CHOQUET-DENY TYPE FUNCTIONAL EQUATIONS
AND CHARACTERIZATIONS OF DISTRIBUTIONS

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Dedicated to My Family
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Preface

Rao and Shanbhag book “Choquet-Deny Type Functional Equations with Applications to Stochastic Models” opened a new vista in functional equations and Paved ways in solving complicated characterization problems, simplifying earlier results and unifying them. We, therefore, in this Dissertation have tried to compile the available results which have used this Choquet-Deny type functional equations in solving characterizations of distributions problems.

The dissertation is based on four Chapters:

Chapter I is introductory in nature, in which we have included concepts, definitions etc., which may be needed in subsequent chapters to grasp the ideas therein.

Chapter II embodies results on characterization of distributions through order statistics.

Chapter III some results on characterizations of distributions based on record statistics through functional equations are given.

Last chapter contain characterization results based on linearity of regressions of generalized order statistics using choquet-Deny type functional equation.

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CHAPTER 1

PRELIMINARIES

1. Order Statistics
If the random variable $X_1, X_2, \cdots, X_n$ are arranged in ascending order of magnitude as $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \cdots \leq X_{i:n} \leq \cdots \leq X_{n:n}$,

Then $X_{i:n}$ or $X_{(i)}$ is called the $i^{th}$ order statistic of a sample of size $n$. The terms $X_{1:n} = \min( X_1, X_2, \cdots, X_n )$ and $X_{n:n} = \max( X_1, X_2, \cdots, X_n )$ are called extremes.

The subject of order statistic deals with the properties and application of these ordered random variables and of function involving them [David, 1981]. It is different from rank order statistics in which the order of value of observation rather than its magnitude is considered.

2. Distribution of order statistics
Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ from a continuous probability density function pdf $f(x)$ and cumulative distribution function df $F(x)$. Then the pdf of $X_{r:n}$ $1 \leq r \leq n$, the $r^{th}$ order statistic, is given by

$$f_{r:n}(x) = C_{r:n}[F(x)]^r[1-F(x)]^{n-r}f(x), \quad -\infty < x < \infty \quad (2.1)$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} = [B(r, n-r+1)]^{-1}, \quad \text{(2.2)}$$
and $df$

$$F_{r:n}(x) = P(X_{r:n} \leq x) = \sum_{i=r}^{n} \binom{n}{i} [F(x)]^{i} [1 - F(x)]^{n-i},$$

(2.3)

$$= C_{r:n} \int_{0}^{F(x)} t^{r-1} (1-t)^{n-r} dt,$$

(2.4)

where equation (2.4) is incomplete beta function. For $X$ continuous, equation (2.1) can be obtained from equation (2.4) by differentiating $w.r.t. x$

In particular:

$$F_{1:n}(x) = 1 - [1 - F(x)]^{n},$$

(2.5)

$$F_{n:n}(x) = [F(x)]^{n},$$

(2.6)

The joint pdf of $X_{r:n}$ and $X_{s:n}$ ($1 \leq r < s \leq n$) is given by

$$f_{r,s:n}(x, y) = C_{r,s:n}[F(x)]^{r} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y),$$

(2.7)

where:

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} = \frac{1}{B(r,s-r,n-s+1)},$$

(2.8)

$$F_{r,s:n}(x, y) = P(X_{r:n} \leq x, X_{s:n} \leq y)$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^{j-i} [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j}$$

(2.9)

The conditional pdf of $X_{s:n}$ given $X_{r:n} = x$ is for $x < y$

$$f_{X_{s:n}|X_{r:n}}(y|x) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{[F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y)}{[1 - F(x)]^{n-r}},$$

(2.10)

$$x < y$$
This is the pdf of $X_{s-r:n-r}$ truncated to the left at $x$.

The conditional distribution of $X_{r:n}$ given $X_{s:n}$

$$f_{X_{r:n} | X_{s:n}}(x|y) = \frac{(s-1)!}{(s-r-1)!(r-1)!} \frac{[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}}{[F(y)]^{s-1}} f(x),$$

$$= \frac{(s-1)!}{(s-r-1)!(r-1)!} \left( \frac{F(x)}{F(y)} \right)^{r-1} \left( 1 - \frac{F(x)}{F(y)} \right)^{s-r-1} \frac{f(x)}{F(y)}, \quad x < y \quad (2.11)$$

which is the pdf of $X_{r:s-1}$ truncated to the right at $y$.

3. Record values and record time (Cramer, 2002)

Record values have been introduced by Chandler (1952) in order to model data of extreme weather conditions. Considering a sequence $(X_j)_{j \in \mathbb{N}}$ of iid random variables with distribution function $F$, record values are defined as a model for successive extremes. Let $L(1) = 1$ and

$$L(n + 1) = \min \{ j > L(n) : X_j > X_{L(n)} \}, \quad n \in \mathbb{N}$$

be the record time of the sequence $(X_j)_{j \in \mathbb{N}}$. Then the random variables $X_{L(n)} , n \in \mathbb{N}$ are called upper record values.

Some times the second or third largest values are of special interest, e.g., in a top-$k$ list. Then, a model of $k$-th record values can be used where $k$ is some positive integer. These random variables have been introduced by Dziubdziela and Kopocinski (1976). They are defined via $k$-th record times $L_k(1) = 1$ and

$$L_k(n + 1) = \min \{ j \in \mathbb{N} : X_{j : j + k - 1} > X_{L_k(n); L_k(n) + k - 1} \}, \quad n \in \mathbb{N}$$

The random variables $X_{L_k(n); L_k(n) + k - 1}, n \in \mathbb{N}$ are called $k$-th record values.

We may be using $R_n$ for $X_{L(n)}$ as the $n$-th record.
4. The Exact Distribution of Record Values

Many properties of the record value can be expressed in terms of the function \( R(x) \), where \( R(x) = -\ln \bar{F}(x) \), \( \bar{F}(x) = 1 - F(x) \). Here "\ln" is used for the natural logarithm.

If we define \( F_n(x) \) as the distribution function of \( R_n \) for \( n \geq 1 \), then we have

\[
F_1(x) = P[R_1 \leq x] = F(x) \quad (4.1)
\]

\[
F_2(x) = P[R_2 \leq x] = \int_{-\infty}^{x} \int_{-\infty}^{y} (F(u))^{i-1} dF(u) dF(y)
\]

\[
= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dF(u)}{1 - F(u)} dF(y)
\]

\[
= \int_{-\infty}^{x} R(y) dF(y) \quad (4.2)
\]

If \( F(x) \) has a density \( f(x) \), then the pdf of \( R_2 \) is

\[
f_2(x) = R(x) f(x) \quad (4.3)
\]

obtained by differentiating (4.2)

The distribution function \( F_3(x) = P(R_3 \leq x) \)

\[
= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=1}^{\infty} [F(u)]^i R(u) dF(u) dF(y)
\]

\[
= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{R(u)}{1 - F(u)} dF(u) dF(y)
\]

\[
= \int_{-\infty}^{x} (R(u))^2 dF(u) \quad (4.4)
\]
The pdf $f_3(x)$ of $R_3$ is

$$f_3(x) = \frac{(R(u))^2}{2!} f(x), \quad -\infty < x < \infty \quad (4.5)$$

Similarly the distribution function $F_n(x)$ of $R_n$ is

$$F_n(x) = P(R_n \leq x)$$

$$= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty \quad (4.6)$$

and the pdf $f_n(x)$ of $R_n$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty \quad (4.7)$$

The joint pdf $f(x_1, x_2, \ldots, x_n)$ of the $n$ record values $R_1, R_2, \ldots, R_n$ is given by

$$f(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \ldots r(x_n), \quad -\infty < x_1 < x_2 < \ldots < x_n < \infty \quad (4.8)$$

where

$$r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{1-F(x)}, \quad 0 < F(x) < 1$$

is known as hazard rate.

The joint pdf of $R_i$ and $R_j$ is

$$f(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \left[ \frac{R(x_j) - R(x_i)}{(j-i-1)!} \right]^{j-i-1} f(x_j), \quad -\infty < x_i < x_j < \infty \quad (4.9)$$

In particular at $i = 1, j = n$, we have

$$f(x_1, x_n) = r(x_1) \left[ \frac{R(x_n) - R(x_1)}{(n-2)!} \right]^{n-2} f(x_n), \quad -\infty < x_1 < x_n < \infty.$$
The conditional distribution of $R_j$ given $R_i = x_i$ is

$$f(R_j | R_i = x_i) = \frac{\left[R(x_j) - R(x_i)\right]^{j-i-1}}{(j-i-1)!} \cdot \frac{f(x_j)}{1 - F(x_i)}, \quad -\infty < x_i < x_j < \infty \quad (4.10)$$

For $j = i + 1$

$$f(R_{i+1} | R_i = x_i) = \frac{f(x_{i+1})}{1 - F(x_i)}, \quad -\infty < x_i < x_{i+1} < \infty \quad (4.11)$$

and

$$f(R_j | R_{i+1} = x_{i+1}) = \frac{i \left(R(x_i)\right)^{j-1} f(x_i)}{\left[R(x_{i+1})\right]^j \left[1 - F(x_i)\right]^{j-1}}, \quad -\infty < x_i < x_{i+1} < \infty \quad (4.12)$$

5. Generalized order statistics (Kamps, 1995)

Let $n \in \mathbb{N}, k \geq 1, m_1, m_2, \ldots, m_{n-1} \in \mathbb{R}, M_r = \sum_{j=r}^{n-1} m_j, 1 \leq r \leq n-1,$ be the parameters, such that $\gamma_r = k + n - r + M_r \geq 1 \quad \forall \ r \in \{1, 2, \ldots, n-1\}$

and let $\bar{m} = (m_1, m_2, \ldots, m_{n-1})$, if $n \geq 2$ ($\bar{m} \in \mathbb{R}$ if $n = 1$)

If the random variable $U(r, n, \bar{m}, k), \ r = 1, 2, \ldots, n.$ possess a joint density function of the form

$$f_U(1, n, \bar{m}, k), U(2, n, \bar{m}, k), \ldots, U(n, n, \bar{m}, k)(u_1, u_2, \ldots, u_n)$$

$$= k \left(\prod_{j=1}^{n} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - u_i)^{m_i} \right) \left(1 - u_n\right)^{k-1} \quad (5.1)$$

on the cone $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq 1$ of $\mathbb{R}^n$, then they are called uniform generalized order statistics.
Case I: When 
\[ m_1 = m_2 = \cdots = m_n = m, \text{ say} \]
The random variables are denoted by \( U(r,n,m,k), \quad r = 1,2,\ldots,n. \)

Let \( F \) be a distribution function, then the random variables 
\[ X(r,n,m,k) = F^{-1}(U(r,n,m,k)), \tag{5.2} \]
are called generalized order statistics.

The joint density function of the generalized order statistics 
\( X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k) \)
is given by 
\[ f_X(x_1,n,m,k),\ldots,X(x_n,n,m,k) = \begin{vmatrix} \prod_{i=1}^{r-1} (1 - F(x_i))^m_i f(x_i) \end{vmatrix} (1 - F(x_n))f(x_n) \tag{5.3} \]
on the cone \( F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1) \)

The pdf of the \( r^{th} \) generalized order statistic based on 
an absolutely continuous df \( F \) given by 
\[ f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!} (1 - F(x))^{k+n-r+M_r-1} f(x) g_m^{-1}(F(x)) \tag{5.4} \]
and the joint pdf of the generalized order statistics \( X(r,n,m,k) \) and 
\( X(s,n,m,k), \) where \( 1 \leq r < s \leq n, \) based on an absolutely continuous df \( F \)
\[ f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = C_{r,s;m} \left[ 1 - F(x) \right]^m g_m^{-1}(F(x)) \]
\[ \times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ 1 - F(y) \right]^{k+(n-s)(m+1)-1} f(x)f(y) \]
\[ x < y, \tag{5.5} \]
where
\[
C_{r,s:n} = \frac{c_{s-1}}{(r-1)!(s-r-1)!}
\]

Thus the conditional pdf of \(X(s, n, m, k)\) given \(X(r, n, m, k) = x\), \(1 \leq r < s \leq n\), is given by
\[
f_{s \mid r}(y \mid x) = \frac{1}{c_{s-1}c_{r-1}} \frac{c_{s-1}}{(s-r-1)!c_{r-1}} \times \frac{[h_m(F(x)) - h_m(F(y))]^{s-r-1}[\bar{F}(y)]^{k+(n-s)(m+1)-1}f(y)}{[\bar{F}(x)]^{k+(n-r-1)(m+1)}}
\]
and the conditional pdf of \(X(r, n, m, k)\) given \(X(s, n, m, k) = y\), \(1 \leq r < s \leq n\), is given by
\[
f_{r \mid s}(x \mid y) = \frac{1}{(s-r-1)!(m+1)^{s-r-1}} \times \frac{[\bar{F}(x)]^{m}g_m^{r-1}(x)[h_mF(y) - h_mF(x)]^{s-r-1}f(x)}{g_m^{s-1}(y)}
\]
where \(\bar{F}(x) = P(X > x) = 1 - F(x)\)

**Case II:** When
\[\gamma_i \neq \gamma_j, m_i \neq m_j\]
then the pdf of \(X(r,n,\tilde{m},k)\) is [Kamps and Cramer, 2001].
\[
f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)[1 - F(x)]^{\gamma_i - 1}
\]
and the joint pdf of \(X(r,n,\tilde{m},k)\) and \(X(s,n,\tilde{m},k)\), \(1 \leq r < s \leq n\) is
\[ f_{X(r,n,m,k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1-F(y)}{1-F(x)} \right)^{y_i} \times \left[ \sum_{i=1}^{r} a_i^{(r)}(s) \left( 1-F(x) \right)^{y_i} \right] \frac{f(x)}{(1-F(x))(1-F(y))} \] (5.9)

where

\[ a_i^{(r)}(s) = \prod_{j=1}^{r} \frac{1}{\left( \gamma_j - \gamma_i \right)}, \quad \gamma_j \neq \gamma_i, \quad 1 \leq i \leq r \leq n \] (5.10)

and

\[ a_i^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{\left( \gamma_j - \gamma_i \right)}, \quad \gamma_j \neq \gamma_i, \quad r + 1 \leq i \leq s \leq n \] (5.11)

The conditional pdf of \( X(s, n, \tilde{m}, k) \) given \( X(r, n, \tilde{m}, k) = x \), \( 1 \leq r < s \leq n \), is given by

\[ f_{X(s,n,m,k) \mid X(r,n,m,k)}(y \mid x) = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{y_i} \frac{f(y)}{\bar{F}(x)}, x < y \] (5.12)

and the conditional pdf of \( X(r, n, \tilde{m}, k) \) given \( X(s, n, \tilde{m}, k) = y \), \( 1 \leq r < s \leq n \), is given by

\[ f_{r \mid x}(x \mid y) = \frac{\left( \sum_{i=1}^{r} a_i^{(r)}(s) [\bar{F}(x)]^{y_i} \right) \left( \sum_{i=r+1}^{s} a_i^{(r)}(s) [\bar{F}(y)]^{y_i} \right) \frac{f(y)}{\bar{F}(x) \bar{F}(y)}}{\left( \sum_{i=1}^{s} a_i(s) [\bar{F}(y)]^{y_i} \right)} \] (5.13)
Particular cases: (i) In the case $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$

(i.e. $\gamma_r = n - r + 1$, $1 \leq r \leq n - 1$), the model can be reduced to the joint density of ordinary order statistics from iid rv's $X_1, X_2, \cdots, X_n$ with distribution function $F$

$$f_{X(1,0,0), X(2,0,0), \ldots, X(n,0,0)}(x_1, x_2, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i)$$

(ii) In the case $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = \alpha - n + 1$ with $n - 1 < \alpha \in \mathbb{R}$ (i.e. $\gamma_r = \alpha - r + 1$, $1 \leq r \leq n - 1$), we describe OS's with non-integral sample size:

$$f_{X(1,0,\alpha-n+1), X(2,0,\alpha-n+1), \ldots, X(n,0,\alpha-n+1)}(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} (\alpha - j + 1)(1 - F(x_n))^\alpha - n \prod_{i=1}^{n} f(x_i)$$

(iii) Given positive real numbers $\alpha_1, \cdots, \alpha_n$, if we put

$$m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1, \quad i = 1, \cdots, n - 1 \text{ and } k_n = \alpha_n$$

(i.e. $\gamma_r = (n - i + 1)\alpha_r$, $1 \leq r \leq n - 1$) to the joint density

$$f_{X(1,\bar{m},\alpha_n), X(2,\bar{m},\alpha_n), \ldots, X(n,\bar{m},\alpha_n)}(x_1, x_2, \ldots, x_n) = n! \prod_{j=1}^{n-1} \left( \prod_{i=1}^{n-1} (1 - F(x_i))^m_i f(x_i) \right) (1 - F(x_n))^\alpha_n \prod_{i=1}^{n} f(x_i)$$

we get sequential order statistics based on

$$F_r(t) = 1 - (1 - F(t))^\alpha_r, \quad 1 \leq r \leq n,$$

where $F$ is an arbitrary, absolutely continuous distribution function
(iv) In the case \( m_1 = m_2 = \cdots = m_{n-1} = -1 \) and \( k \in \mathbb{N} \)
(i.e. \( \gamma_r = k, \ 1 \leq r \leq n-1 \)), we obtain the joint density of the first \( n \) \( k \)-th
record values based on the sequence \((X_i)_{i \in \mathbb{N}}\) of iid r.v's with distribution function \( F \)
\[
f_X(1.n-1,k), X(2.n-1,k), \ldots , X(n,n-1,k), (x_1,x_2,\ldots,x_n)
= k^n \left( \prod_{i=1}^{n-1} \frac{f(x_i)}{1-F(x_i)} \right) (1-F(x_n))^{k-1} f(x_n)
\]

(v) Given positive real numbers \( \beta_1, \ldots, \beta_n \), we choose
\( m_i = \beta_i - \beta_{i+1} - 1 \), \( i = 1, \ldots, n-1 \) and \( k = \beta_n \)
(i.e. \( \gamma_r = \beta_r, \ 1 \leq r \leq n-1 \)) to obtain the joint density
\[
f_X(1.n,\tilde{m},\beta_n), X(2.n,\tilde{m},\beta_n), \ldots , X(n.n,\tilde{m},\beta_n), (x_1,x_2,\ldots,x_n)
= \left( \prod_{j=1}^{n} \beta_j \right) \left( \prod_{i=1}^{n-1} (1-F(x_i))^{m_i} f(x_i) \right) [1-F(x_n)]^{\beta_n-1} f(x_n)
\]
of Pfeifer’s record values from non identically distributed r.v based on
\[F_r(t) = 1-[1-F(t)]^{\beta_r}, \quad 1 \leq r \leq n,\]

(vi) Choosing the parameters \( m_1 = \cdots = m_{r_1} = m_{r_1+1} = \cdots = m_{n-1} = 0 \),
(i.e. \( \gamma_r = \nu - r + 1, \ 1 \leq r \leq r_1 \), \( \gamma_r = \nu - n_1 - r + 1, \ n_1 < r \leq n-1 \)),
we obtain the joint density arising in progressive type II censoring with two stages
\[
f_X(1.n,\tilde{m},k), X(2.n,\tilde{m},k), \ldots , X(n.n,v,k), (x_1,x_2,\ldots,x_n)
= \frac{\nu!}{(\nu-r_1-n_1)!} \left( \prod_{i=1}^{n} f(x_i) \right) [1-F(x_i)]^{m_1} [1-F(x_n)]^{\nu-m_1-n}
\]
6. Choquet-Deny type functional equation

The following results are based on Choquet-Deny type functional equation (Rao and Shanbhag, 1994)

**Theorem: 6.1:** Let \( f \) be a non-negative real locally integrable measurable function on \( \mathbb{R}_+ \), other than the function, which is identically 0, almost everywhere \([L]\), such that it satisfies

\[
 f(x) = \int f(x + y) \mu(dy), \quad \text{for almost all } [L]x \in \mathbb{R}_+ \quad (6.1)
\]

for some \( \sigma \)-finite measure \( \mu \) on (the Borel \( \sigma \)-field of) \( \mathbb{R}_+ \) with \( \mu(\{0\}) < 1 \) (yielding trivially that \( \mu(\{0\}^c) > 0 \)). \( L \) correspond to Lebesgue measure.

Then, either \( \mu \) is arithmetic with some span \( \lambda \) and

\[
 f(x + n\lambda) = f(x)b^n, \quad n = 0, 1, \cdots \quad \text{for almost all } [L]x \in \mathbb{R}_+ \quad \text{with } b \text{ such that}
\]

\[
 \sum_{n=0}^{\infty} b^n \mu(\{n\lambda\}) = 1
\]

or \( \mu \) is non arithmetic and

\[
 f(x) \propto \exp(\eta x) \quad \text{for almost all } [L]x \in \mathbb{R}_+
\]

with \( \eta \) such that

\[
 \int_{\mathbb{R}_-} \exp(\eta x) \mu(dx) = 1
\]

The theorem is due to Lau and Rao (1982) and it has several interesting proofs including that based on exchangeability, given by Alzaid, *et al.*, (1987) [Rao and Shanbhag (1994) for more details.]
Corollary 6.1: Let \( \{(v_n, w_n) : n = 0, 1, \ldots \} \) be a sequence of vectors with non-negative real components such that \( v_n \neq 0 \) for at least one \( n, w_0 < 1 \), and the largest common divisor of the set \( \{n : w_n > 0\} \) is unity. Then

\[
v_m = \sum_{n} v_{m+n} w_n, \quad m = 0, 1, \ldots \tag{6.2}
\]

if and only if

\[
v_n = v_0 b^n, \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad \sum_{n=0}^{\infty} w_n b^n = 1
\]

for some \( b > 0 \)

Corollary 6.2: Let \( X \) be a non-negative random variable with \( P\{X = 0\} < 1 \) and \( h \) be a monotonic right continuous function on \( \mathbb{R}_+ \) such that \( E(|h(X)|) < \infty \) and \( E(h(X)) \neq h(0) \). Then

\[
E\{h(X - x)|X \geq x\} = E(h(X)), \quad x \in \mathbb{R}_+ \text{ with } P\{X \geq x\} > 0 \tag{6.3}
\]

if and only if either \( h^* \) is non-arithmetic and \( X \) is exponential, or for some \( \lambda > 0 \), \( h^* \) is arithmetic with span \( \lambda \) and

\[
P\{X \geq n \lambda + x\} = P\{X \geq x\} \left( P\{X \geq \lambda\} \right)^n, \quad n \in \mathbb{N}_0, x \in \mathbb{R}_+, \text{ where}
\]

\[
h^*(x) = \begin{cases} (h(x) - h(0))/(E(h(x)) - h(0)), & x \geq 0 \\ 0, & x < 0 \end{cases}
\]

Corollary 6.1 is essentially given in Lau and Rao (1982) and is a slight generalization of lemma established by Shanbhag (1977). (Shanbhag takes \( w_1 > 0 \) in place of condition that the largest common divisor of the set of \( n \) for which \( w_n > 0 \) is 1, even though he does not assume priori \( w_0 < 1 \))

Corollary 4.2, in the form that we have given here, has appeared in Rao and
Shanbhag (1994), a somewhat different version of this result was given by Klebanov (1980). If $Y$ is nonnegative random variable and we take

$$h(x) = P\{Y \leq x\}, \quad x \in \mathbb{R}_+,$$

then it follows that for a nonnegative random variable $X$ which is independent of $Y$ with $P\{X \geq Y\} > P\{Y = 0\}$, (4.3) is equivalent to

$$P\{X \geq Y + x \mid X \geq Y\} = P\{X \geq x\}, \quad x \in \mathbb{R}_+. $$

7. Some distributions

(i) Beta distribution:

$$f(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, \quad a, b > 0, \ 0 < x < 1 \quad (7.1)$$

where

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}$$

$$F(x) = \int_0^x \frac{1}{B(a, b)} t^{a-1} (1-t)^{b-1} dt, \quad 0 < x < 1, \ a, b > 0 \quad (7.2)$$

$$E(X^n) = \frac{\Gamma(a + n) \Gamma(a + b)}{\Gamma(a + b + n) \Gamma(a)}$$

(ii) Cauchy distribution:

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty \quad (7.3)$$

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x, \quad -\infty < x < \infty \quad (7.4)$$

For Cauchy distribution the moments of order $<1$ exists, but the moments of order $\geq 1$ do not exist.
(iii) Exponential distribution:

\[ f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0 \]  
\[ F(x) = 1 - e^{-x/\beta}, \quad x > 0 \]  
\[ E(X^n) = n! \beta^n \]

(iv) Gamma distribution:

\[ f(x) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}, \quad 0 < x < \infty, \quad \alpha, p > 0 \]  
\[ F(x) = \frac{\alpha^p}{\Gamma(p)} \int_0^x e^{-\alpha t} t^{p-1} \, dt, \quad 0 < x < \infty \]  

The df of gamma distribution is called incomplete gamma function.

\[ E(X^n) = \frac{\Gamma(p+n)}{\alpha^p \Gamma(p)} \]

(v) Pareto distribution:

\[ f(x) = va^v x^{-v-1}, \quad x > 0, \quad a, v > 0 \]  
\[ F(x) = a^v (1 - x^{-v}), \quad x > 0, \quad a, v > 0 \]  
\[ E(X^n) = \frac{va^v}{v-n} \]
(vi) Power function distribution:

\[ f(x) = va^{-v}x^{v-1}, \quad 0 \leq x \leq a, \quad a, v > 0 \]  

\[ F(x) = a^{-v}x^{v}, \quad 0 \leq x \leq a, \quad a, v > 0 \]  \hspace{1cm} (7.11) \hspace{1cm} (7.12)

\[ E(X^n) = \frac{va^{-v}a^n}{n+v} \]

(vii) Weibull distribution:

\[ f(x) = \alpha px^{p-1}e^{-\alpha x^p}, \quad 0 \leq x < \infty, \quad \theta, p > 0 \]  

\[ F(x) = 1 - e^{-\alpha x^p} \]  \hspace{1cm} (7.13) \hspace{1cm} (7.14)

Moments of standard Weibull distribution are given by

\[ E(X^n) = \alpha^n \Gamma\left(1 + \frac{n}{p}\right) \]
1. Introduction
Ferguson (1964, 1965) and Crawford (1966) were among the earliest authors who characterized geometric and exponential distribution via properties of order statistics. They showed that if $X$ and $Y$ are independent non-degenerate random variables then $\min\{X, Y\}$ is independent of $X - Y$ if and only if for some $\alpha > 0$ and $\beta \in \mathbb{R}$, $\alpha(X - \beta)$ and $\alpha(Y - \beta)$ are either both exponential or both geometric (in usual sense). Rao and Shanbhag (1994) have essentially established the following extended version of the Ferguson-Crawford result.

Let $X_1, X_2, \ldots, X_n$ be a sample from a continuous distribution such that

$$E(X_{k+1:n} \mid X_{k:n}) = aX_{k:n} + b$$

for some $1 \leq k < n$,

then only the following three cases are possible:

1. $a = 1$ and $X_1$ has an exponential distribution.
2. $a > 1$ and $X_1$ has a Pareto distribution.
3. $a < 1$ and $X_1$ has a power distribution.

A general form of distribution $F(x) = 1 - [ah(x) + b]^c$ was characterized through (1.1) by Khan and Abu-Salih (1989)

Let us point out that Ferguson states his result assuming that

$$E(X_{k:n} \mid X_{k+1:n}) = aX_{k+1:n} - b$$

for some $1 \leq k < n$ instead of using the regression $X_{k+1:n}$ on $X_{k:n}$ and arrives at distributions dual to that given in
(1.2) since the duality is obvious (take $Y = -X$). Without losing generality, here we use the regression $X_{k+1:n}$ on $X_{k:n}$.

In Nagaraja (1988) an analogue of this result for discrete distributions is given.

Investigations of characterizations of probability distributions by properties of regression involving different functions of order statistics were lead by many researchers. [Arnold et al., (1992), Johnson et al., (1994)].

A slight refinement of the original Ferguson (1967) result, allowing discontinuity in one of the support ends has been given more recently in Pakes et al., (1996).

Characterization of distributions through conditional expectation, conditioned on non-adjacent order statistics was first of all given by Khan and Ali (1987).

Let $X_1, X_2, \ldots, X_n$ be a sample from an absolutely continuous distribution such that

$$E(X_{k+2:n} \mid X_{k:n}) = aX_{k:n} + b$$

for some $1 \leq k < n - 1$ then the same three cases (1.2) are only possibility.

Dembinska and Wesolowski (1997), has shown that in the absolutely continuous case, instead of a single regression condition, a pair of identities

$$E(X_{k_1+r:n_1} \mid X_{k_1:n_1}) = X_{k_1:n_1} + b_i, i = 1,2,$$

with $r = 3$, $n_1 - k_1 \neq n_2 - k_2$ or with any $r$ and $n_1 - k_1 = n_2 - k_2 + 1$, characterizes the exponential distribution.

Also refer to the related papers of Franco and Ruiz (1997), Lopez Blaqez and Moreno Rebollo (1997), Khan and Aboummoh (2000) and Athar et al. (2003).
2. Characterization through order statistics

Theorem 2.1: Let $X$ and $Y$ be as in the Ferguson-Crafword result and $y_0$ be a point such that there are at least two support points of the distribution of $\min\{X, Y\}$ in $(-\infty, y_0]$. Let $\phi$ be a real valued Borel measurable function on $\mathbb{R}$ such that its restriction to $(-\infty, y_0]$ is non-vanishing and strictly monotonic. Then $X - Y$ and $\phi(\min\{X, Y\})$ are independent if and only if for some $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$, $\alpha(X - \beta)$ and $\alpha(Y - \beta)$ are either both exponential or geometric on $N_0$, in which case $X - Y$ and $\min\{X, Y\}$ are independent.

Proof: The "if" part of the theorem is trivial result. To prove the 'only if' part here one could follow Rao and Shanbhag (1998) to show first that if $P\{X > Y\} > 0$, then the assertion implies that

$l \in (-\infty, y_0], P\{Y = l|X \geq Y, Y \leq y_0\} < 1$, and

\[
P\{X \geq Y + x|X \geq Y, Y \leq y_0\} = P\{X \geq * l + x|X \geq * l, x \in \mathbb{R}_+\}
\]

where $l$ is the left extremity of the distribution of $Y$ and "$\geq *" denotes "$\geq$" if $l$ is a discontinuity point of the distribution of $Y$ and denotes "$>" otherwise. Observe now that if $P\{Y = l\} = 0$, we have

\[
P\{X - Y < 0\} = P\{X - Y < 0|\min\{X, Y\} = l\} = P\{X > l\} = 1,
\]

contradicting the condition that $P\{X \geq Y\} > 0$. In view of the observation that we have made in Corollary (1.6.2), we may then appeal to Corollary (1.6.2) to have the conditional distribution of $X - l$ given that $X \geq l$ is exponential if the conditional distribution of $Y - l$ given that $Y \leq y_0$ is non
arithmetic, and that of \( \lambda[(X-l)/\lambda] \) given that \( X \geq l \), where \([.]\) denotes the integral part, is geometric arithmetic with span \( \lambda \). This in turn implies because of the "independence" condition in the assertion, that the left extremity of the distribution of \( X \) is less than or equal to that of \( Y \) and that \( P\{Y \geq X\} > 0 \). Hence, by symmetry, a further result with the places of \( X \) and \( Y \) interchanged (and the obvious notational change in \( l \)) follows, and one is then led to the result sought.

Rao and Shanbhag (1994) have effectively observed the following two simple corollaries of the Theorem 2.1.

**Corollary 2.1:** If in Theorem 2.1, \( X \) and \( Y \) are additionally assumed to be identically distributed, then the assertions of the theorem hold with \( |X-Y| \) in place of \((X-Y)\)

**Proof:** The corollary follows on noting that, under the assumption, for any \( y \in \mathbb{R} \), \(|X-Y|\) and \( \phi(\min\{X,Y\})I_{\{\min\{X,Y\} \leq y\}} \) are independent if and only if \( X-Y \) and \( \phi(\min\{X,Y\})I_{\{\min\{X,Y\} \leq y\}} \) are independent.

**Corollary:** 2.2: Let \( X \) and \( Y \) be two iid nondegenerate positive random variables and \( y_0 \) be as defined in Theorem 2.1. Then \( \min\{X,Y\}/\max\{X,Y\} \) and \( \min\{X,Y\}I_{\{\min\{X,Y\} \leq y_0\}} \) are independent if and only if for some \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), \( \alpha(\log X - \beta) \) is either geometric or exponential.

**Proof:** Define \( X^* = \log X \), \( Y^* = \log Y \) and \( y_0^* = \log y_0 \) (note that \( y_0 \) to be positive.) Noting that \(-\log(\min\{X,Y\}/\max\{X,Y\}) = |X^* - Y^*|\) and \( \min\{X,Y\}I_{\{\log(\min\{X,Y\}) \leq \log y_0\}} = \exp\{\min\{X^*, Y^*\}\}I_{\{\min\{X^*, Y^*\} \leq y_0^*\}} \)

we get the result from the Corollary 2.1.
Remarks 1: (i) If we replace in Corollary 2.1, the condition on the existence of \( y_0 \) by that there exists a point \( y'_0 \) such that there are at least two support point of the distribution of \( \max\{X, Y\} \) in \([y_0, \infty)\). then the assertions of the corollary with \( \min\{X, Y\} I_{\{\min\{X, Y\} \leq y_0\}} \) replaced by \( \max\{X, Y\} I_{\{\max\{X, Y\} \geq y'_0\}} \) and \( \log X \) replaced by \( -\log X \) holds. This follows because \( \min\{X^{-1}, Y^{-1}\} = (\max\{X, Y\}^{-1}) \) and \( \max\{X^{-1}, Y^{-1}\} = (\min\{X, Y\}^{-1}) \). The result that is observed here is indeed a direct extension of Fisz’s (1958) result, and is yet another result mentioned in the Rao and Shanbhag (1994). (Fisz characterizes the distribution in question via the independence of \( \max\{X, Y\}/\min\{X, Y\} \) and \( \max\{X, Y\} \))

(ii) Under the assumption in Theorem 2.1, the condition that \( X - Y \) and \( \phi(\min\{X, Y\}) I_{\{\min\{X, Y\} \leq y_0\}} \) be independent is clearly equivalent to that for each \( y \in (-\infty, y_0] \), \((X - Y)\) be independent of \( I_{\{\min\{X, Y\} \leq y\}} \)

(iii) Theorem 2.1 remains valid if the “independence” conditions appearing in the assertion is replaced by that conditional upon \( \min\{X, Y\} \in (-\infty, y_0], \) \((X - Y)\) and \( \min\{X, Y\} \) are independent.

(iv) If the assumptions in Theorem 2.1 are met with \( P\{X \geq Y\} > 0 \), then on modifying slightly, the Rao-Shanbhag argument that we have referred to in the proof of theorem proves that conditionally upon \( \min\{X, Y\} \in (-\infty, y_0], (I_{\{X=Y}, (X - Y)^+\}) \) and \( \min\{X, Y\} \) are independent only if \( l \in (-\infty, y_0) \), the conditional distribution of \( X - l \) given that \( X \geq l \) is exponential if the conditional distribution of \( X - l \) given that \( Y \leq y_0 \) is non-arithmetic, and that of \( \lambda \left[ \frac{X - l}{\lambda} \right] \) given that \( X \geq l \) is geometric on
{0, λ, 2λ, ⋯} if the conditional distribution of \( Y-I \) given that \( Y \leq y_0 \) is arithmetic with span \( \lambda \) where \( I \) is the left extremity of the distribution of \( Y \) and \([\cdot]\) denote the integral part.

(v) The version of Theorem 2.1 with \( \min\{X, Y\} \) in place of “two support points” we take “two non zero support point” are in place of “there are...in \((-\infty, y_0]\)” we “take the left extremity of the distribution of \( \min\{X, Y\} \) is non zero and is less than \( y_0 \)” the result in (iii) above and that mentioned here are essentially variations of Rao and Shanbhag (1994).

There is an interesting version of Theorem 2.1 Rossberg (1972), Ramchandran (1980), Rao (1983), Lau and Ramchandran (1991) and Rao and Shanbhag (1994) among others have produced versions of this theorem. A special case of this result for \( n = 2 \) was given in a somewhat restricted form by Puri and Rubin (1970); see also Lau and Rao (1982) Stadje (1994) and Rao and Shanbhag (1995) for comments and extension on the Puri-Robin (1970) result.

**Theorem 2.2** Let \( n \geq 2 \) and \( X_1, X_2, \ldots, X_n \), be iid random variable with \( \text{df } F \) that is not concentrated on \( \{0\} \). Further let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \), denote the corresponding order statistics. Then for some \( 1 \leq i \leq n \),

\[
X_{i+1:n} - X_{i:n} = X_{1:n-i}
\]

where \( X_{1:n-i} = \min\{X_1, X_2, \ldots, X_{n-i}\} \) if and only if one of the following two conditions holds:

(i) \( F \) is exponential

(ii) \( F \) is concentrated on some semilattice of the form \( \{0, \lambda, 2\lambda, 3\lambda, \ldots\} \) with

\[
F(0) = \alpha \quad \text{and} \quad F(j\lambda) - F((j-1)\lambda) = (1 - \alpha)(1 - \beta)\beta^{j-1} \quad \text{for } j = 1, 2, \ldots
\]
some \( \alpha \in \left( 0, \left( \frac{n}{i} \right)^{-1/i} \right] \) and \( \beta \in [0,1) \) such that \( P\{X_{i+1:n} > X_{i:n}\} = (1 - \alpha)^{n-1} \) (which holds with \( \alpha = \left( \frac{n}{i} \right)^{-1/i} \) or \( \beta = 0 \) if and only if

\[
F(0) - F(0-) = \left( \frac{n}{i} \right)^{-1/i},
\]

and

\[
F(\lambda) - F(\lambda-) = 1 - \left( \frac{n}{i} \right)^{-1/i}
\]

for some \( \lambda > 0 \). (The existence of case \( \beta > 0 \) can easily be verified.)

**Proof** We shall first establish the "only" if part of the assertion (2.1) implies that \( P\{X_{1:n-1} \geq 0\} = (1 - F(0-))^{n-i} = 1 \) and hence \( F \) is concentrated on \( \mathbb{R}_+ \). Hence we can conclude from (2.1) that

\[
\left( \frac{n}{i} \right) \int_{\mathbb{R}_+} (\overline{F}(x + y))^{n-i} F^i(dy) = (\overline{F}(x))^{n-i}, \quad x \in \mathbb{R}_+ \quad (2.2)
\]

where \( \overline{F}(x) = 1 - F(x) \), \( x \in \mathbb{R}_+ \). As \( P\{X_1 = 0\} < 1 \), we have then \( \overline{F}(0) > 0 \) and (2.2) implies that \( \left( \frac{n}{i} \right)(F^i(0) \leq 1 \) (where obviously we have used the property of \( F \) that \( F(0-) = 0 \). If \( F \) is nonarithmetic distribution (concentrated on \( \mathbb{R}_+ \)) with \( \left( \frac{n}{i} \right)(F^i(0) < 1 \), then (2.2) implies, in view of Theorem 1.6.1, that (i) holds. On the other hand if \( F \) is arithmetic with span \( \lambda \) and \( \left( \frac{n}{i} \right)(F(0))^i < 1 \), then the equation in question implies, in view of the cited theorem, that \( F \) is such that \( F(0) = \alpha \) and
\[ F(j\lambda) - F((j-1)\lambda) = (1 - \alpha)(1 - \beta)\beta^{j-1} \]

for \( j = 1, 2, \ldots \), some \( \alpha \in (0, \left(\frac{n}{i}\right)^{1/i}) \) and \( \beta \in (0, 1) \) such that \( P\{X_{i+1:n} > X_{i:n}\} = (1 - \alpha)^{n-i} \). Finally, if \( \left(\frac{n}{i}\right)(F(0))^i = 1 \), then (2.2) is not met if \( F \) is nonarithmetic, or arithmetic with some span \( \lambda \) with at least two positive support points, but is met by any \( F \) concentrated on \( \{0, \lambda\} \) for some \( \lambda(> 0) \). This completes the proof of the ‘only if part of the assertion.

It is easy to see that (2.1) is equivalent to the condition that (2.2) holds with \( F(0-) = 1 \). for all the distribution that we have arrived at in the proof of the ‘only if’ part we have \( F(0-) = 1 \) and (2.2) met. Hence we have ‘if’ part of the assertion, and consequently the theorem.

The sketch of the argument that we have produced above to see the validity of Theorem 2.2 tell us further that the following theorem holds. Arnold and Ghosh (1976) and Arnold (1980) have dealt with specialized version of this result; see, also Zijlstra (1983) and Fosam et al. (1993) for further specialized versions and some comments on the earlier literature.

**Theorem 2.3:** Let \( n \geq 2 \) and \( X_1, \ldots, X_n \) be non-degenerate iid random variables with \( df F \). Also, let \( X_{1:n}, \ldots, X_{n:n} \), be order statistics as in Theorem 2.2, then for some \( i \geq 1 \), the conditional distribution of \( X_{i+1:n} - X_{i:n} \) given that \( X_{i+1:n} - X_{i:n} > 0 \) is the same as the distribution of \( X_{1:n-i} \), where \( X_{1:n-i} \), is defined as in Theorem 2.2 if and only if \( F \) is either exponential, or, for some \( \lambda > 0 \), geometric on \( \{\lambda, 2\lambda, 4\lambda, \ldots\} \).

The next two theorems are in the same spirit as Theorem 2.3 and extend slightly the result given in Fosam and Shanbhag (1994). Once again these results follow as corollaries to the strong memory less property.
characterization of the exponential and geometric distributions. The result given in Fosam and Shanbhag (1994) and hence so also the theorem given here, in turn, subsume the specialized results given by Liang and Balakirishnan (1992, 1993).

3. Characterization through conditional distribution of order statistics.

Theorem 3.1: Let \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \) be integers and \( Y_1, Y_2, \ldots, Y_n \) be independent positive random variables such that \( P\{Y_1 > Y_2 > \cdots > Y_n\} > 0 \) and for each \( i = 1, 2, \ldots, k \) the conditional distribution of \( Y_{i+1} \) given that \( Y_{i+1} > Y_{i+2} > \cdots > Y_n \) be non-arithmetic. (The condition on \( Y_i \)'s is clearly met if \( Y_i \)'s are independent positive random variables such that for each \( i = 2, \ldots, n \) and \( y > 0, P\{Y_i > y\} > 0 \).

Then
\[
P\{Y_i - Y_{i+1} > y | Y_1 > Y_2 > \cdots > Y_n\} = P\{Y_i > y | Y_1 > Y_2 > \cdots > Y_n\},
\]
\[
y > 0; \quad i = 1, 2, \ldots, k \tag{3.1}
\]
(where the right hand side of the identity is to be read as \( P\{Y_1 > y\} \) for \( i = 1 \)) if and only if \( Y_i, i = 1, 2, \ldots, k \), are exponential random variables. (The result also holds if "\( >\)" in (3.1) is replaced by "\( \geq\)" with "\( Y_1 > Y_2 > \cdots > Y_n\)" and "\( Y_{i+1} > Y_{i+2} > \cdots > Y_n\)" in the assumption replaced respectively by "\( Y_1 \geq Y_2 \geq \cdots \geq Y_n\)" and "\( Y_{i+1} \geq Y_{i+2} \geq \cdots \geq Y_n\)."

Proof: Defining for each \( i = 1, 2, \ldots, k \), \( X^{(i)} \) and \( Y^{(i)} \) to be independent positive random variable with distribution functions
\[
P\{Y_i \leq x | Y_1 > Y_2 > \cdots > Y_i\} \quad x \in \mathbb{R}_+ \quad \text{and} \quad P\{Y_{i+1} \leq x | Y_{i+1} > Y_{i+2} > \cdots > Y_n\}.
\]
\( x \in \mathbb{R}_+ \), we see that (3.1) can be rewritten as
\[
P\{X^{(i)} > Y^{(i)} + x | X^{(i)} > Y^{(i)}\} = P\{X^{(i)} > x\}, \quad x > 0; \quad i = 1, 2, \ldots, k.
\]
Consequently, in view of the strong memoryless characterization of the exponential distributions, it follows that (3.1) is valid if and only if the distribution functions $P\{Y_i \leq x|Y_1 > Y_2 > \cdots > Y_i\}, x \in \mathbb{R}_+$ are those corresponding to exponential random variables for $i = 1, 2, \cdots, k$. It is easy to see that we have the distribution function $P\{Y_i \leq x|Y_1 > Y_2 > \cdots > Y_i\}, x \in \mathbb{R}_+$ for $i = 1, 2, \cdots, k$ as those corresponding to exponential distributions if and only if the random variables $Y_1, Y_2, \cdots, Y_k$ are exponential. Hence we have theorem.

**Theorem 3.2:** Let $n \geq 2$ and $1 \leq k \leq n-1$ be integers and $Y_1, Y_2, \cdots, Y_n$ be independent non-negative integer-valued random variables such that $P\{Y_1 \geq Y_2 \geq \cdots \geq Y_n\} > 0$ and for each $i = 1, 2, \cdots, k$ the conditional distribution of $Y_{i+1}$ given that $Y_{i+1} > Y_i > \cdots > Y_n$, be arithmetic with span 1. Also let

$$P\{Y_{i+1} = 0|Y_1 \geq Y_2 \geq \cdots \geq Y_n\} < 1, \quad i = 1, 2, \cdots, k.$$ 

(The conditions on $Y_i$'s are clearly made if $Y_i$'s are independent non-negative integer valued random variables such that $P\{Y_i \geq 1\} > 0$ for $i = 1$) if and only if $Y_i, i = 1, 2, \cdots, k$, are geometric random variables.

Theorem 3.2 follows essentially via the argument proof of Theorem 3.1 but with “geometric” in place of “exponential”.

**Remarks 2:** (i) As observed by Fosam and Shanbhag (1994), the specialized version of Theorem 3.1 given by them subsumes the “only if” part (i.e. the major part) of the Liang and Balakrishnan (1992) theorem, note that if $X$ and $Y$ are independent positive random variables such that 0 is a cluster
point of the distribution of $Y$. Then, conditionally upon $X > Y$, the random variables $X - Y$ and $Y$ are independent only if

$$P\{X > Y + x | X > Y\} = \lim_{y \to 0} P\{X > Y + x | X > Y, Y \leq y\}$$

$$= P\{X > x\}, \quad x \in (0, \infty),$$

Consequently, it follows that under the weaker assumption in the Fosam and Shanbhag result in place of original assumption, the Liang and Balakrishnan theorem holds. This improved theorem also holds if $A$ is replaced by $A^* = \{Y_1 \geq Y_2 \geq \cdots \geq Y_n\}$

(ii) if $X$ and $Y$ are independent non negative integer-valued random variables such that $P\{Y = 0\} > 0$, then conditionally upon $X \geq Y$, the random variables $X - Y$ and $Y$ are independent only if

$$P\{X \geq Y + x | X \geq Y\} = \lim_{y \to 0} P\{X \geq Y + x | X \geq Y, Y = 0\}$$

$$= P\{X \geq x\}, \quad x = 0, 1, \cdots$$

In view of this, we have that comments analogous to those on the Liang and Balakrishnan (1992) theorem (but with Theorem 3.2 in place of Theorem 3.1) also apply to the Liang and Balakrishnan (1993) theorem. (Note that in this latter case, we restrict ourselves to the independence conditionally upon $A^*$, where $A^*$ is as in (i)).

(iii) Under a somewhat more assumption, it can be shown that the equation (3.1) with "$\geq" replaced by ">" leads us to characterizations of shifted geometric distributions.
4. Characterization through conditional expectations

Theorem 4.1: Let $Y$ and $Z$ be independent random variables with distributions such that the corresponding supports are equal and $Y$ is continuous. Further, let $\phi$ be a nonarithmetic real monotonic function on $\mathbb{R}_+$ such that $E(|\phi(Y-Z)|) < \infty$. Then for some constant $c \neq \phi(0^+)$,

$$E\{\phi(Y-Z) \mid Y \geq Z, Z\} = c \quad a.s.$$  \hspace{1cm} (4.1)

if and only if $Y$ is exponential, up to a change of location. (By the conditional expectation in (4.1), we really mean the one with $I\{Y \geq Z\}$ in place of $Y \geq Z$; the assertion of theorem is also holds "$Y \geq Z"$ is replaced by "$Y > Z$").

Under the stated assumption in Theorem 4.1, we have (4.1) to be equitant to

$$E\{\phi((Y-Z)^+) \mid Y \geq z\} = c \quad \text{for each } z \in \sup\{G\} \quad \text{with } (P \geq z) > 0.$$  \hspace{1cm} (4.2)

where $G$ is the df $Y$. If $z_1, z_2 \in \sup\{G\}$ such that $z_1 < z_2$ with $P\{Y \geq z_1\} = P\{Y \geq z_2\} > 0$, then from (4.2) it easily follows that the equation in it holds for each $z \in [z_1, z_2]$ consequently, we see that (4.2) is equivalent to the assertion obtained from it by deleting "$\in \sup\{G\}$" and we get Theorem 4.1 as a consequence of Corollary 1.6.2 (one could also arrive at the result directly without appealing to Corollary 1.6.2, from Theorem 1.6.1)

Corollary 4.1 Let $F$ be a continuous and let $X_1, \ldots, X_n$ for $n \geq 2$ be $n$ ordered observations based on random sample of size $n$ from $F$. Further let $i$ be a fixed positive integer less than $n$ and $\phi$ be a non arithmetic real monotonic function on $\mathbb{R}_+$ such that $E[|\phi(X_{i+1:n} - X_{i:n})|] < \infty$. Then for some constant $c \neq \phi(0^+)$,

$$E(\phi(X_{i+1:n} - X_{i:n}) \mid X_{i:n}) = c, \quad a.s.$$  \hspace{1cm} (4.3)

if and only if $F$ is exponential, within a shift.
We can express (4.3) as (4.1) with $Y$ and $Z$ independent random variables such that $Z = X_{i+1}$ and $Y = X_{i-n-1}$; consequently we get, Corollary 4.1 as a corollary to Theorem 4.1.

One could now raise a question how crucial is the assumption of continuity of $Y$ for the validity of Theorem 4.1. The continuity assumption implies that

$$E\{\phi(Y - Z)\} | Y \geq z = c$$

for $a.a. G z \in \mathbb{R}$

(4.4)

is equivalent to

$$E\{\phi((Y - Z) -) | Y \geq z = c$$

for each $z \in \mathbb{R}$ with $P\{Y > z\} > 0$, (4.5)

where $G$ is the $df$ of $Y$. The equivalence mentioned here is the reason as to one is able to get Theorem 4.1 via Corollary 1.6.2 or Theorem 1.6.1.

Suppose we now have the assumptions in Theorem 4.1 made $Y$ non degenerate in place of continuous. Then, if $\phi$ is left continuous and satisfies the condition that

$$G(x +.) = G((y +.)-)$$

a.e. $G(\phi(+)) - \phi(0+)]$ (4.6)

whenever $0 < G(x) = G(y-) < G(y),$

then it easily follows that (4.4) is equivalent to (4.5); thus, we have cases other than those met in Theorem 4.1 under which (4.4) and (4.5) are equivalent.

Taking a clue from the observations made above and using essentially the same arguments as those that led us to Theorem 4.1 and Corollary 2.3 respectively, we can now give the following theorem and corollary. The theorem gives an answer to the question that we have raised above partially

**Theorem 4.2:** Let $Y$ and $Z$ be independent nondegenerate random variables such that the corresponding distributions have the same support and the same set of discontinuity points. Let $\phi$ be a monotonic, real, left
continuous, nonconstant function on $R_+$ for which (4.6) is met (or, more generally, a monotonic, real, non constant function for which (4.4) and (4.5) are equivalent) and $E\phi(Y - Z) < \infty$, where $G$ is the df of $Y$. Then, for some constant $c \neq (0+)$,

$$E\{\phi(Y - Z) | Y > Z, Z\} = c \quad \text{a.s.} \quad (4.8)$$

if and only if the left extreme $l$, of the distribution of $Y$ is finite, and either $\phi$ is nonarithmetic and the conditional distribution of $Y - l$ given that $Y > l$ is exponential, or for some $\lambda > 0, \phi$ is arithmetic with span $\lambda$ and the conditional survivor function, $\bar{G}_l$, of $Y - l$ given that $Y > l$ satisfies for some $\beta \in (0,1)$

$$\bar{G}_l(x + n\lambda) = \beta^n \bar{G}_l(x), \quad x > 0; \quad n = 0, 1, \cdots$$

**Corollary 2.4:** Let $X_{1n}, \ldots, X_{nn}$ be ordered observation base on the random sample of size $n(\geq 2)$ from a nondegenerate distribution with df $F$. Let $1 \leq i \leq n - 1$ be a given integer and $\phi$ be a monotonic real left continuous non constant function on $R_+$ such that $E(|\phi(X_{i+1:n} - X_{i:n})|) < \infty$ and (4.6) met with $F$ in place of $G$. Then, for some constant $c \neq \phi(0+)$

$$E(\phi(X_{i+1:n} - X_{i:n}) | X_{i+1:n} > X_{i:n}, X_{i:n}) = c \quad \text{a.s}$$

if and only if the left extremity, $l$, of $F$ is finite, and either $\phi$ is non arithmetic and (with $X \cap F$) the conditional distribution of $X_1 > l$ is exponential, within a shift, or for some $\lambda > 0, \phi$ is arithmetic with span $\lambda$ and the conditional survivor function, $\bar{F}_l$, of $X_1$ given that $X_1 > l$ satisfies for some $\beta \in (0,1)$

$$\bar{F}_l(x + n\lambda) = \beta^n \bar{F}_l(x), \quad x > l; \quad n = 0, 1, 2, \cdots$$
5. **Linearity of regression for non adjacent order statistics:** (Dembinska and Wesolowski, 1998)

Before starting the main Theorem the following Rao and Shanbhag (1994), result concerning possible solutions of an extended version of the integrated Cauchy functional equation.

**Theorem 5.2:** Consider the integral equation:

$$\int_{R_+} H(x + y)\mu(dy) = H(x) + c \quad \text{for} \ [L] a.a. x \in R_+$$

where $\mu$ is a non-arithmetic $\sigma$-finite measure on $R_+$ and $H : R_+ \to R_+$ is a Borel measurable, either non-decreasing or non-increasing $[L] a.e.$ function that is locally $[L]$ integrable and is not identically equal zero $[L] a.e.$ Then $\exists \eta \in R$

$$\int_{R_+} \exp(\eta x)\mu(dx) = 1,$$

and $H$ has the form

$$H(x) = \begin{cases} \gamma + \alpha(1 - \exp(\eta x)) & \text{for} [L] a.a. x \quad \text{if} \ \eta \neq 0 \\ \gamma + \beta x & \text{for} [L] a.a. x \quad \text{if} \ \eta = 0 \end{cases}$$

where $\alpha, \beta, \gamma$ are some constants. If $c = 0$ then $\gamma = -\alpha$ and $\beta = 0$.

**Theorem 5.2:** Assume that $X_1, \ldots, X_n$ are iid. rv with a common continuous df $F$. Let $E(|X_{k+r:n}|) < \infty$. If for some $k \leq n - r$ and some real $a$ and $b$ $E[X_{k+r:n} | X_{k:n} = x] = ax + b$ holds, then only the following three cases are possible:

1. $a = 1$ and $F$ is a df of an exponential distribution $\bar{F}(x) = e^{-\lambda(x-\mu)}$

2. $a > 1$ and $F$ is a df of a Pareto distribution $\bar{F}(x) = \left(\frac{\mu + \delta}{x + \delta}\right)^{\theta}$
3. \( a < 1 \) and \( F \) is a df of a power function distribution, \( \bar{F}(x) = \left( \frac{y-x}{y-\mu} \right)^\theta \)

**Proof** Using the formulas for the joint distribution of \((X_{i:n}, X_{j:n})\) and the distribution of \(X_{i:n}\)

\[
E(X_{k+r:n} | X_{k:n} = x) = \frac{(n-k)!}{(r-1)!(n-k-r)!} \int_{x}^{\infty} y \left[ \frac{F(x) - F(y)}{y-x} \right]^{r-1} \frac{F^{n-k-r}(y)}{F^{n-k}(x)} d[-F(y)]
\]

\( F \) a.e., where \( \bar{F} = 1 - F \). From (1.1) we get:

\[
\frac{(n-k)!}{(r-1)!(n-k-r)!} \int_{x}^{\infty} y \left[ \frac{\bar{F}(x) - \bar{F}(y)}{\bar{F}(x)} \right]^{r-1} \frac{\bar{F}}{\bar{F}}(y)^{n-k-r} \frac{d[-\bar{F}(y)]}{\bar{F}(x)} = ax + b
\]

(5.5)

where \((l_F, r_F)\) is the support of the distribution defined by \( F \) and \( F \) is strictly increasing in this interval. Notice also that since both sides of (5.5) are continuous with respect to \( x \) we can assume that it holds for any \( x \in (l_F, r_F) \).

Substituting \( t = \bar{F}(y) / \bar{F}(x) \), i.e. \( y = \bar{F}^{-1}(t \bar{F}(x)) \) (observe that \( \bar{F}^{-1} \) exists because \( \bar{F} \) is strictly decreasing in \((l_F, r_F)\)) into equation (5) we get:

\[
\frac{(n-k)!}{(r-1)!(n-k-r)!} \int_{0}^{1} t \bar{F}(x)(1-t)^{r-1} t^{n-k-r} dt = ax + b
\]

Now substitute \( \bar{F}(x) = w \), hence \( x = \bar{F}^{-1}(w) \) and thus

\[
\frac{(n-k)!}{(r-1)!(n-k-r)!} \int_{0}^{1} (tw)(1-t)^{r-1} t^{n-k-r} dt = a\bar{F}(w) + b, \quad w \in (0,1)
\]
Divide both sides by $a$ and substitute once again $t = e^{-u}$ and $w = e^{-v}$. Then

$$\frac{(n-k)!}{a(r-1)!(n-k-r)!} \int_0^\infty F^{-1}(e^{-(u+v)})(1-e^{-u})^{r-1}e^{-(n-k-r)u}e^{-u} du = F^{-1}(e^{-v}) + \frac{b}{a}$$

For any $v > 0$.

Now let $G(v) = F^{-1}(e^{-v})$. Consequently

$$\int_{R_+} G(v+u) \mu(du) = G(v) + \frac{b}{a}, \nu > 0$$

where $\mu$ is a finite measure on $R_+$, which is absolutely continuous with respect to the $[L]$ measure and is defined by

$$\mu(du) = \frac{(n-k)!}{a(r-1)!(n-k-r)!} (1-e^{-u})^{r-1}e^{-(n-k-r+1)u} du.$$

Observe that $G$ is strictly increasing on $(0,\infty)$ since it is a composition of two strictly decreasing functions. Consequently hence the assumptions of Theorem 5.1 are fulfilled. Since $G$ is continuous, it follows that

$$G(v) = \begin{cases} \gamma + \alpha(1 - \exp(\eta v)) & \text{if } \eta \neq 0 \\ \delta + \beta v & \text{if } \eta = 0 \end{cases} \quad (5.6)$$

$\nu > 0$, where $\alpha, \beta, \gamma, \eta$ are some constants and

$$\int_{R_+} \exp(\eta x) \mu(dx) = 1 \quad (5.7)$$

From (5.7) we get:

$$1 = \frac{(n-k)!}{a(r-1)!(n-k-r)!} \int_0^\infty e^{\eta x} (1-e^{-x})^{r-1}e^{-(n-k-r)x}e^{-x} dx$$
After substituting \( t = e^{-x} \) we obtain that \( \eta < n - k - r + 1 \):

\[
1 = \frac{(n-k)!}{a(r-1)!(n-k-r)!} \int_0^1 t^{r-1} (1-t)^{(n-k-r-\eta)} dt
\]

\[
= \frac{1}{a} \frac{B(n-k-r-\eta+1)}{B(n-k-r+1,r)}
\]

where \( B(.,.) \) is the complete beta function defined by

\[
B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p,q > 0
\]

since \( B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \) then

\[
\frac{\Gamma(n-k-r-\eta+1)\Gamma(r)}{\Gamma(n-k-\eta+1)} \cdot \frac{\Gamma(n-k+1)}{\Gamma(n-k-r+1)\Gamma(r)} = a
\]

(5.8)

A slight rearrangement allows to rewrite (5.8) as

\[
a = \frac{n-k}{n-k-\eta}, \quad \frac{n-k-1}{n-k-1-\eta}, \ldots, \frac{n-k-r+1}{n-k-r+1-\eta} = h(\eta),
\]

(5.9)

Observe that

1. \( a < 1 \) if \( \eta < 0 \),
2. \( a > 1 \) if \( 0 < \eta < n - k - r + 1 \),
3. \( a = 1 \) if \( \eta = 0 \).

Moreover there is a unique \( \eta \) that fulfills (5.9), because the function \( h \) is strictly increasing.

Returning to (5.6), for a non-zero \( \eta \), we can write

\[
\bar{F}^{-1}(e^{-v}) = G(v) = \gamma + \alpha(1-e^{\eta v})
\]

which implies

\[
e^{-v} = \bar{F}(\gamma + \alpha(1-e^{\eta v})).
\]
Let us substitute \( z = \gamma + \alpha(1 - e^{\eta v}) \). Then
\[
e^{-v} = \left(1 - \frac{z - \gamma}{\alpha}\right)^{-1/\eta}
\]
Hence \( \bar{F}(z) = \frac{1}{\left(1 - \frac{z - \gamma}{\alpha}\right)^{1/\eta}} \) for \( z > \gamma \).

Consider now three possible cases:

1. If \( \alpha < 1 \) and \( \eta < 0 \) then
\[
\bar{F}(z) = \left(\frac{\alpha + \gamma - z}{\alpha}\right)^{-1/\eta} = \left(\frac{\alpha + \gamma - z}{\alpha + \gamma - \gamma}\right)^{-1/\eta} = \left(\frac{v - z}{v - \mu}\right)^\theta
\]
for \( z \in (\mu, v) \), where \( v = \alpha + \gamma, \mu = \gamma, \theta = -\frac{1}{\eta} > 0 \). Observe that \( \alpha \) has to be positive.

Thus \( X_1 \) is a power function distribution with \( df \) \( (5.10) \) denoted by \( X_1 \sim POW(\theta, \mu, v) \),

where
\[
\theta = -\frac{1}{\eta} \text{ and } \eta \text{ fulfils (5.9)},
\]
\( v \) can be calculated from (5.2) with \( \theta = -\frac{1}{\eta} \),
\( \mu < v \) is a real number.

2. If \( \alpha > 1 \) and \( \eta > 0 \) then
\[
\bar{F}(z) = \left(\frac{-\infty}{z - \infty - \gamma}\right)^{1/\eta} = \left(\frac{\gamma + (-\infty - \gamma)}{z + (-\alpha - \gamma)}\right)^{1/\eta} = \left(\frac{\mu + \delta}{z + \delta}\right)^\theta
\]
for \( z > \mu \), where \( \delta = -\alpha - \gamma, \theta = -\frac{1}{\eta} > 0 \). Observe that \( \alpha \) has to be negative.
Thus $X_1$ is a Pareto distribution with $df$ (5.11) denoted by $X_1 \sim \text{PAR} (\theta, \mu, \delta)$, where:

$$\theta = \frac{1}{\eta} \quad \text{and} \quad \eta \text{ fulfills (5.9)},$$

$$\delta \quad \text{can be calculate from (5.3) with} \quad \theta = \frac{1}{\eta'},$$

$\mu$ is a real number.

Observe that this is the only case in which $\theta = 0$ is allowed. Then $\delta = 0$ and $\mu > 0$.

3. If $\alpha = 1$ and $\eta = 0$ then from (5.6) we get:

$$\overline{F}^{-1}(e^{-y}) = G(y) = \gamma + \beta y$$

$$e^{-y} = \overline{F}(\gamma + \beta y).$$

Let us substitute $z = \gamma + \beta y$. Then $\beta > 0$ and

$$\overline{F}(z) = e^{-(z-\gamma)/\beta} = e^{-\lambda(z-\gamma)}$$

(5.12)

for $z > \gamma$, where $\lambda = \frac{1}{\beta} > 0$.

Hence $X_1$ is an exponential distribution with $df$ (5.12) denoted by $X_1 \sim \text{E}(\lambda, \gamma)$, where

$\lambda$ can be calculated from (5.4)

$\gamma$ is a real number.

Khan and Aboummoh (2000), Athar et al., (2003), have also shown that

$$E[X_{s:n} | X_{r:n} = x] = a^* x + b^*$$

if and only if

$$F(x) = 1 - [ax + b]^c$$
where

\[ a^* = \prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c(n-r-j) + 1} \]

\[ b^* = -\frac{b}{a}(1 - a^*) \]

and with proper choice of \( a, b \) and \( c \) various distributions are characterized.

6. Some results based on Deny's theorem (Rao, 1983)

**Theorem 6.1:** If \( X_{i+1:n} - X_{i:n} \) and \( X_{1:n-1} \) have the same distribution, then,

(a) \( X_1 \) has exponential distribution if \( F \) is not a lattice and

\[ [F(0)]^i < i!(n-1)!/n!. \]

(b) If \( X_1 \) is distributed on integers with \( p_1 = P(X_1 = 1) \neq 0 \) and \( (1 - p_1 - p_0) > 0 \), then the probabilities are of the form

\[ p_0 < \binom{n}{i}^{-i}, p_1 = \alpha(1 - \beta), p_2 = \alpha(1 - \beta)\beta; \]

(6.1)

where \( p_0, \alpha \) and \( \beta \) are determined such that \( \sum p_i = 1 \)

**Proof:**

(a) we have the equation

\[ \left( \begin{array}{c} n \\ i \end{array} \right) \int G^{n-i}(x+y)dF^i(y) = G^{n-i}(x) \quad \forall x \geq 0 \]

where \( G = 1 - F \), and an application of Theorem 1.6.1 gives the result.

(b) We have the equation

\[ \left( \begin{array}{c} n \\ i \end{array} \right)^{-1} G^{n-i}(m) = \sum_{r=0}^{\infty} G^{n-i}(m+r)s_r, \quad m = 0, 1, \ldots, \]

\[ s_r = (p_0 + \cdots + p_r)^i - (p_0 + \cdots + p_{r-1})^i. \]
an application of Corollary 1.6.1 show that \( G(m) = \alpha \beta^m, m = 0, 1, \ldots \),
and \( p_0 i < \binom{n}{i} \).

Note that if the condition \((1 - p_0 - p_1) > 0\) is omitted, then another possible solution is

\[ p_0 i = \binom{i}{n} \quad \text{and} \quad p_1 = (1 - p_0). \]

Other variations of the solutions (6.1) exist depending on the assumption made on \( F \). We also note that the results of Arnold (1980) and Arnold and Ghosh (1976) are again direct consequences of Corollary 1.6.1.

Beg and Kirmani (1979), Kirmani and Alam (1980) proved the proposition mentioned in Theorem 6.2 and 6.3, which follow directly from Theorem 1.6.1.

**Theorem 6.2:** Let \( X_{1:n} < X_{2:n} < \cdots < X_{n:n} \) denote the ordered statistics of a random sample of \( n \) from a continuous distribution \( F(x) \) with \( F(0) = 0 \leq F(x) < 1 \) for all \( x \in [0, \infty) \). Let \( c > 0 \) be a constant and put

\[ V_{i:n}(c) = \min\{X_{i+1:n} - X_{i:n}, c\}. \quad (6.2) \]

Then, upto a change of scale and location

\[ F(x) = 1 - e^{-x}, \quad x \geq 0 \]

if and only if, for some integer \( i \) less than \( n \) and some constant \( \mu \in (0, c) \),

\[ E[V_{i:n}(c) | X_{i:n} = x] = \mu \quad \forall x \in [0, \infty). \quad (6.3) \]

Note that the condition (6.3) implies

\[ \int_0^x G^m(x + t) dt = \mu \quad G^m(x), \quad G(x) = 1 - F(x). \]
Then, applying Theorem 1.6.1, we find

\[ G^m(x) = e^{-\lambda x}, \]

which proves the desired result.

**Theorem 6.3:** Let \( X_{1:n} \leq X_{2:n} \) be ordered statistics in a sample of size 2 from discrete distribution on 0,1,2,\( \cdots \). Then

\[ E(X_{2:n} - X_{1:n} | X_{1:n} = x) = \mu \quad \text{for} \quad x = 0,1,\cdots \quad (6.4) \]

if and only if \( X \) has geometric distribution.

Defining \( \overline{G}_r = p_r + p_{r+1} + \cdots \), where \( P(X = r) = p_r \), the condition (6.4) implies

\[ \overline{G}_{r+1} + \overline{G}_{r+2} + \cdots = \mu \overline{G}_r, \quad r = 0,1,\cdots \quad (6.5) \]

Then, applying Theorem 1.6.1, we have

\[ \overline{G}_r = \alpha \beta^r \quad \text{with} \quad \alpha = 1, \quad (6.6) \]

which proves the desired result.

If we alter the condition (6.4) as

\[ E\left[ \min(X_{2:n} - X_{1:n}, c) | X_{1:n} = x \right] = \mu, \quad \text{for} \quad x = 0,1,\cdots , \quad (6.7) \]

then we have

\[ \overline{G}_{r+1} + \cdots + \overline{G}_{r+c} = \mu \overline{G}_r, \quad r = 0,1,\cdots , \]

and, again by applying Corollary 1.6.1, \( G_r \) is as in (6.6).
CHAPTER 3

CHARACTERIZATION OF DISTRIBUTIONS THROUGH
RECORD VALUES

1. Introduction

2. Characterization through record values

**Theorem 2.1:** Let \( \{R_i; \ i=1,2,\cdots\} \) be a sequence of record values corresponding to a \( dF \). For some \( k \geq 1 \), \( R_{k+1} - R_k = X_1 \), where \( X_1 \) is a random variable with \( dF \), if and only if \( X_1 \) is exponential or, for some \( a > 0 \), \( X_1 \) is geometric on \{a,2a,\cdots\} (i.e. \( a^{-1}X_1 - 1 \) is geometric in the usual sense).

The following condensed version of essentially Witte’s proof is given in Rao and Shanbhag (1994)

**Proof.** Clearly the condition that \( R_{k+1} - R_k \overset{d}{=} X_1 \) is equivalent to \( X_1 > 0 \ a. \ s. \) (i.e. \( F(0) = 0 \)) and
\[ \frac{F_k(dy)}{F(y)} F_k(dy) = \bar{F}(x), \quad x \in (0, \infty), \quad (2.1) \]

where \( F_k \) is the df of \( R_k \) and \( \bar{F}(\cdot) = 1 - F(\cdot) \). Note that if (2.1) holds with \( F(0) = 0 \) then given any point \( s_0 \in \sup[F] \) and hence \( \bar{s}_0 = \sup[F_k] \). Consequently, from the condition, we get that the smallest closed subgroup of \( \mathfrak{R} \) containing \( \sup[F_k] \) equals that containing \( \sup[F] \). In view of Theorem 1.6.1, we have then immediately that if (2.1) holds with \( F(0) = 0 \), then either \( X_1 \) is exponential or for some \( a > 0 \), \( a^{-1}X_1 - 1 \) is geometric (in the usual sense). The converse of the assertion is trivial and hence we have the theorem.

3. Characterization through conditional distribution of records

**Theorem 3.1** Let \( \{R_i : i = 1, 2, \ldots\} \) be as in Theorem 2.1, \( k \) be a positive integer, and \( \phi \) be a non constant real monotonic left continuous function on \( \mathbb{R}_+ \) such that \( \mathbb{E}(|\phi(R_{k+1} - R_k)|) < \infty \) and (2.4.7) is met with \( G \) replaced by \( F_k \), where \( F_k \) is the df of \( R_k \). Then for some \( c \neq \phi(0+) \),

\[ \mathbb{E}\{\phi(R_{k+1} - R_k) | R_k \} = c \quad (3.1) \]

if and only if the left extremity, \( l_k \), of the distribution of \( R_k \) is finite, and either \( \phi \) is nonarithmetic and the conditional distribution of \( X_1 - l_k \) given that \( X_1 > l_k \) is exponential or for some \( \lambda > 0 \), \( \phi \) is arithmetic with span \( \lambda \) and for some \( \beta \in (0, 1) \)

\[ P\{X_1 - l_k \geq x + n\lambda\} = \beta^n P\{X_1 - l_k \geq x\}, \quad x > 0: \quad n = 0, 1, \ldots, \]

where \( X_1 \) is a random variable distributed with df \( F \).
(3.1) can be expressed as (2.4.8) with \( Y \) such that its distribution is given by

the conditional distribution of \( X_1 \) given that \( X_1 \geq l_k \) and \( Z = R_k \), and

hence the theorem follows easily as a Corollary to Theorem 2.2.7. (Note that we allow here the case with \( P\{Y = l_k\} = 0 \) )

**Corollary 3.1** Let the assumption in Theorem 3.1 be met. Then the following assertions hold:

(i) If \( F \) is continuous or has its left extreme one of its continuity points and \( \phi \) is nonarithmetic, then, for some \( c \neq \phi(0+) \), (3.1) is met if and only if \( F \) is exponential, within a shift.

(ii) If \( \phi \) is arithmetic with span \( a \), and \( F \) is arithmetic with span grater than or equal to \( a \) then for some \( c \neq \phi(0+) \), (3.1) is valid if and only if \( F \) has a finite left extremity and the conditional distribution of the residual value of \( X_1 \) over the \( k^{th} \) support point of \( F \) given that this is positive is geometric on \( (a, 2a, \ldots, \} \), where \( X_1 \) is a random variable with \( df F \) (The corollary follows trivially from Theorem 3.1)

**Theorem 3.2:** Let \( \{R_i\} \) be as defined in the previous theorem, but with \( F \) continuous. Let \( k_2 > k_1 \geq 1 \) be fixed integers. Then, on some interval of the type \((-\infty, a)\), with \( a \) greater than the left extremity of the distribution of \( R_{k_1} \),

the conditional distribution of \( R_{k_2} - R_{k_1} \) given \( R_{k_1} = x \) is independent of \( x \) for almost all \( x \) if and only if \( F \) is exponential, within a shift.

It easily follows that the independence in question is equivalent to the condition that for almost all \( [F]_{c \in (\infty, \min\{a, b\})} \), where \( b \) is the right extremity of \( F \), the r.v. \( R^{(c)}_{k_2 - k_1} - c \), where \( R^{(c)}_{k_2 - k_1} \) is \( (k_2 - k_1)^{th} \)
record value corresponding to an iid sequence with df $F(c)$ such that

$$F_c(x) = \begin{cases} \frac{F(x) - F(c)}{1 - F(c)}, & \text{if } x > c \\ 0, & \text{o.w.} \end{cases}$$

is distributed independently of $c$. As $F$ is continuous, the latter condition is seen to be equivalent to the condition that the left extremity $l$, of $F$ is finite and for some $a > l$

$$[1 - F(c + x)] = [1 - F(l + x)][1 - F(c)],$$

$$c \in (-\infty, a) \cap \sup \{F \}, x \in \mathbb{N}_+.$$

(Note that the last equation implies that $b = \infty$). In view of the Marsaglia and Tubilla (1975) result, the assertion of theorem then follows. (Incidentally, the Marsaglia-Tubilla result referred to here could be arrived at as a corollary to either of Theorem 1.6.1.

Remarks 1:

(i) Downton’s (1969) result (with the correction as in Fosam et al. (1993) may be viewed as a specialized version of Theorem 3.1 for $k = 1$. Moreover, if we assume $F$ to be concentrated on \{0,1,2,\ldots\} with $F(1) - F(1-) > 0$ (and the right extremity condition met), then as a corollary to the theorem it follows that, for some $k \geq 1, R_k + 1 - R_k = X_1 + 1$ if and only if $F$ is geometric.

(ii) When $F$ is continuous, Theorem 3.2 holds without the left continuity assumption of $\phi$. (This is also of Corollary 2.2.4)

(iii) A version of Dallas’s (1981) result has also appeared in Nayak (1981). With obvious alternations in its proof, one could easily see that Theorem 3.3
holds with $R_{k2}$ and $R_{k1}$ replaced respectively by $X_{k2,n}$ and $X_{k1,n}$ (assuming of course that $n \geq k_2 > k_1 \geq 1$). A variant of this latter result appears in Gather (1989), the result in this case is that if $F$ is continuous $df$ with support equal to $\mathbb{R}_+$, then $F$ is exponential if and only if

$$X_{j_r:n} - X_{i:n} = X_{j_r-i:n-i}, \ r = 1, 2 \text{ holds for fixed } 1 \leq i < j_1 < j_2 \leq n \text{ and } n \geq 3.$$

4. Characterization of the distribution function by linear regression of upper Record: (Dembinska and Wesolowski, 2000, Athar et al., 2003)

Let $\{X_n, n \geq i\}$ be a sequence of iid continuous random variables with distribution function $F(x)$ and probability density function $f(x)$. Let $R_s$ be the $s^{th}$ upper record value, then the conditional pdf of $R_s$ given $R_r = x, 1 \leq r < s$ is (Ahsanullah, 1995)

$$f(R_s | R_r = x) = \frac{1}{\Gamma(s-r)} [-\ln F(y) + \ln F(x)]^{s-r-1} \frac{f(y)}{F(x)}$$

Here we write the result of Rao and Shanbhag (1994), which will be used

Let

$$\int_{R_+} G(u + v) \mu(du) = G(v) + c^* \text{ a.e.}[L] \text{ for } u \in R_+ = [0, \infty)$$

(4.2)

where $G : R_+ \to R = (-\infty, +\infty)$ is locally integrable Borel measurable function and $\mu$ is a $\sigma$-finite measure on $R_+$ with $\mu(\{0\}) < 1$ then

$$G(r) = \begin{cases} 
\gamma + \alpha'(1 - \exp(\eta x)) & \text{a.e.}[L] \text{ if } \eta \neq 0.
\gamma + \beta' x & \text{a.e.}[L] \text{ if } \eta = 0.
\end{cases}$$

(4.3)

where $\alpha', \beta', \gamma$ are constant and $\eta$ is such that

$$\int_{R_+} \exp(\eta x) \mu(dx) = 1$$

(4.4)
It has been shown that if linear regression is of the form \( a^* x + b^* \), then
if \( a^* < 1 \), the distribution is power function,
if \( a^* = 1 \), the distribution is exponential
if \( a^* > 1 \), the distribution is Pareto.

**Theorem 4.1:** Let \( X \) be an absolutely continuous rv with df \( F(x) \) and
pdf \( f(x) \) on the support \((\alpha, \beta)\), where \( \alpha \) and \( \beta \) may be finite or infinite.
Then for \( r < s \),
\[
E[R_s | R_r = x] = a^* x + b^*
\]
(4.5)
if and only if
\[
\overline{F}(x) = [ax + b]^c
\]
(4.6)
where \( a^* = \left( \frac{c}{c+1} \right)^{s-r} \) and \( b^* = -\frac{b}{a} [1 - a^*] \).

**Proof:** First we prove (4.6) implies (4.5).
We have \( \overline{F}(x) = [ax + b]^c \), \( f(x) = -ac[ax + b]^{c-1} \)
Now, from (4.1)
\[
f(R_s | R_r = x) = \frac{1}{\Gamma(s-r)} \left[ y \ln F(y) + \ln \overline{F}(x) \right]^{s-r-1} \frac{f(y)}{\overline{F}(x)} dy
\]
\[
= \frac{1}{\Gamma(s-r)[ax + b]^c} \left[ y c \ln \left( \frac{ax + b}{ay + b} \right) \right]^{s-r-1} ac[ay + b]^{c-1} dy
\]
Let \( t = \ln \left( \frac{ax + b}{ay + b} \right)^c \), then R.H.S. is
\[
\frac{1}{\alpha \Gamma(s - r)} \int ([ax + b]e^{-t/c - b})^{s-r-1} e^{-t} dt
\]

which reduces to
\[
\left( \frac{c}{c + 1} \right)^{s-r} x + \frac{b}{a} \left[ \left( \frac{c}{c + 1} \right)^{s-r} - 1 \right]
\]

(4.7)

To prove (4.5) implies (4.6), we have
\[
\frac{1}{\Gamma(s - r)} \int y [- \ln \bar{F}(y) + \ln \bar{F}(x)]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy = a^* x + b^*
\]

Now set \( \bar{F}(x) = e^{-x} \) and \( \bar{F}(x) = e^{-(u+v)} \), then
\[
\frac{1}{\Gamma(s - r)} \int_0^\infty \bar{F}^{-1} [e^{-(u+v)}] u^{s-r-1} e^{-u} du = a^* \bar{F}^{-1}(e^{-v}) + b^*
\]

Set \( G(v) = \bar{F}^{-1}(e^{-v}) \) to get
\[
\int G(u + v) \mu du = G(v) + \frac{b^*}{a^*}
\]

(4.8)

where \( \mu(du) = \frac{1}{\Gamma(s - r) a^*} u^{s-r-1} e^{-u} du \)

(4.9)

Now, since (4.8) is of the form (4.2) and therefore in view of (4.2), (4.3) and (4.4), we get
\[
G(v) = \gamma + \alpha'(1 - e^{\eta v}) \quad \text{if} \quad \eta \neq 0
\]

Thus at \( \eta \neq 0 \)
\[
G(v) = \bar{F}^{-1}(e^{-v}) = \gamma + \alpha'(1 - e^{\eta v})
\]

or,
\[
e^{-v} = \bar{F}^{-1}[\gamma + \alpha'(1 - e^{\eta v})]
\]
let \( z = \gamma + \alpha'(1 - e^{\eta v}) \) and therefore,

\[
F(z) = \left[ 1 - \frac{z - \gamma}{\alpha'} \right]^n = [az + b]^c
\]

where \( a = -\frac{1}{\alpha'}, \ b = \frac{\alpha' + \gamma}{\alpha'}, \ c = -\frac{1}{\eta} \)

(4.10)

In view of (4.4) and along with (4.9) and (4.10), we get

\[
a^* = \frac{1}{(1 - \eta)^{s-r}} = \left( \frac{c}{c + 1} \right)^{s-r}
\]

(4.11)

Further at \( v = 0 \), we have from (4.2) and (4.3),

\[
\int G(u) \mu d(u) = G(0) + c^*
\]
or, \( G(0) + c^* = \int (\gamma + \alpha' - \alpha' e^{\eta x}) \mu d(x) \)

i.e. \( \gamma + \frac{b^*}{a^*} = \frac{\gamma + \alpha'}{\alpha'} - \alpha' \Leftrightarrow (\gamma + \alpha') = \frac{b^*}{-a^*} \)

Using (4.10),

\[
\gamma + \alpha' = -\frac{b}{a} = -\frac{b^*}{1 - a^*}
\]

and hence \( b^* = -\frac{b}{a} (1 - a) \).

(4.12)

For \( \eta = 0 \), from (4.3),

\[
G(v) = \frac{1}{F^{-1}(e^{-v})} = \gamma + \beta' v
\]
or \( F(z) = e^{-\lambda(z - \gamma)}, \ \lambda > 0 \)

(4.13)

where \( \lambda = \frac{1}{\beta'} \). Therefore
\[
F(z) = \left[1 - \frac{\lambda(z - \gamma)}{c}\right]^c = [az + b]^c \quad \text{as } c \to \infty \quad (\eta \to 0)
\]

where \( a = -\frac{\lambda}{c}, b = \frac{c + \lambda \gamma}{c} \). So

\[
a^* = \left(\frac{c}{c+1}\right)^{s-r} \to 1 \quad \text{as } c \to \infty
\]

and

\[
b^* = \frac{b}{a} \left(\frac{c}{c+1}\right)^{s-r} - 1
\]

Therefore,

\[
\frac{b}{a} = -\frac{c + \lambda \gamma}{\lambda} = -\frac{t(1-\lambda \gamma) + \lambda \gamma}{(1-t)\lambda}
\]

and with \( t = \frac{c}{c+1} \)

\[
b^* = \frac{t^{s-r+1}(1-\lambda \gamma) + t^{s-r} \lambda \gamma - t(1-\lambda \gamma) - \lambda \gamma}{(t-1)\lambda}
\]

Differentiating numerator and denominator separately \( w.r.t. \ t \) taking limit as \( t \to 1 \), we get

\[
b^* = \frac{(s-r)}{\lambda}
\]

The value of \( a^* \) and \( b^* \) at \( \eta = 0 \) could have also been obtained from (4.3), (4.4), (4.8) and (4.9) as earlier and hence the Theorem.

**Examples**

We have considered here the distributions as given in (Dembinska and Wesolowski, 1998)

**a) Power function distribution**
\[
\bar{F}(x) = \left(\frac{\beta - x}{\beta - \alpha}\right)^\theta = \left[\frac{1}{\beta - \alpha} x + \frac{\beta}{\beta - \alpha}\right]^\theta, \quad \alpha \leq x \leq \beta
\]  
(4.15)

\[a = \frac{1}{\beta - \alpha}, \quad b = \frac{\beta}{\beta - \alpha}, \quad c = \theta
\]

\[a^* = \left(\frac{\theta}{\theta + 1}\right)^{s-r} < 1, \quad b^* = \left[1 - \left(\frac{\theta}{\theta + 1}\right)^{s-r}\right]
\]  
(4.16)

b) Pareto distribution

\[
\bar{F}(x) = \left(\frac{\alpha + \delta}{x + \delta}\right)^\theta = \left(\frac{1}{\alpha + \delta} x + \frac{\delta}{\alpha + \delta}\right)^{-\theta}, \quad \alpha \leq x < \infty, \theta > 0, \alpha + \delta > 0
\]  
(4.17)

\[a = \frac{1}{\alpha + \delta}, \quad b = \frac{\delta}{\alpha + \delta}, \quad c = -\theta
\]

\[a^* = \left(\frac{\theta}{\theta - 1}\right)^{s-r} > 1, \quad b^* = \left[\frac{(\theta)^{s-r}}{(\theta - 1)^{s-r}} - 1\right]
\]  
(4.17)

c) Exponential distribution

\[
\bar{F}(x) = e^{-\lambda(x-\alpha)}, \quad x \geq \alpha; \lambda > 0
\]

\[a = -\frac{\lambda}{c}, \quad b = \frac{c + \lambda \alpha}{c}, \quad c \to \infty
\]  
(4.18)

\[a^* = 1, \quad b^* = \frac{(s-r)}{\lambda}
\]

Remark. If \(h(x)\) be a monotonic function of \(x\) on support \((\alpha, \beta)\), then we have

\[
E[h(R_s)\mid R_r = x] = a^* h(x) + b^*
\]

if and only if

\[
\bar{F}(x) = [ah(x) + b]^c
\]

with the same \(a^*\) and \(b^*\) as given in (4.11) and (4.12). Therefore, with suitable choice of \(a, b, c\) and \(h(x)\), we will get various characterization results of distribution [Khan and Abouammoh, 2000].
5. Characterization by linearity of regression of $R_m$ on $R_{m+k}$

[Dembinska and Wesolowski, 2000]

In this section we will determine the distributions for which

$$E(R_m \mid R_{m+k} = y) = cy + d,$$  \hspace{1cm} (5.1)

where $m$ and $k$ are some positive integers. From (5.1) one gets immediately,

$$\frac{(m+k)!}{m!(k-1)!} \frac{1}{R_{m+k}(y)} \int_{-\infty}^{y} x R^m(x) [R(y) - R(x)]^{k-1} d(R(x)) = cy + d, \hspace{1cm} (5.2)$$

Now apply an idea, relating the conditional distribution of records to distribution of order statistics Arnold et al. (1998) having its origin in Nagaraja (1988). Note that the conditional joint distribution of $R_1, \cdots, R_n$ given $R_{n+1}$ is identical with the joint distribution of the order statistics from a random sample of size $n$ from the df

$$F(x \mid y) = \begin{cases} R(x) / R(y), & x < y, \\ 1, & x \geq y. \end{cases}$$

Therefore

$$\frac{(m+k)!}{m!(k-1)!} \frac{1}{Q_z^{m+k}(y)} \int_{-\infty}^{y} x Q_z^m(x) [Q_z(y) - Q_z(x)]^{k-1} d[Q_z(x)] = cy + d$$

for $y \in (l_F, r_F)$ and $\forall z \in (l_F, r_F)$, where

$$Q_z(y) = \begin{cases} R(y) / R(z), & \text{for } y \leq z \\ 1, & \text{for } y > z \end{cases}$$
is a continuous df. Thus, condition (5.1) is equivalent to the set of conditions:

\[
E(Y_{m+1:n}^{(z)} \mid Y_{m+1+k:n}^{(z)}) = cY_{m+1:n}^{(z)} + d
\]

(5.3)

\( \forall z \in (l_F, r_F) \), where \( Y_1^{(z)}, Y_2^{(z)}, \ldots \) are iid rv's with df \( Q_z \) and \( n \) is a natural number such that \( n \geq k + m + 1 \).

To determine the distributions for which (5.3) holds, we will use the following auxiliary result, which is a version of the main theorem from Dembinska and Wesolowski (1998).

**Lemma 5.1.** Assume that \( X_1, X_2, \ldots \) are iid rv's with a common continuous df \( F \). Let \( E(|X_k:n|) < \infty \). If for some \( k \leq n - r \) and some real \( c \) and \( d \)

\[
E(X_k:n \mid X_{k+r:n}) = cX_{k+r:n} + d.
\]

(5.4)

then only the following three cases are possible:

1. \( c = 1 \) and \( F(x) = e^{\lambda(x-\gamma)} \) for \( x \leq -\gamma \). \( F(x) = 1 \) for \( x > -\gamma \), where \( \gamma \) and \( \lambda > 0 \) are some real numbers (the negative exponential distribution).

2. \( c > 1 \) and \( F(x) = [(\mu + v)/(-x + v)]^\theta \) for \( x < -\mu \), \( F(x) = 1 \) for \( x \geq -\mu \), where \( \theta > 0, v, \mu \) are some real numbers, \( \mu + v > 0 \) (the negative Pareto distribution).

3. \( 0 < c < 1 \) and \( F(x) = 0 \) for \( x \leq -\sigma \), \( F(x) = [(\sigma + x)/(\sigma - \mu)]^\theta \) for \( x \in (-\sigma, -\mu) \) and \( F(x) = 1 \) for \( x \geq -\mu \), where \( \theta > (1/k), \sigma, \mu \) are some real numbers such that \( \mu < \sigma \) (the negative power distribution).

**Proof.** Putting \( Y_k = -X_k \) condition (5.4) turns into

\[
E(Y_{n-k+1:n} \mid Y_{n-k-r+1:n}) = cY_{n-k-r+1:n} - d
\]

(5.5)

(because \( X_{k:n} = -Y_{n-k+1:n} \)) Now the result follows immediately from Dembinska and Wesolowski (1998).
Thus, if (5.3) holds then only the following three cases are possible:

1. \( c = 1 \) and \( Q_z(y) = e^{\lambda(y-z)} \) for \( y \leq z, \forall z \in (l_F, r_F) \). Then \( \forall z \in (l_F, r_F) \) and \( \forall y \leq z \)

\[
e^{-\lambda z} \log[1 - F(z)] = e^{\lambda y} \log[1 - F(y)].
\]

Hence \( \forall z \in (l_F, r_F) \)

\[
e^{-\lambda z} \log[1 - F(z)] = \text{const.}
\]

consequently \( F(y) = 1 - e^{-e^{\lambda(y-z)}} \) \( \forall y \in R \), where \( \lambda > 0 \) and \( y \in R \). Thus \( F \) is a negative Gumbel-type df.

2. \( c > 1 \) and \( Q_z(y) = [(z + v)/(y + v)]^\theta \) for \( y < z, \forall z \in (l_F, r_F) \). Then, as in point 1 it follows that \( F(y) = 1 - e^{-(x-y)}/(x-y)]]^\theta \), for \( y \in (-\infty, \alpha) \), where \( \theta > 1/(m+1) \) and \( \alpha > \gamma \); consequently \( F \) is a negative Frechet-type df.

3. \( 0 < c < 1 \) and \( Q_z(y) = [(\sigma + y)/(\sigma + z)]^\theta \) for \( y \in (-\sigma, z), \forall z \in (l_F, r_F) \).

Then similarly as above it follows that \( F(y) = 1 - e^{-(y-\mu)/(\sigma-\mu)]^\theta} \), for \( y \in (\mu, \infty) \) where \( \sigma > \mu \), i.e. \( F \) is Weibull type df.

The above discussion proves the result dual to Theorem 5.1

**Theorem 5.1:** Let \( X_1, X_2, \ldots \) be iid rvs with a continuous df such that \( E(|R_m|) < \infty \), where \( m \) is a positive integer. If for some positive integer \( k \)

\[
E(R_m | R_{m+k}) = cR_{m+k} + d,
\]

where \( c \) and \( d \) are real numbers, then only the following three cases are possible:

(i). \( c = 1 \) and \( X_1 \) falls Negative Gumble distribution with pdf denoted by \( X_1 \sim NG(\beta, \gamma) \)
\[ f(x) = \beta \exp(\beta(x - \gamma)) / \exp(-e^{\beta(x - \gamma)}) \]

(ii). \( 0 < c < 1 \) and \( X_1 \) follows Negative Frechet distribution with pdf \( f \) denoted by \( X_1 \sim NF(\theta; \gamma, \alpha) \),

\[ f(x) = \frac{\theta(\alpha - \gamma)^{\theta-1}}{(\alpha - x)^\theta} \exp\left( -\frac{\alpha - \gamma}{\alpha - x} \right) \]

(iii). \( c > 1 \) and \( X_1 \) follows Weibull distribution with pdf \( f \) denoted by \( X_1 \sim W(\theta; \mu, \gamma) \).

\[ f(x) = \frac{\theta(x - \gamma)^{\theta-1}}{(\gamma - \mu)^\theta} \exp\left( -\frac{x - \mu}{\gamma - \mu} \right) \]

Related results were also obtained by Franco and Ruiz (1997) and Lopez Blaquex and Moreno Rebollo (1997).
CHAPTER 4
CHARACTERIZATION OF DISTRIBUTIONS THROUGH LINEAR REGRESSION OF GENERALIZED ORDER STATISTICS

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed (iid) random variables (rv) with absolutely continuous distribution function (df) $F(x)$ and probability density function $f(x)$.

Let $n \in \mathbb{N}, n \geq 2, k > 0, \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{N}^{n-1}, M_r = \sum_{j=r}^{n-1} m_j$, such that $r = k + n - r + M_r > 0$ for all $r \in \{1, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k), r = 1, 2, \ldots, n$ are called gos if their joint pdf is given by

$$f(x_1, \ldots, x_n) = \frac{k \prod_{j=1}^{n} \gamma_j \prod_{i=1}^{n-1} (1 - F(x_i))^m_i f(x_i) (1 - F(x_n))^k f(x_n)}{\prod_{i=1}^{n-1} (1 - F(x_i))^m_i f(x_i) (1 - F(x_n))^k f(x_n)}$$

Choosing the parameters appropriately, results for models such as ordinary statistics ($\gamma_i = n - i + 1; \ i = 1, \ldots, n$ i.e. $m_1 = \cdots = m_{n-1} = 0, k = 1$), $k$-th record values ($\gamma_i = k, i.e. m_1 = \cdots = m_{n-1} = -1, k \in \mathbb{N}$), sequential order statistics ($\gamma_i = (n - i + 1)\alpha_i; \ \alpha_1, \ldots, \alpha_n > 0$), order statistics with non integral sample
size \( (\gamma_i = \alpha - i + 1; \ \alpha > 0) \), Pfeifer record values \( (\gamma_i = \beta_i; \ \beta_1, \ldots, \beta_n > 0) \) and progressive type II censored order statistics \( (m_i \in N, k \in N) \) [Kamps (1995) and Kamps and Cramer (2001)] can be obtained.

Here taken two cases are considered.

Case I : \( m_1 = \cdots = m_{n-1} = m \)

Case II : \( m_i \neq m_j, \ \gamma_i \neq \gamma_j, \ i, j = 1, \ldots, n-1 \).

For case I, \( gos \) will be denoted as \( X(r,n,m,k) \) and its pdf is (Kamps, 1995)

\[
f_r(x) = \frac{c_{r-1}}{(r-1)!} [1 - F(x)]^{k+(n-r)(m+1)-1} f(x)[g_m(F(x))]^{r-1}. \tag{1.2}
\]

and the joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k), 1 \leq r < s \leq n \) is

\[
f_{rs}(x) = \frac{c_{s-1}}{(s-r-1)!} [1 - F(x)]^m [g_m(F(x))]^{r-1}
[\gamma_m(F(y) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{k+(n-s)(m+1)-1} f(x)f(y) \tag{1.3}
\]

where

\[
c_{r-1} = \prod_{i=1}^{r} \gamma_i \tag{1.4}
\]

\[
h_m(x) = \begin{cases} 
-\frac{1}{m} (1-x)^{m+1}, & m \neq -1 \\
\log(\frac{1}{1-x}), & m = -1
\end{cases} \tag{1.5}
\]

and

\[
g_m(x) = h_m(x) - h_m(0) \tag{1.6}
\]

For the case II, the pdf of \( X(r,n,\bar{m},k) \) is [Kamps and Cramer, 2001]
\[ f_{X(r,n,m,k)}(x) = c_{r-1} f(x) \sum_{i=1}^{r} a_i (1 - F(x))^\gamma_i - 1 \]  

and the joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k), 1 \leq r < s \leq n \) is

\[ f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1 - F(y)}{1 - F(x)} \gamma_i \]

\[ \times \left[ \sum_{i=1}^{r} a_i(r)(1 - F(x))^\gamma_i \right] \frac{f(x)}{(1 - F(x))} \frac{f(y)}{(1 - F(y))} \]

where

\[ a_i = a_i^{(r)} = \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_j)}, \quad \gamma_j \neq \gamma_j, 1 \leq i \leq r \leq n \]  

\[ a_i^{(r)} = \prod_{j=r+1}^{s} \frac{1}{(\gamma_j - \gamma_j)}, \quad \gamma_j \neq \gamma_j, r+1 \leq i \leq s \leq n \]

Thus, the conditional pdf of \( X(s,n,m,k) \) given \( X(r,n,m,k) = x, 1 \leq r < s \leq n \) is

\[ f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1 - F(y)}{1 - F(x)} \gamma_i \frac{f(y)}{(1 - F(y))}, \quad x \leq y \]

It has been shown by Khan and Alzaid (2004) that for

\[ E[X(s,n,m,k)|X(r,n,m,k) = x] = a^* x + b^* \]

where

(i) \( a^* < 1 \), and the distribution is power function.

(ii) \( a^* > 1 \), and the distribution is Pareto.
The problem is solved using Rao and Shanbhag's result (1994)

Let

$$\int_{R^+} G(u + v) \mu(du) = G(v) + c^*$$  \hspace{1cm} (1.12)

for $u \in R^+ = [0, \infty)$, where $G : R^+ \rightarrow R = (-\infty, \infty)$ is a locally integrable Borel measurable function and $\mu$ is a $\sigma$-finite measure on $R_+$ with $\mu(\{0\}) < 1$ then

$$G\left[\gamma + \alpha'[1 - \exp(\eta x)]\right] \text{ a.e.}[L] \text{ if } \eta \neq 0$$

$$G\left[\gamma + \beta'x\right] \text{ a.e.}[L] \text{ if } \eta = 0$$  \hspace{1cm} (1.13)

where $\alpha', \beta', \gamma$ are constant and $\eta$ is such that

$$\left\{ \begin{array}{ll}
\int_{R^+} \mu(dx) = 1
\end{array} \right.$$  \hspace{1cm} (1.14)

2. Characterization of Distribution when $m_i = m_j; i, j = 1, \ldots, n - 1$

Let $X(r, n, m, k)$, $r = 1, \ldots, n$ be gos, then the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x, 1 \leq r < s \leq n$, in view of (1.2) and (1.3), is

$$f_{s|r}(y | x) = \frac{c_{s-r-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\overline{F}(y)]^{k+(n-s)(m+1)-1} f(y)}{(s-r-1)!c_{r-1} \left[ \overline{F}(x) \right]^{k+(n-r-1)(m+1)}}$$  \hspace{1cm} (2.1)

where $\overline{F}(x) = P[X > x] = 1 - F(x)$.

We have,

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1 - t)^m dt$$

$$= \frac{1 - (1 - x)^{m+1}}{m+1}, \quad m \neq -1$$
Also since
\[ \lim_{m \to -1} g_m(x) = -\log(1-x) \]
Therefore, we will consider only the case
\[ g_m(x) = \frac{1-(1-x)^{m+1}}{m+1} \]
for all \( m \), unless needed otherwise.
Thus (2.1) reduces to
\[
fs(y|x) = \frac{c_{s-1}}{(s-r-1)!c_{r-1}} \left[ 1 - \left( \frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \left[ \frac{F(y)}{F(x)} \right]^{k+m+1(n-s)-1} \frac{f(y)}{F(x)}
\] (2.2)

**Theorem; 2.1** (Khan and Alzaid., 2004): Let \( X \) be an absolutely continuous rv with \( df \ F(x) \) and \( pdf \ f(x) \) on support \((\alpha, \beta)\), where \( \alpha \) and \( \beta \) may be finite or infinite. Then for \( r < s \)
\[ E[X(s,n,m,k)\mid X(r,n,m,k) = x] = a^* x + b^* \] (2.3)
if and only if
\[ F(x) = [ax + b]^c, \quad x \in (\alpha, \beta) \] (2.4)
where
\[
\begin{align*}
a^* &= \prod_{j=1}^{s-r} \frac{c[k + (m+1)(n-r-j)]}{c[k + (m+1)(n-r-j)]+1} = \prod_{j=r+1}^{s} \frac{c_j}{1+c_j} \\
b^* &= -\frac{b}{a}(1-a^*). \end{align*}
\] (2.5)
**Proof:** First we will prove that (2.4) implies (2.3)

We have

\[ \bar{F}(x) = [ax + b]^c \]

\[ f(x) = -ac[ax + b]^{c-1} \]

Now from (2.2),

\[
E[X(s, n, m, k) | X(r, n, m, k) = x] = \frac{c_{s-1}}{c_{r-1}(s - r - 1)!(m + 1)^{s-r-1}} \]

\[
\times \int_y \left[ 1 - \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right]^{s-r-1} \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{k+(m+1)(n-s)-1} \frac{f(y)}{\bar{F}(x)} dy \tag{2.6} \]

Set \( u = \frac{\bar{F}(y)}{\bar{F}(x)} = \left[ \frac{ay + b}{ax + b} \right]^c \), then the RHS reduces to

\[
= \frac{c_{s-1}}{c_{r-1}(s - r - 1)!(m + 1)^{s-r-1}} \int_b^1 \left[ \frac{u^{1/c(ax+b)-b}}{a} \right] [1-u^{m+1}]^{s-r-1} u^{k+(m+1)(n-s)-1} du. \]

Let \( u^{m+1} = t \), then we get

\[
E[X(s, n, m, k) | X(r, n, m, k) = x] = a^* x + b^* \]

where

\[ b^* = -\frac{b}{a}(1 - a^*) \]

and

\[
a^* = \frac{c_{s-1}}{c_{r-1}(s - r - 1)!(m + 1)^{s-r-1}} \int_0^1 \left( \frac{1+c[k+(m+1)(n-s)]}{c(m+1)} \right)^{t-1} (1-t)^{s-r-1} dt \]

which gives
\[ a^* = \prod_{j=1}^{s-r} \frac{c[k + (m + 1)(n - r - j)]}{c[k + (m + 1)(n - r - j)] + 1} = \prod_{j=r+1}^{s} \frac{c^* y_j}{1 + c^* y_j} \]

after simplification. Hence the 'if' part.

To prove that (2.3) implies (2.4), we have from (2.6)

\[ \frac{c_{s-1}}{c_{r-1}(s - r - 1)! (m + 1)^{s-r-1}} \beta \left[ \frac{F(y)}{F(x)} \right]^{s-r-1} \]
\[ \times \left[ \frac{F(y)}{F(x)} \right]^{k+(m+1)(n-s)-1} f(y) dy = a^* x + b^* . \]

Set \( \bar{F}(x) = e^{-x} \) and \( \bar{F}(y) = e^{-(x+y)} \) to get,

\[ \frac{c_{s-1}}{c_{r-1}(s - r - 1)! (m + 1)^{s-r-1}} \int_0^\infty \bar{F}^{-1} \left[ e^{-(x+y)} \right][1 - e^{-u(m+1)}]^{s-r-1} \]
\[ \times e^{-u[k+(m+1)(n-s)]} du = a^* \bar{F}^{-1} (e^{-v}) + b^* \]

Further, if we set \( G(v) = \bar{F}^{-1} (e^{-v}) \), we get,

\[ \int_0^\infty G(u + v) \mu(du) = G(v) + b^* \]
\[ \frac{a^*}{a^*} \]

where

\[ \mu(du) = \frac{c_{s-1}}{a^* c_{r-1}(s - r - 1)! (m + 1)^{s-r-1}} [1 - e^{-u(m+1)}]^{s-r+1} \]
\[ \times e^{-u[k+(m+1)(n-s)]} du \]

Equation (2.7) is of the form (1.13) with \( c^* = \frac{b^*}{a^*} \) and therefore at \( \eta \neq 0 \),

\[ G(v) = \bar{F}^{-1} (e^{-v}) = \gamma + \alpha (1 - e^{\eta v}) \]

or,

\[ e^{-v} = \bar{F} (\gamma + \alpha (1 - e^{\eta v})) \]
Let \( z = \gamma + \alpha'(1 - e^{\eta v}) \)

Therefore,

\[
\bar{F}(z) = \left[1 - \frac{z - \gamma}{\alpha'}\right]^{\frac{1}{\eta}}
\]

\[= [az + b]^c\]

where

\[
a = -\frac{1}{\alpha'}, \quad b = \frac{\alpha' + \gamma}{\alpha'}, \quad c = -\frac{1}{\eta}
\]  

(2.9)

Now to see the relationship between \( a^*, b^* \) and \( a, b, c \), we have from (1.14) and (2.8),

\[
a^* = \frac{c_{s-1}B(s-r, k-\eta/m + n-s)}{c_{r-1}(s-r-1)!(m+1)^{s-r}}
\]

Putting \( \eta = -\frac{1}{c} \) and solving, we get,

\[
a^* = \prod_{j=1}^{s-r} \frac{c[k + (m+1)(n-r-j)]}{c[k + (m+1)(n-r-j)] + 1} = \prod_{j=r+1}^{s} \frac{c\gamma_j}{(1 + c\gamma_j)}
\]

as desired.

For \( b^* \), we have from (1.12) at \( v = 0 \)

\[
G(0) + c^* = \int_0^\infty G(u)\mu(du)
\]

This in view of (1.13) and (2.8) at \( \eta = 0 \) reduces to,

\[
\gamma + \frac{b^*}{a^*} = \frac{\gamma + \alpha'}{a^*} - \alpha'
\]
Therefore,
\[ b^* = -\frac{b}{a}(1 - a^*) \] in view of (2.9).

**Remark 2.1:** In the limiting case as \( c \to \infty \)
(i.e. when \( \eta = 0 \)), we have by Rao and Shanbhag (1994) result

\[ G(\nu) = \bar{F}^{-1}(e^{-\nu}) = \gamma + \beta' \nu \]
or,

\[ \bar{F}(z) = e^{-\lambda(z-\mu)} \]

where \( \lambda = \frac{1}{\beta' \nu}, \quad \mu = \gamma \)

Therefore,

\[ \bar{F}(z) = \left[ 1 - \frac{\lambda(z-\mu)}{c} \right]^c \quad \text{as } c \to \infty \]

\[ = [az + b]^c \]

where, \( a = -\frac{\lambda}{c}, \quad b = \frac{c + \lambda \mu}{c}, \quad c \to \infty \)

Thus \( a^* = 1 \) as \( c \to \infty \)

To find \( b^* \), we have,

\[ \gamma + \frac{b^*}{a} = \int_0^\infty (\gamma + \beta'x) \mu(dx) \]

\[ = \frac{\gamma}{a^*} \frac{\beta c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r+1}} \int_0^1 \ln t \; t^{n-s+k-1} (1-t)^{s-r-1} dt \]
\[
\frac{\gamma}{a^*} - \frac{\beta^* c_{s-1}}{a^* c_{r-1}(s-r-1)!} (m+1)^{s-r+1} B(n-s+\frac{k}{m+1}, s-r).
\]

\[
[\psi(n-s+\frac{k}{m+1}) - \psi(n-r+\frac{k}{m+1})]
\]

where \(\psi(x) = \frac{d}{dx} \ln \Gamma(x)\) [Gradshteyn and Ryzhik, 1980]

Since [Gradshteyn and Ryzhik, 1980]

\[
\psi(x-n) - \psi(x) = -\sum_{k=1}^{n} \frac{1}{x-k}
\]

Therefore,

\[
\frac{\gamma}{a^*} + \frac{b^*}{a^*} = \gamma + \frac{1}{a^*} \sum_{j=1}^{s-r} \frac{1}{\lambda(m+1) j! (n-r)+\frac{k}{m+1} - j}
\]

since \(a^* \to 1\) as \(c \to \infty\)

Hence,

\[
b^* = \frac{1}{\lambda} \sum_{j=1}^{s-r} \frac{1}{k+(n-r-j)(m+1)} = \frac{1}{\lambda} \sum_{j=1}^{s-r} \frac{1}{\gamma_{r+j}}
\]

3. Characterization of distributions when \(\gamma_i \neq \gamma_j, m_i \neq m_j, i, j = 1, \ldots, n-1\)

First we write the lemmas (Khan and Alzaid, 2004)

**Lemma 3.1:** For \(\gamma_j \neq \gamma_i\) and \(c\) any real number

\[
\sum_{i=r+1}^{s} \frac{1}{\prod_{j=r+1}^{s} (\gamma_j - \gamma_i)(1 + c\gamma_i)} = c^{s-r-1} \prod_{i=r+1}^{s} \frac{1}{(1 + c\gamma_i)}
\] \hspace{1cm} (3.1)
**Proof:** Since (1.11) is defined pdf, therefore,

\[ c_{s-1} \sum_{i=r+1}^{s} a_i(r)(s) \left( \frac{F(y)}{F(x)} \right)^{y_i - 1} \int_{x} f(y) \frac{dy}{F(x)} = 1 \]

implying

\[ \sum_{i=r+1}^{s} a_i(r)(s) \frac{c_{s-1}}{\gamma_i} = c_{r-1} \]

or,

\[ \sum_{i=r+1}^{s} \frac{1}{\prod_{j=r+1}^{s} (\gamma_j - \gamma_i)(1 + c \gamma_i)} = c^{s-r-1} \prod_{i=r+1}^{s} \frac{1}{\gamma_i} \]

Replacing \( \gamma_i \) by \((1 + c \gamma_i)\), we have

\[ \sum_{i=r+1}^{s} \frac{1}{\prod_{j=r+1}^{s} (\gamma_j - \gamma_i)(1 + c \gamma_i)} = c^{s-r-1} \prod_{i=r+1}^{s} \frac{1}{(1 + c \gamma_i)} \]

and hence the lemma.

**Theorem 3.1:** Let \( X \) be an absolutely continuous rv with df \( F(x) \) and pdf \( f(x) \) on support \((\alpha, \beta)\) where \( \alpha \) and \( \beta \) may be finite or infinite. Then for \( 1 \leq r < s \leq n, \)

\[ E[X(s, n, \bar{m}, k) | X(r, n, \bar{m}, k) = x] = a^* x + b^* \]

if and only if

\[ \bar{F}(x) = [ax + b]^c, \]
Proof: First we will prove that \( (3.4) \) implies \( (3.3) \).

Now from (1.11),

\[
E[X(s,n,\bar{m},k)|X(r,n,\bar{m},k)=x] = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \left[ \int_{x}^{F(y)} \frac{\beta(F(y))}{F(x)} \right]^{r-i} \frac{f(y)}{F(x)} dy
\]

Using the transformation \( u = \frac{F(y)}{F(x)} = \left( \frac{ay + b}{ax + b} \right)^{c} \), the RHS reduces to

\[
= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \int_{0}^{1} \left[ \frac{u^{1/c}(ay + b) - b}{a} \right] u^{r-i-1} du
\]

\[
= a^{*} x + b^{*}
\]

where

\[
b^{*} = -\frac{b}{a}(1-a^{*})
\]

and

\[
a^{*} = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \int_{0}^{1} u^{-1/c+y_{i}^{-1}} du
\]

\[
= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_{i}^{(r)}(s)}{(\gamma_{i+1/c}^{-1})^{c}}
\]

(3.6)

Since \( \frac{c_{s-1}}{c_{r-1}} = \prod_{i=r+1}^{s} \gamma_{i} \), we get using Lemma 3.1,
\[ a^* = \left( \prod_{i=r+1}^{s} \gamma_i \right) \sum_{l=r+1}^{s} c \frac{\gamma_j - \gamma_i}{\prod_{j=r+1, j \neq i}^{s} (1 + c \gamma_j)} \]

\[ = \prod_{i=r+1}^{s} \frac{c \gamma_i}{(1 + c \gamma_i)} \quad (3.7) \]

and hence the 'if' part.

To prove that (3.3) implies (3.4), we have

\[ \int \frac{f(y)}{F(x)} \, dy = a^* x + b^* \]

Set \( F(x) = e^{-v}, \quad F(y) = e^{-(u+v)} \) and then \( G(v) = F^{-1}(e^{-v}), \) to get,

\[ \int_0^\infty G(u + v) \mu(du) = G(v) + b^* \frac{v}{a^*} \quad (3.8) \]

where

\[ \mu(du) = \frac{c_{s-1}}{a^* c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) e^{-\gamma_i u} \quad (3.9) \]

Therefore, if \( \eta \neq 0, \) we get,

\[ \bar{F}(z) = [az + b]^c \]

where

\[ a = -\frac{1}{c}, \quad b = \frac{a' + \gamma}{a'}, \quad c = -\frac{1}{\eta}. \quad (3.10) \]

Now to see the relationship between \( a^*, b^* \) and \( a, b, c \) we have from (1.14) and (3.9),

\[ \frac{c_{s-1}}{a^* c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) = 1 \]
or,

\[ a^* = \frac{c_{s-1}}{a} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{c_{r-1}(i + c_{r-1})} \]

\[ = \prod_{i=r+1}^{s} \frac{c_{r-1}^{(i)}}{(1 + c_{r-1})}, \quad \text{from Lemma 3.1} \]

and

\[ b^* = -\frac{b}{a} (1 - a^*) \]

as explained in section 2.

Remark 3.1: In the limiting case as \( c \to \infty \) (i.e. when \( \eta = 0 \)), we have

\[ F(z) = e^{-\lambda(z-\mu)} = [az + b]^c \]

where \( a = -\frac{\lambda}{c}, \quad b = \frac{c + \lambda \mu}{c}, \quad c \to \infty. \)

Therefore,

\[ a^* = 1 \quad \text{as} \quad c \to \infty. \]

To find \( b^* \), since we have for any finite \( a \), [Khan and Abouammoh, 2000].

\[ \frac{1}{(a_1 + 1)} + \frac{a_1}{(a_1 + 1)(a_2 + 1)} + \cdots + \frac{a_1 \cdots a_{r-1}}{(a_1 + 1)(a_2 + 1) \cdots (a_{r-1} + 1)} + \frac{a_1 \cdots a_r}{(a_1 + 1) \cdots (a_{r-1} + 1)} = 1 \]

we get,

\[ 1 - a^* = 1 - \prod_{i=r+1}^{s} \frac{c_{r-1}^{(i)}}{1 + c_{r-1}} \]

\[ = \frac{1}{(c_{r+1})^{(1)}} + \frac{c_{r+1}}{(c_{r+1})^{(1)}(c_{r+2})^{(1)}} + \cdots + \frac{(c_{r+1})^{(1)} \cdots (c_{r-1})^{(1)}}{(c_{r+1})^{(1)}(c_{r+2})^{(1)} \cdots (c_{r-1})^{(1)}} \]

\[ \to \frac{1}{\lambda} \left[ \frac{1}{\gamma_{r+1}} + \frac{\gamma_{r+1}}{\gamma_{r+1} \gamma_{r+2}} + \cdots + \frac{\gamma_{r+1} \cdots \gamma_{r-1}}{\gamma_{r+1} \cdots \gamma_{r-1} \gamma_{r}} \right] = \frac{1}{\lambda} \sum_{i=r+1}^{s-r} \frac{1}{\gamma_{r+i}} \]
4. Some examples: For both the cases

1. Power function distribution

\[ F = \left( \frac{v-x}{v-\mu} \right)^{\theta} = \left[ -\frac{1}{v-\mu} x + \frac{v}{v-\mu} \right]^{\theta}, \quad \mu \leq x \leq v \]

\[ a = \frac{i}{v-\mu}, \quad b = \frac{v}{v-\mu}, \quad c = \theta \]

\[ a^* = \prod_{i=r+1}^{s} \frac{\theta y_i}{\theta y_i + 1} < 1 \]

\[ b^* = v(1 - a^*) \]

2. Pareto distribution

\[ F(x) = \left( \frac{\mu + \delta}{x + \delta} \right)^{\theta} = \left( \frac{1}{\mu + \delta} x + \frac{\delta}{\mu + \delta} \right)^{-\theta}, \quad \mu \leq x < \infty \]

\[ a = \frac{1}{\mu + \delta}, \quad b = \frac{\delta}{\mu + \delta}, \quad c = -\theta. \]

\[ a^* = \prod_{i=r+1}^{s} \frac{\theta y_i}{\theta y_i - 1} > 1 \]

\[ b^* = \delta(a^* - 1) \]

3. Exponential distribution

\[ F(x) = e^{-\lambda(x-\mu)}, \quad \geq \mu \]

\[ = \left[ 1 - \left( \frac{\lambda(x-\mu)}{c} \right)^c \right]^c, \]

\[ a = -\frac{\lambda}{c}, \quad b = \frac{c + \lambda \mu}{c}, \quad c \to \infty \]
Let \( h(x) \) be a monotonic function of \( x \).

Then

\[ E[X(s, n, \bar{m}, k) \mid X(r, n, \bar{m}, k) = x] = a^* \cdot h(x) + b^* \]

if and only if

\[ F(x) = [ah(x) + b]^c \]

with the same \( a^* \) and \( b^* \) as given in (2.5) and (3.5).

Therefore, the following more examples may be appended [Khan and Abouammoh, 2000]

<table>
<thead>
<tr>
<th>Distributions</th>
<th>( F(x) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i). Power function</td>
<td>( 1 - a^{-p} x^p )</td>
<td>( -a^{-p} )</td>
<td>1</td>
<td>1</td>
<td>( x^p )</td>
</tr>
<tr>
<td></td>
<td>( 0 \leq x \leq a )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(ii). Pareto</td>
<td>( a^p x^{-p} )</td>
<td>( a^p )</td>
<td>0</td>
<td>1</td>
<td>( x^{-p} )</td>
</tr>
<tr>
<td></td>
<td>( 0 \leq x &lt; \infty )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a^{-p} )</td>
<td>( 0 )</td>
<td>( -1 )</td>
<td>( x^p )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a^{-1} )</td>
<td>( 0 )</td>
<td>( -p )</td>
<td>( x )</td>
<td></td>
</tr>
<tr>
<td>(iii). Beta of first kind</td>
<td>( (1 - x)^p )</td>
<td>( -1 )</td>
<td>1</td>
<td>( p )</td>
<td>( x )</td>
</tr>
<tr>
<td></td>
<td>( 0 \leq x \leq 1 )</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
(iv). Weibull \[ \exp[-\theta(x - \mu)^p], \quad -\theta/c \quad 1 \quad \infty \quad (x - \mu)^p \]
\[ x \geq \mu \]

(v). Inverse Weibull \[ \exp[-\theta(x - \mu)^p], \quad -1 \quad 1 \quad 1 \quad e^{-\theta x^{-p}} \]
\[ 0 \leq x < \infty \]

(vi). Burr type XII \[ (1 + \theta x^p)^{-m}, \quad \theta \quad 1 \quad -m \quad x^p \]
\[ 0 \leq x < \infty \]

(vii). Cauchy \[ \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x \]
\[ \frac{1}{\pi} \quad \frac{1}{2} \quad 1 \quad \tan^{-1} x \]
\[ -\infty < x < \infty \]

5. Some remarks

Remark 5.1: Generalized order statistics reduces to ordinary statistics at 
\[ m_r = 0, \quad k = 1 \text{ and } \gamma_r = n - r + 1. \]
Therefore for ordinary order statistics
\[ a^* = \prod_{i=r+1}^{s} \frac{c(n-i+1)}{c(n-i+1)+1} = \prod_{j=0}^{s-r-1} \frac{c(n-r-j)}{c(n-r-j)+1} \]
\[ b^* = \frac{b}{a}(1-a^*) \]

For adjacent order statistics \((s = r + 1)\)
\[ a^* = \frac{c(n-r)}{c(n-r)+1}, \quad b^* = -\frac{b}{a} \frac{1}{c(n-r)+1} \]

was obtained by Ferguson (1967) and Khan and Abu-Salih (1989).

For \( s = r+2 \), the result was given by Wesolowski and Ahsanullah (1997).

**Remark 5.2:** For \( m_r = -1 \) and \( \gamma_r = k \) (i.e. when \( \gamma_i = \gamma_j \)), we have for the \( k-th \) records

\[ a^* = \left( \frac{ck}{1+ck} \right)^{s-r} \]

At \( k = 1 \),

\[ a^* = \left( \frac{c}{1+c} \right)^{s-r} \]

\[ b^* = -\frac{b}{a} (1 - a^*) \]

was obtained by Dembinska and Wesolowski (2000) and Athar et al. (2003).

For adjacent record values \( (s = r+1) \), the result was given by Nagraja (1988).

**Remark 5.3:** For adjacent \( gos \) \( (s = r+1) \) at \( m_i = m_j \), characterizing results were obtained by Keseling (1999), Cramer et al. (2004).

**Theorem 5.1.** Khan et al. (2005) have shown that

\[ E[\xi^k X(r,n,m,k) \mid X(s,n,m,k) = y] = g_{r,s}(y) \] \hspace{1cm} (5.1)

if and only if

\[ F(x) = \left[ 1 - \exp \left( \int_x^{\infty} A(y)dy \right) \right]^{m+1} \quad m \neq 1 \] \hspace{1cm} (5.2)

and

\[ F(x) = \exp \left[ \exp \left( \int_x^{\infty} B(y)dy \right) \right] \quad m = -1 \] \hspace{1cm} (5.3)
where

\[ A(y) = \frac{g'_{r|s}(y)}{(s-1)[g_{r|s-1}(y) - g_{r|s}(y)]}, \quad m \neq -1 \]

\[ B(y) = -\frac{f(y)}{F(y) \log F(y)}, \quad m = -1 \]

**Theorem 5.2** Let for \(1 \leq r < s \leq n\)

\[ g_{r|s}(y) = E[\varepsilon \{X(r,n,\tilde{m},k)\} | X(s,n,\tilde{m},k) = y] \quad (5.4) \]

then

\[ \frac{f(y)}{F(y)} + \frac{B'_s(y)}{\gamma_s B_s(y)} = \frac{g'_{r|s}(y)}{\gamma_s[r_{r|s-1}(y) - g_{r|s}(y)]} \]

and

\[ \frac{[B_s(x)]^{\gamma_s}}{F(x)} = \exp \left[ \int_x^\beta \frac{g'_{r|s}(y)}{\gamma_s[r_{r|s-1}(y) - g_{r|s}(y)]} dy \right] \]

where

\[ B_s(y) = \sum_{i=1}^s a_i(s)[F(y)]^{\gamma_i} \quad (5.5) \]
REFERENCES


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