MATRIX TRANSFORMATIONS ON CESARO
SEQUENCE SPACES OF A ABSOLUTE
AND NON-ABSOLUTE TYPE

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The present dissertation entitled 'Matrix Transformations on Cesaro sequence spaces of a absolute and non-absolute type', containing an upto-date account of the results obtained by various researchers in this line, has been prepared under the esteemed supervision of Dr. F.M. Khan, Reader, Department of Mathematics, Aligarh Muslim University, Aligarh.

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The dissertation consists of Four Chapters. In Chapter Zero, which is introductory, I have given the definitions and notations used in the subsequent chapters. The chapters one, two and third have been devoted to the detailed account of the results on Cesaro sequence space of a non-absolute type, matrix transformations on Cesaro sequence spaces of a non-absolute type
and Cesaro sequence space of absolute type and related results respectively.

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CHAPTER-ZERO

INTRODUCTION

1. EXPLANATORY REMARKS:

The study of sequence spaces has been of great interest. Recently number of books have been published in this area over the last few years (See for example, [5], [4], and [6]).

In this dissertation we shall introduce two classes of sequence spaces one of which is of a non-absolute type and another of an absolute type. By absolute we mean that if a sequence \( x = \{x_k\}_k \) belongs to a given space so does its absolute value \( |x| = \{|x_k|\}_k \). Otherwise the space is said to be non-absolute. Non-Absolute type of sequence space has some undesirable properties. First, it does not necessarily contain all finite sequences, secondly, it is not solid. We see that many results remain valid without the above two properties which are usually assumed in the study of sequence spaces.
One area of study in sequence spaces is matrix transformations. Not much have been studied regarding non-absolute type of sequence spaces and their matrix transformations, which are free from above mentioned undesirable properties. So we study in the present dissertation such type of sequence spaces and their matrix transformations with main emphasis on Cesaro sequence spaces.

Let $X$ and $Y$ be sequence spaces, and $A$ an infinite matrix. A standard problem is to find necessary and sufficient condition on $A$ s.t. $A : X \rightarrow Y$. There are many known results in this direction.

In Chapter one we have phrased them in terms of associated norms and further two basic theorems are given which shows that many known results are then their easy consequences. In the end of this Chapter some further generalizations of these results are also mentioned.

Ng and Lee determined the associate norms and the associates spaces of Cesaro sequence spaces of a
non-absolute type. In the second Chapter we have mentioned these results and the necessary and sufficient conditions under which an infinite matrix will transform the Cesaro sequence spaces of non-absolute type into respectively the space $l_\infty$ of all bounded sequences and the space $C$ of all convergent sequences.

In Chapter third we have considered the Cesaro sequence spaces of an absolute type and their $\alpha$-duals. Further we have mentioned the results of matrix transformations on those spaces. At the end of this Chapter we have also introduced Cesaro Function Spaces and their duals. Here also we have listed some problems.

2. **DEFINITIONS AND NOTATIONS:**

For convenience, we list some of the well known spaces as follows:

$C = \text{The space of all convergent sequences } x = \{x_k\}$

such that $|x_k - l| \rightarrow 0$ for some $l$.

$C_0 = \text{The space of all null sequences } x = \{x_k\}$

Such that $|x_k| \rightarrow 0$. 
\[ l_p = \text{The space of all sequences } x = \sum_{k=1}^{\infty} x_k \text{ such that} \]
\[ (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} < \infty \text{ where } 1 \leq p < \infty. \]

\[ l_\infty = \text{The space of all bounded sequences } x = \sum_{k=1}^{\infty} x_k \text{ such that} \]
\[ \text{Sup } |x_k| < \infty. \]

\[ bV = \text{The space of all sequence } x = \sum_{k=1}^{\infty} x_k \text{ such that} \]
\[ \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty, \]

\[ bV_0 = bV \cap C_0 = \left\{ x \in bV : \lim_{n \to \infty} x = 0 \right\}. \]

For suitable norms, the above are all complete normed linear spaces that is Banach spaces [5].

\[ * \]

\[ \text{Ces}(p) = \text{The space of all continuous linear functional on } \text{Ces}(p) \text{ that is the dual space of } \text{Ces}(p). \]

A linear topological space \( X \) is called a paranormed space if there exists a subadditive function \( g : X \to \mathbb{R}^+ \) such that \( g(0) = 0, g(x) = g(-x) \) and the multiplication is continuous, that is, \( \lambda_n \to \lambda \) and \( g(x_n - x) \to 0 \), implies that \( g(\lambda_n x_n - \lambda x) \to 0 \) for \( \lambda, x_n \in \mathbb{C} \) and \( x_n' \in X. \)
A set $X$ is said to be solid if $x \in X$ and $|y| \leq |x|$ imply that $y \in X$ where $X$ is a sequence space.

Let $A = (a_{n,k})$ be an infinite matrix of complex numbers $a_{n,k} (n,k = 1, 2, \ldots)$ and $P, Q$ two subsets of the space $S$ of complex sequences. We say that the matrix $A$ defines a matrix transformation from $P$ into $Q$, if for every sequence $x = (x_k) \in P$ the sequence $A(x) = A_n(x)$ is in $Q$, where $A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$.

The class of all such matrix transformations from $P$ into $Q$ will be denoted by $(P, Q)$. 
CHAPTER-I

CESARO SEQUENCE SPACES OF A NON-ABSOLUTE TYPE

1.1

In this Chapter we shall introduce a class of sequence spaces which are non-absolute type. This sequence space was first introduced by P.N. Nung [7] in 1978. Further characterization among others the dual spaces of these sequence spaces are also given here. Some further generalization are also included in this Chapter.

1.2 DEFINITION:

Let $A$ be an infinite matrix then we define

$$C_A = \{ x ; A x \in C \}.$$

This sequence space has been studied by many; see, for example [3].

In general, let $Y$ be a given sequence space then

$$X = \{ x ; A x \in Y \}.$$

In what follows, we assume that the mapping of $A$ from $X$ to $Y$ is one-one and onto.
In particular, when \( A \) is a Cesaro matrix \( C \) where

\[
C = \begin{bmatrix}
1 \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

and \( Y = \ell_p \) for \( 1 \leq p \leq \infty \). Then \( X \) is called Cesaro sequence space of a non-absolute type \([7]\) and it is denoted by \( X_p \).

In other words, \( x \in X_p \) for \( 1 \leq p < \infty \) if and only if

\[
\left\{ \frac{1}{n} \sum_{k=1}^{n} x_k \right\}^{1/p} < \infty.
\]

and similarly for \( p = \infty \).

In this connection we shall show that these sequence spaces have some undesirable properties (see \([8]\)).

(i) It does not necessarily contain all finite sequences.

Example (1.1) \( e^1 = (1,0,0,\ldots) \notin X_1 \); however \( X_1 \) is not empty since \( \sum_{k=1}^{(k+1)-1} \frac{k^{-1}}{k} \geq 1 \in X_1 \).

(ii) It is not solid

A set \( X \) is said to be solid if \( x \in X \) and \( |y| \leq |x| \implies y \in X \).
Note that if $X$ is solid then $X$ is also absolute.

Now let $x = \{x_k\}$ and $x_k = (-1)^k$

Then $x \in X_p$ but $|x| \notin X_p$ for $1 < p < \infty$.

That is $X_p$ is non-absolute for $1 < p < \infty$, indeed also for $p = 1$ and $\infty$, therefore $X_p$ cannot be solid.

We shall see that many results remain valid without the above two properties which are usually assumed in the study of sequence spaces.

1.3 One area of study in sequence spaces is matrix transformations. Let $T = (t_{n,k})$ be an infinite matrix mapping a sequence space $X$ into another sequence space $Z$. That is, $y = Tx \in Z$ whenever $x \in X$ where

$$y_n = \sum_{k=1}^{\infty} t_{n,k} x_k$$

The problem is to find necessary and sufficient condition on $T$ such that $T$ maps $X$ into $Z$.

A necessary condition is that for each $n$ and each $x \in X$ the series $\sum_{k=1}^{\infty} t_{n,k} x_k$ must converge.

In other words, $\{t_{n,k}\} k \geq 1$ belongs to the $\beta$-dual or Kothe dual of $X$ for each $n$. Hence, to find
the necessary and sufficient condition on $A$ s.t $A : X \longrightarrow Z$ where $Z$ is another sequence space, it is essential to characterize the Kothe-dual of $X$.

We shall consider the sequence space $X$ s.t. $A : X \longrightarrow Z$ in which $A$ is one-one and onto. In other words, $X$ is defined in terms of $A$ and $Z$. Many sequence spaces can be expressed in this form.

Example (1.2). The space $bv$ of all sequences $x$ such that

$$\sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty$$

can be regarded as such a space with $Z = \ell_1$ and

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ 0 & \end{pmatrix}$$

$$\|x\|_X = \|Ax\|_Z$$

The $\beta$-dual or Kothe-dual of $X$, denoted by $X^\beta$, is the space of all sequences $y = \{y_k\}$ such that

$$\sum_{k=1}^{\infty} x_k y_k$$

converges for all $x = \{x_k\} \in X$. 
Example (1.3). \( \ell_p^\beta = \ell_q \) where \( 1 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

The \( \alpha \)-dual of \( X \), denoted by \( X^\alpha \), is the space of all sequences \( y = \{ y_k \} \) such that \( \sum_{k=1}^{\infty} |x_k y_k| \) converges for all \( x = \{ x_k \in X \} \).

REMARK: If the space \( X \) is solid then \( X^\alpha = X^\beta \).

For example, \( \ell_p^\alpha = \ell_p^\beta = \ell_q \) where \( 1 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). For convenience, we shall always write \( x = \{ x_k \} \), \( y = \{ y_k \} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Let \( y \in X^\beta \), and define the associate norm as follows.

\[
\|y\|^\beta = \operatorname{Sup} \{ \phi(n) ; n \geq 1 \}.
\]

where \( \phi(n) = \operatorname{Sup} |\sum_{k=1}^{n} x_k y_k| ; x \in X \) and \( \|x\| \leq 1 \).

Here we assume implicitly that for every \( k \) there is a sequence \( x \) such that \( \|x\| \leq 1 \) and \( x_k \neq 0 \).

Then the Kothe-dual \( X^\beta \) with the associate norm is again a normed linear space. The Kothe dual is also called the associate space by Zaanen and Luxemburg [20].
A norm is said to be absolute if \(||x|| = ||x||_x||\)
where \(|x| = \left\{ |x_k| \right\}\). Obviously, if the norm in \(X\) is absolute then for \(y \in X^\beta\)

\[ ||y||^\beta = \text{Sup} \left\{ \sum_{k=1}^{\infty} |x_k y_k| \div ||x|| \leq 1 \right\} \]

**Example (1.4).** When \(y \in l_p^\beta\) the associate norm \(||y||^\beta\)
is the usual norm in \(l_q^\beta\). Also let \(C\) be the space of all convergent sequences then the associate norm in \(C^\beta = l_1\) is simply the usual norm in \(l_1\).

For \(y \in X^\beta\), we define the associate seminorm in the classical sense as follows.

\[ ||y||^\ast = \text{Sup} \left\{ \sum_{k=1}^{\infty} x_k y_k \div x \in X \text{ and } ||x|| \leq 1 \right\} \]

Note that if the norm in \(X\) is absolute then

\[ ||y||^\beta = ||y||^\ast \]. Otherwise, \(||y||^\ast\) may only be a seminorm (See[10]). The well-known Kojima-Schur theorem [1] can now be rephrased as follows, in which \(e = \{1,1,\ldots,\ldots\}\) and \(e^i = \{e_{ik}\}\) with \(e_{11} = 1\) and \(e_{ik} = 0\) for \(k \neq i\).

**Theorem 1.1:** Let \(A = (a_n,k)\) be an infinite matrix.
Then $A : C \longrightarrow C$ if and only if $A e^k \in C$ for $k = 1, 2, \ldots, A \in C$ and

$$\operatorname{Sup} \left\{ \left\| (a_n, k) \right\|_{1}^{\beta+1} n \geq 1 \right\} < \infty.$$ 

This theorem can easily in a lightly more general form.

A sequence space $X$ is said to be have the AD property if the linear hull of $\{ e^k ; k = 1, 2, \ldots \}$ is dense in $X$, and to have the AE property if the linear hull of $\{ e, e^k ; k = 1, 2, \ldots \}$ is dense in $X$.

Now we state a theorem due to P.Y. Lee [9].

**Theorem 1.2:** Let $A = (a_n, k)$ be an infinite matrix and $X$ a sequence space satisfying the AE property. Then $A : X \longrightarrow C$ if and only if $A e^k \in C$ for $k = 1, 2, \ldots, A \in C$ and $\{ a_n, k \} \supseteq 1 \in X^\beta$ for all $n$,

$$\operatorname{Sup} \left\{ \left\| (a_n, k) \right\|_{1}^{\ast} \right\} n \geq 1 \in X^\beta \left\{ n \geq 1 \right\} < \infty.$$ 

**Theorem 1.3:** Theorem 1.2 holds with the condition $A e^k \in C$ omitted if the AE property of $X$ is replaced by the AD property.
The following theorem characterizes the β-dual or Kothe-dual of a space.

**Theorem 1.4:** Let $X$ and $Y$ be sequence spaces with $Y$ having the AD property and $A$ an infinite matrix such that $A : X \to Y$ is one-one and onto with $A^{-1} = (a_{n,k}^{-1})$ then $y \in \beta X$ if and only if $yA^{-1}$ exists and

$$||y||^\beta = \text{Sup} \left\{ \left( \sum_{k=1}^{n} \left( \sum_{i=1}^{\infty} y_k a_{ki} \right) \right)^* \mid y \in \beta X \right\} < \infty.$$ 

**Proof:** We write $S = Ax$ and $x = A^{-1}S$. Then

$$\lim_{n \to \infty} \sum_{k=1}^{n} x_k y_k = \lim_{n \to \infty} \sum_{k=1}^{n} y_k \sum_{i=1}^{\infty} a_{ki}^{-1} S_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \left( \sum_{k=1}^{n} y_k a_{ki}^{-1} \right) S_i$$

Hence by using Theorem 1.3 we get the result.

Applying above theorem, we see that $(bv)^\beta$ is the space of all $y \in CS$ such that

$$\text{Sup} \left\{ \left( \sum_{k=1}^{n} y_k \right) ; n \geq 1 \right\} < \infty.$$ 

Where $CS$ is the space of all convergent sequences.

We shall give two more examples.
**Example (1.5).** Take $A$ to be $\sigma^r$ where $r$ is a positive integer, $\sigma^r(x) = \sigma(\sigma^{r-1} x)$ and 

$$(\sigma x)_n = \sum_{k=1}^{n} x_k$$

Then the inverse of $A$ is $\bigtriangleup^r$ where $\bigtriangleup x = \bigtriangleup (\bigtriangleup^{r-1} x)$ and $(\bigtriangleup x)_n = x_n - x_{n-1}$. With $x_0 = 0$. If we define $H(r, p)$ to be the space of all sequences $x$ such that $\sigma^r x \in l_p$ for $1 \leq p < \infty$, then its $\beta$-dual is the space of all bounded sequences $y$ such that

$$\bigtriangleup y \in l_q.$$ 

**Example (1.6).** Now replace $A$ in the above example by $a_{n,k} = \frac{1}{n}$ for $1 \leq k \leq n$ and $a_{n,k} = 0$ for $k > n$. $X_p$ the space of all $x$ such that $Ax \in l_p$ $(1 \leq p \leq \infty)$. 

Now the first natural question to ask is what is the $\beta$-dual of $X_p$. In this section we shall give the results of Lee-Peng Yee [8] relating to the $\beta$-dual of $X_p$.

Suppose $y \in X_p^\beta$ where $1 < p < \infty$. Consider

$$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n-1} \left( \frac{1}{k} \sum_{i=1}^{k} x_i \right) (k(y_k - y_{k+1})) + \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) (n y_n)$$

$$= \sum_{k=1}^{n} b_{n,k} S_k$$
Where
\[ b_{n,k} = \begin{cases} 
\frac{k}{n} (y_k - y_{k+1}) & \text{when } 1 \leq k \leq n-1 \\
ny_n & \text{when } k = n \\
0 & \text{when } k > n
\end{cases} \]

and
\[ S_k = \frac{1}{k} \sum_{i=1}^{k} x_i. \]

Let \( B = (b_{n,k}) \). Then for every \( x \in X_p \) and \( S = \sum_{k \in \ell_p} S_k \), we have \( BS \in C \). It is well known [2] that \( B \) maps \( \ell_p \) into \( C \) if and only if
\[
\lim_{n \to \infty} b_{n,k} \quad \text{exists for each } k, \quad \text{and}
\]
\[
\sup \left\{ \left( \sum_{k=1}^{n} |b_{n,k}|^q \right)^{1/q} \ ; \ n \geq 1 \right\} < \infty
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Note that the first condition is trivially satisfied.

The second condition gives the following [7].

**Theorem 1.5:** The \( \ell \)-dual of \( X_p \) for \( 1 < p < \infty \) is the space of all sequences \( y \) such that
\[
\sup \left\{ \left( \sum_{k=1}^{n-1} |k(y_k - y_{k+1})|^q + |ny_n|^q \right)^{1/q} \ ; \ n \geq 1 \right\} < \infty
\]

we remark that similar results hold for \( p = 1 \) and \( \infty \).
Alternatively, we may split the condition into two as follows:

\begin{align*}
(1.5.1) \quad & \sup_{n \geq 1} |n y_n| < \infty \\
(1.5.2) \quad & \bigg( \sum_{k=1}^{\infty} |k(y_k - y_{k+1})|^q \bigg)^{\frac{1}{q}} < \infty.
\end{align*}

In other words we can say that the $\beta$-dual for $1 \leq p < \infty$ is the space of all $y$ such that $\{k y_k\}$ is bounded and $\{k(y_k - y_{k+1})\} \in l_q$.

When $p = \infty$, its $\beta$-dual is the space of all null sequences $y$ such that $\{k(y_k - y_{k+1})\} \in l$.

**REMARK:** A similar result holds if the AD property of $Y$ is replaced by the AE property.

Thus the revised version will include the case when $Y = C$ which has the AE property but not the AD property.

The following theorem is due to Zeller, and is an easy consequence of [25, Theorem 17, vii].

A sequence space $X$ is said to have the AK property if $X$ has the AD property and

$$\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$$

converges to $x$ as $n \to \infty$ for each $x \in X$. 
**Theorem 1.6:** Let $X,Y$ and $Z$ be sequence spaces with $Y$ having the AK property and $A$ an infinite matrix such that $A : X \to Y$ is one-one and onto.

Then $T = \{ t_{n,k} \}_{k \geq 1}$ maps $X$ into $Z$ if and only if

$$\text{(1.6.1)} \quad \left\{ t_{n,k} \right\}_{k \geq 1} \subseteq X^Y$$

for each $n$,

$$\text{(1.6.2)} \quad TA^{-1} \text{ maps } Y \text{ into } Z.$$

**Proof:** The proof of this theorem depends on the following associative property.

$$T(A^{-1}y) = (TA^{-1})y \text{ for all } y \in Y,$$

which follows from the AK property of $Y$. A similar result proved by Jakimovski and Livne [24]. There they assume that $Y$ satisfies the following property:

For each infinite matrix $A = (a_{n,k})$ satisfying $YC \subseteq C_A$

we have for each $y \in Y$

$$\lim_{m \to \infty} (Ay)_m = \sum_{k=1}^{\infty} a_{k,y} k$$

where

$$a_k = \lim_{n \to \infty} a_{n,k}$$

Obviously, if $Y$ has the AK property, then the above condition is satisfied.
In the same way, we can characterize

\( T : Z \rightarrow X \) where \( Z \) has the AK property and \( X \) is defined as in theorem 1.5.

**Theorem 1.7:** Let \( X, Y \) and \( A \) be defined as in theorem 1.5 and \( Z \) a sequence space having the AK property.

Then \( T : Z \rightarrow X \) if and only if

\[
(1.7.1) \sum_{n,k \geq 1} t_{n,k}^2 \in Z^\beta \quad \text{for each } n
\]

\( (1.7.2) AT : Z \rightarrow Y \)

Now combining theorem 1.5 and 1.6 we have

**Theorem 1.8:** Let \( X \) be defined as in theorem 1.5.

Then \( T : X \rightarrow X \) if and only if

\[
(1.8.1) \sum_{n,k \geq 1} t_{n,k}^2 \in X^\beta \quad \text{for each } n,
\]

\( (1.8.2) A(TA^{-1}) : Y \rightarrow Y. \)

Using the theorems 1.6, 1.7 and 1.8 we can characterize easily \( T : X \rightarrow Z \) or \( T : Z \rightarrow X \) or \( T : X \rightarrow X \) as long as we can characterize \( TA^{-1} : Y \rightarrow Z \) or \( AT : Z \rightarrow Y \) or \( A(TA^{-1}) : Y \rightarrow Y. \)

Hence we may deduce as many theorems as there are in
Stiglitz and Tietz [23]. As expected, many known results follow easily from the above theorems, for example, those involving \( H(r,p) \) and \( X_p \) as given in section 1.3. In particular, we shall consider the results of Ishiguro and Zeller [22] and Dawson [21].

Let \( X = C_A \); i.e. \( X = \{ x; Ax \in C \} \). Assume that \( A \) is a lower semi-matrix with non-zero diagonal satisfying the following condition [1; p.68].

\[
\sup \left\{ \left| \sum_{k=1}^{n} a_{n,k} - a_{n,k-1} \right|; \ n \geq 1 \right\} < \infty
\]

\[
\lim_{n \to \infty} a_{n,k} = 1 \text{ for each } k
\]

with \( a_{n,-1} = 0 \). Assume further that \( A \) is perfect in the sense that \( e^i, i = 1, 2, \ldots, \) form a basis in \( X \). Then every continuous linear functional \( F \) on \( X \) is of the form

\[
F(x) = g \lim_{n \to \infty} S_n + \sum_{n=1}^{\infty} g_n S_n
\]

where \( S_n = \sum_{k=1}^{n} a_{n,k} x_k \) and \( \sum_{n=1}^{\infty} |g_n| < \infty \).

Hence the conjugate space \( X^* \) of \( X \) is isomorphic to the space of all \( y \) such that

\[
y_k = F(e^k) = g + \sum_{n=1}^{\infty} g_n a_{n,k}
\]
with norm \(|y| = |F| = |g| + \sum_{n=1}^{\infty} |g_n|\).

The following theorem due to Ishiguro and Zeller [22, Theorem 1] is an easy consequences of theorem 1.3 and 1.4.

**THEOREM 1.9:** Let \(X = C_A\) with \(A\) given as above. Then an infinite matrix \(T\) maps \(X^*\) into \(C\) if and only if

\[(1.9.1) \sup \left\{ \left| \sum_{k=1}^{\infty} a_{n,k} t_{m,k} \right| : n \geq 1 \text{ and } m \geq 1 \right\} < \infty.

\[(1.9.2) \lim_{m \to \infty} \sum_{k=1}^{\infty} t_{m,k} \text{ exists.}

\[(1.9.3) \lim_{m \to \infty} t_{m,k} \text{ exists for each } k.

Dawson [21] consider the case when \(X^*\) is the space of all convergent sequence \(y\) satisfying

\[
\sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} y_{k,a_{ki}} \right| < \infty.

In the next theorem due essentially to Dawson [21], we shall assume that both \(X^*\) and \(Y^*\) are such spaces, in which \(Y = C_B\) and \(B\) is a lower semi-matrix having the same properties as \(A\) above.
**Theorem 1.10**: Let $X = C_A$ and $Y = C_B$ with $A$ and $B$ given as above. Then an infinite matrix $T$ maps $X^*$ into $Y^*$ if and only if

\[(1.10.1) \lim_{m \to \infty} t_{m,k} \text{ exists for each } k,\]

\[(1.10.2) \sum_{k=1}^{\infty} t_{m,k} \in Y^*, \]

\[(1.10.3) \sup \left\{ \sum_{k=1}^{\infty} t_{m,k} a_{n,k} ; m, n \geq 1 \right\} < \infty, \]

\[(1.10.4) \sup \left\{ \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} b_{m,i} \sum_{k=1}^{\infty} t_{m,k} a_{n,k} ; n \geq 1 \right\} < \infty. \]

The proof follows from theorem 1.5 and 1.6.

**1.4 Some Remarks:**

We shall make some remarks on how theorem 1.6 and 1.7 may be extended. Consider the situation in theorem 1.6 with $Y$ replaced by $C$. Then an infinite matrix $T$ maps $X$ into $Z$ if and only if $\sum_{k=1}^{\infty} t_{n,k} a_{n,k} \in X^B$ for each $n$, $TA^{-1}$ maps $C_0$ into $Z$ and $T(A^{-1})e \in Z$ when $C_0$ denotes the space of all null sequences.

Note that $C = C_0 + [e]$ where $[e]$ stands for the linear space spanned by $e$, and that $C_0$ has the AK property. In fact, the above applies to any space $Y$. 
which has the form $Y_e + [e]$ such that $Y_e$ has the
AK property. Similarly, we may extend theorem 1.7 with
$Z$ having the AK property replaced by $Z = Z_0 + [e]$
where $Z_0$ has the AK property.

The results in section (1.2) can also be generalized
slightly with the AK property replaced by a weaker condi­
tion, namely, the AL property, and the usual infinite sum
by a generalized sum, namely,

$$A - \sum_{k=1}^{\infty} x_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_{n,k} x_k$$

where $A = (a_{n,k})$ is given and fixed.

A sequence space $X$ is said to have the AL property
(with respect to $A = (a_{n,k})$) if for every $x \in X$ the
sequence $\{a_{n1} x_1, a_{n2} x_2, \ldots, a_{nn} x_n, 0, \ldots\}$ belonging
to $X$ for each $n$, converges to $x$ in the norm of $X$.
The idea of considering the AL property was suggested by
Zeller [11, P.45]. Then we define

$A$-associate space of $X$ to be the set of all $y$
such that for each $y$, $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} x_k y_k$ exists
for all $x \in X$. 

Obviously, if \( X \) has the AL property, then

\( \text{A- associate space of } X \text{ coincides with its conjugate space.} \)

Hence we can rephrase the above theorems, in particular, theorems 1.6 and 1.7 in terms of the AL property.
CHAPTER-II

MATRIX TRANSFORMATIONS ON CESARO SEQUENCE SPACES OF A NON-ABSOLUTE TYPE

2.1 In [17] Ng and Lee consider the space \( H \) of all real sequences \( \{x_k\} \) for which the series \( \sum_{k=1}^{\infty} x_k \) is convergent. Here we shall determine the associate space of \( H \) in the sense of Kothe theory, and determine the necessary and sufficient conditions under which a matrix transformation will map \( H \) respectively into the space \( l_\infty \) of all bounded sequences and the space \( C \) of all convergent sequences. Further in this note we shall use these results to find the necessary and sufficient conditions under which an infinite matrix will transforms the Cesaro sequence spaces into the space \( l_\infty \) and the space \( C \) respectively.

2.2 NORMED KOTHE SEQUENCE SPACES:

Let \( X \) be the set of all real sequences \( x = \{x_k\} \).

A functional \( \ell \) from \( X \) into the extended non-negative
real number system is called a semi-norm if for every \( x \) and \( y \) in \( X \),

(a) \( \rho(0) = 0 \);

(b) \( \rho(\alpha x) = |\alpha| \rho(x) \); for every real number \( \alpha \).

(c) \( \rho(x+y) \leq \rho(x) + \rho(y) \).

If instead of (a), \( \rho \) satisfies the condition that
\( \rho(x) = 0 \) if and only if \( x = 0 \), then \( \rho \) is called a norm. We denote by \( X_\rho \) the collection of all sequences \( x \) satisfying \( \rho(x) < \infty \). Obviously \( X_\rho \) is a linear space and it is called normed \( K \) the sequence space of non-absolute type with the seminorm \( \rho \). If \( X_\rho \) is complete with respect to the norm \( \rho \), then \( X_\rho \) is called a Banach sequence space of non-absolute type, since we did not assume the absolute property
\( \rho(x) = \rho(|x|) \) where \( |x| = \sum |x_k|^2 \). From now on, we shall always assume that \( X_\rho \) is a Banach sequence space of non-absolute type.

Give a seminorm \( \rho \), we define a new seminorm \( \rho' \) as follows:
\( \ell'(x) = \operatorname{Sup} \left\{ \sum_{k=1}^{\infty} x_k y_k \mid \ell(y) \leq 1 \right\} \)

and put \( \ell'(x) = \infty \) if the series \( \sum_{k=1}^{\infty} x_k y_k \) does not converge for some \( y \) satisfying \( \ell(y) \leq 1 \). The seminorm \( \ell' \) is called the associate semi-norm of \( \ell \).

The space \( X_{\ell'} \) consisting of all sequences \( x \in X \) with \( \ell'(x) < \infty \) is called the associate space of \( X_{\ell} \). For any \( x \in X_{\ell} \) and any \( y \in X'_{\ell} \) we always have

\[
\left| \sum_{k=1}^{\infty} x_k y_k \right| \leq \ell(x) \ell'(y).
\]

A semi-norm \( \ell \) is said to be saturated, if for every non-empty subset of positive integers, there exists a non-empty subset \( F \) of \( E \) such that \( \ell(x_F) < \infty \), where the sequence \( x_F = \{x_k\} \) is defined as

- \( x_k = 1 \) if \( k \in F \)
- \( x_k = 0 \) if \( k \notin F \).

It is easy to see that \( \ell \) is saturated if and only if \( X_{\ell} \) contains all finite sequences that is all sequences having only finitely many non-zero terms.

The following is a consequence of the Banach-Steinhaus Theorem.
**Theorem 2.1:** Let $\mathcal{P}$ be saturated, and $y \in X$. Then $y \in X_\mathcal{P}$ if and only if the series $\sum_{k=1}^\infty x_k y_k$ is convergent for every $x \in X_\mathcal{P}$.

**2.3 The Associate Space of the Space $H$:**

Let $H$ be the space of all sequences $x \in X$ such that the series $\sum_{k=1}^\infty x_k$ convergent. We define a norm $\rho$ in $H$ by

$$
\rho(x) = \sup \left\{ \left| \sum_{k=1}^n x_k \right| ; n \geq 1 \right\}.
$$

**Theorem 2.2:** The space $H$ is a Banach sequence space of non-absolute type.

**Proof:** Let $\{x^{(i)}\}$ be a Cauchy sequence in $H$, so that given $\varepsilon > 0$ we have $\rho(x^{(i)} - x^{(j)}) < \varepsilon$ for all $i, j \geq n_0(\varepsilon)$. We write $x^{(i)} = \{x_k^{(i)}\}$. Then for fixed $k$, $\{x_k^{(i)}\}$ is convergent if $\lim_{i \to \infty} x_k^{(i)} = x_k$, then

$$
\left| \sum_{k=1}^m (x_k^{(i)} - x_k) \right| < \varepsilon \text{ for } i \geq n_0(\varepsilon) \text{ and all } m = 1, 2, \ldots.
$$

Hence we have $\rho(x^{(i)} - x) \leq \varepsilon$ for all
$1 \geq n_0 (\in )$.

This proves that $H$ is complete.

**THEOREM 2.3:** The space $H$ is separable.

**PROOF:** For each $x \in H$, where $x = \sum x_k e_k$, we let

$$x^N = \{x_1, x_2, \ldots, x_N, 0, \ldots\}$$

Then it is easy to see that $\rho (x - x^N) \to 0$ as $N \to \infty$. If $A$ is a countable dense subset of the real number system, the finitely non-zero sequences with entries in $A$ form a countable dense subset of $H$.

In what follows, we shall determine the associate space of $H$. Let $V$ be the space of all $y \in X$ such that

$$\sum_{k=1}^{\infty} |y_k - y_{k+1}| < \infty.$$

**THEOREM 2.4:** The associate space $H'$ of the space $H$ coincides with the space $V$, and the associate norm $\rho'$ of $\rho$ is equivalent to the norm

$$||y|| = |y_1| + \sum_{k=1}^{\infty} |y_k - y_{k+1}|,$$

for all $y \in H'$. 
PROOF: First we shall prove that \( H \subseteq V \). For any \( y \in H \) and any \( x \in H \), the series \( \sum_{k=1}^{\infty} x_k y_k \) is convergent, therefore we have \( x_k y_k \to 0 \) as \( k \to \infty \), for all \( x \in H \). Now we claim that \( y \) is a bounded sequence. Suppose that \( y \) is not bounded, then for every integer \( i > 0 \) there exists a subsequence \( \{ y_{k_i} \} \) of \( y \) such that \( |y_{k_i}| > 1 \).

Now we define an element \( x = \sum_{i=1}^{\infty} x_i \in H \) by
\[
x_k = \begin{cases} (-1)^{i+1} & , \quad k = k_i, i = 1,2, \ldots \, \\ 0 & , \quad \text{elsewhere} \end{cases}
\]

Then we see that \( x_{k_i} y_{k_i} \geq 1 \) which does not tends to zero. Next, for any positive integer \( n \), we have
\[
\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} (S_k - S_{k-1}) y_k = \sum_{k=1}^{n} (y_k - y_{k+1}) (S_k - S) + y_1 (S_n - S) y_n
\]

where
\[
S_n = \sum_{k=1}^{n} x_k \quad \text{and} \quad \lim_{n \to \infty} S_n = S.
\]
Since \( y \) is bounded, it follows that

\[
\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} (y_k - y_{k+1})(S_k - S) + S y_1
\]

is convergent for all sequences \( \{(S_k - S)\} \) which converges to zero. Hence we have

\[
\sum_{k=1}^{\infty} |y_k - y_{k+1}| < \infty
\]

which implies that \( H' \subseteq V \). Now for each \( y \in H' \), we have

\[
\rho'(y) = \sup \left\{ \left| \sum_{k=1}^{\infty} x_k y_k \right| : \rho(x) \leq 1 \right\}
\]

\[
= \sup \left\{ \left| \sum_{k=1}^{\infty} (y_k - y_{k+1})(S_k - S) + S y_1 \right| : \sup_{k \geq 1} |S_k| \leq 1 \right\}
\]

\[
\leq 2 \left( \sum_{k=1}^{\infty} |y_k - y_{k+1}| + |y_1| \right)
\]

\[
= 2 \|y\|.
\]

Next we shall prove that \( 2 \rho(y) \geq \|y\| \) for all \( y \in H' \). For each positive integer \( n \), choose an element \( \{S_k^*\} \) in \( \ell_\infty \) as follows
Then we have

\[ \sum_{k=1}^{n} |y_k - y_{k+1}| = |\sum_{k=1}^{\infty} (S_k - 0)(y_k - y_{k+1})| \]

\[ \leq \sup_{k} \left\{ |\sum_{k=1}^{\infty} (S_k - 0)(y_k - y_{k+1})| : \right\} \]

\[ \sup_{k \geq 1} |S_k| \leq 1 \text{ and } S_k \rightarrow 0 \right\}. \]

It follows that

\[ \sum_{k=1}^{\infty} |y_k - y_{k+1}| \leq \sup_{k} \left\{ |\sum_{k=1}^{\infty} (S_k - S)(y_k - y_{k+1}) + \delta y_1| : \right\} \]

\[ \sup_{k \geq 1} |S_k| \leq 1 \text{ and } S_k \rightarrow 0 \right\} \}

\[ = \sup_{k} \left\{ |\sum_{k=1}^{\infty} \delta_k y_k| : \rho(x) \leq 1 \right\} \]

\[ = \rho'(y). \]

Obviously we have \( |y_1| \leq \rho'(y) \) if \( y = \{y_k\} \in H'. \)

Therefore \( ||y|| \leq 2 \rho'(y) \), hence \( \frac{1}{2} ||y|| \leq \rho'(y) \leq 2 ||y|| \).
for all \( y \in H' \) and the two norms \( \| \cdot \| \) and \( \| \cdot \|' \) are equivalent. Finally, it is not difficult to see that \( V \subseteq H' \), by the fact that the associate norm \( \| \cdot \|' \) is saturated and by theorem 2.1. This completes the proof of the theorem.

**THEOREM 2.5:** The associate space \( H' \) of the space \( H \) coincides with the conjugate space (Banach dual) \( H^* \) of the space \( H \) algebraically and Isometrically.

**PROOF:** For any \( y \in H' \), then we see that

\[
T_y(x) = \sum_{k=1}^{\infty} x_k y_k
\]

defines a linear continuous functional on \( H \) with norm \( \| T_y \| = \langle y \rangle' \).

Conversely, if \( T \in H^* \), let \( e^k \) denote the sequence with 1 in the kth coordinate and zero elsewhere. For any \( x \in H \) let also

\[
x^N = \{ x_1, x_2, \ldots, x_N, 0, \ldots \}
\]

Then we have

\[
x^N = \sum_{k=1}^{N} x_k e^k
\]

and \( \rho(x - x^N) \to 0 \) as \( N \to \infty \). Since \( T \) is continuous,
we have \( T(x) = \lim_{N \to \infty} T(x^N) = \lim_{N \to \infty} \sum_{k=1}^{N} x_k T(e^k) \)

\[ = \sum_{k=1}^{\infty} x_k T(e^k) \]

which is convergent for all \( x \in H \). This implies that the sequence \( \{ T(e^k) \} \) is an element in \( H' \) by theorem 2.1 and

\[ ||T|| = \text{Sup} \left\{ \sum_{k=1}^{\infty} x_k T(e^k) \mid \ell^p(x) \leq 1 \right\} \]

\[ = \ell^p(T(e^k)). \]

This shows that every \( T \in H^* \) can be represented by an element \( \{ T(e^k) \} \) in \( H' \). Thus if we identify each \( T \in H^* \) with \( \{ T(e^k) \} \) in \( H' \), we see that \( H' = H^* \) algebraically and isometrically.

2.4 MATRIX TRANSFORMATIONS ON H:

In this section we shall mention the results of Nung [17] to determine the necessary and sufficient conditions under which an matrix transformation

\[ A = \{ a_{n,k} \} \]

will map the space \( H \) into respectively
the space $l_\infty$ of all bounded sequences and the space $C$ of all convergent sequences.

Now first of all we state a lemma due to Zeller [11].

**Lemma 2.1:** If a matrix $A$ transforms a BK-space $E$ into BK-space $F$, then the transformation is linear and continuous.

Here a BK-space is a Banach sequence space in which every coordinate mapping $x \rightarrow x_k$ is continuous.

For example, $\ell_p (1 \leq p = \infty)$, $C$ and the space $C_0$ of all null sequences with uniform norms are all BK-spaces.

**Lemma 2.2:** The space $H$ is a BK-space.

**Proof:** Since the associate space $H'$ contains all the finite sequences, if $x(n) = \{x_k^{(n)}\} \in H$, with

$$P(x(n)) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

then

$$|x_j^{(n)}| = \left| \sum_{k=1}^{\infty} x_k^{(n)} e_j^{(k)} \right| \leq P(x(n)) P(e_j^{(k)}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

where $e_j^{(k)}$ is the sequence 1 at the $j$-th
place and zero elsewhere.

2.4.1 MATRIX TRANSFORMATION OF $H$ INTO $l_{\infty}$.

Let us consider the matrix transformation given by

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \ldots$$

**Theorem 2.6:** A matrix transformation $A = (a_{n,k})$ maps the space $H$ into the space $l_{\infty}$ of all bounded sequences if and only if

(2.6.1) $\sup_{n \geq 1} \left( \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| \right) < \infty$

(2.6.2) $\sup_{n \geq 1} |a_{n,k}| < \infty$ for every fixed $k$.

**Proof:** Suppose the conditions hold. For each $x \in H$, with $S = \sum_{k=1}^{\infty} x_k$ and $S_k = \sum_{i=1}^{k} x_i$, then for each positive integer $n$, we have

$$| \sum_{k=1}^{\infty} a_{n,k} x_k | = |a_{n,1} S + \sum_{k=1}^{\infty} (a_{n,k} - a_{n,k+1})(S_k - S)|$$

$$\leq \sup_{n \geq 1} \left( |a_{n,1}| + \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| \right) 2\rho(x).$$

This shows that $Ax$ is in $l_{\infty}$ and $A = (a_{n,k})$ maps $H$ into $l_{\infty}$. 
Conversely, suppose $A = (a_{n,k})$ maps $H$ into $\ell_\infty$.

For each $k = 1, 2, \ldots$, the sequence $e^k$ with 1 at the $k$th place and zero elsewhere is an element in $H$.

Thus

$$\sup_{n \geq 1} |a_{n,k}| = \sup_{n \geq 1} \left| \sum_{j=1}^{\infty} a_{n,j} e^k \right| < \infty$$

For every fixed $k$, therefore condition (2.6.2) holds.

Next by Lemma 2.1 and Lemma 2.2, the mapping is linear and continuous, therefore there exists real constant $K$ such that

$$\sup_{n \geq 1} \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right| \leq K \| x \| \text{ for all } x \in H.$$ 

Now for each $x \in H$ with $S_n = \sum_{k=1}^{n} x_k$ and $\lim_{n \to \infty} S_n = 0$

We have

$$\sup_{n \geq 1} \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right| \leq \sup_{n \geq 1} \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n,k+1}) S_k \right|$$

$$\leq K,$$

and this is true for all null sequence $\{S_k\} \in C_0$. 

since the mapping defined by \( \sigma x = \left\{ \sum_{k=1}^{n} x_k \right\} \) is a one-to-one continuous linear operator from \( H \) onto the space \( C \) of all convergent sequences. Since the associate space of \( C_0 \) is the space \( \ell_1 \), therefore we have

\[
\text{Sup} \left( \sum_{n \geq 1}^{\infty} \left| a_{n,k} - a_{n,k+1} \right| \right) < \infty
\]

This completes the proof of the theorem.

2.4.2 MATRIX TRANSFORMATION OF \( H \) INTO \( C \):

THEOREM 2.7: A matrix transformation \( A = (a_{n,k}) \) maps \( H \) into the space \( C \) of all convergent sequences if and only if

(2.7.1) \( \text{Sup} \left( \sum_{n \geq 1}^{\infty} \left| a_{n,k} - a_{n,k+1} \right| \right) < \infty \),

(2.7.2) \( \lim_{n \to \infty} a_{n,k} = \xi_k \) for every fixed \( k \).

Moreover,

(2.7.3) \( \lim_{n \to \infty} (Ax)_n = \xi_1 S + \sum_{k=1}^{\infty} (\xi_k - \xi_{k+1}) (S_k - S) \)
where
\[ S = \sum_{k=1}^{\infty} x_k \quad \text{and} \quad S_k = \sum_{i=1}^{k} x_i \]
and
\[ (Ax)_n = \sum_{k=1}^{n} a_{n,k} x_k. \]

**Proof:** This is a well-known theorem about the matrix transformation \( \Lambda = (a_{n,k}) \) which maps \( C \) into \( C \).

For the proof we refer to (Cooke [1], p. 65-66).

**Remark:** The space \( H \) can be generalized to the space \( H^{(r)} (r \geq 2) \) which consists of all real sequences \( x = \{x_k\} \) such that the series \( \sum_{n=1}^{\infty} (\sigma^{r-1} x)_n \) is convergent,

where
\[ (\sigma x)_n = \sum_{k=1}^{n} x_k, \quad (\sigma^2 x)_n = \sum_{k=1}^{n} (\sigma x)_k \ldots, \]
\[ (\sigma^r x)_n = \sum_{k=1}^{n} (\sigma^{r-1} x)_k. \]

This associate space of the space \( H^{(r)} (r \geq 2) \) can be obtained in a similar way, and the necessary and sufficient conditions under which a matrix transformation
will maps the space $H^r$ into the space $l_\infty$ and $C$ respectively can also be obtained.

2.5 This section consists of results of Nung [18] related to the matrix transformation of Cesaro sequence spaces of non-absolute type into respectively the space $l_\infty$ of all bounded sequences and the space $C$ of all convergent sequences. In [7] Ng and Lee determined the associate norms and the associate spaces of the Cesaro sequence spaces of a non-absolute type. First we shall describe required sequence space and then we give the proofs of main theorems.

2.6 CESARO SEQUENCE SPACES:

Let $X_p (1 \leq < \infty)$ and $X_\infty$ be respectively the spaces of all $x \in X$ with

$$||x||_p = \left( \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty$$

and

$$||x||_\infty = \text{Sup} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p ; k = 1, 2, \ldots \right\} < \infty.$$
Note that the above norms are saturated except for \( P = 1 \). By Shiue [16] and Leibowitz [14], the Cesaro sequence spaces defined as

\[
\text{Ces}_p = \left\{ a = \{a_n\}_{n=1}^\infty : \|a\|_p = \left( \sum_{n=1}^\infty \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \right) < \infty \right\}
\]

For \( 1 \leq p < \infty \),

and \( \text{Ces}_\infty = \left\{ a = \{a_n\}_{n=1}^\infty : \|a\|_\infty = \sup_n \left( \sum_{k=1}^n |a_k| \right) < \infty \right\} \).

We note also that the spaces \( X_p \) defined above are different from the Cesaro sequence spaces \( \text{Ces}_p(1 \leq p < \infty) \).

In fact \( \text{Ces}_p \subset X_p \) (\( 1 \leq p < \infty \)) and \( \text{Ces}_p \not\subset X_p \) (see [7]).

Now by using theorem 2.2 we can prove that \( X_p \) are Banach sequence spaces of a non-absolute type.

Now we shall give the results due to Ng and Lee [7].

**Theorem 2.8:** Let \( Y_q \) be the space of all \( y \in X \) such that

\[
(2.8.1) \quad |ky_k| \leq M \quad \text{for all} \quad k = 1, 2, \ldots, \\
(2.8.2) \quad \lambda_q(y) = \left( \sum_{k=1}^\infty |k(y_k - y_{k+1})|^q \right)^{1/q} < \infty \quad \text{for} \quad 1 \leq q < \infty
\]
and
\[ \lambda_{\infty}(y) = \sup \left\{ |y_k - y_{k+1}| : k = 1, 2, 3, \ldots \right\} < \infty. \]

**Theorem 2.9:** The associate space \( X'_p \) of \( X_p \) is the space \( Y_q \) with the norm \( \lambda_q \), where \( \frac{1}{p} + \frac{1}{q} = 1. \)

---

**2.7 Matrix Transformations on \( X'_p \):**

In this section we shall find the necessary and sufficient conditions under which an infinite matrix will transforms the Cesaro sequence spaces \( X_p \), into respectively the space \( l_\infty \) of all bounded sequences and the space \( C \) of all convergent sequences.

**2.7.1. Matrix Transformation of \( X'_p \) into \( l_\infty \):**

Let us consider the matrix transformation given by
\[ y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \ldots \]
provided the series on the right is convergent.

Now we shall prove a theorem due to Nung [18].
**THEOREM 2.10:** A matrix transformation $A = (a_{n,k})$ maps the space $X_p (1 \leq p \leq \infty)$ into the space $\ell_\infty$ if and only if

\[(2.10.1) \quad \sup_{n \geq 1} \| \sum_{k=1}^{\infty} k (a_{n,k} - a_{n,k+1}) \|^q_{(q)} < \infty\]

\[(2.10.2) \quad \sup_{k \geq 1} |k a_{n,k}| < \infty, \text{ for every fixed } n,\]

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\| \cdot \|_{(q)}$ is the $\ell_q$ norm.

**PROOF:** First we prove that the conditions are necessary.

Suppose $A = (a_{n,k})$ maps $X_p$ into $\ell_\infty$, then the series

\[(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k\]

is convergent for every $x \in X_p$.

Then by theorem 2.8 the sequence $\{a_{n,k}^2\}_{k \geq 1}$ is an element in $\ell_q$ for every $n$, it follows that the condition (2.10.2) holds and $\| \{k(a_{n,k} - a_{n,k+1})^2\}_{k \geq 1} \|^q_{(q)} < \infty$.

Since $X_p$ and $\ell_\infty$ are BK-spaces by lemma 2.1

\[\|Ax\|_{\ell^\infty} \leq K \|x\|_p\]

for some real constant $K$, and all $x \in X_p$ or

\[\sup_{n \geq 1} \| (Ax)_n \| \leq K \|S\|_{(p)}\]
for all \( x \in X_p \) with \( S = \{ S_k \} \) where \( S_k = \frac{1}{k} \sum_{i=1}^{k} x_i \).

It follows that

\[
\sup_{n \geq 1} \left| \frac{\sum_{k=1}^{\infty} k(a_{n,k} - a_{n,k+1}) S_k}{||S||_l(p)} \right| \leq K.
\]

Hence we have

\[
\sup_{n \geq 1} \left| \sum_{k=1}^{\infty} k(a_{n,k} - a_{n,k+1}) S_k \right| \leq K.
\]

Thus condition (2.10.1) holds.

Conversely, suppose conditions (2.10.1) and (2.10.2) hold, then for each \( n = 1, 2, 3, \ldots \), the sequence \( \{ a_{n,k} \} \) is an element of \( Y_q \), therefore

\[
(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k
\]

is convergent for each \( x \in X_p \) and by Hölder inequality, we have

\[
\sup_{n \geq 1} |(Ax)_n| = \sup_{n \geq 1} \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right| \\
= \sup_{n \geq 1} \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n,k+1}) S_k \right| \\
= \sup_{n \geq 1} \left| \sum_{k=1}^{\infty} k(a_{n,k} - a_{n,k+1}) S_k \right| \\
\leq K.
\]
Which shows that $Ax \in \ell_\infty$ and $A = (a_{n,k})$ maps $X_p$ into $\ell_\infty$.

This completes the proof of the theorem.

2.7.2. MATRIX TRANSFORMATION OF $X_p$ INTO $C$:

Let us consider the matrix transformation given by

$$\varphi_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \ldots.$$

**Theorem 2.11:** A matrix transformation $A = (a_{n,k})$ maps $X_p$ into the space $C$ if and only if

(2.11.1) $\sup_{n \geq 1} \left\{ \sum_{k=1}^{\infty} k^q (a_{n,k} - a_{n,k+1}) \right\} \leq I (q) < \infty$,

(2.11.2) $\sup |k a_{n,k}| < \infty$ for every fixed $n$,

(2.11.3) $\lim_{n \to \infty} k(a_{n,k} - a_{n,k+1}) = \delta_k$ for every fixed $k$,

where $\frac{1}{p} + \frac{1}{q} = 1$. 
PROOF: First we prove that the conditions are necessary. Suppose \( A = (a_{n,k}) \) maps \( X_p \) into \( C \), the condition (2.11.1) and (2.11.2) follows similarly as in the proof of Theorem 2.10. To prove that condition (2.11.3) is necessary, we take, for each fixed \( k \), as sequence \( x(k) \) in \( X_p \) with

\[
x_j^{(k)} = \begin{cases} 
  k & \text{if } j = k \\
  -k & \text{if } j = k + 1 \\
  0 & \text{if } j \neq k, k+1
\end{cases}
\]

Then we see that

\[
S_k = \frac{1}{k} \sum_{j=1}^{k} x_j^{(k)} = 1,
\]

and \( S_j = 0 \), if \( j \neq k \).

For this sequence \( x^{(k)} \) we have

\[
(Ax^{(k)})_n = \sum_{j=1}^{\infty} a_{n,j} x_j^{(k)}
\]

\[
= \sum_{j=1}^{\infty} j(a_{n,j} - a_{n,j+1}) S_j
\]

\[
= k (a_{n,k} - a_{n,k+1}) \rightarrow \delta_k \text{ as } n \rightarrow \infty.
\]
This shows that condition (2.11.3) is necessary.

Conversely, suppose the conditions (2.11.1), (2.11.2) and (2.11.3) hold. Then by conditions (2.11.1) and (2.11.2) the series

\[(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k \text{ is convergent for every } x \in X_p.\]

By condition (2.11.3) we have

\[|k(a_{n,k} - a_{n,k+1})|^q \longrightarrow |\xi_k|^q \text{ as } n \longrightarrow \infty\]

and since for every positive integer \(m\)

\[\left\{ \sum_{k=1}^{m} |k(a_{n,k} - a_{n,k+1})|^q \right\}^{1/q} \leq \sup_{n \geq 1} \left\{ \sum_{k=1}^{\infty} |k(a_{n,k} - a_{n,k+1})|^q \right\}^{1/q} = \beta^{1/q}.
\]

By letting \(n \longrightarrow \infty\), we get

\[\left\{ \sum_{k=1}^{m} |\xi_k|^q \right\}^{1/q} \leq \sup_{n \geq 1} \left\{ \sum_{k=1}^{\infty} |k(a_{n,k} - a_{n,k+1})|^q \right\}^{1/q}.
\]

Since this is true for every positive integer \(m\), it follows that

\[\left\{ \sum_{k=1}^{\infty} |\xi_k|^q \right\}^{1/q} < \infty.
\]

Now for every sequence \(x \in X_p\), we have

\[S_n = \frac{1}{n} \sum_{k=1}^{n} x_k \longrightarrow 0 \text{ as } n \longrightarrow \infty.\]
Given any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left\{ \sum_{k=N}^{\infty} |S_k|^{1/p} \right\} \leq \varepsilon / 4 \beta$$

and by condition (2.11.3), there exists integer $N'$ such that

$$\left| \sum_{k=1}^{N} \left\{ k(a_n,k - a_n,k+1) - \delta_k \right\} S_k \right| < \varepsilon / 2$$

for all $n \geq N'$. Now for all $n \geq N'$

$$\left| \sum_{k=1}^{\infty} \left\{ k(a_n,k-a_n,k+1) - \delta_k \right\} S_k \right| \leq \sum_{k=1}^{N} \left\{ k(a_n,k-a_n,k+1) - \delta_k \right\} S_k$$

$$+ \sum_{k=N+1}^{\infty} \left\{ k(a_n,k-a_n,k+1) - \delta_k \right\} S_k$$

$$< \varepsilon / 2 + \left( \sum_{k=N+1}^{\infty} \left| k(a_n,k-a_n,k+1) \right| + \left| \delta_k \right| \right)^{1/q} \left( \sum_{k=N+1}^{\infty} |S_k|^p \right)^{1/p}$$

$$< \varepsilon / 2 + 2 \beta \cdot \varepsilon / 4 \beta$$

$$= \varepsilon$$

So we have

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} k(a_n,k - a_n,k+1) S_k = \sum_{k=1}^{\infty} \delta_k S_k$$
It follows that

\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} x_k
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{\infty} k(a_{n,k} - a_{n,k+1}) S_k
\]

\[
= \sum_{k=1}^{\infty} \xi_k S_k.
\]

This shows that $Ax \in C$, and $A = (a_{n,k})$ maps $\chi_p$ into $C$, and the proof is complete.

**Remark:** $\xi_k = 0$ for all $k$ if $C$ above is replaced by the space $C_0$ of all null sequences.
CHAPTER-III

CESARO SEQUENCE SPACES OF AN ABSOLUTE TYPE

3.1 In this Chapter we shall mention a class of sequence spaces which is of an absolute type. These sequence spaces was first defined by J.S. Shine [16] and its a-dual given by Jagers [15] for $1 < p < \infty$ and by Ng and Lee [12] for $p = \infty$. By absolute we mean that if a sequence $x = \{x_k\}$ belongs to a given space so does its absolute value $|x| = \{|x_k|\}$. Here we also give a function version of Cesaro sequence spaces due to J.S. Shiue [19]. In this connection we shall study the results of K.P. Lim [31] relating to matrix transformation on Cesaro sequence spaces of an absolute type. In the last we list the some problems.

3.2 DEFINITION:

Let $A$ be infinite matrix and $Y$ a sequence space. We consider $X = \{x ; A |x| \subseteq Y\}$. In particular, when $A$ is a Cesaro matrix $C$ and
Y = ℓ_p for 1 < p < ∞. We call X Cesaro sequence space of an absolute type and it is denoted by Ces_p.

Now we can define in otherwords, x ∈ Ces_p for 1 < p < ∞ if and only if

\[ \left( \sum_{n=1}^{\infty} \left( \frac{k}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right) < \infty. \]

and similarly for p = ∞.

3.3 This section consists of results of S.K. Lim [26] regarding α-dual of Ces_p. Now the first problem is to find the α-dual of Ces_p.

Now we observe that,

\[ \sum_{k=1}^{n} |x_k y_k| = \sum_{k=1}^{n-1} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^{k(|y_k|-|y_{k+1}|)+(\frac{1}{n} \sum_{i=1}^{n} |x_i|)(n|y_n|)} \]

\[ = \sum_{k=1}^{n-1} S_k t_k + \left( \frac{1}{n} \sum_{i=1}^{n} |x_i| \right) (n|y_n|) \]

where \( S_k = \frac{1}{k} \sum_{i=1}^{k} |x_i| \) and \( t_k = k (|y_k| - |y_{k+1}|). \)
If \( \left( \frac{1}{n} \sum_{i=1}^{n} |x_i| \right) (n \ |y_n|) \rightarrow 0 \) and \( t = \{ t_k \}_{k \in \mathbb{N}} \)

with \( \frac{1}{p} + \frac{1}{q} = 1 \) then \( y \in \text{Ces}_p^\alpha \).

But the converse does not hold because the series

\[
\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} S_k t_k
\]

if exists, converges for some \( S = \{ S_k \}_{k \in \mathbb{N}} \) only.

Hence a different approach is required.

Suppose \( y \in \text{Ces}_p^\alpha \). Now using functional analytic method [25, Th. 17 iii], we can show that for every \( x \in \text{Ces}_p \),

\[
\sum_{k=1}^{\infty} |x_k y_k| \leq M \ |x|.
\]

Here the norm is defined to be

\[
||x|| = \left[ \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^p \right]^{1/p}
\]

It is easy to show that \( y \in C_0 \) and

\[
\left( \frac{1}{n} \sum_{i=1}^{n} |x_i| \right) (n \ |y_n|) \rightarrow 0.
\]
Then we obtain,

\[
\sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)(k |y_k| - |y_{k+1}|) \leq M \left( \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^{p} \right)^{1/p}
\]

or alternatively,

\[
\sum_{k=1}^{\infty} S_k t_k \leq M \left( \sum_{k=1}^{\infty} S_k^p \right)^{1/p}
\]

In order to say \( t = \{ t_k \}_{k \in \mathbb{Q}^1} \), we require \( S_k = t_k^q \).

In other words, we want to choose \( x \in \mathcal{C}_p \) such that

\[
\left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right) = (k (|y_k| - |y_{k+1}|))^{q-1}
\]

or equivalently,

\[
\sum_{i=1}^{k} |x_i| = k^q (|y_k| - |y_{k+1}|)^{q-1}
\]

\[
= [k^p (|y_k| - |y_{k+1}|)]^{q-1}
\]

This is possible only when \( k^p(|y_k| - |y_{k+1}|) \) is non-negative and increasing in \( k \).

Therefore we have proved.
LEMMA 3.1: The space $C_{sp}$ contains all $y$ such that

$$y \in C_0 \text{ and } \sum_{k=1}^{\infty} [k \left| y_k \right| - |y_{k+1}|]^q < \infty$$

where $k^p \left| y_k \right| - |y_{k+1}|$ is non-negative and increasing; and its solid hull.

Rewriting the above lemma, we have

THEOREM 3.1: The $\alpha$-dual of $C_{sp}$ for $1 < p < \infty$ is the space of all sequences $y$ such that $y \in C_0$ and

$$\sum_{k=1}^{\infty} |k(\tilde{y}_k - \tilde{y}_{k+1})|^q < \infty$$

where $\tilde{y}$ denotes the infimum of all $y^* \leq |y|$ with $k^p (y_k^* - y_{k+1}^*)$ being non-negative and increasing.

Again, let $X = \{x; A|x| \in Y\}$. Lim [26] considered the cases for other special $A$ and $Y$.

In particular when $A = C$, Cesaro matrix, and $Y = \omega_0$ where $\omega_0 = \{x; \text{C}|x| \in C_0\}$.
In this section we shall give a function version of Cesaro sequence spaces due to Shiue [19]. Cesaro function spaces, denoted by $\text{CES}_P$ for $1 < p < \infty$, is defined to be the set of all real valued measurable functions defined on $(0,\infty)$ such that

$$\left[ \int_0^\infty \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \, dx \right]^{1/p}$$

converges.

In otherwords, $f \in \text{CES}_P$ if and only if

$$T |f| \in L_p (0, \infty) \quad \text{where} \quad (T|f|)(x) = \frac{1}{x} \int_0^x |f(t)| \, dt.$$  

Obviously, the operator $T$ plays the role of matrix $A$ or $C$ as in the sequence version.

Given a function space $X$ with functions defined on $(0,\infty)$ the Kothe dual of $X$ is the space of all real valued measurable functions of defined on $(0,\infty)$ such that

$$\int_0^\infty f(x) g(x) \, dx \quad \text{exists for every} \quad f \in X.$$  

Unfortunately, it is difficult to express the inverse of $T$ precisely. Hence the technique used in the sequence
version to find the dual does not work here. However, it is still possible to find the dual of $\text{CES}_p$. We do it by first of all converting the problem in function spaces to one in sequence spaces, and then solve the problem in sequence spaces.

We observe that for $n \leq x \leq n+1$

$$\frac{1}{2n} \int_0^n \leq \frac{1}{n+1} \int_0^x |f(t)| \, dt \leq \frac{1}{n} \int_0^{n+1} \leq \frac{2}{n+1} \int_0^{n+1}$$

In otherwords, the integral

$$\int_1^\infty \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \, dx$$

converges if and only if the series

$$\sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n S_k \right)^p$$

converges.

Where $S_k = \int_{k-1}^k |f(t)| \, dt$.

That is, $S = \{S_k \in \text{CES}_p\}$.

Next we write for $(m+1)^{-1} \leq x \leq m^{-1}$

$$\frac{m+1}{2} \int_0^{1/(m+1)} \leq m \int_0^{1/(m+1)} \leq \frac{1}{x} \int_0^x |f(t)| \, dt \leq (m+1) \int_0^{1/m} \leq 2m \int_0^{1/m}.$$
In otherwords, the integral
\[ \int_{0}^{1} \left( \int_{0}^{x} |f(t)| \, dt \right)^{p} \, dx \]
converges if and only if
the series \( \sum_{m=1}^{\infty} \left( \sum_{k=m}^{\infty} t_{k} \right)^{p} (1 - \frac{1}{m+1}) \)
converges
where \( t_{k} = \int_{1/k}^{1/(k+1)} |f(t)| \, dt \).

We say that \( t = \{ t_{k} \} \) belongs to a reverse Cesaro sequence space or \( t \in d_{p} \).

Now combining the above, we obtain that \( f \in \text{CES}_{p} \)
where \( 1 < p < \infty \) if and only if \( S = \{ S_{k} \} \in \text{CES}_{p} \) and
\( t = \{ t_{k} \} \in d_{p} \) where,
\[ S_{k} = \int_{k-1}^{k} |f(t)| \, dt \quad \text{and} \quad t_{k} = \int_{1/k}^{1/(k+1)} |f(t)| \, dt. \]

Since the \( \alpha \)-dual of \( \text{CES}_{p} \) is known and that of \( d_{p} \)
can be found therefore we have solved the problem of
the Kothe dual of \( \text{CES}_{p} \).

**Theorem 3.2**: The Kothe dual of \( \text{CES}_{p} \) for \( 1 < p < \infty \)
is the space of all real valued measurable functions
g such that \( u = \{ u_k \} \in \text{Ces}_p^g \) and \( v = \{ v_k \} \in d_p^g \)

where \( u_k = \text{ess-Sup} \left\{ |g(x)| ; k-1 \leq x \leq k \right\} \)

\( v_k = \text{ess-Sup} \left\{ |g(x)| ; (k+1)^{-1} \leq x \leq k^{-1} \right\} \).

3.5 **MATRIX TRANSFORMATION ON Ces(p):**

In this section we shall study the results of K.P. Lira [31] relating to matrix transformation on Cesaro sequence spaces of an absolute type. First we shall describe the required sequence space and then we give the proofs of main theorems. We further determine the dual space of Cesaro sequence space Ces(p).

3.6 **DEFINITION:** For \( p = (p_r) \) with \( \inf p_r > 0 \)

\[ \text{Ces}(p) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left( \frac{1}{2^r} \sum_r \sum_{r} |x_k| \right)^{p_r} < \infty \right\} \]

where \( \sum_r \) denotes a sum over the ranges \( 2^r \leq k < 2^{r+1} \).

First of all we have to show that the space Ces(p) is paranormed by

\[ (3.6.1) \quad g(x) = \left( \sum_{r=0}^{\infty} \left( \frac{1}{2^r} \sum_r \sum_{r} |x_k| \right)^{p_r} \right)^{1/M} \]
If $|t| = \text{Sup } p_r < \infty$ and $M = \max (1, H)$.

**PROOF:** It is clear that

\[ g(\theta) = 0 \text{ and } g(x) = g(-x) \text{ where } \theta = (0, 0, \ldots). \]

\( g \) is subadditive:

Since \( p_r \leq M \), we have

\[
\left( \sum_{r} \frac{p_r}{M} \right)^M \leq \left( \sum_{r} \frac{p_r}{M} \right)^M + \left( \sum_{r} \frac{p_r}{M} \right)^M
\]

and since \( M \geq 1 \), we see by Minkowski's inequality that \( g \) is subadditive.

Now finally we have to check continuity of multiplication. For any complex \( \lambda \), we have

\[
g(\lambda x) = \left( \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \sum_{r} \left| \lambda x_k \right| \right)^{p_r} \right)^{1/M}
\]

\[
\leq \text{Sup } |\lambda| \cdot g(x).
\]

Now let \( \lambda \rightarrow 0 \), for any fixed \( x \) with \( g(x) \neq 0 \).

Since \( \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \sum_{r} |x_k| \right)^{p_r} < \infty \), so for
\[ |\lambda| < 1 \text{ and } \varepsilon > 0, \text{ there exists an integer } m_0 > 0 \text{ s.t.} \]

\[(3.6.2) \sum_{r=m_0}^{\infty} \left( \frac{1}{2^r} \sum_{j} |\lambda x_k| \right)^{p_r} < \varepsilon. \]

Now taking \(|\lambda|\) sufficiently small such that \(|\lambda| < \varepsilon/g(x)\) for \(r = 0, 1, \ldots, m_0 - 1\), we then have

\[(3.6.3) \sum_{r=0}^{m_0-1} \left( \frac{1}{2^r} \sum_{j} |\lambda x_k| \right)^{p_r} < \varepsilon. \]

Together with (3.6.2) and (3.6.3), we obtain

\[g(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0.\]

Hence continuity of multiplication is exist.

We have thus shown that \(C_0(p)\) is paranormed by (3.6.1).

The completeness of \(C_0(p)\) may be proved by using the same kind of argument to that in [30].

Note that, since we have \(\left( \sum \frac{1}{2^r} \sum |x_k| \right)^{p_r} \leq (2^r) \sum |x_k|^{p_r} \) for \(p_r \geq 1\), it is clear that \(l(p)\) (See[27]) is contained in \(C_0(p)\).
Now we write \( A(n) = \max_k |a_{n,k}| \)

where for each \( n \) the maximum is taken for \( k \) in 
[2^n, 2^{n+1}]. We state the following inequality
(See [28]) which will be used later. For any \( C > 0 \)
and any two complex numbers \( a, b \),

\[
(3.6.4) \quad |ab| \leq C \left( |a|^q C^{-q} + |b|^p \right)
\]

where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

3.5.1 \hspace{1cm} \text{MATRIX TRANSFORMATION OF CES}(p) \hspace{1cm} \text{INTO} \hspace{1cm} L_\infty.

In this section we shall introduce an important
result of K.P. Lim [31]. In which we shall determine
the necessary and sufficient condition on an infinite
matrix \( A = (a_{n,k}) \) so that it should transform Ces(p)
into \( L_\infty \).

Let us consider the following Matrix transformation.

\[
Y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, 3, \ldots
\]

**THEOREM 3.3:** Let \( 1 < p_\tau \leq \sup_{n} p_\tau < \infty \). Then
\( A \in (\text{Ces}(p), L_\infty) \) if and only if there exists an
integer $E > 1$ such that

$U(E) < \infty$, where $U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r-q_r}$

and $\frac{1}{p_r} + \frac{1}{q_r} = 1$, $r = 0, 1, 2, \ldots$.

**Proof:** Suppose there exists an integer $E > 1$ such that $U(E) < \infty$. Then by inequality (3.6.4), we have

$$\forall n, k \quad |a_n, k x_k| = \sum_{r=0}^{\infty} \left| a_n, k x_k \right|$$

$$\leq \sum_{r=0}^{\infty} 2^r A_r(n) \frac{1}{2^r} \sum_r \left| x_k \right|.$$  

$$\leq E \left( \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r-q_r} + \sum_{r=0}^{\infty} \left( \frac{1}{2^r} \sum_r \left| x_k \right| \right)^{p_r} \right)$$

$$< \infty.$$  

Therefore, $A \in (\text{Ces}(p), \ell_\infty)$.  

Conversely, suppose $A \in (\text{Ces}(p), \ell_\infty)$ and there does not exist $E > 1$ such that $U(E) < \infty$. Since $\lim_n \sup \left| A_n(x) \right| < \infty$, so we may use the same kind of argument to that in [29] to prove that each $A_n(x)$ defined $\sum a_n, k x_k$ is in $\text{Ces}(p)$. Since $\text{Ces}(p)$ is
complete and \( \sup_{n} |A_n(x)| < \infty \), by uniform boundedness principle there exists a number \( D \) independent of \( n \) and \( x \) and a number \( \delta < 1 \) such that

\[
(3.6.5) \quad |A_n(x)| \leq D.
\]

For every \( x \in S[\Theta, \delta] \) and every \( n \), where \( S[\Theta, \delta] \) is the closed sphere in \( \text{Ces}(p) \) with centre the origin \( \Theta \) and radius \( \delta \).

Now choose an integer \( G > 1 \) such that

\[
G \delta^M > D.
\]

Since \( U(G) = \infty \), there exists an integer \( m_0 > 1 \) such that

\[
(3.6.6) \quad R = \sum_{r=0}^{m_0} \left( 2^r A_r(n) \right) 2^q \delta^{M/p} > 1.
\]

Define a sequence \( x = (x_k) \) as follows, \( x_k = 0 \) if \( k \leq 2^{m_0+1} \) and \( x_{N(r)} = 2^{\frac{rq}{p}} \delta^{M/p} (\text{Sgn} \ a_{n,N(r)}) |a_{n,N(r)}|^{q} \delta^{-q/r/p} \),

\[
x_k = 0 \text{ (} k \notin N(r) \text{)}, \text{ for } 0 \leq r \leq m_0.
\]
Where \( N(r) \) is the smallest integer such that

\[ |a_{n,N(r)}| = \max_r |a_{n,k}|. \]

Then by (3.6.6), we have,

\[
g(x) = \left( \sum_{r=0}^{m_0} 2^r (q_r - 1) p_r \ |a_{n,N(r)}| \right)^{1/M}
\]

\[
= \left( \sum_{r=0}^{m_0} \left( 2^r |a_{n,N(r)}| \right)^{q_r - p_r - q_r} \right)^{1/M}
\]

\[
\leq \sum_{r=0}^{m_0} \frac{1}{2} \left( \sum_{r=0}^{m_0} 2^r |a_{n,N(r)}| \right)^{q_r - q_r} \]

Hence, \( x \in S[\Theta, \delta] \). Moreover,

\[
|A_n(x)| = \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right|
\]

\[
= \sum_{r=0}^{m_0} 2^r \left( \sum_{r=0}^{m_0} 2^r |a_{n,N(r)}| \right)^{q_r - q_r/p_r}
\]

\[
= \sum_{r=0}^{m_0} 2^r \left( \sum_{r=0}^{m_0} 2^r |a_{n,N(r)}| \right)^{q_r - q_r}
\]

\[
\leq \sum_{r=0}^{m_0} \frac{1}{2} \left( \sum_{r=0}^{m_0} 2^r |a_{n,N(r)}| \right)^{q_r - q_r} \]

\[
= \sum_{r=0}^{m_0} \frac{1}{2} \left( \sum_{r=0}^{m_0} 2^r |a_{n,N(r)}| \right)^{q_r - q_r} \]

\[
\leq D.\]
Which contradicts (3.6.5). This completes the proof of the theorem.

Consider the well-known summability matrix $A$

deefined by
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

for any $x = (x_k) \in \text{Ces}(p)$,

$Ax = (x_1, \frac{1}{2}(x_1 + x_2), \ldots, \frac{1}{n} \sum_{i=1}^{n} x_i, \ldots)$

It is routine to check that $Ax \in l_\infty$, so $A \in (\text{Ces}(p), l_\infty)$.

Moreover, for each $n$, we have

$A_r(n) = \begin{cases} 
\frac{1}{n} & \text{if } 2^r \leq n \\
0 & \text{if } 2^r > n
\end{cases}$
Therefore,

\[
\sum_{r=0}^{\infty} (2^r A_r(n)) q_r e^{-q_r} = \sum_{r=0}^{k} \frac{2^r}{n^r} q_r - q_r
\]

for some \( k \) such that \( 2^k \leq n \).

Take any integer \( E \geq 2^2 \), we get

\[
\sum_{r=0}^{\infty} (2^r A_r(n)) q_r e^{-q_r} = \sum_{r=0}^{k} \frac{2^r}{n^r} q_r
\]

\[
\leq \sum_{r=0}^{k} \left( \frac{2^r}{2^n} \right)
\]

\[
\leq \sum_{r=0}^{k} \left( \frac{2^r}{2^n} \right)
\]

\[
= \frac{1}{4n} (2^{k+1} - 1) < 1
\]

as \( 2^k \leq n \).

This shows that there exists an integer \( E > 1 \)

such that \( U(E) < \infty \) for this summability matrix \( A \).
3.5.2 MATRIX TRANSFORMATIONS OF CES(p) INTO C:

Here we shall characterize the class \((\text{Ces}(p), C)\).

**Theorem 3.4:** Let \(1 < p_r \leq \text{Sup } p_r < \infty\). Then

\(A \in (\text{Ces}(p), C)\) if and only if,

\[ (3.4.1) \quad a_{n,k} \longrightarrow a_k \quad (n \longrightarrow \infty, k \text{ fixed}) \]

\[ (3.4.2) \quad \text{There exists an integer } E > 1 \text{ such that } U(E) < \infty, \text{ where } U(E) \text{ is defined as in Theorem (3.3).} \]

**Proof:** Suppose \(A \in (\text{Ces}(p), C)\). Then \(A_n(x)\) exists for each \(n \geq 1\) and \(\lim_{n \to \infty} A_n(x)\) exists, for every \(x \in \text{Ces}(p)\). Therefore a similar argument to that in Theorem 3.3, we have the condition (3.4.2). The condition (3.4.1) is obtained by taking \(x = e_k \in \text{Ces}(p)\), where \(\Omega_k\) is a sequence with 1 in the \(k\)th place and zero elsewhere.

On the other hand, suppose that the conditions (3.4.1) and (3.4.2) are satisfied. Then the conditions (3.4.1) and (3.4.2) imply that

\[ (3.6.7) \quad \sum_{r=0}^{q_r} 2^r \max_r |a_k| \quad E \leq U(E) < \infty. \]
By using (3.6.7), it is easy to check that \( \sum_{k=1}^{\infty} a_k x_k \) is absolutely convergent, for each \( x \in \text{Ces}(p) \).

Moreover, for each \( x \in \text{Ces}(p) \), there exists an integer \( m_0 \geq 1 \) such that

\[
g_{m_0}(x) = \sum_{r=m_0}^{\infty} \left( \frac{1}{2^r} \sum_{k} |x_k| \right)^{p_r} < 1.
\]

If \( g_{m_0}(x) \neq 0 \), we have

\[
\sum_{r=m_0}^{\infty} 2^r B_r(n) \frac{1}{2^r} \sum_{k} |x_k|/(g_{m_0}(x)) \leq \sum_{r=m_0}^{\infty} \frac{q_r^{p_r}}{2^r} \sum_{k} |x_k|/(g_{m_0}(x))
\]

\[
\leq E \left( \sum_{r=m_0}^{\infty} \frac{q_r^{p_r}}{2^r} \right) E \left( \frac{1}{2^r} \sum_{k} |x_k| \right) / (g_{m_0}(x))
\]

\[
\leq E \left( \sum_{r=m_0}^{\infty} \frac{q_r^{p_r}}{2^r} \right) E (q_r + 1)
\]

Where \( B_r(n) = \max_{r} |a_{n,k} - a_k| \).
Thus

\[ (3.6.8) \sum_{m_0}^{\infty} |a_{n,k} - \alpha_k| x_k \leq E \left( \sum_{r=m_0}^{\infty} (2^{rB(x(n)})^{q_r} \sum_{r=q_{r+1}}^{1/M} g_m(x) \right) \]

Clearly (3.6.8) holds if \( g_m(x) = 0 \).

Since \( \sum_{m_0}^{\infty} (2^{rB(x(n)})^{q_r} \sum_{r=q_{r+1}}^{1/M} g_m(x) \leq 2U(E) < \infty \), from (3.6.8),

it follows immediately that \( \lim_{n \to \infty} \sum_{m_0}^{\infty} a_{n,k} x_k = \sum_{m_0}^{\infty} \alpha_k x_k \).

This shows that \( A \in (\text{Ces}(p), C) \). Which proves the theorem.

**REMARK**: In this direction I have characterized the classes \( (\text{Ces}(p), \hat{C}) \) and \( (\text{Ces}(p), \hat{l}_\infty) \) which are appearing in Tamkang J. Math. Vol. 19, No.4, 1988. Where \( \hat{C} \) denote the space of almost convergent sequences and \( \hat{l}_\infty \) is the space of almost boundedness.

3.7 This sequence space has been generalized by Malkowsky and Eberhard [39] in the following way.

For \( \alpha > 0 \) and \( 0 < p \leq \infty \),
$$\text{CES}_p(\alpha) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \left[ \sum_{i \in N} \left( \frac{\alpha - 1}{2^{k+1-1}} \right)^{\alpha} x_i \right] < \infty \right\}$$

where $N^{(k)} = \left\{ m \in N : 2^k \leq m \leq 2^{k+1} - 1 \right\}$

$$A_s = \left( \frac{r + g}{r} \right).$$

He investigates various properties of the spaces $\text{CES}_p(\alpha)$, determines their continuous and Kothe-Toeplitz duals and characterizes the sets $(\text{CES}_p(\alpha), l_1), (\text{CES}_p(\alpha), C)$ for $0 < p \leq \infty$ and the sets $(l_\infty, \text{CES}_p(\alpha))$, $(l_1, \text{CES}_p(\alpha)), (C, \text{CES}_p(\alpha))$ for $1 \leq p < \infty$.

It is also stated that the known Cesaro sequence spaces $\text{Ces}_p = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \sum_{k=1}^{n} |x_k| < \infty \right\}$ are closely connected with the spaces $\text{CES}_p(1)$ for $1 < p \leq \infty$, and thus the author's results contain two theorems of K.P. Lim [30] concerning the sets $(\text{Ces}_\infty, l_\infty)$ and $(\text{Ces}_p, C)$.

3.8 **CONTINUOUS LINEAR FUNCTIONALS ON CES(p):**

In this section we shall determine the dual space of
Ces(p), that is the space of all continuous linear functionals on Ces(p) and it is denoted by Ces(p)*.

(See [31]).

**Theorem 3.5:** Let $1 < p < \sup r p < \infty$. Then Ces(p)* is isomorphic to $\mu(q)$, where

$$\mu(q) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left(2^r \max_{r} |x_k|\right)^q r^{-q} < \infty \right\}$$

For some integer $E > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof:** It is easy to check that each $x \in \text{Ces}(p)$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$. Then, for any $f \in \text{Ces}(p)$,

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} a_k x_k$$

where $a_k = f(e_k)$.

We show that the convergence of $\sum_{k=1}^{\infty} a_k x_k$ implies that $a = (a_k) \in \mu(q)$. For, if $a \notin \mu(q)$, we can determine integers $0 = n(0) < n(1) < n(2) < \ldots$ such that

$$G_s = \sum_{r=n(s)}^{n(s+1)-1} \left(2^r \max_{r} |a_k|\right)^q r^{-q}/p > 1$$

for $s = 0, 1, 2, \ldots$.
Define a sequence \( x = (x_k) \) as follows: For each \( S, \)

\[
x_{N(r)} = 2^{rq} |a_{N(r)}|^{q_r-1} \text{Sgn} \ a_{N(r)}(S+2)^{G_S-1}
\]

for \( n(S) \leq r < n(S+1) - 1, \)

and

\( x_k = 0 \) for \( k \notin N(r) \)

where \( N(r) \) is such that \( |a_{N(r)}| = \max_r |a_k|, \)

The maximum is taken for \( k \) in \([2^r, 2^{r+1})\).

Therefore we have,

\[
\Sigma_{r=n(S)}^{n(S+1)-1} a_k x_k = \sum_{r=n(S)}^{n(S+1)-1} \left( 2^r |a_{N(r)}| \right)^{q_r} (S+2)^{G_S-1} = (S+2)^{-1}
\]

It follows that \( \Sigma_{k=1}^{\infty} a_k x_k = \Sigma_{k=0}^{\infty} (S+2)^{-1} \) diverges. Moreover,
\[
\sum_{r=n(S)}^{n(S+1)-1} \frac{\frac{1}{2^r} \sum_{k=1}^{\infty} |x_k|}{p_r} = \sum_{r=n(S)}^{n(S+1)-1} \frac{r(q-1)}{a_N(r)} \frac{q_r-1}{(s+2)G_s} - q_r p_r
\]

\[
\leq G_s^{-1} \sum_{r=n(S)}^{n(S+1)-1} 2^r |a_N(r)| \frac{q_r-1}{(s+2)} - q_r p_r
\]

\[
\leq (s+2)^{-2} G_s^{-1} \sum_{r=n(S)}^{n(S+1)-1} 2^r |a_N(r)| \frac{q}{r} - q_r p_r
\]

\[
\leq \frac{1}{(s+2)^2}
\]

Hence, \( \sum_{r=0}^{\infty} \frac{\frac{1}{2^r} \sum_{k=1}^{\infty} |x_k|}{p_r} \leq \sum_{s=0}^{\infty} \frac{1}{(s+2)^2} < \infty \).

That is, \( x \in \text{Ces}(p) \). This shows that, if \( a \in \mu(q) \), there exists \( x \in \text{Ces}(p) \) such that \( \sum_{k=1}^{\infty} a_k x_k \) diverges, which proves the assertion that the convergence of

\[ \sum_{k=1}^{\infty} a_k x_k \] implies that \( a \in \mu(q) \).
On the other hand, suppose \( a \in \mu(q) \). Then, a similar argument to that in Theorem 3.3, we know that
\[
\sum_{k=1}^{\infty} a_k x_k < \infty \text{ for each } x \in \text{Ces}(p). \quad \text{Hence}
\]
\[
f(x) = \sum_{k=1}^{\infty} a_k x_k \text{ defines a linear functional on Ces}(p)
\]
using the argument of Theorem 3.4, we have,
\[
\sum_{k=1}^{\infty} |a_k x_k| \leq E \left( \sum_{r=0}^{\infty} (2^r \max |a_k|)^q r^{q-1} g(x) \right)
\]
whenever \( g(x) \leq 1 \). Hence \( \sum_{k=1}^{\infty} a_k x_k \) defines an element of \( \text{Ces}(p) \).

Furthermore, it is easy to see that the representation (3.6.9) is unique.

Hence we can define a mapping \( T : \text{Ces}(p) \longrightarrow \mu(q) \) by
\[
T(f) = (a_1, a_2, \ldots)
\]
where \( a_k \) appear from the representation in (3.6.9).

It is evident that \( T \) is linear and bijection.

Hence \( \text{Ces}(p) \) is isomorphic to \( \mu(q) \), which proves the theorem.
In this section we list the following some problems.

**Problem 1:** What is the Banach dual of $X_p$?

For an absolute sequence space, very often its Banach dual that is the space of all continuous linear functionals coincides with the $\alpha$-dual, and also with the $\beta$-dual as a proper sub space.

**Problem 2:** What is the second $\beta$-dual of $X_p$?

We have characterized the $\beta$-dual $X_p^{\beta}$. The conditions on the elements in $X_p^{\beta}$ are not symmetrical with respect to those in $X_p$. Hence the space $X_p$ is not perfect that is $X_p^{\beta \beta} \neq X_p$. Then what is $X_p^{\beta \beta}$?

**Problem 3:** Let $X = \{ x; A \mid x \in Y \}$ and $Y$ be solid.

What is the $\alpha$-dual of $X$?

Lim [26] has solved the problem for some special $A$ and $Y$. The corresponding problem has also been solved completely for the non-absolute case.
It would be of interest to characterize the $\alpha$-dual of $X$ for more general $A$ and $Y$. If $Y$ is not solid, the problem seems to be harder.

**Problem 4:** Let $X$ be a Banach space. Study the space $C_p(X)$ of all $X$-valued sequences $x$ such that

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} ||x_k|| \right)^p < \infty,$$

and similarly for the non-absolute case.

The problem was proposed and is being studied by Wu-Bo-er. For similar results on $C_0(X)$, $C(X)$, $\ell_p(X)$ for $0 < p \leq \infty$, and $w_p(X)$ for $0 < p < \infty$, see [33].

**Problem 5:** Let $\Gamma$ be a family of infinite matrices and $Y$ a sequence space. Let $X$ be the space of all sequences $x$ such that $Tx \in Y$ whenever $T \in \Gamma$ and

$$\text{Sup} \left\{ \frac{1}{2} ||Tx|| : T \in \Gamma \right\} < \infty.$$

What is the $\beta$-dual of $X$?

The above space $X$ was studied by Lim and Lee [35].
**Problem 6:** Let $\omega_p$ denote the space of all sequences $x$ such that there is a number $L$ satisfying
\[
\frac{1}{n} \sum_{k=1}^{n} |x_k - L|^p \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\]

What are the necessary and sufficient conditions on $A$ such that an infinite matrix $A$ maps $\omega_1$ into $\omega_p$ for $1 < p < \infty$?

This is a special case of a problem raised in Kuttner and Thorpe [34]. An attempt has been made in Lee and Lim [32].

**Problem 7:** Characterize the continuous orthogonally additive functionals on $X_p$.

Given an sequence space $X$, a functional $f$ on $X$ is said to be orthogonally additive if
\[
f(x+y) = f(x) + f(y)
\]

Whenever $x, y \in X$ and $x_k y_k = 0$ for all $k$. 
**PROBLEM 8:** Characterize the continuous orthogonally additive functionals on $\text{Ces}_p$.

For interest, we include in the references some standard reference books on the subject: [1], [36], [37], [38] and [25].
B I B L I O G R A P H Y


[21]. D.F. Dawson

[22]. Kazuo Ishiguro und Karl Zeller.


[26]. S.K. Lim


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