are mutually independent for exponential distribution. The converse of this result was first investigated by Fisz(1958), who showed that for \( n = 2 \), the independence of \( X_{1:2} \) and \( X_{2:2} - X_{1:2} \) characterizes the
exponential distribution. Ferguson (1964) proved that if independent random variable X and Y have absolutely continuous distribution and if min(X, Y) and X - Y are independent, then both X and Y have exponential distribution with the same value of the location parameter. Tanis (1964) extended the result of Fisz (1958) to arbitrary n ≥ 2 and proved the theorem which is stated below;

THEOREM 2.2.1: (Tanis, 1964)
Let F(x) be an absolutely continuous distribution function of the random variable X with F(a) = 0 and F(x) > 0 for x > a. Then X_{1:n} and \( \sum_{j=2}^{n} (X_{j:n} - X_{1:n}) \) are independent if and only if the probability density function of X is \( f(x) = b e^{-b(x-a)}, x \geq a \), where b > 0 and a are finite constants.

Rossberg (1972) gave a very general extension of Tanis, but his proof was analytic and very complicated, whereas Ouyang (1982) generalized the result of Tanis as given below.

THEOREM 2.2.2: (Ouyang, 1982)
Let \( X_1, X_2, \ldots, X_n \) be a random sample of size n from a population with continuous distribution function F(x). Then for some 1 ≤ k < n, \( X_{k:n} \) and \( U_k = \sum_{k+1}^{n} (X_{j:n} - X_{k:n}) \) are independent if and only if F(x) = 1 - \( e^{-b(x-a)} \), x ≥ a, where b > 0 and a are
finite constants.

It is well known (Kawata and Sakamoto, 1949), that the independence of the sample mean and the sample variance characterizes the normal distribution, whereas the independence of the smallest order statistic $X_{1:n}$ and $\sum (X_i - X_{1:n})$ under suitable conditions characterizes the exponential distribution (Tanis, 1969; Ferguson, 1967) which we have seen earlier.

Goria (1987) characterized the uniform distribution through the independence between,

(i) $X_{n:n}$ and $X_{1:n}/X_{n:n}$

(ii) $X_{n:n}$ and $\bar{X}/X_{n:n}$

which is given as below,

**THEOREM 2.2.3**: (Goria, 1987)

Let $X$ be a non-negative random variable with continuous distribution function $F(x)$ which is twice differentiable from the right in the interior of the unit interval $[0,1]$ and non vanishing first derivative at the origin. Furthermore let the conditions

1. $F(1) = 1$
2. $E(x)$ is finite
3. $F^{-1}$ is unique and is absolutely continuous in $[0,1]$ 

hold. For an ordered sample $X_{1:1}, \ldots, X_{n:n}$ writing $U_{i:n} = X_i/n:X_{n:n}$,
i = 1, 2, ...., (n-1) and \( U_n = X_{n:n} \), then \( (U_1, U_2, \ldots, U_{n-1}) \) and 
\( U_n \) are independent if and only if \( F \) is uniform on \([0,1]\).

**PROOF:** It is fairly straightforward to see that the conditional density \( \{U_1, \ldots, U_{n-1}\} \) given \( U_n = u \) is of the form

\[
g(u_1, \ldots, u_{n-1}/u) = (n-1) \prod_{i=1}^{n-1} \left( \frac{dF(u_i)F(u)}{F(u)} \right), \text{ for } 0 < u_1 < \ldots < u_{n-1} < 1
\]

\[\text{...............(2.2.1)}\]

Now if \( F \) is uniform on \([0,1]\) then the equation (2.2.1) is clearly free of \( u \). As a result \( (U_1, \ldots, U_{n-1}) \) and \( U_n \) are independent. Conversely suppose that \( \{U_1, \ldots, U_{n-1}\} \) and \( U_n \) are independent, namely that the right side of (2.2.1) is free of \( u \). This implies that the quantities appearing in \{ \} of equation (2.2.1) are independent of \( u \) for each \( i = 1, 2, \ldots, (n-1) \). Now we have

\[
\int_0^w u_i dF(u_i)/F(u) = F(uw)/F(u) \quad \text{...............(2.2.2)}
\]

as \( u \to 1 \), (2.2.2) reduces to \( F(w) \).

implies that \( F(uw) = F(u) \cdot F(w) \).

As \( F(x) \) is uniform on \([0,1]\) if and only if any two numbers \( t_1 \) and \( t_2 \) such that \( 0 \leq t_2 \leq t_1 \leq 1 \),

\[
P\{X \leq t_1 \mid X \leq t_2\} = P\{X \leq t_2\} \quad \text{(Goria, 1987)}
\]

Hence \( F \) is uniform.
THEOREM 2.2.4: (Goria, 1987)

Under the assumptions of Theorem 2.2.3, $X_{n:n}$ and $X/X_{n:n}$ are independent if and only if $F$ is uniform on $[0,1]$.

PROOF: Suppose that $F$ is uniform. Now by the Theorem 2.2.3, $U_n$ and $\{U_1, U_2, \ldots, U_{n-1}\}$ are independent, which yields the independence of $X_{n:n}$ and $X/X_{n:n}$.

The characteristic function of $nX/X_{n:n}$ given that $X_{n:n} = x$ is

$$h_n(t) = \exp(it) \left\{ \int_0^x \exp \left[ it \frac{x}{(n-1)} \right] \frac{dF(x)}{F(x)} \right\} (n-1)$$

Because of the independence of $X_{n:n}$ and $X/X_{n:n}$ the right side of (2.2.3) does not involve $x$. This is to say that quantity within $\{ \}$ is free of $x$.

When $n = 2$, we note that the characteristic function of $X_{1:2}/X_{2:2}$ given that $X_{2:2} = x$ is

$$\left\{ \int_0^x \exp \left[ it \frac{x}{x} \right] \frac{dF(x)}{F(x)} \right\}$$

By theorem (2.2.3) with $n = 2$ we then have the desired result.

2.3 CHARACTERIZATION VIA COVARIANCE BOUNDS:

Let $X_{(1)}$, $X_{(2)}$ be the order statistics of a sample of size 2 from a continuous random variable $X$ with distribution function $F(x)$ and density function $f(x)$. Abdelhamid(1985) obtained an upper
bound for the covariance of $X_{(1)}, X_{(2)}$ to characterize the uniform distribution, which is stated in the following Theorem.

**THEOREM 2.3.1:** (Abdelhamid, 1985)

Let $\left( X_{(1)}, X_{(2)} \right)$ be the order statistics of a sample of size 2 from a continuous distribution $F$ with finite variance. Let $f$ be the density of $F$. Then

$$\text{Cov}\left[ X_{(1)}, X_{(2)} \right] \leq \frac{1}{2} \int_{0}^{1} \left[ f(F^{-1}(u)) \right]^{-2} du - \frac{1}{2} \left( \sigma_{x_{(1)}}^2 + \sigma_{x_{(2)}}^2 \right)$$

with equality if and only if $f$ is uniform density on $(0,1)$.

His main result, although correct, is based on an incorrect lemma, which was pointed by Arnold and Brockett (1988).

Arnold and Brockett (1988) also gave upper bounds for the variance of the smallest and largest order statistics from a sample of size $n$.

Papathanasiou (1990) also obtained the upper bounds for the covariance $\text{Cov}\left[ X_{(1)}, X_{(2)} \right]$ and characterized uniform distribution.

**THEOREM 2.3.2:** (Papathanasiou, 1990)

Let $X_{(1)}, X_{(2)}$ be the order statistics from a sample of size 2 from an absolutely continuous distribution $F$ with density $f$, variance $\sigma^2$ and mean $\mu$. Then we have

$$\text{Cov}\left[ X_{(1)}, X_{(2)} \right] \leq \frac{1}{2} \sigma^2 \quad \ldots \ldots (2.3.1)$$
where equality holds if and only if $f$ is a uniform distribution.

**Proof:** We observe that

$$E[X(1), X(2)] = E^2[X],$$

where $X$ is the r.v. with density $f$ and thus covariance can be written as

$$\text{Cov}[X(1), X(2)] = \left( \int_{-\infty}^{\infty} x f(x) \, dx \right)^2 - 4 \int_{-\infty}^{\infty} x f(x) \, dx \int_{-\infty}^{\infty} x f(x)[1-F(x)] \, dx$$

Multiplying the first integral by $F(x) + [1-F(x)]$ and expanding the square we can find after simple calculations that

$$\text{Cov}[X(1), X(2)] = \left( \int_{-\infty}^{\infty} [2F(x) - 1] f(x) \, dx \right)^2$$

Since $\int_{-\infty}^{\infty} [2F(x) - 1] f(x) \, dx = 0$

we can write

$$\text{Cov}[X(1), X(2)] = \left( \int_{-\infty}^{\infty} (x-\mu) [2F(x) - 1] f(x) \, dx \right)^2 \quad \text{......(2.3.2)}$$

Applying the Cauchy–Schwarz inequality in (2.3.2) we obtain

$$\text{Cov}[X(1), X(2)] \leq \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \, dx \int_{-\infty}^{\infty} (2F(x) - 1)^2 f(x) \, dx$$

and thus the required result (2.3.1). The equality holds if and only if $2F(x)-1 = c(x-\mu)$.

implies that bound is attained if and only if $f$ is a uniform distribution.
THEOREM 2.3.3: (Papathanasiou, 1990)

Let the assumption be as in Theorem (2.3.2). Moreover suppose $F$ is strictly monotonic and $F^{-1}$ differentiable. Then we have

$$\text{Cov}\left[ X_1, X_2 \right] = \left( \int_0^1 u(1-u) h'(u) \, du \right)^2 \quad \ldots \ldots (2.3.3)$$

PROOF: From (2.3.2) we observe that

$$\text{Cov}\left[ X_1, X_2 \right] = 4 \, \text{Cov}^2 \left[ X, F(X) \right]$$

setting $F(x) = u$ and $F^{-1}(u) = h(u)$ the Cov[$X, F(X)$] can be written as

$$\text{Cov}\left[ X, F(X) \right] = \int_0^1 \int_0^v (v - u)(h(v) - h(u)) \, du \, dv$$

After some algebra, we get the required result.

Applying the Cauchy–Schwarz inequality to equation (2.3.3) we have the following corollary.

Corollary 2.3.1:

Under the assumption of Theorem (2.3.3), an upper bound for the covariance of $X_1, X_2$ is

$$\text{Cov}\left[ X_1, X_2 \right] \leq B(p,q) \int_0^1 u^{3-p} (1-u)^{3-q} (h'(u))^2 \, du \quad \ldots \ldots (2.3.4)$$

where $B(p,q)$ is the beta function. Equality holds if and only if

$$c u^{p/2-1/2} (1-u)^{q/2-1/2} = u^{3/2-p/2} (1-u)^{3/2-q/2} h'(u). \quad c$$

where $c = \text{constant}$
PROOF: From (2.3.5) equality holds if and only if

\[ h'(u) = cu^{p-2}(1-u)^q-2 \]

or,

\[ \frac{dF^{-1}(u)}{du} = cu^{p-2}(1-u)^q-2 \]

\[ \ldots \ldots (2.3.6) \]

Since the function \( cu^{p-2}(1-u)^q-2 \) determines \( F \) up to translation (See Arnold and Brockett, 1988) we obtain characterizations for certain distributions.

Specific choices of \( p \) and \( q \) characterize several distributions.

(i) If \( p=2 \) and \( q=2 \) then \( f \) is Uniform distribution.

(ii) If \( p=2 \) and \( q=1 \) then \( f \) is a translated Exponential distribution.

(iii) If \( p=1 \) and \( q=1 \) then \( f \) is a Logistic distribution.

(iv) If \( p=2 \) and \( q<1 \), then \( f \) is a translated classical Pareto distribution.
CHAPTER-3

CHARACTERIZATION THROUGH SINGLE MOMENTS OF ORDER STATISTICS

3.1 INTRODUCTION:
Moments of order statistics are extensively used in characterization of specific distribution. Various approaches are available in literature. Renyi (1953) has not only characterized the exponential distribution but also gave a representation for the exponential order statistics in a random sample of size n, which enables one to compute the moments of exponential order statistics. Epstein and Sobel (1954) gave certain theorems which are helpful for computing the moments of exponential order statistics. Govindrajulu (1975), Lin (1988) and Kamps (1991) used recurrence relations of moments of single order statistics to characterize some specific distributions.

Hoeffding (1953) pointed out that if $X_{k:n}$ be the Kth smallest order statistics of a random sample of size n from a distribution $F(x)$ with $E|X| < \infty$. Then the sequence of expected values of order statistics,

$$\left\{ \mu_{k:n} : 1 \leq k \leq n, n \geq 1 \right\} \quad \ldots (3.1.1)$$

The first one to study the order statistic characterization problem is Chan (1967), who showed that if $E|X| < \infty$ then $F$ is
uniquely determined by the sequence,

\{ \mu_n : n = 1,2,3,\ldots \} \quad \ldots (3.1.2)

or equivalently, by

\{ \mu_n : n = 1,2,3,\ldots \} \quad \ldots (3.1.3)

Interestingly, in the same journal, the same result is obtained independently by Konheim (1971) using a different proof. Gupta (1974) reproved Chan's result for the lattice distribution. As an immediate consequence of Chan's theorem, we have the following.

1. \( \mu_n = 1/(n+1) \) for \( n=1,2,3,\ldots \) iff \( F \) is uniform \((0,1)\),
   \[ F(x) = x, \quad 0<x<1. \]

2. \( \mu_n = 1/n \) for \( n=1,2,3,\ldots \) iff \( F \) is unit exponential,
   \[ F(x) = 1-\exp(-x), \quad x>0. \]

3. \( \mu_n = n/(n-1) \) for \( n=2,3,4,\ldots \) iff \( F \) is Pareto,
   \[ F(x) = 1-1/x, \quad x>1. \]

4. \( \mu_n = 2n/(2n+1) \) for \( n=1,2,3,\ldots \) iff \( F \) is triangular,
   \[ F(x) = x^2, \quad 0<x<1. \]

Chan's theorem is not limited to the continuous distributions. Result (2) has an interesting application. It is well known that the minimum order statistics (suitably normalized) from an exponential distribution are all identically distributed. That the
converse is also true is obtained by Desu(1971).

Pollak(1973) extended Chan's result to a more general sequence,

\[ \{ \mu_{k,n} : n=1,2,3,... \} \]

where integers \( k=k(n) \) are allowed to vary with \( n \):

\[ 1 \leq k(n) \leq n \]

Chan's result is thus a special case of Pollak's: \( K(n)=1 \) or \( K(n)=n \). Using Pollak's result, one can have the following,

\[ (5) \mu_{(n+1)/2,n} = 0 \text{ for } n=1,3,5,... \quad \text{and} \quad \mu_{n/2,n} = -2/n \text{ for } n=2,4,6,... \text{ iff } F \text{ is logistic, } F(x)=1/(1+\exp(-x)). \]

Arnold and Meeden(1975) studied the conditions under which an arbitrary subset of (3.1.1) characterizes \( F \). Ali(1976) provided yet another proof of Chan's. Lin(1984) showed that for each odd positive integer \( n \), knowledge of \( \mu_{k,n} \) for two distinct \( k \) is enough to characterize \( F \). A general survey on characterization of distribution via expectation of order statistics can be found in Glambos(1975), Glambos and Kotz(1978), Arnold(1983), Azlarov and Volodin(1986) and Huang(1989).

Lin(1988a) assumed that \( X_1 \) and \( X_2 \) be two i.i.d random variables having arbitrary distribution \( F(x) \) and further \( X_{2,2}=\max\{X_1,X_2\} \). Then he showed \( \mu_2^2 = 1/2 \) and \( \mu_{2;2} = 1/2 \) iff \( F \) is the uniform over interval \((0,1)\). He has also given the general result about power distribution and Weibull distributions.

Ali and Khan(1994b) characterized a class of truncated and
non-truncated distributions using the recurrence relations of the expectations of a function of one and two order statistics. The specific distributions as a particular case of the general class of distributions are Power function, Weibull, Pareto, Beta of the first kind, Burr type XII, Rectangular, Rayleigh, Exponential, Lomax, Log logistic, Cauchy and inverse Weibull.

3.2 CHARACTERIZATION OF EXPONENTIAL DISTRIBUTION:

**THEOREM 3.2.1:** (Govindrajulu, 1975; Azlarov and Volodin, 1986)

For $i=0,1,\ldots$; $\mu_{i+1:n}^{(2)} - \mu_{i:n}^{(2)} = 2(n-i)^{-1} \theta \mu_{i+1:n}^{(2)}$; $n=i+1, i+2, \ldots$

if and only if $F(x) = 1 - \exp(-x/\theta)$, $x, \theta > 0$.

**PROOF:** After integrating by parts once we obtain

$$\mu_{i+1:n}^{(2)} = \mu_{i:n}^{(2)} + 2 \sum_{i=0}^{\infty} x I(F;i,n-i)dx \quad \text{.....(3.2.1)}$$

Thus the characterizing property is equivalent to

$$\int_0^\infty x F^i(1-F)^{n-i-1}dx = \theta \int_0^\infty x F^i(1-F)^{n-i-2}dF \quad \text{.....(3.2.2)}$$

or, for $i=0,1,\ldots$

$$\int_0^1 u^i(1-u)^{n-i-1} H(u) \left\{(1-u) H'(u) - \theta\right\} du = 0, \ n=i+1, i+2, \ldots$$

Now, the only continuous function which is orthogonal to $u^i(1-u)^{n-i-1}$ (a linear combination of $u^0, u, \ldots, u^{n-1}$) for $n=i+1, i+2, \ldots$ is the zero function itself.

Hence $(1-u) H'(u) = \theta$ for almost all $u(0 < u < 1)$, which implies that

$$F(x) = 1 - \exp(-x/\theta), \ x, \theta > 0.$$
Also, if \( F(x) = 1 - \exp(-x/\theta) \) then (3.2.1) is trivially satisfied.

**THEOREM 3.2.2:** (Govindrajulu, 1975; Azlarov and Volodin, 1986)

The relation
\[ n(\mu_{i:1:n}^{(2)} - \mu_{i-1:1:n-1}^{(1)}) = 2\theta \mu_{i:1:n}^{(1)} \]

holds if and only if \( F(x) = 1 - \exp(-x/\theta) \); \( x, \theta > 0 \).

**PROOF:**

(Case I: \( i = 1 \)):

After integrating by parts we have
\[ \mu_{i:1:n}^{(2)} = 2 \int_0^\infty x I(F;0,n) \, dx \quad \ldots \ldots (3.2.3) \]

Hence the characterizing property is equivalent to
\[ \int_0^\infty x I(F;0,n) \, dx = \theta \int_0^\infty x I(F;0,n-1) \, dF \quad \ldots \ldots (3.2.4) \]

or
\[ \int_0^1 H(u) (1-u)^{n-1} \left\{ (1-u) H'(u) - \theta \right\} \, du = 0, \quad n=1,2,\ldots \]

That is \( H'(u) = \theta (1-u)^{-1} \), for almost all \( u(0<u<1) \).

Thus, as noted earlier \( F(x) = 1 - \exp(x/\theta) \); \( x, \theta > 0 \).

(Case II: \( i > 1 \)):

Using the recurrence relation
\[ (n-i)\mu_{i:1:n}^{(k)} + i\mu_{i+1:1:n}^{(k)} = n\mu_{i:1:n-1}^{(k)} \quad \ldots \ldots (3.2.5) \]

we get
\[ n(\mu_{i+1:1:n}^{(2)} - \mu_{i:1:n-1}^{(2)}) = (n-i)(\mu_{i+1:1:n}^{(2)} - \mu_{i:1:n}^{(2)}) \]

Hence the proof is completed by using theorem 3.2.1.

Let \( x_{1:1:n} \leq x_{2:1:n} \leq \ldots \leq x_{n:1:n} \) be the order statistics of a random
sample of size $n$ from a distribution $F(x)$. Let $\left\{n_j\right\}_{j=1}^{\infty}$ be a sequence of integers satisfying

$$0 < n_1 < n_2 < \ldots \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty \quad \ldots (3.2.6)$$

then it is well known that $F(x) = 1 - \exp(-x)$, $x \geq 0$,

iff $\mu_{1:n_j} = 1/n_j$ for all $j \geq 1$. (See, e.g. Galambos and Kotz, 1978, pp. 55-57). Under suitable conditions on the inverse function $F^{-1}$ defined by $F^{-1}(t) = \inf\left\{x : F(x) \geq t\right\}$, $t \in (0,1)$.

Lin(1988) extended the above result by considering the relationship between the $(m-1)$th and $m$th moments of order statistics.

**THEOREM 3.2.3:** (Lin, 1988a)

Let $X$ be a random variable (r.v.) with distribution $F(x)$ and $E|X|^m < \infty$ for some constant $m \geq 1$. Let $k \geq 1$ be an integer and $\left\{n_j\right\}_{j=1}^{\infty}$ a sequence of integers satisfying (3.2.6). Assume that $F^{-1}(0^+) = 0$, that $F^{-1}$ is positive if $m>1$ is used, and further that $F^{-1}$ is absolutely continuous on $(0,1)$. Then for given constant $\lambda > 0$, $F(x) = 1 - \exp(-x/\lambda)$, $x \geq 0$, if and only if

$$\mu^{(m)}_{k:n_j} = \mu^{(m)}_{k-1:n_j} + \frac{m\lambda}{n_j-k+1} \mu^{(m-1)}_{k:n_j} \quad \text{for all } n \geq k \quad \ldots (3.2.7)$$
3.3 CHARACTERIZATION OF UNIFORM, LOGISTIC AND PARETO DISTRIBUTIONS:

THEOREM 3.3.1: (Lin, 1988a)
Assume that all the conditions of the Theorem 3.2.3 hold. Then for given constant \( \lambda > 0 \), \( F(x) = x/\lambda \), \( x \in (0, \lambda) \), if and only if,

\[
\mu_k:n_j = \mu_{k-1}:n_j + \frac{m\lambda}{n_j+1} \mu_{k:n_j+1} \text{ for all } n_j \geq k
\]

\[\text{......(3.3.1)}\]

Proof of the above Theorem can be seen in the given reference.

THEOREM 3.3.2: (Hwang and Lin, 1988)
Let \( F_0 \) be an arbitrary distribution with finite expectation and let \( F_{F_0} \) is as,

\[
F_{F_0} = \{ F_{\mu,\sigma} : F_{\mu,\sigma}(x) = F_0\left(\frac{x-\mu}{\sigma}\right) \text{ on } \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0 \}
\]

Then the pair \( \mu_k:n \) and \( \mu_{1:m} \) characterizes \( F \) in the class \( F_{F_0} \) if and only if \( \Delta = EX_{k:n}^{(0)} - EX_{1:m}^{(0)} \neq 0 \), in which superindex "o" means that the corresponding random sample is drawn from \( F_0 \).

THEOREM 3.3.3: (Hwang and Lin, 1988)
Let \( F_u = \{ F_{\mu,\sigma} : F_{\mu,\sigma}(x) = (x-\mu)/\sigma \text{ on } (\mu, \mu+\sigma), \mu \in \mathbb{R}, \sigma > 0 \} \), then the pair \( \mu_k:n \) and \( \mu_{1:m} \) characterize \( F \) in the class \( F_u \) if and only if

\[
\frac{k}{n+1} \neq \frac{1}{m+1}.
\]
PROOF: Notice that \( F = F_0 \), where \( F = x \) on \((0,1)\) and that for \( F \)

\[
\begin{align*}
EX_{k:n}^{(0)} &= \frac{k}{n+1} \\
and EX_{1:m}^{(0)} &= \frac{1}{m+1}
\end{align*}
\]

(See, e.g. David 1981, pp.35)

Hence the desired result follows immediately from lemma.

**THEOREM 3.3.4:** (Lin, 1988b)

Let \( EX < \infty \). Then for fixed integers \( 2 \leq k \leq n \),

\[
\left( EX_{k,n} \right)^2 \leq \frac{nk}{(n+1)(k-1)} \mu_{k-1:n-1}^{(2)}
\]

with equality if and only if either \( F \) is degenerate at \( x = 0 \) or \( F = U(0,c) \),

where

\[
c = \left\{ \frac{n(n+1)}{k(k-1)} \mu_{k-1:n-1}^{(2)} \right\}^{1/2} > 0.
\]

**PROOF:** Applying Cauchy-Schwarz inequality we have,

\[
\begin{align*}
\left( EX_{k,n} \right)^2 &= \left\{ k \binom{n}{k} \right\}^2 \left\{ \int_0^1 F^{-1}(t) t^{k-2} (1-t)^{(n-k)/2} dt \right\}^2 \\
&\leq \left\{ k \binom{n}{k} \right\}^2 \int_0^1 \left[ F^{-1}(t) \right]^2 t^{k-2} (1-t)^{n-k} dt \int_0^1 t^k (1-t)^{n-k} dt \\
&= \frac{nk}{(n+1)(k-1)} \mu_{k-1:n-1}^{(2)}
\end{align*}
\]

Since the inverse function \( F^{-1} \) is non-decreasing and left continuous on \((0,1)\) we obtain that equality in (3.3.2) holds if
and only if \( F^{-1}(t) = 0 \), \( t \in (0,1) \).

The first case is equivalent to that \( F \) is degenerate at \( x=0 \) and the latter case \( F = U(0,c) \) with \( c = \left[ \frac{n(n+1)}{k(k-1)} \right]^{1/2} \left( \frac{\mu_k}{\mu_{k-1:n-1}} \right) > 0 \).

The proof is completed.

**COROLLARY:**

\( \mu^2 = \frac{1}{3} \) and \( \mu_{2:2} = \frac{2}{2} \)

if and only if \( F = U(0,1) \).

**THEOREM 3.3.5: (Logistic Distribution):** (Lin, 1988b)

Let \( X \) be a r.v. with distribution \( F(x) \) and \( E|X|^m < \infty \) for some integer \( m \leq 1 \). Let \( k \geq 2 \) be an integer and \( \{n_j\}_{j=1}^\infty \) be a sequence of integers satisfying (3.2.6). Assume that \( P(X=0) = 0 \) if \( m \neq 0 \) is used, and \( F^{-1} \) is absolutely continuous on \((0,1)\). Then for given constant \( \lambda > 0 \),

\[
F(x) = \left\{ 1 - \exp\left(-\frac{x-\mu}{\lambda}\right) \right\}^{-1}
\]

on \( \mathbb{R} \) for some constant \( \mu \in \mathbb{R} \), if and only if for all \( n_j \geq k \),

\[
\mu_{k:in_j}^{(m)} = \mu_{k-1:n_j}^{(m)} + \frac{m \lambda n_j}{(k-1)(n_j-k+1)} \mu_{k-1:n_j-1}^{(m-1)} \]  \((3.3.3)\)

**THEOREM 3.3.6: (Pareto Distribution):** (Lin, 1988b)

Let \( X \) be a r.v. with distribution \( F(x) \) and \( E|X_{k-1,n_0}|^m < \infty \) for some constant \( m \geq 1 \) and some positive integers \( n_0 \geq k \geq 2 \). Let
\( \{n_j\}_{j=1}^{\infty} \) be a sequence of integers satisfying (3.2.6). Assume that 
\( F^{-1}(0^+) = \lambda > 0 \) and \( F^{-1} \) is absolutely continuous on \((0,1)\). Then
given integers \( 1 \geq 1 \), \( F(x) = 1 - (x/\lambda)^{-1/1} \), \( x \geq \lambda \) if and only if
for all \( n_j = n_0 + 1 \)

\[
\mu_{k:n_j}^{(m)} = \mu_{k-1:n_j}^{(m)} + \frac{\lambda 1(n_j)!(n_j-k-1)!}{(n_j-1)!(n_j-k+1)!} \mu_{k:n_j-1}^{(m-1)} \quad (3.3.4)
\]

Proof of the Theorems (3.3.4) and (3.3.5) can be seen in the given references.

THEOREM 3.3.7: (Khan, 1987)

Let \( X \) be an absolutely continuous r.v. with df \( F(x) \) such that
\( F(x) < 1 \) for \( x \geq 1 \). Then

\[
1 - F(x) = x^{-p}, \quad p > 0, \quad x \geq 1
\]

if and only if

\[
\mu_{1:n}^{(-k)} = \frac{np}{np+k}, \quad k > 1
\]

PROOF: Necessary part is obvious. Sufficiency part gives,

\[
1 - k \int_{1}^{\infty} x^{-k-1} (1 - F(x))^n \, dx = \frac{np}{np+k}
\]

which reduces to,

\[
\int_{1}^{\infty} x^{-k-1} \left[ \left(1-F(x)\right)^n - x^{-np} \right] \, dx = 0
\]

implying that (Galambos and Kortz, 1975)

\[
1 - F(x) = x^{-p}
\]

which completes the proof.
3.4 CHARACTERIZATION OF GENERAL CLASS OF DISTRIBUTION:

(a) Characterization of $F_j(x) = 1 - [ah(x) + b]^C$.

THEOREM: 3.4.1 (Ali and Khan, 1994c)

Let $F(.)$ be the distribution function with $E|X| < \infty$ for some $j \geq 1$. Let the inverse function $F^{-1}(.)$ be positive on $[0,1]$ and $F^{-1}(0^-) = Q_1$, $F^{-1}(1) = P_1$. Assume further that \( \{n_i\}, i = 1, 2, \ldots \) be a sequence of integers satisfying $2 \leq n_1 < n_2 < \ldots$ and $\sum_{i=1}^{\infty} 1/n_i = \infty$.

Then for $1 \leq r < s \leq n_i$

$$E \left\{ g \left( X_{r,n_i} \right) \right\} - E \left\{ g \left( X_{r-1,n_i} \right) \right\} = - \frac{n_1 P_2}{(n_1 - r + 1)} \left[ E \left\{ g \left( X_{r-1,n_i} - 1 \right) \right\} - E \left\{ g \left( X_{r-2,n_i} - 1 \right) \right\} \right] - \frac{1}{(n_1 - r + 1)c} E \left\{ m \left( X_{r,n_i} \right) \right\}$$

\[\ldots \ldots (3.4.1)\]

if and only if

$$F(x) = Q_2 - \frac{[ah(x) + b]^C}{(P - Q)}; \quad x \in (Q_1, P_1) \quad \ldots \ldots \ldots (3.4.2)$$

where $m(x) = [ah(x) + b] \frac{g'(x)}{h(x)}$.

THEOREM 3.4.2: (Ali and Khan, 1994c)

Under the condition of Theorem 3.4.1 the following statements are equivalent.
(i) $F(x) = \frac{[ah(x)+b]^c}{(P-Q)}; x \in (Q_1, P_1)$  

(ii) $E\left\{g(x_{r:n_1})\right\} = E\left\{g(x_{r-1:n_1-1})\right\} - \frac{(P-Q)(n_1-r+1)}{n_1(n_1+1)ca} E\left\{Z(x_{r:n_1+1})\right\}$  

(iii) $E\left\{g(x_{r:n_1})\right\} = E\left\{g(x_{r-1:n_1})\right\} - \frac{(P-Q)(n_1+1)}{n_1(n_1+1)ca} E\left\{Z(x_{r:n_1+1})\right\}$  

(iv) $E\left\{g(x_{r-1:n_1-1})\right\} = E\left\{g(x_{r-1:n_1})\right\} - \frac{(P-Q)(r-1)}{n_1(n_1+1)ca} E\left\{Z(x_{r:n_1+1})\right\}$  

where $Z(x,y) = \left[ah(y)+b\right]^{1-c} \frac{g'(x,y)}{h'(y)}$  

and $g'(x,y) = \frac{\partial}{\partial y} g(x,y)$  

Proof of the Theorem 3.4.1 and Theorem 3.4.2 can be seen in the given references.  

For proper choice of $a,b,c$ and $h(x)$, various distributions can be obtained as given in the Table 3.1.
<table>
<thead>
<tr>
<th>Distribution Function</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>h(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>$-\lambda^{-p}$</td>
<td>1</td>
<td>1</td>
<td>$x^p$</td>
</tr>
<tr>
<td>$F_1(x) = \lambda^{-p}x^p$; $0 \leq x \leq \lambda$</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>$\lambda^{-p}x^p$</td>
</tr>
<tr>
<td>2. Pareto function</td>
<td>$\lambda^p$</td>
<td>0</td>
<td>1</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td>$F_1(x) = 1-\lambda^p x^{-p}$; $\lambda \leq x &lt; \infty$</td>
<td>$\lambda$</td>
<td>0</td>
<td>p</td>
<td>$x^{-1}$</td>
</tr>
<tr>
<td>3. Beta of the first kind</td>
<td>1</td>
<td>0</td>
<td>p</td>
<td>$\frac{\lambda-x}{\lambda-\beta}$</td>
</tr>
<tr>
<td>$F_1(x) = 1 - \left(\frac{\lambda-x}{\lambda-\beta}\right)^p$; $\beta \leq x \leq \lambda$</td>
<td>$-1$</td>
<td>$\frac{\lambda}{\lambda-\beta}$</td>
<td>p</td>
<td>$\frac{x}{\lambda-\beta}$</td>
</tr>
<tr>
<td>4. Weibull</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$e^{-\theta x}$</td>
</tr>
<tr>
<td>$F_1(x) = 1-e^{-\theta x^p}$; $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$\theta$</td>
<td>$e^{-x^p}$</td>
</tr>
<tr>
<td>5. Burr type XII</td>
<td>$\theta$</td>
<td>1</td>
<td>$-\lambda$</td>
<td>$x^p$</td>
</tr>
<tr>
<td>$F_1(x) = 1-(1+\theta x^p)^{-\lambda}$; $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$-\lambda$</td>
<td>$1+\theta x^p$</td>
</tr>
<tr>
<td>6. Rectangular</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\frac{\lambda-x}{\lambda-\beta}$</td>
</tr>
<tr>
<td>$F_1(x) = \frac{x-\beta}{\lambda-\beta}$; $\beta \leq x \leq \lambda$</td>
<td>$-1$</td>
<td>$\frac{\lambda}{\lambda-\beta}$</td>
<td>1</td>
<td>$\frac{x}{\lambda-\beta}$</td>
</tr>
<tr>
<td>7. Rayleigh</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$e^{-\theta x^2}$</td>
</tr>
<tr>
<td>$F_1(x) = 1-e^{-\theta x^2}$; $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$\theta$</td>
<td>$e^{-x^2}$</td>
</tr>
<tr>
<td>8. Exponential</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$e^{-\theta x}$</td>
</tr>
<tr>
<td>$F_1(x) = 1-e^{-\theta x}$; $0 \leq x &lt; \infty$</td>
<td>1</td>
<td>0</td>
<td>$\theta$</td>
<td>$e^{-x}$</td>
</tr>
</tbody>
</table>
TABLE - 3.1 (Continued)

<table>
<thead>
<tr>
<th>Distribution Function</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>h(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. Lomax</td>
<td>θ</td>
<td>1</td>
<td>-λ</td>
<td>x</td>
</tr>
<tr>
<td>( F_1(x) = 1 - (1 + \theta x)^{-\lambda} ); ( 0 \leq x &lt; \infty )</td>
<td>1</td>
<td>0</td>
<td>-λ</td>
<td>1 + \theta x</td>
</tr>
<tr>
<td>10. Loglogistic</td>
<td>θ</td>
<td>1</td>
<td>-1</td>
<td>x^p</td>
</tr>
<tr>
<td>( F_1(x) = 1 - (1 + \theta x^p)^{-1} ); ( 0 \leq x &lt; \infty )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1 + \theta x^p</td>
</tr>
<tr>
<td>11. Cauchy</td>
<td>( \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{x-\theta}{\lambda} \right) )</td>
<td>( \frac{1}{\pi} )</td>
<td>( \frac{1}{2} )</td>
<td>1 tan^{-1} \left( \frac{x-\theta}{\lambda} \right)</td>
</tr>
</tbody>
</table>

(b) Characterization of \( F_4(x) = [ah(x)+b]^c \):

THEOREM 3.4.3 (Ali and Khan, 1994c)

Under the condition of Theorem 3.4.1

\[
E\left\{ g\left(X_{r:n_i}\right) \right\} - E\left\{ g\left(X_{r-1:n_i}\right) \right\} = \frac{n_i P_3}{(n_i - r + 1)} \left[ E\left\{ g\left(X_{r:n_i-1}\right) \right\} - E\left\{ g\left(X_{r-1:n_i-1}\right) \right\} \right] - \frac{1}{(n_i - r + 1)c_a} E\left\{ m\left(X_{r:n_i}\right) \right\}
\]

\[..........(3.4.7)\]

if and only if

\[
F(x) = -Q_3 + \frac{[ah(x)+b]^c}{P-Q}; \quad x \in (Q_1, P_1)
\]

\[..........(3.4.8)\]
THEOREM 3.4.4 (Ali and Khan, 1994c)

Under the condition of Theorem 3.4.1 the following statements are equivalent.

(i) \( F(x) = -Q_3 + \frac{[ah(x)+b]^c}{P-Q} \); \( x \in (Q_1, P_1) \) ............(3.4.9)

(ii) \( E\left[g\left(X_{r:n_1}\right)\right] = E\left[g\left(X_{r-1:n_{1-1}}\right)\right] + \frac{(P-Q)(n_1-r+1)}{n_1(n_1+1)ca} E\left[z\left(X_{r:n_1+1}\right)\right] \) ............(3.4.10)

(iii) \( E\left[g\left(X_{r:n_1}\right)\right] = E\left[g\left(X_{r-1:n_{1-1}}\right)\right] + \frac{(P-Q)}{n_1(n_1+1)ca} E\left[z\left(X_{r:n_1+1}\right)\right] \)

(iv) \( E\left[g\left(X_{r-1:n_{1-1}}\right)\right] = E\left[g\left(X_{r-1:n_{1-1}}\right)\right] + \frac{(P-Q)(r-1)}{n_1(n_1+1)ca} E\left[z\left(X_{r:n_1+1}\right)\right] \) ............(3.4.12)

For proper choice of \( a, b, c \) and \( h(x) \), various distributions can be obtained as given in the Table 3.2.
<table>
<thead>
<tr>
<th>Distribution Function</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>h(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td>( \lambda^{-p} )</td>
<td>0</td>
<td>1</td>
<td>( x^p )</td>
</tr>
<tr>
<td>( F^1(x) = \lambda^{-p} x^p ); ( 0 \leq x \leq \lambda )</td>
<td>( \lambda^{-1} )</td>
<td>0</td>
<td>( p )</td>
<td>( x )</td>
</tr>
<tr>
<td>2. Pareto function</td>
<td>( -\lambda^{-p} )</td>
<td>1</td>
<td>1</td>
<td>( x^{-p} )</td>
</tr>
<tr>
<td>( F^1(x) = 1 - \lambda^{-p} x^{-p} ); ( \lambda \leq x &lt; \infty )</td>
<td>( \lambda )</td>
<td>0</td>
<td>( p )</td>
<td>( e^{-\theta x^{-p}} )</td>
</tr>
<tr>
<td>3. Inverse Weibull</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( e^{-\theta x^{-p}} )</td>
</tr>
<tr>
<td>( F^1(x) = e^{-\theta x^{-p}} ); ( 0 \leq x &lt; \infty )</td>
<td>( \lambda )</td>
<td>0</td>
<td>( p )</td>
<td>( e^{-x^{-p}} )</td>
</tr>
<tr>
<td>4. Burr type III</td>
<td>( \theta )</td>
<td>1</td>
<td>-( \lambda )</td>
<td>( x^{-p} )</td>
</tr>
<tr>
<td>( F^1(x) = (1+\theta x^{-p})^{-\lambda} ); ( 0 \leq x &lt; \infty )</td>
<td>( 1 )</td>
<td>1</td>
<td>-( \lambda )</td>
<td>( \theta x^{-p} )</td>
</tr>
<tr>
<td>5. Rectangular</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( x-\beta )</td>
</tr>
<tr>
<td>( F^1(x) = \frac{x-\beta}{\lambda-\beta} ); ( \beta \leq x \leq \lambda )</td>
<td>( \lambda )</td>
<td>0</td>
<td>( p )</td>
<td>( \frac{x}{\lambda-\beta} )</td>
</tr>
<tr>
<td>6. Cauchy</td>
<td>( \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\theta}{\lambda}\right) )</td>
<td>( \frac{1}{\pi} )</td>
<td>( \frac{1}{2} )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( -\infty &lt; x &lt; \infty )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.1 INTRODUCTION:
Several recurrence relations and identities for product moments of order statistics from a distribution function (df) $F$ have been obtained by Joshi (1971), Khan et al (1983b) and Balakrishnan (1986). Govindrajulu (1975) characterized the exponential distribution using the relations among variances and co variances of exponential order statistics. See also Azlarov and Volodin (1986). Lin (1989) used product moments of two order statistics to characterize Power function, Exponential and Normal distributions. He also extended the result obtained by Govindrajulu (1966) regarding the characterizations of the normal and truncated normal distributions. So, for each integer $k \leq 1$, define the df

$$
\phi_k(x) = C_k^{-1} \int_{-\infty}^{x} \exp\left(-t^{2k}/(2k)\right)dt, \quad x \in (-\infty, \infty)
$$

where the constant

$$
C_k = \int_{-\infty}^{\infty} \exp\left(-t^{2k}/(2k)\right)dt
$$
He called $\phi_k$ as an extended normal df. To study the characterization of $\phi_k$ and its truncated form, he used two lemmas, which can also be seen in this chapter.

Ali and Khan (1994c) characterized two general class of distribution by expectation of function of one and two order statistics. Incidentally, some of the result obtained by Lin (1988), Kamps (1991) also follow from their result.

4.2 CHARACTERIZATION OF EXPONENTIAL DISTRIBUTION:

THEOREM 4.2.1: (Azlarov and Volodin, 1986 and Govindrajulu, 1975)

For $i = 0,1,2,3, \ldots$; $\sigma_{i+1,i+1:n} - \sigma_{i,i:n} = (\mu_{i+1:n} - \mu_{i:n})^2$; $n = i+1,i+2,\ldots$ if and only if $F(x) = 1 - \exp(-x/\theta)$, $x, \theta > 0$.

PROOF: The characterizing property can be rewritten as

$$\frac{(\mu^{(2)}_{i+1:n} - \mu^{(2)}_{i:n})}{(\mu_{i+1:n} - \mu_{i:n})} = 2 \mu_{i+1:n} \quad \text{........................(4.2.1)}$$

Also, integrating by parts, we obtain

$$\mu_{i+1:n} - \mu_{i:n} = c_{i,n-i-1:n} \int_{0}^{\infty} I(F;i,n-i)dx \quad \text{........................(4.2.2)}$$

Using (3.2.1) and (4.2.2) in (4.2.1) we have

$$\int_{0}^{\infty} x I(F;i,n-i)dx / \int_{0}^{\infty} I(F;i,n-i)dx = \int_{0}^{\infty} x I(F;i,n-i-1)dx / \int_{0}^{\infty} I(F;i,n-i-1)dx \quad \text{........................(4.2.3)}$$
Thus we have
\[ \int_0^\infty x I(F; i, n-i) \, dx = \theta \int_0^\infty x I(F; i, n-i-1) \, dF \] ................................(4.2.4)
and
\[ \int_0^\infty I(F; i, n-i) \, dx = \int_0^\infty I(F; i, n-i-1) \, dF \] ................................(4.2.5)
for some \( \theta > 0 \). In the \( H \)-notation (4.2.4) becomes
\[ \int_0^1 u^i (1-u)^{n-i-1} H(u) \left[ H'(u)(1-u) - \theta \right] \, du = 0 \]
Thus \( H'(u) = \theta (1-u)^{-1} \) for almost all \( u, 0 < u < 1 \).
Hence \( F(x) = 1 - \exp(-x/\theta) \).
Similarly (4.2.5) also leads to the same solution.

**THEOREM 4.2.2:** (Govindrajulu, 1975)

For \( i = 1, 2, 3, \ldots \); \( \sigma_{i, i:n} - \sigma_{i-1, i-1:n-1} = (\mu_{i:n} - \mu_{i-1:n-1})^2 \); \( n = i, i+1, \ldots \) if and only if \( F(x) = 1 - \exp(-x/\theta) \).

**PROOF: Case (i=1)**

The characterizing property becomes
\[ \mu_{1:n}^{(2)} = 2 \mu_{1:n}^2 \quad \text{or} \quad \mu_{1:n}^{(2)} / \mu_{1:n} = 2 \mu_{1:n} \] ................................(4.2.6)

Now performing integration by parts in each integral on the left hand side ratio in (4.2.6) we obtain
\[ \int_0^\infty x I(F; 0, n) \, dx / \int_0^\infty I(F; 0, n) \, dx = \int_0^\infty x I(F; 0, n-1) \, dF / \int_0^\infty I(F; 0, n-1) \, dF \] ................................(4.2.7)
Consequently, proceeding as we did in the proof of Theorem 4.2.1, we are led to solution \( F(x) = 1 - \exp(-x/\theta) \).
Case II \((i > 1)\)

One can rewrite the characterizing property as

\[
\mu_{i+1:n}^{(2)} - \mu_{i:n-1}^{(2)} = 2\mu_{i+1:n}^{(2)}(\mu_{i+1:n}^{(2)} - \mu_{i:n-1}^{(2)})
\]  \hspace{1cm} \text{(4.2.8)}

Applying the recurrence relation (3.2.5) to both sides of (4.2.8) we obtain,

\[
\mu_{i+1:n}^{(2)} - \mu_{i:n}^{(2)} = 2\mu_{i+1:n}^{(2)}(\mu_{i+1:n}^{(2)} - \mu_{i:n}^{(2)})
\]  \hspace{1cm} \text{(4.2.9)}

Now, the proof is completed after applying Theorem 4.2.1.

**THEOREM 4.2.3** (Govindrajulu, 1975)

For \(i = 1,2,\ldots; \sigma_{i,i:n}^{(2)} - \sigma_{i,i+1:n}^{(2)} = 0\); \(n = i,i+1,i+2\ldots\)

if and only if \(F(x) = 1-\exp(-x/\Theta)\).

**THEOREM 4.2.4** (Govindrajulu, 1975)

For \(i = 0,1,2,3\ldots; \sigma_{i,i:n}^{(2)} - (n-i)^{-1}\sum_{j=i+1}^{n} \sigma_{i,j:n}^{(2)} = 0\); \(n = i,i+1,i+2\ldots\)

if and only if \(F(x) = 1-\exp(-x/\Theta)\).

**THEOREM 4.2.5** (Govindrajulu, 1975)

For \(i = 0,1,2,\ldots\) and \(k = i,i+1,\ldots; \sigma_{i,k:n}^{(2)} = \sigma_{i,k+1:n}^{(2)}\); \(n = k,k+1\ldots\)

if and only if \(F(x) = 1-\exp(-x/\Theta); x,\Theta > 0\).

**THEOREM 4.2.6** (Govindrajulu, 1975)

Let \(\mu_{i:n} = \Theta\). For \(i = 1,2,\ldots; \sum_{j=1}^{n} \sigma_{i,j:n} = \Theta \mu_{i:n}\); \(n = i,i+1,\ldots\)
if and only if \( F(x) = 1 - \exp(-x/\theta); \ x, \theta > 0. \)

**THEOREM 4.2.7 (Lin, 1989)**

Let \( F \) be a df with \( E|X|^l < \infty \) for some \( l \geq 1 \) and \( \{n_i\}_{i=1}^{\infty} \) be the sequence of integers satisfying
\[
2 \leq n_1 < n_2 \ldots \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} 1/n_i = \infty
\]

Assume further that \( F^{-1} \) is absolutely continuous on \((0,1)\). Then, for any given constant \( \lambda > 0 \) and integers \( r \geq 1, j \geq 0, k \geq 1 \) satisfying \( j+k \leq 1 \)
\[
\mu_{r,r+1:n_i}^{(j,k)} - \mu_{r,r:n_i}^{(j,k)} = \frac{k\lambda}{n_i-r} \mu_{r,r+1:n_i}^{(j,k-1)} \quad \text{for all} \quad n_i \geq r+1
\]

if and only if \( F(x) = 1 - \exp(-x/\lambda) \) on \((0,\infty)\).

Proof of the above Theorems can be seen in the given References.

**4.3 CHARACTERIZATION OF UNIFORM, EXTENDED NORMAL AND PARETO DISTRIBUTIONS:**

**THEOREM 4.3.1: (Lin, 1989)**

Let \( F \) be a df with \( E|X|^l < \infty \) for some \( l \leq 1 \). Let the inverse function \( F^{-1} \) be positive on \((0,1)\) and \( F^{-1}(0^+) = 0 \). Assume further that \( \{n_i\}_{i=1}^{\infty} \) is a sequence of integers satisfying
Then for any given constant $\lambda > 0$ and non-negative integers $j, k$ satisfying $j+k+1 \leq 1$

$$\mu_{1,2:n_1}^{(j,k)} = \frac{n_i}{\lambda(j+1)} \mu_{1:n_1-1}^{(j+k+1)}$$

for all $i \geq 1$ if and only if $F(x) = x/\lambda$ on $(0, \lambda)$.

**THEOREM 4.3.2:** (Lin, 1989)

Let $F$ and $\{n_i\}_{i=1}^\infty$ be the same as in Theorem 4.3.1. Assume further that $F^{-1}$ is absolutely continuous on $(0,1)$. Then, for any given constant $\lambda > 0$ and integers $r \geq 1$, $j \geq 0$, $k \geq 1$ satisfying $j+k \leq 1$

$$\mu_{r,r+1:n_i}^{(j,k)} - \mu_{r,r:n_i}^{(j,k)} = \frac{k\lambda}{n_1+1} \mu_{r,r+1:n_1+1}^{(j,k-1)}$$

for all $n_1 \geq r+1$ if and only if $F(x) = x/\lambda$ on $(0, \lambda)$.

**THEOREM 4.3.3:** (Govindrajulu, 1966)

$F(x)$ is normal if and only if for $i = 1, 2, \ldots$; $\sum_{j=1}^n s_{i,j:n} = 1$; $n = i, i+1, \ldots$.

For proof of the Theorem 4.3.1, 4.3.2 and 4.3.3 one can see the given references.

**LEMMA 4.3.1:** (Lin, 1989)

Let $F$ be a df with the inverse function $F^{-1}$ differentiable on
Then for any given integer \( k \geq 1 \),
\[
(F^{-1})'(u) \int_{u}^{1} \left( F^{-1}(v) \right)^{2k-1} dv = 1 \quad \text{a.e. on } (0,1) \quad \ldots\ldots(4.3.1)
\]
if and only if \( F(x) = \left\{ \phi_k(x) - \phi_k(A) \right\} / \left\{ 1 - \phi_k(A) \right\} \) on \( (A, \infty) \)
\[
\ldots\ldots(4.3.2)
\]
for some extended real number \( A \geq -\infty \).

**THEOREM 4.3.4:** (Lin, 1989)

Let \( F \) be a df with \( \mathbb{E}X^{2k} < \infty \) for some integer \( k \geq 1 \). Assume further that \( \left\{ n_i \right\}_{i=1}^{\infty} \) is a sequence of integers satisfying
\[
2 \leq n_1 < n_2 < \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} 1/n_i = \infty
\]
Then,
\[
\mu_{n_i:n_i}^{(2k)} - \mu_{n_{i-1}:n_{i-1}}^{(1,2k-1)} = 1 \quad \text{for all } i \geq 1
\]
if and only if (4.3.2) holds for some \( A \geq -\infty \).

**PROOF:** Please see the given reference.

**LEMMA 4.3.8:** (Lin, 1989)

Let \( F \) be a df with \( \mathbb{E}X^{2k} < \infty \) for some integer \( k \geq 1 \), and \( \mathbb{E}X^{2k-j} = 0 \) for some positive odd integer \( j < 2k \). Then for all \( n \geq 2 \) and \( 1 \leq r \leq n \)

\[
\sum_{s=1}^{n} \mu_{r,s:n}^{(j,2k-j)} = \binom{n}{r} \int_{0}^{1} \left\{ \int_{u}^{1} \left( F^{-1}(v) \right)^{2k-j} dv \right\} u^{r-1}(1-u)^{n-r}d(F^{-1}(u))^j
\]
\ldots\ldots(4.3.3)
THEOREM 4.3.5: (Lin, 1989)

Let $F$ and $\left\{ n_i \right\}_{i=1}^{\infty}$ be the same as in Theorem 4.3.3. Assume further that $\sum_{i=1}^{\infty} n_i = 0$ and $F^{-1}$ is differentiable on $(0,1)$. Then for any given integer $r \geq 1$, 

\[ \sum_{s=1}^{n_i} \mu_{r,s;n_i}^{(1,2k-1)} = 1 \quad \text{for all } n_i \geq r+1 \]

if and only if $F(x) = \phi_k(x)$ on $(-\infty, \infty)$.

PROOF: It follows from (4.3.3) with $j = 1$ that (4.3.4) is equivalent to

\[ \int_0^1 (1-u)^{n_i-r-1} g(u) du = 0 \quad \text{for all } n_i \geq r+1 \]

where $g(u) = \left\{ (F^{-1})'(u) / \int_0^u (F^{-1}(v))^{2k-1} dv - 1 \right\} u^{r-1}(1-u), \; u \in (0,1)$

Suppose $F = \phi_k$; then, by Lemma 4.3.1, $g(u) = 0$ on $(0,1)$ and hence (4.3.5) holds.

Conversely, suppose (4.3.5) holds; then we want to prove $F = \phi_k$. Notice first that $g \in L(0,1)$ by the assumption $\sum_{i=1}^{\infty} n_i = 0$. Then employing the Muntz_Szasz Theorem to (4.3.5) yields $g(u) = 0$ a.e. on $(0,1)$ and hence (4.3.1) holds. Finally, Lemma 4.3.1 and the condition $\sum_{i=1}^{\infty} n_i = 0$ together complete the proof.

THEOREM 4.3.6: (Lin, 1989)

Let $F$ and $\left\{ n_i \right\}_{i=1}^{\infty}$ be the same as in Theorem 4.3.3. Assume further
that $F^{-1}(0^+)=0$ and $F^{-1}$ is differentiable on $(0,1)$. Then
\[
\sum_{s=1}^{n_i} \mu_{1,s}^{(1,2k-1)} = 1 \quad \text{for all } i \geq 1
\]
if and only if (4.3.2) holds with $A = 0$, namely,
\[
F(x) = 2\phi_k(x) - 1 \quad \text{on } (0,\infty).
\]

**PROOF:** Please see the given reference.

**THEOREM 4.3.7 :** (Khan, 1987)

Let $X$ be an absolutely continuous r.v. with df $F(x)$. Let $F(x) > 0$ for $x > 1$ with $F(1) = 0$ and $F(\infty) = 1$, then for $k > 0$, $1 \leq r < s \leq n$,
\[
1 - F(x) = x^{-p}, \quad p > 0, \quad x \geq 1
\]
if and only if \( \mu^{(-k,k)}_{r,s,n} = \mu^{(k)}_{s-r,n-r} = E(x^k_{s:n} | X_{r:n} = 1) \)

**PROOF:** The 'only if' part can be proved by noting that
\[
1-F_1(u) = P\left[ \frac{X_{s:n}}{X_{r:n}} > u \right] = \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{s-r-1} \binom{n-r}{j} \int_1^\infty (F(v))^{r-1} \left( F(uv) - F(v) \right)^j (1 - F(uv))^{n-r-j} dF(v)
\]
\[
1-F_2(u) = P(X_{s-r:n-r} > u) = \sum_{j=0}^{s-r-1} \binom{n-r}{j} (F(u))^j (1 - F(u))^{n-r-j}
\]

\[ \cdots \]

\[ 55 \]
\[ j = 0 \]

Hence \[ \mu_{r,s:n}^{(k)} = \mu_{s-r:n-r}^{(k)} \]

implies

\[
\sum_{j=0}^{s-r-1} \binom{n-r}{j} \int_1^{\infty} \int_1^{\infty} u^{k-1}(F(v))^{r-1} \left[ (F(uv) - F(v))^j (1 - F(uv))^{n-r-j} - (F(u))^j (1-F(u))^{n-r-j} (1-F(v))^{n-r} \right] dF(v) du = 0
\]

This gives (Galambos and Kotz, 1975)

\[
(1 - F(uv)) = (1 - F(u)) (1 - F(v))
\]

implying that \( 1 - F(u) = u^{-p} \), \( p > 0 \).

4.4 CHARACTERIZATION OF GENERAL CLASS OF DISTRIBUTION:

(a) Characterization of \( F_1(x) = 1 - [ah(x) + b]^c \)

**THEOREM 4.4.1**: (Ali and Khan, 1994c)

Let \( F(.) \) be the distribution function with \( E|X|^l < \infty \) for some \( l \geq 1 \).

Let the inverse function \( F^{-1}(.) \) be positive on \([0,1]\) and

\( F^{-1}(0^+) = q_1, F^{-1}(1) = p_1 \).

Assume further that \( \{n_i\}, i=1,2,3,\ldots \)

be a sequence of integers satisfying \( 2 \leq n_1 < n_2 < \ldots \)

and \( \sum_{i=1}^{\infty} \frac{1}{n_i} = \infty \).

Then for \( l \leq r < s \leq n_i \),

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\begin{align*}
E\left\{ g\left(x_{r:n_1}, x_{s:n_1}\right) \right\} - E\left\{ g\left(x_{r:n_1}, x_{s-1:n_1}\right) \right\} \\
= -\frac{n_1 p_2}{(n_1 - s + 1)} \left[ E\left\{ g\left(x_{r:n_1-1}, x_{s:n_1-1}\right) \right\} - E\left\{ g\left(x_{r:n_1-1}, x_{s-1:n_1-1}\right) \right\} \right] \\
- \frac{1}{(n_1 - s + 1) c a} E\left\{ m\left(x_{r:n_1}, x_{s:n_1}\right) \right\} \quad \cdots \quad \text{(4.4.1)}
\end{align*}

if and only if

\[ F(x) = Q_2 - \frac{[ah(x) + b]^c}{(P - Q)} ; \quad x \in (Q_1, P_1) \quad \cdots \quad \text{(4.4.2)} \]

where,

\[ m(x, y) = [ah(y) + b] \frac{g'(x, y)}{h'(y)} ; \quad g'(x, y) = \frac{\partial}{\partial y} g(x, y) ; \]

\[ h'(y) = \frac{\partial}{\partial y} h(y) ; \quad P_2 = \frac{1 - P}{P - Q} \text{ and } Q_2 = \frac{1 - Q}{P - Q} . \]

PROOF: Ali and Khan (1994b) have shown that for distribution (4.4.2), relation (4.4.1) holds and hence the "if" part.

To prove the "only if" part, we have in view of (1.3.26) and (1.3.27).

\[
\frac{C_{r,s:n_1}}{(n_1 - s + 1)} \int \int g\left( F^{-1}(u), F^{-1}(v) \right) u^{r-1} (v-u)^{s-r-1} (1-v)^{n_1 - s + 1} \left. du \, d(F^{-1}(v)) \right|_{0 \leq u \leq v \leq 1}
\]
\[
\begin{align*}
&= - \frac{n_i P_2 c_{r,s,n_i-1}}{(n_i-s+1)(n_i-s)} \int \int g\left(F^{-1}(u), F^{-1}(v)\right) u^{r-1} (v-u)^{s-r-1} (1-v)^{n_i-s} du \, d(F^{-1}(v)) \\
&\quad - \frac{c_{r,s,n_i}}{(n_i-s+1)c a} \int \int m\left(F^{-1}(u), F^{-1}(v)\right) u^{r-1} (v-u)^{s-r-1} (1-v)^{n_i-s} dudv
\end{align*}
\]

that is, \[
\int_0^1 (1-v)^{n_i-s} g^*(v) \, dv = 0
\]

Where
\[
g^*(v) = \left\{ \left(1-v+P_2\right)\left(F^{-1}(v)\right)' + \frac{ah(F^{-1}(v))+b}{cah'(F^{-1}(v))} \right\}
\]

\[
\cdot \int_0^v g\left(F^{-1}(u), F^{-1}(v)\right) u^{r-1} (v-u)^{s-r-1} du
\]

Under the given condition \{(1-v)^{n_i}\} is complete in L(0,1) (See Lin, 1989 and Kamps, 1991).

Since \[
\int_0^v g\left(F^{-1}(u), F^{-1}(v)\right) u^{r-1} (v-u)^{s-r-1} du \neq 0
\]

therefore, \(g^*(v) = 0\)

which implies,

\[
\frac{cah'(F^{-1}(v))\left(F^{-1}(v)\right)'}{ah'(F^{-1}(v)) + b} = - \frac{1}{(1-v+P_2)}
\]

that is

\[
\log_e[a h(F^{-1}(v)) + b] = \log_e[1-v+P_2] + k
\]
where $k$ is constant of integration which at $v=0$ gives $k = \log_e(P-Q)$.

Therefore
\[
[ah(F^{-1}(v))+b]^c = (1-v+P_z)(P-Q)
\]
or,
\[
v = \frac{1-Q}{P-Q} - \frac{[ah(F^{-1}(v))+b]^c}{(P-Q)}
\]
that is
\[
F(x) = Q_2 - \frac{[ah(x)+b]^c}{P-Q}; \quad x \in (Q_1, P_1)
\]
and hence the result.

**THEOREM 4.4.2**: (Ali and Khan, 1994c)

Under the condition of Theorem 4.4.1
\[
E\left\{g\left(x_{r:n_1}, x_{s:n_1}\right)\right\} - E\left\{g\left(x_{r:n_1}, x_{s-1:n_1}\right)\right\}
\]
\[
= -\frac{(P-Q)}{(n_1+1)ca} E\left\{z\left(x_{r:n_1}, x_{s:n_1+1}\right)\right\} \quad \text{(4.4.3)}
\]
if and only if
\[
F(x) = Q_2 - \frac{[ah(x)+b]^c}{(P-Q)}; \quad x \in (Q_1, P_1) \quad \text{(4.4.4)}
\]
where $z(x,y) = [ah(y)+b]^{1-c} \frac{g'(x,y)}{h'(y)}$, $Q_2 = \frac{1-Q}{P-Q}$,
\[
g'(x,y) = \frac{\partial}{\partial y} g(x,y) \quad \text{and} \quad h'(y) = \frac{\partial}{\partial y} h(y).
\]

**PROOF**: The "if" part follows from Ali and Khan (1994b).

To prove "only if" part, proceed on the same lines of Theorem 4.4.1, to get the required result.
Remark: The particular distribution of (4.4.2) are given in Table 3.1.

(b) Characterization of \( F(x) = \{ah(x)+b\}^c \)

THEOREM 4.4.3: (Ali and Khan, 1994c)

Under the condition of Theorem 4.4.1

\[
E\left\{ g\left(x_{r:n_i},x_{s:n_i}\right)\right\} - E\left\{ g\left(x_{r:n_i},x_{s-1:n_i}\right)\right\} 
\]

\[
= \frac{n_i P_3}{(n_i-s+1)} \left[ E\left\{ g\left(x_{r:n_i-1},x_{s:n_i-1}\right)\right\} - E\left\{ g\left(x_{r:n_i-1},x_{s-1:n_i-1}\right)\right\} \right]
\]

\[
- \frac{1}{(n_i-s+1)ca} E\left\{ m\left(x_{r:n_i},x_{s:n_i}\right)\right\} \quad \quad \text{.....(4.4.5)}
\]

if and only if

\[
F(x) = -Q_3 + \frac{[ah(x)+b]^c}{P-Q} \quad \quad x \in (Q_1,P_1) \quad \quad \text{.....(4.4.6)}
\]

where

\[
Q_3 = \frac{Q}{P-Q} \quad \quad \text{and} \quad \quad P_3 = \frac{P}{P-Q} 
\]

PROOF: For the distribution function (4.4.6), it is easy to show that (4.4.5) exits (Ali and Khan, 1994b).

To prove the "only if" part, we get after simplification

\[
\int_0^1 (1-v)^{n_i-s} g^*(v) \, dv = 0
\]

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where
\[ g^*(v) = \left\{ \left(1-v-P_1\right)(F^{-1}(v))' + \frac{ah(F^{-1}(v)) + b}{ah'(F^{-1}(v))} \right\}' \int_0^v g'(F^{-1}(u),F^{-1}(v)) u^{r-1} (v-u)^{s-r-1} \, du \]

Under the given conditions \( \{(1-v)^{n_1}\} \) is complete, implies that
\[ g^*(v) = 0 \]

which yields the solution
\[ F(x) = -Q_3 + \frac{[ah(x)+b]^c}{P-Q}; \ x \in (Q_1,P_1) \]

**THEOREM 4.4.4:** (Ali and Khan, 1994c)

Under the condition of Theorem 4.4.1
\[ E\left\{g\left(x_{r:n_1},x_{s:n_1}\right)\right\} - E\left\{g\left(x_{r:n_1},x_{s-1:n_1}\right)\right\} = \frac{P-Q}{(n_1+1)c_{a}} E\left\{x_{r:n_1+1},x_{s:n_1+1}\right\} \quad \ldots \ldots (4.4.7) \]

if and only if
\[ F(x) = -Q_3 + \frac{[ah(x)+b]^c}{P-Q}; \ x \in (Q_1,P_1) \quad \ldots \ldots (4.4.8) \]

**PROOF:** For distribution (4.4.8) relation (4.4.7) was obtained by Ali and Khan(1994b).

Remaining part of the proof follows like Theorem 4.4.3

**Remark:** The particular distribution of (4.4.6) are given in Table 3.2.
5.1 INTRODUCTION:
Characterization through conditional moments have been studied by many authors. Shanbhag (1970) characterized the exponential and geometric distribution in terms of conditional expectation. His result for exponential distribution is further generalized by Hamdan (1972). Ferguson (1967) characterized a continuous distribution function $F$ for single order gap. Beg and Kirmani (1978) showed that the conditional variance of $X_{r+1:n}$ given $X_{r:n} = x$ does not depend on $x$ if and only if $X$ has exponential distribution. Khan and Beg (1987) extended the result of Beg and Kirmani and showed that variance of $X_{r+1:n}^p$ given $X_{r:n} = x$ does not depend on $x$ if and only if $X$ has Weibull distribution for $p > 0$. Khan and Khan (1987) obtained recurrence relations for single and product moments of order statistics from doubly truncated Burr distribution (Burr type XII) utilizing the results developed by Khan et al. (1983 a,b) and then used some results to characterize the Burr distribution. Khan and Abu_Salih (1988) characterized the Weibull and inverse Weibull
distribution for double order gap. Khan and Ali (1987) used the conditional moments of order statistics with higher order gap to characterize Weibull, Exponential, Raleigh, Burr, Log_logistic, Pareto and Power function distributions. Khan et al. (1988) obtained the moments of order statistics from a symmetrically truncated Logistic distribution and used the result to characterize the Logistic distribution.

Talwalkar (1977) gave characterizations for the general class of absolutely continuous distributions in terms of conditional expectations. On the basis of certain generalizations of Talwalker's result, Ouyang (1983) characterized a general class of continuous distribution including the Power, Weibull, Beta distributions and also two forms of Burr's distribution. Khan and Abu_Salih (1989) extended the results of Talwalkar (1977) and Ouyang (1983) and obtained several new characterizing results.

5.2 CHARACTERIZATION THROUGH SINGLE ORDER GAP:

THEOREM 5.2.1 : (Shanbhag, 1970)

The non-negative valued $X$ is distributed according to a exponential distribution if and only if

$$E\{X \mid X > y\} = y + E(X) \text{ for every } y \quad ....(5.2.1)$$
PROOF: (5.2.1) is equivalent to
\[ \int_0^\infty z \, dB(z) = (x + \mu) \, \bar{B}(x) \text{ for every } x \quad \ldots \quad (5.2.2) \]
where \( B(z) \) denotes the distribution function of \( X \), \( \bar{B}(x) \) denotes \( 1 - B(x) \) and \( \mu \) denotes \( E(X) \). Clearly, this implies that (5.2.1) is equivalent to
\[ \int_0^\infty (z-x) \, dB(z) = \mu \, \bar{B}(x) \text{ for every } x \quad \ldots \quad (5.2.3) \]
Since the left hand side of (5.2.3) is continuous, it follows that its right hand side is continuous. Now, it can be easily seen that left hand side is differentiable. This implies the differentiability of the right hand side. The differentiation yields.
\[ \frac{d}{dx} \bar{B}(x) = - \frac{1}{\mu} \bar{B}(x) \]
It immediately follows that the equation has the solution
\[ \bar{B}(x) = \exp(-x/\mu), \]
because \( \mu \) denotes \( E(X) \). Clearly, this satisfies (5.2.3) and thus we have the Theorem.

COROLLARY 5.2.1: (Shanbhag, 1970)
The non-negative valued \( X \) has a modified exponential distribution with non-zero probability mass at 0, i.e., a distribution with the distribution function of the type
\[ G(z) = 1 - K \exp(-\lambda z) \quad (z > 0) \]
\[ (0 < k < 1, \lambda > 0) \]

if and only if

\[ E\left[ X \mid X > y \right] = \frac{E(X)}{1 - P(X=0)} \vee y \]
\[ P(X=0) > 0 \]

\[ \therefore \quad E\left[ X \mid X > y \right] = y + E(X \mid X > 0) \vee y \]
\[ P(X=0) > 0 \]

\[ \therefore \quad (5.2.4) \]

The corollary follows at once because the theorem establishes that \( X \) has a modified distribution.

if and only if

\[ E\left[ X \mid X > y \right] = y + E(X \mid X > 0) \vee y \]
\[ P(X=0) > 0 \]

\[ \therefore \quad (5.2.5) \]

THEOREM 5.2.2 : (Ferguson, 1967)

If \( Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n} \) are the order statistics of a sample of size \( n \) from a distribution with continuous distribution function \( F \), and if for some positive integer \( m \) less than \( n \),

\[ E[ Y_{m:n} \mid Y_{m+1:n} = y ] = ay - b \quad \text{a.s. for some numbers } a \text{ and } b, \]

then the distribution function is, except for change of location and scale,

(i) \( F(x) = e^x \), for \( x < 0 \), if \( a = 1 \)
(ii) \( F(x) = x^\theta \), for \( 0 < x < 1 \), if \( 0 < a < 1 \)
(iii) \( F(x) = (-x)^\theta \), for \( x < -1 \), if \( a > 1 \)

where \( \theta = a/(m(1-a)) \).
PROOF: Since the expectation of the maximum of a sample of size \( m \) from the distribution of \( F \) truncated at \( y \) is,

\[
E\left[ Y_{m:n} \mid Y_{m+1:n} = y \right] = \begin{cases} 
\int_{\infty}^{y} x dF(x)^{m} / F(y)^{m} & \text{if } F(y) \neq 0 \\
y & \text{if } F(y) = 0 
\end{cases}
\] ....(5.2.6)

from (5.2.6) we have

\[
\int_{\infty}^{y} x dF(x)^{m} = (ay - b) F(y)^{m} \text{ a.s. } F \] ....(5.2.7)

Find numbers \( c \) and \( d \), possibly infinite, s.t. \( \{ x : 0 < F(x) < 1 \} \) is the interval \((c, d)\). There does not exist a subinterval \((c_{1}, d_{1})\), \( c < c_{1} < d_{1} < d \) over which \( F \) is constant since the left side of (5.2.7) is constant in such an interval and the right side is increasing, while both sides are continuous, so that they could not possibly be equal at the next points of increase of \( F \). Thus (5.2.7) is valid for all \( y \in (c, d) \). The left hand side may be written as

\[
\int_{\infty}^{y} x dF(x)^{m} = y F(y)^{m} - \int_{\infty}^{y} F(x)^{m} dx \] ....(5.2.8)

The existence of integral on the left implies the existence of integral on the right.

Let \( H(y) = \int_{\infty}^{y} F(x)^{m} dx \); then \( H'(y) \) exists for all \( y \) and is equal to \( F(y)^{m} \). Equation (5.2.7) may be rewritten as

\[
\frac{d}{dy} \log H(y) = ((1-a)y + b)^{-1} \text{ for all } y \in (c, d) \] ....(5.2.9)
we may solve this differential equation separately for the three cases mentioned in the theorem.

**Case(i)**: \( a = 1 \). Since \( \log H(y) \) is increasing in \((c,d)\), \( b \) must be positive. By integrating (5.2.9) we find \( H(y) = K e^{y/b} \), and by differentiating, we find \( F(y) = K b^{-1} e^{y/b} \). Clearly \( c = -\infty \) and \( d < \infty \); and since \( F(d) = 1 \), \( F(y) = e^{(y-d)/(mb)} \) for \( y < d \). This is the distribution under (i) with change of location and scale.

**Case(ii)**: \( 0 < a < 1 \). Integrating (5.2.9), we find that \( H(y) = K (1-a)y + b)^{1/(1-a)} \) for \( y \in (c,d) \). In this case, \( c \) and \( d \) are finite and the restrictions \( F(c) = 0 \) and \( F(d) = 1 \) give

\[
F(y) = \left[ \frac{y-c}{d-c} \right]^{\theta} \quad \text{for} \quad c < y < d ,
\]

where \( \theta = a/((1-a)m) \) and \( c = -b/(1-a) \).

**Case(iii)**: \( a > 1 \). Integrating (5.2.9), we find that \( H(y) = K (b-(a-1)y)^{-1/(a-1)} \) for \( y \in (c,d) \). In this case \( c = -\infty \) and \( d < \infty \). The restriction \( F(d) = 1 \) gives

\[
F(y) = \left[ \frac{y-y_d}{y} \right]^{\theta} \quad \text{for} \quad y < d
\]

where \( \theta = -a/((a-1)m) < 0 \), and \( y_d = b/(a-1) > d \).

This completes the proof.

**THEOREM 5.2.3** (Beg and Kirmani, 1978)

Let \( X \) be a random variable having distribution function \( F(x) \) with \( F(0) = 0 \) and \( E(X^2) < \infty \). If \( F(x) < 1 \) for all \( x < \infty \) and \( f(x) = dF(x)/dx > 0 \)
for all $x > 0$, then

$$F(x) = 1 - \exp(-x/\theta), \ x \geq 0 \quad \ldots (5.2.10)$$

for some $\theta > 0$ if and only if, for some positive integer $r$ less than $n$,

$$\text{Var}[X_{r+1:n} | X_{r:n} = x] = c, \text{ for all } x > 0 \quad \ldots (5.2.11)$$

where $c$ is a positive constant.

**PROOF:** Please see the given references.

**THEOREM 5.2.4:** (Weibull) (Khan and Beg, 1987)

Let $X$ be a continuous random variable with d.f. $F$ such that $F(0) = 0$ and $F$ has second derivative on $(0, \infty)$ and its first derivative is non-vanishing on $(0, \infty)$ so that in $F(x) < 1$ for all $x \geq 0$. Let $0 < p < \infty$ and $F$ has moment of order $2p$. If for some integer $0 < r < n$

$$\text{Var}[X_{r+1:n}^p | X_{r:n}^p = x] = c \quad \text{(constant)}$$

then $F(x) = 1 - \exp(-\theta x^p)$ for $x \geq 0$ where $\theta > 0$ is given by $\theta^{-2} = c(n-r)^2$; and conversely.

**PROOF:** It may be seen that conditional p.d.f. of $X_{s:n}$ given $X_{r:s} = x \ (r < s)$ is

$$\frac{(n-r)!}{(s-r-1)!(n-s)!} \left\{ \frac{F(y) - F(x)}{F(x)} \right\}^{s-r-1} \left\{ 1 - F(y) \right\}^{n-s} \frac{f(y)}{1 - F(x)}^{n-r}, \ y \geq x \quad \ldots (5.2.12)$$

which is just the unconditional distribution of the $(s-r)$th order
statistics in a sample of \((n-r)\) drawn from \(\frac{f(y)}{1-F(x)}\), \(y \geq x\), that is from the parent distribution truncated on the left at \(x\). Therefore,

\[
E\left[X_{r+1:n} | X_{r:n} = x\right] = \frac{(n-r)}{1-F(x)} \int_{x}^{\infty} y^k \left\{1-F(y)\right\}^{n-r-1} f(y) \, dy \quad \ldots (5.2.13)
\]

Thus for the Weibull distribution

\[F(x) = 1 - e^{-\theta x^p}, \quad x \geq 0, \ \theta > 0\]

we have

\[
E\left[X_{r+1:n} | X_{r:n} = x\right] = \sum_{m=0}^{k} \frac{k!}{m!} x^m \theta^m a^p(k-m) \quad \ldots \ldots (5.2.14)
\]

where \(a^{-p}=(n-r)\theta\), obtained by Khan et al (1983a). Thus,

\[
\text{Var}[X_{r+1:n} | X_{r:n} = x] = \theta^2 = \frac{1}{(n-r)^2 \theta^2} = c
\]

This proves the necessary part.

For sufficiency part, we have

\[
\frac{(n-r)}{n-r} \int_{x}^{\infty} y^{2p} \left\{1-F(y)\right\}^{n-r-1} f(y) \, dy - \frac{(n-r)^2}{\left(1-F(x)\right)^2} \int_{x}^{\infty} y^p \left\{1-F(y)\right\}^{n-r-1} f(y) \, dy = c
\]

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\[ i.e., b \left\{ 1 - F(x) \right\} b \int_x^\infty y^p \left\{ 1 - F(y) \right\}^{b-1} f(y) \, dy - \left[ b \int_x^\infty y^p \left\{ 1 - F(y) \right\}^{b-1} f(y) \, dy \right]^2 \]
\[ = c \left\{ 1 - F(x) \right\}^{2b} \quad \text{....(5.2.15)} \]

Differentiating (5.2.15) once w.r.t. \( x \), cancelling out \( f(x) \left\{ 1 - F(x) \right\}^{b-1} \) on both sides and differentiating both sides of the resulting relation again w.r.t. \( x \), we get

\[ b p \int_x^\infty y^p \left\{ 1 - F(y) \right\}^{b-1} f(y) \, dy = c b x^{p-1} \left\{ 1 - F(x) \right\}^{b-1} f(x) + p x^p \left\{ 1 - F(x) \right\}^b \quad \text{....(5.2.16)} \]

Differentiating (5.2.16) again w.r.t. \( x \), we have,

\[ cb x^{1-p} \left\{ 1 - F(x) \right\}^{b-1} \left[ f'(x) - (b-1) \frac{f^2(x)}{1-F(x)} - \frac{(p-1)f(x)}{x} \right. \]
\[ \left. + \frac{1}{cb} x^{2(p-1)} \left\{ 1 - F(x) \right\}^p \right] = 0 \]

giving us

\[ \frac{f'(x)}{1-F(x)} - (b-1) \left[ \frac{f(x)}{1-F(x)} \right]^2 - \frac{p-1}{x} \left[ \frac{f(x)}{1-F(x)} \right] + b \theta^2 x^{2(p-1)} = 0 \]
\[ \text{....(5.2.17)} \]

The solution of (5.2.17) in view of (Lemma 2.1, Khan and Beg, 1987) is \( F(x) = 1 - e^{-\theta x^p} \).

and hence the Theorem.
For $p=1$, Theorem 5.2.4 reduces to the result of Beg and Kirmani (1978) (i.e. Theorem 5.2.3).

Also, from (5.2.12), it can be seen that

$$E\left[ x_{n:n}^{pk} | x_{n-1:n} = x \right] = E\left[ x^{pk} | x > x \right]$$

and

$$Var\left[ x_{n:n}^{pk} | x_{n-1:n} = x \right] = Var\left[ x^{pk} | x > x \right]$$

Thus, we have the result

$$Var\left[ x^{pk} | x > x \right] = 1/\theta^2$$

if and only if $F(x) = 1 - e^{-\theta x}$.

**THEOREM 5.2.5**: (Burr) : (Khan and Khan, 1987)

Let $X$ be a random variable having df $F(x)$ with $F(0)=0$ and $E(X^k)<\infty$. If $F(x)<1$ for all $x<\alpha$ and $f(x) = \frac{\partial F(x)}{\partial x} > 0$ for all $x>0$ then,

$$F(x) = 1 - (1+\theta x^p)^{-m} \quad \ldots (5.2.18)$$

for $\theta>0$, $x\geq 0$, $p>0$ and $m>0$, if and only if for $r<n$,

$$E\left[ x_{r+1:n}^{pk} | x_{r:n} = x \right] = \frac{a}{a-1} \left[ x^p + \frac{1}{a\theta} \right] \quad \ldots (5.2.19)$$

where $a = m(n-r)$ is independent of $x$.

**PROOF**: The necessary part follows from (1.5.5). To prove the sufficiency part, we have from (1.5.1), (1.5.5) and (5.2.19),

$$\int_0^\infty (n-r) \cdot \left\{ y^p \{ 1-F(y) \} \right\}^{n-r-1} f(y) dy = \frac{a}{a-1} \left[ x^p + \frac{1}{a\theta} \right] \left[ 1-F(x) \right]^{n-r}$$
differentiating both sides w.r.t. \( x \) and solving we get

\[
\frac{f(x)}{1-F(x)} = \frac{m \theta x^{p-1}}{1+\theta x^p}
\] ....(5.2.20)

The solution of (5.2.20) is

\[
F(x) = 1 - (1+\theta x^p)^{-m}
\]

and hence the Theorem.

5.3 CHARACTERIZATION THROUGH DOUBLE ORDER GAP:

THEOREM 5.3.1: (Weibull) : (Khan and Abu_Salih, 1988)

Let \( X \) be a continuous random variable with d.f. \( F(x) \) on \((0,\infty)\) such that \( F(0)=0, F(x)<1 \) for \( x>0 \) and \( E(X^p)<\infty \). Let \( F(x) \) be twice differentiable with \( F'(x)\neq 0 \). Then

\[
F(x) = 1-e^{-\theta x^p}, \quad \theta, p > 0, \quad x \geq 0
\]

if and only if for integer \( r, 1 \leq r \leq n-2, \)

\[
E\left[ X_{r+2:n}^p \mid X_{r:n} = x \right] = x^{p+c} \quad \text{......(5.3.1)}
\]

where \( c = \frac{1}{\theta} \left( \frac{1}{b} + \frac{1}{b-1} \right) \) and \( b = (n-r) \).

PROOF: Necessary part: We have from (1.5.7) and (1.5.10)

\[
E\left[ X_{r+2:n}^p \mid X_{r:n} = x \right] = x^{p+c} + \theta \left( \frac{1}{b} + \frac{1}{b-1} \right) \int_0^\infty y^{p-1} \left\{ \left( \frac{1-F(y)}{1-F(x)} \right)^b \left[ \frac{F(y)-F(x)}{1-F(x)} \right] \left( \frac{1-F(y)}{1-F(x)} \right)^{b-1} \right\} dy
\]
Now for a given $1-F(x) = e^{-\theta x^p}$, it is easy to see that

\[
(b-1)p \int_{x}^{\infty} y^{p-1} \left( \frac{1-F(y)}{1-F(x)} \right)^b \, dy = \frac{(b-1)}{b \theta} 
\]

\[
E\left[x^{p} \mid x_{r:n} = x\right] = x^p + \frac{1}{\theta} \left( \frac{1}{b-1} - \frac{b-1}{b} \right) 
\]

\[
= x^p + \frac{1}{\theta} \left( \frac{1}{b-1} + \frac{1}{b} \right) 
\]

**Sufficiency part:** We have from (5.3.1) and (5.3.2)

\[
x^p + bp \int_{x}^{\infty} y^{p-1} \left( \frac{1-F(y)}{1-F(x)} \right)^b \, dy - (b-1)p \int_{x}^{\infty} y^{p-1} \left( \frac{1-F(y)}{1-F(x)} \right)^b \, dy = x^p + c 
\]

this gives

\[
bp(1-F(x)) \int_{x}^{\infty} y^{p-1}(1-F(y))^{b-1} \, dy - (b-1)p \int_{x}^{\infty} y^{p-1}(1-F(y))^{b} \, dy = c(1-F(x))^{b} 
\]

Differentiating both sides w.r.t. $x$ and rearranging, we get

\[
\int_{x}^{\infty} y^{p-1}(1-F(y))^{b-1} \, dy = \frac{c}{p} (1-F(x))^{b-1} - \frac{x^{p-1}}{b} \frac{(1-F(x))^{b}}{F'(x)} 
\]

Differentiating again and solving we get

\[
\frac{F''(x)}{1-F(x)} + b \left[ \frac{F'(x)}{1-F(x)} \right] - \frac{b^* - 1}{p \theta} x^{1-p} \left[ \frac{F'(x)}{1-F(x)} \right]^{3} - \frac{p-1}{x} \frac{F'(x)}{1-F(x)} = 0 
\]

where $b^* = 2b = 2(n-r)$
Let \( \frac{F'(x)}{1-F(x)} = u \)

then \( \frac{F''(x)}{1-F(x)} = \frac{du}{dx} - u^2 \)

this reduces (5.3.3) to

\[
\frac{du}{dx} - \frac{b^* - 1}{p \theta} \frac{u^3}{x^{p-1}} + (b^* - 1)u^2 - \frac{p-1}{x} u = 0 \quad \text{.....(5.3.4)}
\]

The solution of the differential equation

\[
\frac{dy}{dx} = f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \quad \text{.....(5.3.5)}
\]

for \( f_2(x) \neq 0 \) is given by Kamke (1943), we get

Hence using Kamke's result (1943), we get

\[
u = \frac{F'(x)}{1-F(x)} = p \ \theta \ x^{p-1}
\]

which gives,

\[ F(x) = 1-e^{-\theta x^p} \]

This completes the proof.

**THEOREM 5.4.2 : (Inverse Weibull) : (Khan and Abu_Salih, 1988)**

Let \( X \) be a continuous r.v. with d.f. on \((0,\infty)\) such that \( F(x) > 0 \) for \( 0 < x < \infty \) and \( E(X^{-p}) < \infty \). Let \( F(x) \) be twice differentiable with \( F'(x) \neq 0 \), then

\[ F(x) = e^{-\theta x^{-p}}, \ \theta, p > 0, \ x \geq 0 \quad \text{.....(5.3.6)}
\]

if and only if for integer \( r, 1 \leq r \leq n-2, \)

\[ E\left[x^{-p} \mid X_{r+1:n} = x\right] = x^{-p} + c \quad \text{.....(5.3.7)}
\]

where \( c = \frac{1}{\theta}\left(\frac{1}{r} + \frac{1}{r-1}\right) \)
PROOF: For $F(x)$ given in (5.3.6), necessary part follows in view of (1.5.9) and (1.5.11). To prove the sufficiency part, note that from (1.5.9), (1.5.11) and (5.3.7), we have

$$\int_0^x y^{-p-1} \left( \frac{F(y)}{F(x)} \right)^{r+1} dy - \int_0^x y^{-p-1} \left( \frac{F(y)}{F(x)} \right)^{r+1} dy = c$$

Proceeding on the lines of Theorem (5.3.1), we get the differential equation

$$\frac{F''(x)}{F(x)} + \frac{(2r+1)}{p^\theta} x^{p+1} \left( \frac{F'(x)}{F(x)} \right)^3 - 2(r+1) \left( \frac{F'(x)}{F(x)} \right)^2 + \frac{(p+1)}{x} \frac{F'(x)}{F(x)} = 0$$

If we set $\frac{F'(x)}{F(x)} = u$, then (5.3.8) reduces to

$$\frac{du}{dx} + \frac{(2r+1)}{p^\theta} x^{p+1} u^3 - (2r+1)u^2 + \frac{(p+1)}{x} u = 0$$

which is again of the form (5.3.5). Hence Kamke's (1943) result will give

$$u = \frac{F'(x)}{F(x)} = p \theta x^{-p-1}$$

The solution of which is

$$F(x) = e^{\theta x-p}.$$

5.4 CHARACTERIZATION THROUGH HIGHER ORDER GAP:

Lemma 5.4.1: (Khan and Ali, 1987)

Let $X$ be a continuous r.v. having df $F(x)$, then $F(x)$ is unique if

$$E[X_{r+i:n}^{k} | X_{r:n} = x], \quad 1 \leq i \leq n-r, \quad 1 \leq r \leq n$$

exists.
Proof: From (1.5.2), we have
\[ E\left[X_{r+i:n}^k \mid X_{r:n} = x\right] = E\left[X_{i:n-r}^k \mid X_{i:n-r} \geq x\right] \]
Thus, if it exists, then from the result of Kotlarski (1972), it uniquely determines the df of \( X_{i:n-r} \), which in turn uniquely determines \( F(x) \).

Lemma 5.4.2 : (Khan and Ali, 1987)
Let \( X \) be a continuous r.v. having df \( F(x) \). Then \( F(x) \) is unique if
\[ E\left[X_{r+i:n}^k \mid X_{r+i:n} = x\right], \quad 1 \leq i \leq n-r; \quad 1 \leq r \leq n \text{ exists.} \]
Proof: The proof follows in view of (1.5.4) and the result of Kotlarski (1972).

Theorem 5.4.1 : (Weibull) : (Khan and Ali, 1987)
Let \( X \) be a continuous r.v. having df \( F(x) \) with \( F(0)=0 \) and \( E(X^k) < \infty, \quad k > 0 \). If \( F(x) < 1 \) for all \( x < \infty \), then \( F(x) = 1 - e^{-\theta x^p}, \quad x \geq 0, \quad p > 0 \)
if and only if for \( r < n \), \( 1 \leq i \leq n-r \),
\[ E\left[X_{r+i:n}^p \mid X_{r:n} = x\right] = x^p + \frac{1}{\theta} \sum_{1=0}^{i-1} \frac{1}{n-r-1} \]
\[ \text{.....(5.4.1)} \]
Proof: Khan et al. (1983a) have shown that for doubly truncated Weibull distribution
\[ F(x) = Q_2 - \frac{e^{-\theta x^p}}{P-Q}, \quad -\ln(1-Q) \leq x^p \leq -\ln(1-P), \]
\[ \mu_{r:n}^{(k)} = Q_2 \mu_{r-1:n-1}^{(k)} - P_2 \mu_{r:n-1}^{(k)} + \frac{k}{np\theta} \mu_{r:n}^{(k-p)}, \]
where \( Q_2 = \frac{1-Q}{P-Q} \) and \( P_2 = \frac{1-P}{P-Q} \)
In case of left truncation \((P=1)\), \(Q_2=1\), \(P_2=0\).

Thus, \(\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + \frac{k}{np\theta} \mu_{r:n}^{(k-p)}\),
\[\text{i.e., } \mu_{1:n-r}^{(p)} = \mu_{1-1:n-r-1}^{(p)} + \frac{1}{(n-r)\theta},\]
and the necessary part is proved by noting that
\[\mu_{0:n-r-i}^{(p)} = \frac{Q_{i}^{p}}{1} = x^p\]
Sufficiency part follows from Lemma 5.4.1.

For \(i=1\), the Theorem was proved by Khan and Beg (1987). Also, if we put \(p=1\), \(i=1\), we get the result obtained by Ferguson (1967). At \(p = 2\), it characterizes Raleigh distribution.

**THEOREM 5.4.2 : (Burr distribution) : (Khan and Ali, 1987)**

Let \(X\) be a r.v. having continuous df \(F(x)\) with \(F(0)=0\) and \(E(x^k) < \infty, k > 0\). If \(F(x) < 1\) for all \(x < \omega\), then

\[F(x) = 1 - (1 + \theta x^p)^{-m}, \theta > 0, x \geq 0, p > 0, m > 0,\]
if and only if for \(r < n\), \(1 \leq i \leq n-r\),
\[
E[x_{r+i:n}^p | x_{r:n} = x] = \begin{cases}
\frac{x^p m(n-r)}{m(n-r)-1} + \frac{1}{\theta_{m(n-r)-1}}, & \text{for } i=1 \\
x^p \prod_{i=0}^{i-1} \frac{m(n-r-1)}{m(n-r-1)-1} + \frac{1}{\theta_{m(n-r)-1}} \\
\left[1 + \sum_{j=0}^{i-2} \prod_{i=0}^{j} \frac{m(n-r-1)}{m(n-r-1)-1}\right], & \text{for } i \geq 2
\end{cases}
\]

PROOF: Khan and Khan (1987) have shown that for doubly truncated Burr distribution with d.f.

\[
F(x) = Q_2 - \frac{1}{P - Q} \frac{1}{(1+\theta x^p)^m}, \quad Q_1 \leq x \leq P_1,
\]

\[
\left[1 - \frac{k}{mnp}\right]^{\mu_r:n} = Q_2^{\mu_r-1:n-1} - P_2^{\mu_r-1:n-1} + \frac{k}{mnp\theta^{\mu_r:n}},
\]

for \(k \neq mnp, 1 \leq r \leq n,\) where

\[
\Theta_1^p = [(1-Q)^{-1/m} - 1], \quad \Theta_1^p = [(1-p)^{-1/m} - 1], \quad Q_2 = \frac{1-Q}{P-Q}, \quad P_2 = \frac{1-P}{P-Q}
\]

Therefore, for the left truncation only, \(P_2=0\) and \(Q_2=1.\)

This gives,

\[
\mu_{i:n-1}^{(p)} = \frac{m(n-r)}{m(n-r)-1} \mu_{i-1:n-1}^{(p)} + \frac{1}{\theta_{m(n-r)-1}}
\]

noting that \(\mu_{0:n}^{(p)} = Q_1^{p} = x^p,\) we prove the necessary part.

Sufficiency part follows from Lemma (5.4.1).

If we put \(i=1,\) we get the result of Khan and Khan (1987). For \(m=1,\) it characterizes the log-logistic distribution.
Theorem 5.4.3: (Pareto distribution) (Khan and Ali, 1987)

Let X be a continuous r.v. having df F(x) < 1 for all x < ∞ with F(a) = 0 and E(X^k) < ∞, k > 0. Then

\[ F(x) = 1 - a^p x^{-p}, a > 0, p > 0, x \geq a \]

if and only if for r < n, 1 ≤ i ≤ n-r,

\[ E\left[X_{r+i:n}^k | X_{r:n} = x\right] = x^{k-1} \prod_{i=0}^{r} \frac{p(n-r-i)}{p(n-r-1)-k} \]

(5.4.3)

Proof: Khan et al. (1983) have shown that for doubly truncated Pareto distribution

\[ F(x) = -Q_2 - \frac{a^p x^{-p}}{P - Q}, a(1-Q)^{-1/p} \leq x \leq a(1-P)^{-1/p}, \]

with P = \( \frac{p-1}{p-Q} \), Q = \( \frac{Q-1}{P-Q} \).

For left truncation, P = 0, Q = -1. Therefore,

\[ (k) \mu_{i:n-r}^{(k)} = (k) \mu_{i-1:n-r-1}^{(k)} \frac{(n-r)p}{(n-r)p-k} \]

Again noting that \( (k) \mu_{0:n}^{(k)} = Q_p = \frac{p}{k} \), we prove the necessary part.

The sufficiency part follows from Lemma 5.4.1.

For i=1, it reduces to the result of Khan and Khan (1987).

Theorem 5.4.4: (Power function) (Khan and Ali, 1987)

Let X be a continuous r.v. having df F(x) > 0 for 0 < x < a with F(a) = 1 and E(X^k) < ∞, k > 0. Then

\[ F(x) = a^{-p} x^p, 0 \leq x \leq a, a > 0, p > 0 \]
if and only if for \( 1 \leq r < n, 1 \leq i \leq n-r, \)

\[
E\left[X_{r:n}^i \mid X_{r+1:n} = x\right] = x^p \prod_{i=0}^{i-1} \frac{p(r+1)}{p(r+1)+k} \quad \ldots \ldots (5.4.4)
\]

PROOF: For doubly truncated power function distribution

\[
F(x) = \frac{a^p x^p}{P - Q}, \quad aQ^{1/p} \leq x \leq aP^{1/p}, \quad a > 0, \quad p > 0.
\]

Khan et al. (1983a) have shown that

\[
\mu_{r:n}^{(k)} = \left[P_2 \mu_{r:n-1}^{(k)} - Q_2 \mu_{r-1:n-1}^{(k)}\right] \frac{np}{np+k}
\]

where \( P_2 = \frac{P}{P - Q} \) and \( Q_2 = \frac{Q}{P - Q} \).

Thus, in case of right truncation \((Q = 0)\), we have \( P_2 = 1 \) and \( Q_2 = 0 \), giving us,

\[
\mu_{r:n}^{(k)} = \mu_{r:n-1}^{(k)} \frac{np}{np+k}
\]

which in view of (1.5.4) and (1.3.11) proves (5.4.4).

The sufficiency part follows from Lemma 5.4.2.

For \( i = 1 \), the Theorem was proved by Khan and Khan (1986).

THEOREM 5.4.5 : (Khan et al., 1988)

Let \( X \) be a continuous random variable having a symmetric df \( F(x) \) and pdf \( f(x) \) with \( E(X) < \infty \). If \( 0 < F(x) < 1 \) for \(-\infty < x < \infty\), then

\[
F(z) = \left(1 + e^{-z}\right)^{-1}
\]

if and only if for \( 1 \leq r \leq n-3 \)

\[
E\left[X_{r+1:n} \mid X_{r:n} = x, X_{r+3} = z\right] = \frac{2z F(z) [1-F(z)]}{[2F(z)-1]^2} - \frac{1}{[2F(z)-1]} \quad \ldots \ldots (5.4.5)
\]
PROOF: Note that as a convention (Khan et al., 1983b)

\[ x_{n:n-1} = P_1, \quad n \geq 1 \]
\[ x_{0:n} = Q_1, \quad n \geq 0 \]

and therefore,

\[ \mu_{1:0} = P_1 \quad \text{and} \quad \mu_{0:0} = Q_1 \]

and for symmetric distribution

\[ \mu_{1:1} = 0 \quad \text{and} \quad \mu_{2:2} = -\mu_{1:2} \]

Thus the necessary part follows from (1.3.29) and (1.5.15) at \( r=2, \quad n=2 \) and \( k=1 \).

To prove the sufficiency part, we have from (1.5.16) and (5.4.5)

\[
\int_{-z}^{z} y[F(z)-F(y)]f(y)dy = z[F(z)][1-F(z)] - \frac{[2F(z)-1]}{2}
\]

Differentiating both sides w.r.t. \( z \) and solving we get

\[ F(z)[1-F(z)] - f(z) = 0 \]

The solution which leads to

\[ F(z) = (1 + e^{-z})^{-1} \]

and hence the proof.

5.5 CHARACTERIZATION OF GENERAL CLASS OF DISTRIBUTION:

THEOREM (5.5.1) : (Hamdan, 1972)

Let \( h \) be a strictly increasing differentiable function from the interval \([a,b]\) onto \([0,\infty)\) and let \( c \) be a positive constant
satisfying \( a < c < b \). The random variable \( X \) has its cumulative distribution given by

\[
P(X \leq x) = \begin{cases} 
0, & x < a \\
1 - e^{-h(x)/h(c)}, & a < x < b \\
1, & x \geq b 
\end{cases} \quad \text{.....(5.5.1)}
\]

if and only if

\[
\forall y \in [a, b), \mathbb{E}[h(X) | X > y] = h(y) + h(c) \quad \text{.....(5.5.2)}
\]

If \( [a, b) \) is taken to be \( [0, \infty) \) and \( h(x) = x^d, x \in [0, \infty), \ d>0, \) then Theorem 5.5.1 gives a characterization of the Weibull distribution.

\[
P(X \leq x) = \begin{cases} 
0, & x < 0 \\
1 - e^{-x^d/c^d}, & x \geq 0 
\end{cases} \quad \text{.....(5.5.3)}
\]

If we put additionally \( d=1 \), the Weibull distribution reduces to the exponential distribution.

**THEOREM 5.5.2**: (Ouyang, 1983)

Let \( X \) be a random variable with continuous distribution function \( F(x) \) and \( F(x) < 1 \) for all \( x \in [\alpha, \beta) \). Then

\[
F(x) = \begin{cases} 
1 - [u(x) + d]^c, & \text{for } x \in [\alpha, \beta) \\
0, & \text{for } x < \alpha \\
1, & \text{for } x \geq \beta 
\end{cases} \quad \text{.....(5.5.4)}
\]

where \( c \in \{-1, 0\} \) and \( d \) are constants, \( u(x) \) is a real valued continuous function defined on \( [\alpha, \beta) \), possessing continuous
derivative on \((a, \beta]\) with \(u(\alpha) = 1 - d\) and limit \(u(x) = -d\), \(x \to \beta\)

if and only if

\[
E \left[ u(x) | X > y \right] = \frac{1}{c+1} \left[ cu(y) - d \right], \text{ for all } y \in [\alpha, \beta]
\]

\[\ldots (5.5.5)\]

The interval \([\alpha, \beta]\) is open on the left, whenever \(\alpha = -\infty\).

PROOF: The necessary of this condition can be verified directly. To prove the sufficiency, we note that

\[
E \left[ u(x) | X > y \right] = \frac{\int_{y}^{\beta} u(x) dF(x)}{1 - F(y)} = \frac{E \left[ u(x) \right] + \int_{\alpha}^{y} u(x) d[1 - F(x)]}{1 - F(y)}
\]

Hence, formula (5.5.5) may be put in the form

\[
[1 - F(y)][cu(y) - d] = (c+1) \left\{ E \left[ u(x) \right] + \int_{\alpha}^{y} u(x) d[1 - F(x)] \right\}
\]

\[
= (c+1) \left\{ E \left[ u(x) \right] + u(y)[1 - F(y)] - u(\alpha)
\right. \\
\left. - \int_{\alpha}^{y} u'(x) [1 - F(x)] dx \right\}
\]

or equivalently,

\[
- [u(y)+d][1-F(y)] = (c+1) \left\{ E \left[ u(x) \right] - u(\alpha) - \int_{\alpha}^{y} u'(x)[1-F(x)] dx \right\}
\]

\[\ldots (5.5.6)\]

Since by assumption \(F(x)\) and \(u'(x)\) are continuous, the R.H.S. of (5.5.6) is differentiable. Consequently, so is the L.H.S. But \(u'(y)\) exists by assumption and thus the existence of \(F'(y)\) follows. We can now differentiate (5.5.6) w.r.t. \(y\) and get
- \left[ u(y) + d \right] [1 - F(y)]' - u'(y) [1 - F(y)] = -(c+1)u'(y) [1 - F(y)]

or equivalently,

\[
\frac{[1 - F(y)]'}{1 - F(y)} = \frac{cu'(y)}{u(y) + d} \quad \ldots (5.5.7)
\]

Integrate (5.5.7) from \(a\) to \(x\), then the assumption of continuity of \(F(x)\) yields.

\[
\ln[1 - F(x)] - \ln[1 - F(a)] = c \left\{ \ln[u(x) + d] - \ln[u(a) + d] \right\} \quad \ldots (5.5.8)
\]

Since \(u(a) = 1 - a\) and \(F(a) = 0\), (5.5.8) can be rewritten as

\[
F(x) = 1 - \left[ u(x) + d \right]^c
\]

This completes the proof.

**Corollary 5.5.1** Let \(X\) be a random variable with continuous distribution function \(F(x)\) and \(F(x) < 1\) for all \(x \in (0,1)\). If

\[
E [ -x^a | X > y ] = - \frac{1}{2} \left[ y^a + 1 \right] \quad \text{for } a > 0 \text{ and } y \in [0,1),
\]

then \(F(x) = x^a\) for \(x \in [0,1)\).

In particular, if \(a = 1\), then \(F(x) = x\) for \(x \in [0,1)\).

**Corollary 5.5.2** Let \(X\) be a random variable with continuous distribution function \(F(x)\) and \(F(x) < 1\) for all \(x \in [0,\infty)\). If

\[
E [ e^{-bx^a} | X > y ] = \frac{1}{2} \ e^{-by^a} \quad \text{for } a > 0, \ b > 0 \text{ and } y \in [0,\infty),
\]

then \(F(x) = 1 - e^{-bx^a}\) for \(x \in [0,\infty)\).

When \(a = 1\), \(F(x) = 1 - e^{-bx}\) for \(x \in [0,\infty)\).

**Corollary 5.5.3** Let \(X\) be a random variable with continuous distribution function \(F(x)\) and \(F(x) < 1\) for all \(x \in [1,\infty)\). If

\[
E \left[ x^{-a} | X > y \right] = \frac{1}{2} \ y^{-a} \quad \text{for } a > 0 \text{ and } y \in [1,\infty),
\]

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then \( F(x) = 1 - x^{-a} \) for \( x \in [1, \infty) \).

**Corollary 5.5.4:** Let \( X \) be a random variable with continuous distribution function \( F(x) \) and \( F(x) < 1 \) for all \( x \in [0,1) \). If
\[
E[-X|X > y] = \frac{-1}{b+1} [by + 1] \quad \text{for} \quad b > 0 \quad \text{and} \quad y \in [0,1),
\]
then \( F(x) = 1 - (1-x)^b \) for \( x \in [0,1) \).

When \( b=1 \), \( F(x) = x \) for \( x \in [0,1) \).

**Corollary 5.5.5:** Let \( X \) be a random variable with continuous distribution function \( F(x) \) and \( F(x) < 1 \) for all \( x \in (-\infty, 0) \). If
\[
E[-(1+e^{-x})^{-a}|X > y] = -\frac{1}{2}[(1+e^{-y})^{-a}+1] \quad \text{for} \quad a > 0 \quad \text{and} \quad y \in (-\infty, 0),
\]
then \( F(x) = (1+e^{-x})^{-a} \) for \( x \in (-\infty, 0) \).

**Theorem 5.5.3:** (Ouyang, 1983)

Let \( X \) be a random variable with continuous distribution function \( F(x) \) and \( F(x) > 0 \) for all \( x \in (\alpha, \beta] \). Then
\[
F(x) = \begin{cases} 
[d-u(x)]^c, & \text{for} \ x \in (\alpha, \beta] \\
0, & \text{for} \ x \leq \alpha \\
1, & \text{for} \ x > \beta
\end{cases} \quad \text{(5.5.9)}
\]

where \( c \in (-1,0) \) and \( d \) are constants, \( u(x) \) is a real valued function continuously differentiable on \( (\alpha, \beta] \) with \( u(\beta) = d-1 \) and
\[
\lim_{x \to \alpha^+} u(x) = d, \quad \text{if and only if}
\]
\[
E[u(X)|X < y] = \frac{1}{c+1} [cu(y)+d], \quad \text{for} \ y \in (\alpha, \beta]. \quad \text{(5.5.10)}
\]

The interval \( (\alpha, \beta] \) is open on the right, whenever \( \beta = +\infty \). Indeed, in
this case,

\[ F(x) = \begin{cases} \left[ \frac{d - u(x)}{d} \right]^c, & \text{for } x \in (\alpha, \infty) \\ 0, & \text{otherwise} \end{cases} \] ....(5.5.11)

where \( d \neq 0 \) and \( \lim_{x \to +\infty} u(x) = 0 \).

PROOF: The necessity of the condition can be easily verified. Proof of sufficiency which is similar to Theorem 5.5.2.

The condition (5.5.10) implies that,

\[ F(y)[cu(y)+d] = (c+1)\left\{ E[u(x)] - \int_{\beta}^{y} u(x)dF(x) \right\} \]

\[ = (c+1)\left\{ E[u(x)] - u(\beta) + u(y)F(y) + \int_{\beta}^{y} u'(x)F(x)dx \right\} \]

or equivalently,

\[ [d - u(y)]F(y) = (c+1)\left\{ E[u(X)] - u(\beta) + \int_{\beta}^{y} u'(x)F(x)dx \right\} \]

....(5.5.12)

By assumption, we can differentiate (5.5.12) w.r.t. \( y \), and get

after some calculations.

\[ \frac{F'(y)}{F(y)} = -\frac{c}{d - u(y)}u'(y) \] ....(5.5.13)

Integrate (5.5.13) from \( x \) to \( \beta \), we obtain

\[ F(x) = [d - u(x)]^c, \]

which is formula (5.5.9).

When \( \beta = +\infty \), integrate (5.5.13) from \( x \) to \( \infty \) and we get

\[ F(x) = \left[ \frac{d - u(x)}{d} \right]^c, \]

which is the formula (5.5.11). Hence the proof.
Corollary 5.5.6: Let $X$ be a random variable with continuous distribution function $F(x)$ and $F(x) > 0$ for all $x \in (0,1]$. If
\[ E [-x^a | X \leq y] = -\frac{1}{2} y^a \text{ for } a > 0 \text{ and } y \in (0,1], \]
then $F(x) = x^a$ for $x \in (0,1]$.
When $a=1$, $F(x) = x$ for $x \in (0,1]$.

Corollary 5.5.7: Let $X$ be a random variable with continuous distribution function $F(x)$ and $F(x) > 0$ for all $x \in (0,\omega)$. If
\[ E [e^{-bx^a} | X \leq y] = \frac{1}{2} [e^{-by^a} + 1] \text{ for } a > 0, b > 0 \text{ and } y \in (0,\omega), \]
then $F(x) = 1 - e^{-bx^a}$ for $x \in (0,\omega)$.
When $a=1$, $F(x) = 1 - e^{-bx}$ for $x \in (0,\omega)$.

Corollary 5.5.8: Let $X$ be a random variable with continuous distribution function $F(x)$ and $F(x) > 0$ for all $x \in (1,\omega)$. If
\[ E [x^{-a} | X \leq y] = \frac{1}{2} [y^{-a} + 1] \text{ for } a > 0 \text{ and } y \in (1,\omega), \]
then $F(x) = 1 - x^{-a}$ for $x \in (1,\omega)$.

Corollary 5.5.9: Let $X$ be a random variable with continuous distribution function $F(x)$ and $F(x) > 0$ for all $x \in (0,1]$. If
\[ E [(1-x)^b | X \leq y] = \frac{1}{2} [(1-y)^b + 1] \text{ for } b > 0 \text{ and } y \in (0,1], \]
then $F(x) = 1 - (1-x)^b$ for $x \in (0,1]$.
When $b=1$, $F(x) = x$ for $x \in (0,1]$.

Corollary 5.5.10: Let $X$ be a random variable with continuous distribution function $F(x)$ and $F(x) > 0$ for all $x \in (0,\omega)$. If
\[ E [(1+x^a)^{-b} | X \leq y] = \frac{1}{2} [(1+y^a)^{-b} + 1] \text{ for } a > 0, b > 0 \text{ and } y \in (0,\omega), \]
then \( F(x) = 1 - (1+x^a)^{-b} \) for \( x \in (0, \infty) \).

When \( a=1 \), \( F(x) = 1-(1+x)^{-b} \) for \( x \in (0, \infty) \).

i.e. Beta distribution of the second kind with parameters \( a=1 \) and \( b>0 \).

**THEOREM 5.5.4**: (Khan and Abu_Salih, 1989)

Let \( X \) be an absolutely continuous random variable (r.v.) with d.f. \( F(x) \) and p.d.f. \( f(x) \).

Suppose \( F(x)<1 \) for all \( x \in (\alpha, \beta) \), \( F(\alpha)=0 \) and \( F(\beta) = 1 \).

Then \( F(x) = 1 - [ah(x) + b]^c \) for \( x \in (\alpha, \beta) \).

if and only if for \( r < n \),

\[
E\left[h(X_{r+1:n}) | X_{r:n} = x\right] = \frac{ac(n-r)h(x) - b}{a[(n-r)c + 1]}, \quad \ldots(5.5.14)
\]

where \( h() \) is a monotonic, continuous and differentiable function on \( (\alpha, \beta) \), \( a>0 \), \( (n-r)c+1 \neq 0 \).

**PROOF**: Note that

\[
1 - F(x) = [ah(x)+b]^c = -\frac{ah(x)+b}{ach'(x)} f(x) \quad \ldots(5.5.15)
\]

As we know that the conditional p.d.f. of \( X_{s:n} \) given \( X_{r:n} = x \), \( 1 \leq r < s \leq n \), is

\[
\frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)}{[1-F(x)]^{n-r}}, \quad x \leq y \quad \ldots(5.5.16)
\]

Now, in view of (5.5.16),

\[
E\left[h(X_{r+1:n}) | X_{r:n} = x\right] = \frac{(n-r)}{[1-F(x)]^{n-r}} \int_x^\beta h(y)[1-F(y)]^{n-r-1}f(y)dy
\]

Integrating by parts and noting the (5.5.15), it is easy to prove
the necessary part. To prove the sufficiency part, we have

$$\frac{\beta}{(n-r)} \int \frac{h(y)[1-F(y)]^{n-r-1}}{x} f(y)dy = \frac{[ac(n-r)h(x)-b]}{a[c(n-r)+1]} [1-F(x)]^{n-r}$$

Differentiating both sides w.r.t. $x$ and rearranging we get

$$-\frac{f(x)}{1-F(x)} = \frac{ach'(x)}{ah(x)+b}$$

which gives $1-F(x) = [ah(x)+b]^c$.

Remark 5.5.4 a: At $r=n-1$

$$a = \frac{f(k)-1}{(f(k)-1)h(\alpha)+g(k)}, \quad b = \frac{g(k)}{(f(k)-1)h(\alpha)+g(k)}$$

and

$$c = \frac{f(k)}{1-f(k)},$$

Theorem 5.5.4 reduces to the result of Talwalker (1977).

Remark 5.5.4 b: If we put $r=n-1$, $a=1$, $b=d$, $h(x)=u(x)$ in the Theorem 5.5.4, we get the result of Ouyang (1983).

Remark 5.5.4 c: Power function distribution was also characterized by Khan and Khan (1986) at $r=n-1$. At $a=1$, $p=1$, power function reduces to Burr type I (uniform) distribution.

Remark 5.5.4 d: Burr type XII was characterized by Khan and Khan (1987). At $m=1$, it is log_logistic distribution. Whereas at $k=1$, Burr type II distribution is logistic distribution.

Lemma 5.5.1: (Khan and Abu_Salih, 1989)

Let $X$ be a absolutely continuous r.v. with d.f. $F(x)$ and p.d.f. $f(x)$. Suppose $F(x)>0$ for all $x \in (\alpha, \beta)$, $F(\alpha)=0$ and $F(\beta)=1$. Then

$$F(x) = [ah(x)+b]^c$$
if and only if for $r < n$, 
\[
E \left[ h(X_{n-r+1:n}) | X_{n-r+1:n} = x \right] = \frac{ac(n-r)h(x)+b}{a[(n-r)c+1]}
\]
where $h()$ is a monotonic, continuous and differentiable function on $(\alpha, \beta)$, $a \neq 0$, $(n-r)c+1 \neq 0$.

**PROOF**: The proof follows from the fact that the conditional distribution of $X_{r+1:n}$ given $X_{r:n}$ from $F()$ is the same as the conditional distribution of $X_{n-r+1:n}$ given $X_{n-r+1:n}$ from $1-F()$.

**Remark 5.5.1 a**: If we put in Lemma 5.5.1, 
\[
a = \frac{(1-f(k))}{(1-f(k))h(\beta)-g(k)}, \quad b = -\frac{g(k)}{(1-f(k))h(\beta)-g(k)}
\]
and 
\[
c = \frac{f(k)}{1-f(k)}, \quad r = n-1,
\]
we get Theorem 2 of Talwalker (1977).

**Remark 5.5.1 b**: Put $a = -1$, $b = d$, $r = n-1$, $h(x) = u(x)$ in Lemma 5.5.1 to get Theorem 5.5.3 (i.e. Theorem 2 of Ouyang, 1983).

**THEOREM 5.5.5**: (Khan and Abu_Salih, 1989)

Let $X$ be an absolutely continuous r.v. with d.f. $F(x)$ and p.d.f. $f(x)$. Suppose $F(x)<1$ for all $x \in (\alpha, \beta)$ and $F(\alpha)=0$. Then 
\[
F(x) = 1 - \exp(-ah(x) + b)
\]
if and only if for $r < n$, 
\[
E \left[ h(X_{r+1:n}) | X_{r:n} = x \right] = h(x) + \frac{1}{a(n-r)}, \quad a \neq 0.
\]
where $h()$ is a monotonic, continuous and differentiable function on $(\alpha, \beta)$.
PROOF: Note that

\[ 1 - F(x) = \exp(-ah(x)+b) = \frac{f(x)}{ah'(x)} \]

The rest of the proof is straightforward.

Remark 5.5.5: If we put \( a = \frac{1}{h(\theta)} \), \( b = 0 \) and \( r = n-1 \), we get the result obtained by Hamdan (1972).

Lemma 5.5.2: (Khan and Abu_Salih, 1989)

Let \( X \) be a absolutely continuous r.v. with d.f. \( F(x) \) and p.d.f. \( f(x) \). Suppose \( F(x) > 0 \) for all \( x \in (\alpha, \beta) \) with \( F(\beta) = 1 \). Then

\[ F(x) = \exp(-ah(x)+b) \]

if and only if for \( r < n \),

\[ E \left[ h(X_{n-r:n}) \mid X_{n-r+1:n} = x \right] = h(x) + \frac{1}{a(n-r)}, \; a \neq 0. \]

where \( h() \) is a monotonic, continuous and differentiable function on \( (\alpha, \beta) \).

PROOF: Refer Lemma 5.5.1 and Theorem 5.5.5.
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