ON SOME PROBLEMS OF SPECIAL FUNCTIONS

ABSTRACT OF THE THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN APPLIED MATHEMATICS

BY

MOHAMMAD ASIF

UNDER THE SUPERVISION OF

PROF. MUMTAZ AHMAD KHAN

M.Sc., Ph.D. (Lucknow)

DEPARTMENT OF APPLIED MATHEMATICS

FACULTY OF ENGINEERING

ALIGARH MUSLIM UNIVERSITY

ALIGARH - 202002 (U.P.), INDIA

2010
Abstract

The present thesis embodies the researches carried out at the Aligarh Muslim University, Aligarh. The thesis has been divided into seven chapters. Besides the introductory first chapter, second chapter deals with Jacobi type and Gegenbauer type generalization of certain polynomials, third and fourth chapters concern with $q$-analogues of the Jacobi type and Gegenbauer type generalized polynomials and certain classical polynomials respectively. The last three chapters i.e. fifth, sixth and seventh deal with the generalization of the exponential operators obtained by Dattoli and his collaborators.

Each chapter is divided into a number of sections. Definitions and equations have been numbered chapter wise. The section number is followed by number of equation e.g. (4.3.2) refers to equation number 2 of section 3 of chapter 4.

A brief review of some important special functions, the classical orthogonal polynomials, some generating functions, integral transformations, exponential shift operators, Hermite Kampé de Fériet polynomials, basic hypergeometric series, basic numbers, $q$-analogues of some polynomials, $q$-analogues of some identities and the definition and notations that commonly arise in practices are given in chapter-I.

Chapter-II considers Jacobi type and Gegenbauer type generalization of certain polynomials and their generating functions. Relationships among generalized polynomials have also been included.
Chapter-III refers to $q$-analogues of Jacobi type and Gegenbauer type generalization of certain polynomials studied in chapter-II. Moreover, $q$-analogues of their generating functions have also been established.

In chapter-IV, $q$-analogues of Bateman’s polynomials, Pasternack’s polynomials, Shively’s pseudo-Laguerre polynomials, Cesàro polynomials, Gottlieb polynomials, generalized Hypergeometric polynomial set obtained by S. D. Bajpai and M. S. Arora, have been studied. Further $q$-analogues of their generating functions have also been derived.

Chapter-V deals with the generalization of exponential operators used by Dattoli and Levi for translation and diffusive operator which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

The aim of Chapter-VI is to introduce and use the generalized exponential shift operators, operators on the base $a$ ($a > 0, \neq 1$), to deal with the families of Pseudo-Kampé de Fériet polynomials, which can be viewed as the natural complement for the theory of fractional derivatives and partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of exponential operators, operators on the base $a$ ($a > 0, \neq 1$), providing generalized fractional forms of shift operators.

The objective of the last chapter is to introduce the generalized exponential operators and to use it for dealing with the families of partial differential
equations of evolution type and to treat the problems involving fractional operators. Further, the properties of the families of special polynomials or special functions (like the Riemann ζ function) are naturally associated with the proposed formalism.

In the end an exhaustive and up to date list of writings and original papers of the subject matter of this thesis have been provided in the form of a bibliography.
ON SOME PROBLEMS OF SPECIAL FUNCTIONS

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

APPLIED MATHEMATICS

BY

MOHAMMAD ASIF

UNDER THE SUPERVISION OF

PROF. MUMTAZ AHMAD KHAN
M.Sc., Ph.D. (Lucknow)

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING
ALIGARH MUSLIM UNIVERSITY
ALIGARH - 202002 (U.P.), INDIA

2010
PRAYER

I begin in the name of Almighty God, the Most Gracious and the Most Merciful without whose will this thesis would not have seen the daylight.

All praise be to the Almighty God, sustainer of the worlds, and may the blessings and peace be upon the Master of Prophets, on his progeny and all his companions.

The primary aim of education is to prepare and qualify the new generation for participation in the cultural development of the people, so that the succeeding generations may benefit from the knowledge and experience of their predecessors.
A Humble Tribute
To
Architect of My Alma Mater

Sir Syed Ahmad Khan (Oct 17, 1817 to Mar 27, 1898)

Sir your Vision, Dynamism and Courage
Guide me in every step I take.
Dedicated To
My Beloved
Mother
Mrs. Nafees Begum
And
Esteemed Father
Mr. Abdul Aziz
With Heartfelt Gratitude
And
Indebtedness
To
My Esteemed
Supervisor
Prof. Mumtaz Ahmad Khan
CERTIFICATE

Certified that the thesis entitled "On Some Problems Of Special Functions" is based on a part of research work Mr. Mohammad Asif carried out under my guidance and supervision in the Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh. To the best of my knowledge and belief the work included in the thesis is original and has not been submitted to any other University or Institution for the award of a degree.

This is to further certify that Mr. Mohammad Asif has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of Aligarh Muslim University, Aligarh.

Chaired
Applied Maths Dept.
E.H. College of Engg. & Tech
A.M.U., ALIGARH

Ph.No. 0571270027  Mob.No.09897686959
Table of Contents

Acknowledgements v

Preface ix

1 Introduction 1
  1.1 Special Functions and its Growth ...................... 1
  1.2 Orthogonal Polynomials ............................ 4
  1.3 Definitions, Notations and Results Used ............... 5
      1.3.1 Gaussian Hypergeometric Function and Generalization 7
      1.3.2 Hypergeometric Function of Two and Several Variables 9
      1.3.3 The Classical Orthogonal Polynomials ............. 10
      1.3.4 Hypergeomertic Representations of the Polynomials . 12
      1.3.5 Integral Transforms ........................... 13
  1.4 Jacobi Type and Gegenbauer Type Generalization of Certain Polynomials ............................... 18
  1.5 On Some Generating Functions of $q$- Analogues of Jacobi Type and Gegenbauer Type Generalized Polynomials ............... 22
  1.6 On Some Generating Functions of Certain $q$-Polynomials . 22
  1.7 Generalized Exponential Operators and Difference Equations 27
  1.8 Shift Operators On The Base $a$ ($a > 0, \neq 1$), Pseudo-Polynomials and Monomial Type Functions ....................... 32
1.9 Generalized Operational Methods, Fractional Operators and
Special Polynomials ........................................... 35

2 Jacobi Type And Gegenbauer Type Generalization Of Cer-
tain Polynomials ............................................. 43
  2.1 Introduction ............................................... 43
  2.2 Jacobi Type Generalization of Certain Polynomials and Their
       Generating Functions ....................................... 47
    2.2.1 The Sister Celine’s Polynomial ...................... 47
    2.2.2 The Bateman’s Polynomials .......................... 51
    2.2.3 The Pasternack’s Polynomial ....................... 54
    2.2.4 The Bateman’s Polynomial $Z^m(x)$ .................. 56
  2.3 Gegenbauer Type Generalization of Certain Polynomials and
       Their Generating Functions ............................... 59
    2.3.1 The Sister Celine’s Polynomial ...................... 60
    2.3.2 The Bateman’s Polynomials .......................... 61
    2.3.3 The Pasternack’s Polynomial ....................... 63
    2.3.4 The Rice’s Polynomials ............................. 65
    2.3.5 The Hahn Polynomials ............................... 66

3 On Some Generating Functions Of $q$-Analogues Of Jacobi
Type And Gegenbauer Type Generalized Polynomials ........ 67
  3.1 Introduction ............................................... 67
  3.2 The $q$-Analogue of The Jacobi Type Generalization of Certain
       Polynomials and Their Generating Functions ............ 68
    3.2.1 The Sister Celine’s Polynomial ...................... 68
    3.2.2 The Bateman’s Polynomials .......................... 76
    3.2.3 The Pasternack’s Polynomials ....................... 80
    3.2.4 The Rice’s Polynomials ............................. 81
3.2.5 The Hahn’s Polynomials ........................................ 82

3.3 The $q$-Analogue of the Gegenbauer Type Generalization of the
Certain Polynomials and Their Generating Functions .......... 83

3.3.1 The Sister Celine’s Polynomials ............................. 83

3.3.2 The Bateman’s Polynomials ................................. 85

3.3.3 The Pasternack’s Polynomials .............................. 87

3.3.4 The Rice’s Polynomials .................................... 89

3.3.5 The Hahn’s Polynomials ...................................... 90

4 On Some Generating Functions Of Certain $q$-Polynomials 92

4.1 Introduction ..................................................... 92

4.1.1 Shively’s Pseudo-Laguerre Polynomials .................. 93

4.1.2 Cesàro Polynomials ......................................... 93

4.1.3 Gottlieb Polynomials ....................................... 94

4.1.4 Generalized Hypergeometric Polynomial Set ............. 95

4.2 $q$-Analogue of Certain Polynomials .......................... 97

4.2.1 $q$-Bateman’s Polynomial .................................. 97

4.2.2 $q$-Pasternack’s Polynomials .............................. 98

4.2.3 $q$-Shively’s Pseudo-Laguerre Polynomials ............... 98

4.2.4 $q$-Cesàro Polynomial ....................................... 98

4.2.5 $q$-Gottlieb Polynomials .................................... 98

4.2.6 $q$-Generalized Hypergeometric Polynomial Set .......... 99

4.3 Generating Functions of Certain $q$-Polynomials ............. 99

4.3.1 Generating Functions of $q$-Bateman Polynomials ...... 100

4.3.2 $q$-Pasternack Polynomials’ Generating Functions ...... 100

4.3.3 Generating Function of Pseudo-Laguerre Set ............. 101

4.3.4 Generating Functions of $q$-Cesàro’s Polynomials ...... 101

4.3.5 Generating Functions of $q$-Gottlieb Polynomials ...... 101
4.3.6 Generating Function of $q$-Generalized Hypergeometric Polynomial Set ........................................... 102

5 Generalized Exponential Operators And Difference Equations ................................................................. 117
  5.1 Introduction ........................................................................................................................................ 117
  5.2 Generalized Difference Equations ...................................................................................................... 123
  5.3 Generalized Shift Operators and Jackson Derivatives ....................................................................... 128
  5.4 The Remarks ...................................................................................................................................... 138

6 Shift Operators On The Base $a$ ($a > 0, \neq 1$), Pseudo-Polynomials And Monomial Type Functions ........ 143
  6.1 Introduction ........................................................................................................................................ 143
  6.2 Exponential Operators on Base $a$ ($a > 0, \neq 1$) ............................................................................ 146
  6.3 Fraction Order Exponential Operators $a$ ($a > 0, \neq 1$) ................................................................. 153
  6.4 Concluding Remarks ......................................................................................................................... 156

7 Generalized Operational Methods, Fractional Operators And Special Polynomials ..................................... 160
  7.1 Introduction ........................................................................................................................................ 160
  7.2 Fractional Operators and a New Class of Special Polynomials .......................................................... 166
  7.3 Concluding Remarks ......................................................................................................................... 177

Bibliography ................................................................................................................................................. 187

Appendix ....................................................................................................................................................... 209
Acknowledgements

Words and lexicons can not do full justice in expressing a deep sense of gratitude and indebtedness which flows from my heart towards my esteemed supervisor Professor Dr. Mumtaz Ahmad Khan, Chairman Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh. Under his benevolence and able guidance, I learned and ventured to write. His inspiring benefaction bestowed upon me so generously and his unsparing help and precious advice have gone into the preparation of this thesis. He has been a beckon all long to enable me to present this work. I shall never be able to repay for all that I have gained from him.

I also take this opportunity to thank every staff member of the Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh, India who have contributed in anyway towards the completion of this task.

My thanks are due to my seniors Dr. Abdul Hakim Khan, Dr. (Mrs.) Muqaddas Najmi, Dr. Ajay Kumar Shukla, Dr. Khursheed Ahmad, Dr. Ghazi Salama Abukhammash, Dr. Abdul Rahman Khan, Dr. Bhagwat Swaroop Sharma, Dr. Mukesh Pal Singh, Dr. Bijan Rouhi, Dr. Naseem Ahmad Khan, Dr.(Mrs.) Rihana Parveen and Dr. Bahman Alidad all of whom
obtained their Ph.D. degrees under the supervision of my supervisor Prof. Mumtaz Ahmad Khan for their help and encouragement in completion of my Ph.D. work.

I am also thankful to my senior scholars Mr. Syed Mohd. Amin, Mr. Ajeet Kumar Sharma, Mr. Shakeel Ahmad Alvi, Dr. Khursheed Ahmad and Dr. B. S. Sharma all of whom obtained their M.Phil degrees under the supervision of my supervisor Prof. Mumtaz Ahmad Khan.

My thanks are also due to my fellow research workers Mr. Mohd. Akhlaq, Mr. Isthirar Ahmad, Mr. Nisar K. S., Mr. Mohd. Khalid Rafat Khan and Mr. Shakeel Ahmad who obtained their M.Phil degrees and all except Mr. Isthirar are pursuing their Ph.D. work under the supervision of Prof. Mumtaz Ahmad Khan. Out of them one Mr. Mohd. Akhlaq has already submitted his Ph.D. thesis.

I feel pride and privilege for my beloved mother Mrs. Nafees Begum and esteemed father Mr. Abdul Aziz who proved, needless to say, a symbol of sacrifice, without their love, support and cooperation I would not have been able to pursue this research work.

I would also like to express many thanks to my younger brother Mr. Mohammad Rashid, (M. C. A.), Lecturer, Department of Computer Science, D. S. (P.G.) College, Aligarh and my dear youngest brother Mr. Mohammad Shakir, M.Sc.(Physics) final year, D. P. B. S. (P.G.) College, Anupshahr for cooperation and being a source of inspiration to me. I wish to express my heart-felt gratitude to my elder sister Mrs. Afsana Begum, younger sister Miss Shabana and other related family members for their constant encour-
agement and blessings. They have always been the source of inspiration through out my research work.

I shall be failing in my duties if I do not express my gratitude to the wife of my supervisor Mrs. Aisha Bano (M.Sc., B.Ed.), T.G.T., Zakir Husain Model Senior Secondary School, Aligarh for her kind hospitality and prayer to God for my welfare.

I am also thankful to my supervisor’s children Miss Ghazala Perveen Khan (daughter) pursuing Ph.D. ( Electrical) from Jamia Millia Islamia, New Delhi, Mr. Rashid Imran Ahmad Khan (Son), M. Tech. final year (Chemical), Faculty of Engineering, A. M. U., Aligarh and to his youngest son Mr. Abdul Razzaq Khan, a class XII student of Senior Secondary School Certificate, A. M. U., Aligarh for enduring the preoccupation of their father with this work.

Thanks are also due to my friends Dr. Danish Lohani, Post Doctor Fellow, Department of Mathematics, A. M. U., Aligarh for revitalizing my thoughts and emotions during the time of despair and I shall also remain indebted to Mr. Akhlaq Hussain, Indian Institute of Technology, Kanpur for providing/mailing me research papers/materials time to time during the exposition of this work.

In the end I would like to thank Mr. Mohammad Rashid, my younger brother, for helping me in typing the manuscript of my Ph.D. thesis in latex format.
I wish to express my heartfelt thanks to the Human Resource Development Group Council of Scientific & Industrial Research of India for awarding Senior Research Fellowship (NET) (F. No. 10-2(5)/2005(i)-E.U.II).

Finally, I owe a deep sense of gratitude to the authorities of Aligarh Muslim University, my Alma Mater, for providing me adequate research facilities.

(Mohammad Asif)
Department of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University, Aligarh 202 002, India.
Preface

The present thesis embodies the researches carried out at the Aligarh Muslim University, Aligarh. The thesis has been divided into seven chapters. Besides the introductory first chapter, second chapter deals with Jacobi type and Gegenbauer type generalization of certain polynomials, third and fourth chapters concern with $q$-analogues of the Jacobi type and Gegenbauer type generalized polynomials and certain classical polynomials respectively. The last three chapters i.e. fifth, sixth and seventh deal with the generalization of the exponential operators obtained by Dattoli and his collaborators.

Each chapter is divided into a number of sections. Definitions and equations have been numbered chapter wise. The section number is followed by number of equation e.g. (4.3.2) refers to equation number 2 of section 3 of chapter 4.

A brief review of some important special functions, the classical orthogonal polynomials, some generating functions, integral transformations, exponential shift operators, Hermite Kampé de Fériet polynomials, basic hypergeometric series, basic numbers, $q$-analogues of some polynomials, $q$-analogues of some identities and the definition and notations that commonly arise in practices are given in chapter-I.

Chapter-II considers Jacobi type and Gegenbauer type generalization of certain polynomials and their generating functions. Relationships among
generalized polynomials have also been included.

Chapter-III refers to $q$-analogues of Jacobi type and Gegenbauer type generalization of certain polynomials studied in chapter-II. Moreover, $q$-analogues of their generating functions have also been established.

In chapter-IV, $q$-analogues of Bateman’s polynomials, Pasternack’s polynomials, Shively’s pseudo-Laguerre polynomials, Cesàro polynomials, Gottlieb polynomials, generalized Hypergeometric polynomial set obtained by S. D. Bajpai and M. S. Arora [4], have been studied. Further $q$-analogues of their generating functions have also been derived.

Chapter-V deals with the generalization of exponential operators used by Dattoli and Levi [34] for translation and diffusive operator which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

The aim of Chapter-VI is to introduce and use the generalized exponential shift operators, operators on the base $a$ ($a > 0, a \neq 1$), to deal with the families of Pseudo-Kampé de Fériet polynomials, which can be viewed as the natural complement for the theory of fractional derivatives and partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of exponential operators, operators on the base $a$ ($a > 0, a \neq 1$), providing generalized fractional forms of shift operators.

The objective of the last chapter is to introduce the generalized exponential operators and to use it for dealing with the families of partial differential equations of evolution type and to treat the problems involving fractional operators. Further, the properties of the families of special polynomials or
special functions (like the Riemann \( \zeta \) function) are naturally associated with the proposed formalism.

In the end an exhaustive and up to date list of writings and original papers of the subject matter of this thesis have been provided in the form of a bibliography.
Chapter 1

Introduction

1.1 Special Functions and its Growth

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as \(x^n, e^x, \log x, \text{ and } \sin x\), "higher" functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma
function long before Euler reached it. Wallis had also the first encounter with elliptic integrals while using Cavalieri’s primitive forerunner of the calculus. [It is curious that two kinds of special functions encountered in the seventeenth century, Wallis’ elliptic integral and Newton’s elementary symmetric functions, belongs to the class of hypergeometric functions of several variables, which was not studied systematically nor even defined formally until the end of the nineteenth century]. A more sophisticated calculus, which made possible the real flowering of special functions, was developed by Newton at Cambridge and by Leibnitz in Germany during the period 1665-1685. Taylor’s theorem was found by Scottish mathematician Gregory in 1670, although it was not published until 1715 after rediscovery by Taylor.

In 1703 James Bernoulli solved a differential equation by an infinite series which would now be called the series representation of a Bessel function. Although Bessel functions were met by Euler and others in various mechanics problems, no systematic study of the functions was made until 1824, and the principal achievements in the eighteenth century were the gamma function and the theory of elliptic integrals. Euler found most of the major properties of the gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century. Other significant developments were the discovery of Vandermonde’s theorem in 1722 and the definition of Legendre polynomials and the discovery of their addition theorem by Laplace and Legendre during 1782-1785. In a slightly different form the polynomials had already been met by Liouville in 1722.

The golden age of special functions, which was centered in nineteenth
century German and France, was the result of developments in both mathematics and physics: the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) and their property of double periodicity was published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, specially in the hands of Jacobi and Weierstrass.

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss's memoir on the $\binom{2}{1}$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The $\binom{3}{2}$ series was studied by Clausen (1928) and the $\binom{1}{1}$ series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are $\binom{0}{1}$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrödinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss's hypergeometric function $\binom{2}{1}$. Various generalizations of $\binom{2}{1}$ were introduced by Horn, Kampé de Fériet, MacRobert, and Mijer. From another new view point, that of a differential difference equation discussed much
earlier for polynomials by Appell (1880), Truesdell (1948) made a partly successful effort at unification by fitting a number of special functions into a single framework.

1.2 Orthogonal Polynomials

Orthogonal polynomials constitute an important class of special functions in general and hypergeometric functions in particular. The subject of orthogonal polynomials is a classical one whose origins can be traced to Legendre’s work on planetary motion with important applications to physics and to probability and statistics and other branches of mathematics, the subject flourished through the first third of the century. Perhaps as a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in orthogonal polynomials has revived in recent years.

The ordinary hypergeometric functions have been the subject of extensive researches by a number of eminent mathematicians. These functions play a pivotal role in mathematical analysis, physics, Engineering and allied sciences. Most of the special functions, which have various physical and technical applications and which are closely connected with orthogonal polynomial and problems of mechanical quadrature, can be expressed in terms of generalized hypergeometric functions. However, these functions suffer from a shortcoming that they do not unify various elliptic and associated functions. This drawback was overcome by E. Heine through the definition of a generalized basic hypergeometric series.
1.3 Definitions, Notations and Results Used

Frequently occurring definitions, notations and results used in this thesis are as given under:

**The Gamma Function and Related Functions**

The Gamma function has several equivalent definitions, most of which are due to Euler,

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \quad \text{Re}(z) > 0 \quad (1.3.1)
\]

upon integrating by part, equation (1.3.1) yields the recurrence relation

\[
\Gamma(z + 1) = z\Gamma(z). \quad (1.3.2)
\]

The relation (1.3.2) yields the useful result

\[
\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots
\]

which shows that gamma function is the generalization of factorial function

**The Beta Function**

Beta function \(B(p, q)\) are defined by

\[
B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, \quad \text{Re}(p) > 0, \ \text{Re}(q) > 0 \quad (1.3.3)
\]

Gamma function and Beta function are related by the following relation

\[
B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q \neq 0, -1, -2, \ldots \quad (1.3.4)
\]
Chapter 1: Introduction

The Pochhammer Symbol

The Pochhammer symbol \((\lambda)_n\) is defined by

\[
\begin{align*}
(\lambda)_n &= \begin{cases} 
1, & \text{if } n = 0 \\
\lambda(\lambda + 1) \cdots (\lambda + n - 1), & \text{if } n = 1, 2, 3, \cdots 
\end{cases}
\end{align*}
\]

(1.3.5)

In terms of Gamma function, we have

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, \cdots
\]

(1.3.6)

Further,

\[
(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n
\]

(1.3.7)

\[
(\lambda)_{-n} = \frac{(-1)^n}{(1 - \lambda)_n}, \quad n = 1, 2, 3, \cdots, \lambda \neq 0, \pm 1, \pm 2,
\]

(1.3.8)

\[
(\lambda)_{n-m} = \frac{(-1)^m (\lambda)_n}{(1 - \lambda - n)_m}, \quad 0 \leq m \leq n.
\]

(1.3.9)

For \(\lambda = 1\), equation (1.3.9) reduces to

\[
(n - m)! = \frac{(-1)^m n!}{(-n)_m}, \quad 0 \leq m \leq n.
\]

(1.3.10)

Another useful relation of Pochhammer symbol \((\lambda)_n\) is included in Gauss’s multiplication theorem:

\[
(\lambda)_{mn} = (m)_m \prod_{j=1}^{m} \left(\frac{\lambda + j - 1}{m}\right)_n, \quad n = 0, 1, 2, \cdots
\]

(1.3.11)

where \(m\) is a positive integer.

For \(m = 2\) the equation (1.3.11) reduces to Legendre’s duplication formula

\[
(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda + \frac{1}{2}}{2}\right)_n, \quad n = 0, 1, 2, \cdots
\]

(1.3.12)

In particular, one has

\[
(2n)! = 2^{2n} \left(\frac{1}{2}\right)_n n! \quad \text{and} \quad (2n + 1)! = 2^{2n} \left(\frac{3}{2}\right)_n n!
\]

(1.3.13)
Chapter 1: Introduction

The Error Function

The Error function \( \text{erf}(z) \) is defined for any complex \( z \) by

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) \, dt
\]  

(1.3.14)

and its complement by

\[
\text{erfc}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) \, dt.
\]  

(1.3.15)

Note that

\[
\begin{align*}
\text{erf}(0) &= 0, \quad \text{erfc}(0) = 1 \\
\text{erf}(\infty) &= 1, \quad \text{erfc}(\infty) = 0
\end{align*}
\]  

(1.3.16)

1.3.1 Gaussian Hypergeometric Function and Generalization

The second order linear differential equation

\[
z(1-z) \frac{d^2w}{dz^2} + [c - (a + b + 1)z] \frac{dw}{dz} - abw = 0
\]  

(1.3.17)

has a solution

\[
w = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}
\]

where \( a, b, c \) are parameters independent of \( z \) for \( c \) neither zero nor a negative integer and is denoted by \( \text{$_2F_1(a,b;c;z)$} \) i.e.

\[
\text{$_2F_1(a,b;c;z)$} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},
\]  

(1.3.18)

which is known as hypergeometric function. The special case \( a = c, b = 1 \) or \( b = c, a = 1 \) yields the elementary geometric series \( \sum_{n=0}^{\infty} z^n \), hence the term hypergeometric.

If either of the parameter \( a \) or \( b \) is negative integer, then in this case, equation (1.3.18) reduces to hypergeometric polynomials.
Chapter 1: Introduction

Generalized Hypergeometric Function

The hypergeometric function defined in equation (1.3.18) can be generalized in an obvious way.

\[ pF_q \left[ \begin{array}{c} \alpha_1, \alpha_2, \cdots, \alpha_p \\ \beta_1, \beta_2, \cdots, \beta_q \end{array} \right; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} 
\]

where \( p, q \) are positive integer or zero. The numerator parameter \( \alpha_1, \cdots, \alpha_p \) and the denominator parameter \( \beta_1, \cdots, \beta_q \) take on complex values, provided that

\[ \beta_j \neq 0, -1, -2, \cdots, j = 1, 2, \cdots, q \]

Convergence of \( pF_q \)

The series \( pF_q \)

(i) converges for all \(| z | < \infty \) if \( p \leq q \)

(ii) converges for \(| z | < 1 \) if \( p = q + 1 \) and

(iii) diverges for all \( z, z \neq 0 \) if \( p > q + 1 \)

Further more if we set

\[ \omega = \text{Re} \left( \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \right) > 0, \]

the \( pF_q \) series with \( p = q + 1 \) is

(i) Absolutely convergent for \(| z | = 1 \) if \( \text{Re}(\omega) > 0 \)

(ii) Conditionally convergent for \(| z | = 1, z \neq 1 \) if \(-1 < \text{Re}(\omega) < 0 \)
(iii) Divergent for $|z| = 1$ if $\Re(\omega) \leq -1$.

An important special case when $p = q = 1$, the equation (1.3.19) reduces to the confluent hypergeometric series $\, _1F_1$ named as Kummers series [182], (see also Slater [197]) and is given by

$$1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.$$  \hspace{1cm} (1.3.20)

When $p = 2, q = 1$, equation (1.3.19) reduces to ordinary hypergeometric function $\, _2F_1$ of second order given by (1.3.18).

### 1.3.2 Hypergeometric Function of Two and Several Variables

#### Appell Function

In 1880, Appell [2] introduced four hypergeometric series which are generalization of Gauss hypergeometric function $\, _2F_1$ and are given below:

$$F_1[a, b, b'; c, x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m+n}m!n!}, \text{ (max}\{\text{max} \{|x|, |y|\} < 1)$$  \hspace{1cm} (1.3.21)

$$F_2[a, b, b'; c, c', x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m} (c')_{n}m!n!}, \text{ (max}\{\text{max} \{|x|, |y|\} < 1)$$  \hspace{1cm} (1.3.22)

$$F_3[a, a', b, b'; c, x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m}(a')_{n}(b)_{m}(b')_{n}x^{m}y^{n}}{(c)_{m+n}m!n!}, \text{ (max}\{\text{max} \{|x|, |y|\} < 1)$$  \hspace{1cm} (1.3.23)

$$F_4[a, b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}x^{m}y^{n}}{(c)(c')_{n}m!n!}, \text{ (max}\{\text{max} \{|x|, |y|\} < 1).$$  \hspace{1cm} (1.3.24)
The standard work on the theory of Appell series is the monograph by and
Kampé de Fériet [3]. See also Slater ([197] and Exton [[47]; p.23(28)) for a
review of the subsequent work.

Humbert Function

In 1920, Humbert has studies seven confluent form of the four Appell func-
tions and denoted by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, E_1, E_2$ four of them are given below
(see, [199]):

$$\Phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (|x| < 1, |y| < \infty) \quad (1.3.25)$$

$$\Phi_2[\beta', \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (|x| < \infty, |y| < \infty) \quad (1.3.26)$$

$$\Phi_3[\beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (|x| < \infty, |y| < \infty) \quad (1.3.27)$$

$$\Psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)(\gamma')_n} \frac{x^m y^n}{m! n!}, \quad (|x| < 1, |y| < \infty) \quad (1.3.28)$$

$$\Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)(\gamma')_n} \frac{x^m y^n}{m! n!}, \quad (|x| < \infty, |y| < \infty). \quad (1.3.29)$$

1.3.3 The Classical Orthogonal Polynomials

The hypergeometric representation of classical orthogonal polynomial such as
Jacobi polynomial, Gegenbauer polynomial, Legendre polynomial, Hermite
polynomial and Laguerre polynomial and their orthogonality properties, Rodrigues formula, recurrence relation and the differential equation satisfied by them are given in detail in Szegö, [202], Rainville [191]. We mention few of them:

**Jacobi Polynomial**

The Jacobi Polynomials $P_{n}^{(\alpha, \beta)}(x)$ are defined by generating relation

$$
\sum_{n=0}^{\infty} P_{n}^{(\alpha, \beta)}(x) t_n = [1 + 1/2(x + 1)t]^\alpha [1 + 1/2(x - 1)t]^\beta \quad (1.3.30)
$$

$$
\text{Re}(\alpha) > -1, \text{Re}(\beta) > -1
$$

The Jacobi Polynomials have a number of finite series representation ([191] p.255) one of them is given below:

$$
P_{n}^{(\alpha, \beta)}(x) t_n = \sum_{k=0}^{n} \frac{(1 + \alpha)_n(1 + \alpha + \beta)_{n+k}}{k! (n-k)! (1 + \alpha)_k (1 + \alpha + \beta)_n} \left( \frac{x - 1}{2} \right)^k. \quad (1.3.31)
$$

For $\beta = \alpha$ the Jacobi Polynomial $P_{n}^{(\alpha, \alpha)}(x)$ is known as ultraspherical polynomial which is connected with the Gegenbauer polynomial $C_{n}^{(\alpha)}(x)$ by the relation ([1]; p.191)

$$
P_{n}^{(\alpha, \alpha)}(x) = \frac{(1 + \alpha)_n C_{n}^{\alpha+1/2}(x)}{(1 + 2\alpha)_n}. \quad (1.3.32)
$$

For $\alpha = \beta = 0$, equation (1.3.31) reduces to Legendre Polynomial $P_n(x)$.

**Hermite Polynomial**

Hermite Polynomial are defined by means of generating relation

$$
\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}, \quad (1.3.33)
$$

valid for all finite $x$ and $t$ and one can easily obtain

$$
H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n! (2x)^{n-2k}}{k! (n-2k)!}. \quad (1.3.34)
$$
Chapter 1: Introduction

Associated Laguerre Polynomial

The associated Laguerre Polynomial $L_n^{(\alpha)}(x)$ are defined by means of generating relation.

$$
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1-t)^{-(\alpha+1)} \exp \left( \frac{xt}{t-1} \right).
$$

(1.3.35)

For $\alpha = 0$, the above equation (1.3.35) yield the generating function for simple Laguerre Polynomial $L_n(x)$.

A series representation of $L_n^{(\alpha)}(x)$ for non negative integers $n$, is given by

$$
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+\alpha)!}{k! (n-k)! (\alpha+k)!} x^k.
$$

(1.3.36)

for $\alpha = 0$, equation (1.3.36) gives the definition of Laguerre polynomial.

Laguerre Polynomial $L_n^{(\alpha)}(x)$ is also the limiting case of Jacobi Polynomial

$$
L_n^{(\alpha)}(x) = \lim_{|\beta| \to \infty} \left\{ P_n^{(\alpha,\beta)} \left( 1 - \frac{2x}{\beta} \right) \right\}.
$$

(1.3.37)

1.3.4 Hypergeometric Representations of the Polynomials

Some of the orthogonal polynomials and their connections with hypergeometric function used in our work are given below:

Jacobi Polynomial

$$
P_n^{(\alpha,\beta)}(z) = \binom{\alpha+n}{n} \ _2F_1 \left[ \begin{array}{c} -n, \alpha + \beta + n + 1 \\ \alpha + 1 \end{array} ; \frac{1-z}{2} \right].
$$

(1.3.38)
Chapter 1: Introduction

Gagenbauer Polynomial

\[ C_n^\gamma(z) = \left(\frac{n + 2\gamma - 1}{n}\right)_{\gamma + 1/2}^{2F_1} \left[ \begin{array}{c} -n, 2\gamma + n \\ \gamma + 1/2 \end{array} \right] \]  
(1.3.39)

Legendre Polynomial

\[ P_n(z) = P_n^{(0,0)}(z) = \left(\frac{\Gamma(n + 1)}{\Gamma(n + 1)}\right)^{1/2} \left[ \begin{array}{c} -n, n + 1 \\ 1 \end{array} \right] \]  
(1.3.40)

\[ P_n^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z + 1}{z - 1}\right)^{\mu/2} \left[ \begin{array}{c} -n, n + 1 \\ 1 - \mu \end{array} \right] \]  
(1.3.41)

Hermite Polynomial

\[ H_n(z) = (2z)^n \left([\frac{1}{2} - \frac{n}{2}] - z^{-2}\right) \]  
(1.3.42)

Laguerre Polynomial

\[ L_n^{(\alpha)}(z) = \frac{(1 + \alpha)^n}{n!} \left[iF_1\left[-n; 1 + \alpha; z\right]\right] \]  
(1.3.43)

1.3.5 Integral Transforms

Integral transforms play an important role in various fields of applied mathematics and physics. The method of solution of problems arising in physics lie at the heart of the use of integral transform.

Let \( f(t) \) be a real or complex valued function of real variable \( t \), defined on interval \( a \leq t \leq b \), which belongs to a certain specified class of functions and
let $F(p, t)$ be a definite function of $p$ and $t$, where $p$ is a complex quantity, whose domain is prescribed, then the integral equation

$$
\phi[f(t); p] = \int_a^b F(p, t)f(t) \, dt \quad (1.3.44)
$$

where the class of functions to which $f(t)$ belongs and the domain of $p$ are so prescribed that the integral on the right exists.

$F(p, t)$ is called the kernel of the transform $\phi[f(t), p]$, if we can define an integral equation

$$
f(t) = \int_c^d F(t)\phi[f(t), p] \, dt \quad (1.3.45)
$$

then (1.3.45) defines the inverse transform for (1.3.44). By given different values to the function $F(p, t)$, different integral transforms are defined by various authors like Fourier, Laplace, Hankel and Mellin transforms et cetera.

**Fourier Transform**

Fourier transform is defined as

$$
\mathcal{F}[f(x); \xi] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)e^{i\xi x} \, dx \quad (1.3.46)
$$

the Fourier transform of $f(x)$ and regard $x$ as complex variable.

**Laplace Transform**

Laplace transform is defined as

$$
\mathcal{L}[f(t); p] = \int_0^{\infty} f(t)e^{-pt} \, dt \quad (1.3.47)
$$

the Laplace transform of $f(t)$ and regard $p$ as complex variable.
Chapter 1: Introduction

Hankel Transform

Hankel transform is defined as

\[ \mathcal{H}_{\nu}[f(t); \xi] = \int_0^\infty f(t) t J_{\nu}(\xi t) \, dt \]  

(1.3.48)

where \( \xi \) is a complex variable.

The following definitions, notations, etc. shall be used in the developments of \( q \)-analogues of the polynomials.

\textbf{\( q \)-Hypergeometric series} \( \phi_s \)

The \( q \)-analogue of the hypergeometric series or \( \phi_s \) basic hypergeometric series is defined as [50; eq.(1.3.22), p.(4)]

\[ \phi_s(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_s; q, z) \equiv \phi_s \left[ \begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} ; q, z \right] \\
= \sum_{n=0}^{\infty} \frac{(a_1; q)_n, (a_2; q)_n, \cdots, (a_r; q)_n}{(q; q)_n, (b_1; q)_n, \cdots, (b_s; q)_n} \left( -1 \right)^n q^{\frac{n(n-1)}{2}} \frac{n!}{z^n}, \]  

(1.3.49)

with \( \binom{n}{2} = \frac{n(n-1)}{2} \), where \( q \neq 0 \) when \( r > s+2 \). Parameters \( b_1, b_2, \cdots, b_s \), are such that the denominator factors in the terms of the series are never zero.

\textbf{\( q \)-Shifted Factorial}

\[ (a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, \cdots, \end{cases} \]  

(1.3.50)

is the \( q \)-shifted factorial and it is assumed that the denominator parameters \( b \neq q^{-m} \) for \( m = 0, 1, \cdots \).
Chapter 1: Introduction

$q$-Number

A $q$-number or basic number is defined as

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \neq 1. \quad (1.3.51)$$

$q$-Factorial

The $q$-number factorial of $n!$ is defined for a nonnegative integer $n$ by

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad (1.3.52)$$

$q$-Number Shifted Factorial

The corresponding $q$-number shifted factorial is defined by

$$[a]_{q,n} = \prod_{k=0}^{n-1} [a + k]_q, \quad (1.3.53)$$

Clearly,

$$\lim_{q \to 1} [n]_q! = n!, \quad \lim_{q \to 1} [a]_q = a,$$

and

$$[a]_{q,n} = \frac{(q^a; q)_n}{(1 - q)^n}, \quad \lim_{q \to 1} [a]_{q,n} = (a)_n.$$ 

Corresponding to

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n$$

the compact notation is used

$$[a_1, a_2, \cdots, a_m]_{q,n} = [a_1]_{q,n}[a_2]_{q,n} \cdots [a_m]_{q,n}.$$ 

Similarly, the compact notation for the eq. (1.3.1) is given as

$$\sum_{n=0}^{\infty} \frac{[a_1, a_2, \cdots, a_r]_{q,n}}{[n]_q!} \frac{[b_1, \cdots, b_s]_{q,n}}{[n]_q!} \left[ (-1)^n q^{\binom{n}{2}} \right] z^n = r \phi_s(q^{a_1}, q^{a_2}, \cdots, q^{a_r}, q^{b_1}, q^{b_2}, \cdots, q^{b_s}; q, z) \quad (1.3.54)$$
Chapter 1: Introduction

$q$-Binomial Theorem

$q$-analogue of the binomial theorem is defined as (see [[50]; eq.(1.3.2), p.(8)])

\[ \psi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1, \quad |q| < 1, \quad (1.3.55) \]

$q$-Exponential Functions

Gasper and Rahman [[50]; eqs.(1.3.15), (1.3.16), p.(10-11)] define two $q$-analogues of the exponential function as given below

\[ e_q(z) = \psi_0(0; -; q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1, \quad (1.3.56) \]

and

\[ E_q(z) = \psi_0(-; -; q, -z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{(q; q)_n} = (-z; q)_\infty \quad (1.3.57) \]

than

\[ e_q(z) E_q(-z) = 1, \quad e_{q^{-1}}(z) = E_q(-qz). \quad (1.3.58) \]

The $q$-Identities

The following easily verified identities shall be used in this exposition

\[ (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad (1.3.59) \]

\[ (a^{-1}q^{1-n}; q)_n = (a; q)_n (-a^{-1})^n q^{-\left(\frac{n}{2}\right)}, \quad (1.3.60) \]

\[ (a; q)_{n-k} = \frac{(a; q)_n}{(a^{-1}q^{1-n}; q)_k} (-qa^{-1})^k q^{\left(\frac{n}{2}\right)-nk}, \quad (1.3.61) \]

\[ (a; q)_{n+k} = (a; q)_n (aq^n; q)_k, \quad (1.3.62) \]
Chapter 1: Introduction

\[(aq^n; q)_k = \frac{(a; q)_k(aq^k; q)_n}{(a; q)_n}, \quad (1.3.63)\]

\[(aq^k; q)_{n-k} = \frac{(a; q)_n}{(a; q)_k}, \quad (1.3.64)\]

\[\left(q^{-n}; q\right)_k = \frac{(q; q)_n}{(q; q)_{n-k}}(-1)^k \left(\frac{k}{2}\right)^{-nk}, \quad (1.3.65)\]

\[\left(aq^{-n}; q\right)_k = \frac{(a; q)_n(qa^{-1}; q)_n}{(a^{-1}q^{1-k}; q)_n}q^{-nk}, \quad (1.3.66)\]

1.4 Jacob! Type and Gegenbauer Type Generalization of Certain Polynomials

In 1947, Sister Celine (Fasenmyer[48]) obtained some basic formal properties of the hypergeometric polynomials. Sister Celine’s Polynomials are defined by the following generating relation

\[(1 - t)^{-1} \sum_{n=0}^{\infty} f_n \left[ \begin{array}{c} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \\ \frac{4\pi t}{(1-t)^2} \end{array} \right] t^n = \sum_{n=0}^{\infty} f_n \left[ \begin{array}{c} a_1, \ldots, a_p; \\ b_1, \ldots, b_s; \\ x \end{array} \right] t^n \quad (1.4.1)\]

which yields

\[f(a_i; b_i; x) \equiv f_n(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; x)\]

\[\equiv \sum_{n=0}^{\infty} f_n \left[ \begin{array}{c} -n, n+1, a_1, \ldots, a_r; \\ 1, \frac{1}{2}, b_1, \ldots, b_s; \\ x \end{array} \right] \quad (1.4.2)\]

\(n\) is non-negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention.
Her polynomials include as special cases of Legendres polynomials $P_n(1 - 2x)$, some special Jacobi polynomials, Rice’s polynomials $H_n(\xi, p, \nu)$, Bateman’s $Z_n(x)$, $F_n(z)$ and Pasternaks $F^m_n(z)$ which is a generalization of Bateman’s polynomials $F_n(z)$.

The Bateman’s Polynomials

Bateman defined the following polynomials (see [191], [199])

\[ F_n(z) = \, _3F_2 \left[ \begin{array}{c} -n, n + 1, \frac{1}{2}(1 + z); \\ 1, 1; \end{array} \right] \]

Bateman ([7], [8]) obtained the following generating functions

\[ \sum_{n=0}^{\infty} F_n(z) t^n = \frac{1}{1-t} \, _2F_1 \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2} + \frac{1}{2}z; \\ 1; \end{array} \right] \left( \frac{-4t}{(1-t)^2} \right) \]

\[ \sum_{n=0}^{\infty} [F_n(z - 2) - F_n(z)] t^n = \frac{2t}{(1-t)^3} \, _2F_1 \left[ \begin{array}{c} \frac{3}{2}, \frac{1}{2} + \frac{1}{2}z; \\ 2; \end{array} \right] \left( \frac{-4t}{(1-t)^2} \right) \]

The Pasternack’s Polynomials

The generalization of the Bateman’s Polynomial due to Pasternack is given below:

\[ F^m_n(z) = \, _3F_2 \left[ \begin{array}{c} -n, n + 1, \frac{1}{2}(z + m + 1); \\ 1, m + 1; \end{array} \right] \]

which is a generalization of Bateman’s polynomials $F_n(z)$. Generating function of generalization of Pasternack’s polynomials is given below:

\[ \sum_{n=0}^{\infty} F^m_n(z) t^n = \frac{1}{(1-t)} \, _2F_1 \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}(z + m + 1); \\ m + 1; \end{array} \right] \left( \frac{-M}{(1-t)^2} \right) \]
In 1936, Bateman [191] was interested in constructing inverse Laplace transforms. For this purpose he introduced the following polynomial

\[ Z_n(x) = \binom{-n, n+1;}{1, 1; x} \]  

Rainville (see [191], p. 137, Theorem 48) writes the generating function as follows:

\[ \sum_{n=0}^{\infty} Z_n(x)t^n = \frac{1}{1-t} \binom{1/2, n+1; -4xt}{1; \quad \frac{-4xt}{(1-t)^2}}. \]

In 1939, Pasternack [191] obtained the following generating function of Bateman’s polynomials:

\[ \sum_{n=0}^{\infty} F_m(-2n-1) \frac{(-t)^n}{n!} = e^{-t} Z_m(t) \]

\[ \sum_{n=0}^{\infty} F_n(-2m-1)t^n = (1-t)^{-m-1}(1+t)^m P_m \left( \frac{1+t^2}{1-t^2} \right). \]

**The Rice’s Polynomials**

S. O. Rice made a considerable study of the polynomials defined by

\[ H_n(\xi, p, \nu) = \binom{-n, n+1, \xi;}{1, p; \nu} \]

The generalized Rice’s Polynomial due to Khandekar [178] is given below:

\[ \frac{n!}{(1+\alpha)_n} H_n^{(\alpha, \beta)}(\xi, p, \nu) = \binom{-n, n+\alpha+\beta+1, \xi;}{1+\alpha, p; \nu} \]

Generating function of the generalized Rice polynomial due to Khandekar [178] is given below:
The Hahn Polynomials

Hahn polynomial is defined as

\[ Q_n(x; \alpha, \beta, N) = \binom{-n}{n + \alpha + \beta; -x; 1 + \alpha, -N} \]

\[ \alpha, \beta > -1, \ n, x = 0, 1, \ldots, N. \]  

The following generating functions are satisfied by the Hahn polynomial (1.4.15):

\[ \sum_{n=0}^{\infty} \frac{(-N)^n}{(\beta + 1)_n n!} Q_n(x; \alpha, \beta, N) t^n = \binom{-x; -t}{\alpha + 1; \beta + 1} \binom{x - N; t}{\alpha; \beta + 1} \]

\[ x = 0, 1, \ldots, N, \]  

and

\[ \sum_{n=0}^{\infty} \frac{(-N)^n}{(\beta + 1)_n n!} Q_n(x; \alpha, \beta, N) t^n = \binom{-x, -x + \beta + N + 1; -t}{x - N, -x + \alpha + 1; t} \]

\[ \times \binom{-x, -x + \beta + N + 1; -t}{x - N, -x + \alpha + 1; t} \]

\[ x = 0, 1, \ldots, N. \]
Motivated by the Jacobi type generalization of the Rice’s polynomials obtained by Khandekar [178], we shall study Jacobi type and Gegenbauer type generalization of certain polynomials and their generating functions in chapter 2.

1.5 On Some Generating Functions of $q$-Analogues of Jacobi Type and Gegenbauer Type Generalized Polynomials

The chapter 3 deals with $q$-analogues of the polynomials studied in the chapter 2. Thus, chapter-3 has been divided into two broad sections. The section 3.2 of this chapter deals $q$-analogues of the Jacobi type generalized polynomials and their generating functions relation while section 3.3 and their generating functions treats $q$-analogues of the Gegenbaur type generalized polynomials.

1.6 On Some Generating Functions of Certain $q$-Polynomials

In chapter 4, a study of the $q$-analogue of the Bateman’s polynomials $F_n(z)$, $Z_n(x)$, Pasternack’s generalization of Bateman’s polynomial $F^n_m(z)$, Shively’s pseudo-lagurre and other polynomials $R_n(a, x)$, Cesàro’s polynomials $g^{(s)}(x)$, Gottlieb polynomials $l_n(x; \lambda)$, and generalized Hypergeometric polynomial sets has been made. Further $q$-analogue of their generating functions relations have also been determined. In addition to the polynomials noted in the introduction section of chapter 2, some more polynomials and their generating functions are also included whose $q$-analogues shall be studied in the next consecutive sections of this chapter.
Chapter 1: Introduction

Shively’s Pseudo-Laguerre Polynomials

Shively studied the following pseudo-Laguerre set

\[ R_n(a, x) = \frac{(a)_{2n}}{n!(a)_n} \; _1F_1(-n; a + n; x) \] (1.6.1)

which are related to the proper simple Laguerre polynomial

\[ L_n(x) = \; _1F_1(-n; 1; x) \] (1.6.2)

by

\[ R_n(a, x) = \frac{1}{(a - 1)_n} \sum_{k=0}^{n} \frac{(a - 1)_{n+k} L_{n-k}(x)}{k!} \] (1.6.3)

Shively obtained Toscano’s other generating relation

\[ \sum_{n=0}^{\infty} \frac{R_n(a, x) t^n}{(\frac{1}{2} + \frac{1}{2}a)^n} = e^{xt} \; _0F_1 \left[ -\frac{1}{2}; t^2 - xt \right] \] (1.6.4)

Cesàro Polynomials

Cesàro polynomials [199] are defined as

\[ g_n^{(s)}(x) = \binom{s+n}{n} \; _2F_1 \left[ -n, 1; x \right] \]

\[ = \frac{(1+s)_n}{n!} \; _2F_1 \left[ -n, 1; x \right] \] (1.6.5)

and eq. (1.6.5) satisfy the following generating functions

\[ \sum_{n=0}^{\infty} g_n^{(s)}(x) t^n = (1-t)^{-s-1}(1-xt)^{-1} \] (1.6.6)

\[ \sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(s)}(x) t^n = (1-t)^{-s-1-k}(1-xt)^{-1} g^{(s)}_k \left( \frac{x(1-t)}{1-xt} \right) \] (1.6.7)
Chapter 1: Introduction

Gottlieb Polynomials

Gottlieb polynomials are given below (see [5], [191], [199]):

\[ l_n(x; \lambda) = e^{-n\lambda} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{k} \right) (1 - e^{\lambda})^k \]  

(1.6.8)

or

\[ l_n(x; \lambda) = e^{-n\lambda} {}_2F_1 \left[ \begin{array}{c} -n, \quad -x; \\ 1; \end{array} \right] \left( 1 - e^{\lambda} \right) \]  

(1.6.9)

satisfy the following generating functions (see [191], [199]):

\[ \sum_{n=0}^{\infty} l_n(x; \lambda) t^n = (1 - t)^x (1 - te^{-\lambda})^{-x-1}, \quad |t| < 1 \]  

(1.6.10)

\[ \sum_{n=0}^{\infty} l_n(x; \lambda) \frac{e^{\frac{n\lambda}{2}} t^n}{n!} = \exp(te^{-\frac{\lambda}{2}}) {}_1F_1 \left[ \begin{array}{c} -x; \\ 1; \end{array} \right] \left( e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}} \right) t \]  

(1.6.11)

\[ \sum_{n=0}^{\infty} l_n(x; \lambda) \frac{t^n}{n!} = e^t {}_1F_1 \left[ \begin{array}{c} x + 1; \\ 1; \end{array} \right] \left( 1 - e^{-\lambda} \right) t \]  

(1.6.12)

\[ \sum_{n=0}^{\infty} \frac{(c)_n}{n!} l_n(x; \lambda) t^n = (1 - te^{-\lambda})^{-c} {}_2F_1 \left[ \begin{array}{c} c, \quad -x; \\ 1; \end{array} \right] \left( \frac{1 - e^{-\lambda} t}{1 - te^{-\lambda}} \right) \]  

(1.6.13)

\[ \sum_{n=0}^{\infty} \binom{n+k}{k} l_{n+k}(x; \lambda) t^n = (1 - t)^{x-k}(1 - te^{-\lambda})^{-x-1} l_k(x, \alpha), \]  

(1.6.14)

where \( \alpha = \log \left[ \frac{e^{\lambda} - 1}{1 - t} \right] \)
Generalized Hypergeometric Polynomial Set

In 1994, S. D. Bajpai and M. S. Arora [4] studied some properties of the generalized hypergeometric polynomial set, given below:

\[ U_n(\beta; \gamma; x) = x^n \, _2F_1 \left[ -n, \beta; \gamma; \frac{1}{x} \right] \]  

(1.6.15)

where \( n \) is a non-negative integer and \( x \) is any non-zero complex variable and \( \beta, \gamma \) are independent of \( n \) for if \( \beta, \gamma \) dependent upon \( n \) then many properties which are valid for \( \beta, \gamma \) independent of \( n \) fail to be valid for \( \beta, \gamma \) dependent upon \( n \).

In 1997, I. K. Khanna and V. Srinivasa Bhagavan [179] derived generating function of generalized hypergeometric polynomial set (1.6.15) in terms of Gottlieb Polynomials, is given below:

\[ U_n(-x; 1; (1 - e^\lambda)^{-1}) = (e^{-\lambda} - 1)^{-n} l_n(x; \lambda) \]  

(1.6.16)

Recently, M. A. Khan and M. Akhlaq [137] defined two variable and three variable analogues of the Gottlieb Polynomials. Of which, the two variable analogue of Gottlieb polynomials is given below:

\[ l_n(x, y; \lambda, \mu) = e^{-n(\lambda+\mu)} F \left[ \begin{array}{ccc} -n & -x & -y \\ 1 & - & - \\ \end{array} \right] \]  

(1.6.17)

or in other words,

\[ l_n(x, y; \lambda, \mu) = e^{-n(\lambda+\mu)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_r(-x)_r(-y)_s(1-e^\lambda)_r(1-e^\mu)_s}{r!s!(1)^{r+s}} \]  

(1.6.18)

and the three variable analogue as follows [137]:
Chapter 1: Introduction

\[ l_n(x, y, z; \lambda, \mu, \eta) = \]

\[
e^{-n(\lambda+\mu+\eta)} F \left[ \begin{array}{c} -n; -; -; -x; -y; -z; \\ 1; -; -; -; -; -; -; -; 1 - e^\lambda, 1 - e^\mu, 1 - e^\eta \end{array} \right] \]

or in other words,

\[ l_n(x, y, z; \lambda, \mu, \eta) = \]

\[
e^{-n(\lambda+\mu+\eta)} \sum_{r=0}^{n} \sum_{s=0}^{n-r-s} \sum_{k=0}^{r-s} (-n)_r s_k (-x)_r (-y)_s (-z)_k (1 - e^\lambda)^r (1 - e^\mu)^s (1 - e^\eta)^k \]

\[
= \frac{r! s! k! (1)_{r+s+k}}{n!} \]

(1.6.20)

They [137] obtained the following generating functions similar to (1.6.14)

\[
\sum_{n=0}^{\infty} l_n(x, y; \lambda, \mu) t^n = (1 - te^{-\mu})^x (1 - te^{-\lambda})^y (1 - te^{-\lambda+\mu})^{x-y-1}, \quad |t| < 1 
\]

(1.6.21)

\[
\sum_{n=0}^{\infty} (c)_n l_n(x, y; \lambda, \mu) \frac{t^n}{n!} = 
\]

\[
= (1 - te^{-(\lambda+\mu)-c}) F \left[ \begin{array}{c} c; -x; -y; \\ 1; -; -; \frac{t(e^\lambda-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}}, \frac{t(e^\mu-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}} \end{array} \right] 
\]

(1.6.22)

and for result (1.6.18) as

\[
\sum_{n=0}^{\infty} l_n(x, y, z; \lambda, \mu, \eta) t^n = (1 - te^{-(\mu+\eta)})^x (1 - te^{-(\lambda+\eta)})^y 
\times (1 - te^{-(\lambda+\mu+\eta)})^{x-y-z-1}, \quad |t| < 1 
\]

(1.6.23)
\[
\sum_{n=0}^{\infty} (c)_n l_n(x, y, z; \lambda, \mu, \eta) \frac{t^n}{n!} = (1-te^{-(\lambda+\mu+\eta)-c})
\]

\begin{align*}
\times F\begin{bmatrix}
\epsilon \cdots -\epsilon \cdots -\epsilon \cdots & -\epsilon \cdots -\epsilon \cdots -\epsilon \cdots \\
1 \cdots -1 \cdots -1 \cdots -1 \cdots -1 \\
\frac{t(e^{c-1})e^{-(\lambda+\mu+\eta)}}{1-e^{-(\lambda+\mu+\eta)}} & \frac{t(e^{c-1})e^{-(\lambda+\mu+\eta)}}{1-e^{-(\lambda+\mu+\eta)}} & \frac{t(e^{c-1})e^{-(\lambda+\mu+\eta)}}{1-e^{-(\lambda+\mu+\eta)}} \\
\end{bmatrix}.
\end{align*}

(1.6.24)

1.7 Generalized Exponential Operators and Difference Equations

In 2000, Dattoli and Levi [34] discussed general methods for the solution of difference equations, arising in physical and biological problems. Their technique play crucial role in unifying the generalized families of the difference equations.

The chapter 5 deals with the generalization of exponential operators used in [34] to operators of the type \(a^{\lambda q(x)} \frac{dx}{x}\), where base \(a \ (a > 0, \ a \neq 1)\) is a real number. In particular when \(a = e\), the operator reduces to the operators used by Dattoli et al. [34].

The action of the generalized exponential operator on a generic function \(f(x)\) is defined as

\[
a^{\lambda q(x)} \frac{dx}{x} f(x) = e^{(\lambda \ln(a) q(x))} \frac{dx}{x} f(x)
\]

\[
= f(F^{-1}(\lambda \ln(a) + F(x))). \tag{1.7.1}
\]

where \(F(x)\) (called the Similarity Factor (S.F.)) denotes the function
and \( F^{-1}(\sigma) \) is its inverse.

For \( q(x) = 1 \), the SF is given by

\[
F(x) = \int^x \frac{d\xi}{q(\xi)},
\]

therefore \( F^{-1}(x) = x \), then the operator (1.7.1) reduces to the ordinary translation or shift operator as follows:

\[
a^\frac{\lambda^z}{x} f(x) = f(F^{-1}(\lambda \ln(a) + x))
\]

\[
= f(\lambda \ln(a) + x).
\]

Another example of application of the operator (1.7.1), for \( q(x) = x \), the SF is given by

\[
F(x) = \int^x \frac{d\xi}{\xi} = \ln(x),
\]

so that \( F^{-1}(x) = e^x \), and hence the operator (1.7.1) reduces to the dilatation operator

\[
a^{\lambda \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + \ln(x)))
\]

\[
= f(e^{\lambda \ln(a) + \ln(x)}) = f(a^\lambda x).
\]

The ordinary shift operators and their properties play a central role within the context of the theory of difference equations [54]. One can, therefore, suspect that the above generalized exponential operators and the wealth of their properties can be exploited to develop tools which allow the solution of
different forms of difference equations.

**Particular case:** The substitution of $a = e$, into the eqs. (1.7.1), (1.7.3) and (1.7.5) reduce to the eqs. (1), (2') and (3) of Dattoli et al. [34].

A simple example of how the exponential operators can help us to solve difference equations may be illuminating. Let us consider the linear dilatation difference equation of the type

$$b_1 f(a^2x) + b_2 f(ax) + b_3 f(x) = 0, \quad (1.7.6)$$

which, according to eq. (1.7.5), eq. (1.7.6) can be written in the following form

$$\left[b_1 \, a^{2x} \frac{df}{dx} + b_2 a^x \frac{df}{dx} + b_3\right] f(x) = 0. \quad (1.7.7)$$

Suppose $f(x) = R^{\ln(x)}$, we have

$$a^{\lambda x} \frac{df}{dx} R^{\ln(x)} = e^{\lambda \ln(a) x} \frac{df}{dx} R^{\ln(x)},$$

where

$q(x) = x$, so that $F(x) = \ln(x)$ and $F^{-1}(x) = e^x$

or

$$F^{-1}(\lambda \ln(a) + \ln(x)) = e^{\lambda \ln(a) + \ln(x)} = xa^\lambda,$$

therefore,

$$a^{\lambda x} \frac{df}{dx} R^{\ln(x)} = R^{\ln(xa^\lambda)} = R^{\lambda \ln(a)} R^{\ln(x)}. \quad (1.7.8)$$

Hence we can associate with eq. (1.7.7) the characteristic equation
\[ [b_1 R_{2\ln(a)}^2 + b_2 R_{\ln(a)}^2 + b_3] R_{\ln(x)}^2 = 0 \]

or

\[ b_1 R_{2\ln(a)}^2 + b_2 R_{\ln(a)}^2 + b_3 = 0, \tag{1.7.9} \]

whose roots \( R_1^{\ln(a)} \), and \( R_2^{\ln(a)} \) allow to write \( f(x) \) in terms of the following linear combination of independent solutions:

\[ f(x) = c_1 R_1^{\ln(x)} + c_2 R_2^{\ln(x)} = \sum_{\alpha=1}^{2} c_\alpha R_\alpha^{\ln(x)}. \tag{1.7.10} \]

The above example indicates that we can extend well-established methods of solutions of difference equations to other types of equations reducible to ordinary difference equations, after a proper change of variable implicit in eqs. (1.7.1), (1.7.3).

**Particular case:** The replacement of \( a \) with \( e \) in the eqs. (1.7.6), (1.7.7), (1.7.8) and (1.7.9) give raise to the eqs. (5), (6), (7), and (8) of Dattoli et al. [34].

To give a further example of the flexibility of the formalism associated with exponential operators, let us consider the generalized Heat Equation of the following type

\[
\begin{cases}
\frac{\partial}{\partial \lambda} Q(x, \lambda \ln(a)) = \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 Q(x, \lambda \ln(a)), \\
Q(x, 0) = g(x),
\end{cases} \tag{1.7.11}
\]

which can formally be solved by rewriting eq (1.7.11) as

\[
\frac{\partial}{\partial \lambda} Q(x, \lambda \ln(a)) - \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 Q(x, \lambda \ln(a)) = 0,
\]
which can formally be solved by considering this as ordinary linear differential equation of order one, whose I.F. is determined as

\[ e^{-\int \ln(a)\left(q(x)\frac{\partial}{\partial x}\right)^2 d\lambda} = e^{-\ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2} = a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2}, \]

we can, therefore, find its general solution as

\[ Q(x, \lambda \ln(a))a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2} = C, \]

where \( C \) is any constant and using the given initial condition, we get

\[ Q(x, 0) = g(x) = C, \]

and finally, we obtain the solution of the Heat equation (1.7.11) as

\[ Q(x, \lambda \ln(a)) = a^{\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2} g(x). \] (1.7.12)

The use of the identity

\[ e^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2b\xi} d\xi, \]

replacing \( b^2 \) with \( \lambda \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 \), we have

\[ e^{\lambda \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2} = a^{\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2} \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\sqrt{\lambda \ln(a)}\xi} d\xi, \] (1.7.13)

with the use of the eq. (1.7.1), finally yields the solution of eq (1.7.11) in the form of an integral transform, which can be viewed as a generalized Gauss transform.
\[ a^{\lambda q(x)} g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(\lambda \ln(a) + F(x))) d\xi. \]  \hspace{1cm} (1.7.14)

or, in other words, we have

\[ Q(x, \lambda \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(\lambda \ln(a) + F(x))) d\xi. \]  \hspace{1cm} (1.7.14)'

It is evident that the formalism associated with generalized exponential operators can be exploited in many flexible ways in finding the general solution of a large number of problems. This chapter is devoted to the discussion of methods which provide the solution of the classes of “difference” and generalized “Heat” equations and we shall see that the techniques we propose offer reliable analytical tools and efficient numerical algorithms.

**Particular case:** To put \( a = e \), in the eqs. (1.7.11), (1.7.12), and (1.7.14) give raise the same forms of the eqs. (10), (11) and (13) respectively of Dattoli et al. [34].

### 1.8 Shift Operators On The Base \( a \ (a > 0, \neq 1) \), Pseudo-Polynomials and Monomial Type Functions

In what follows, we consider analytic function \( f(x) \) so that the corresponding Taylor expansion

\[ f(x + \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f^{(k)}(x), \]  \hspace{1cm} (1.8.1)

converges to corresponding value of \( f \) in a suitable neighborhood of \( x \).

In 2003, Dattoli et al. [39] discussed the exponential operators, the operators on the natural base \( e \).
In chapter 6, we generalize the exponential operators \([39]\) on the base \(a\) \((a > 0, \neq 1)\), as follows:

\[
\hat{A}_m = a^\lambda \left( \frac{\partial}{\partial x} \right)^m
\]  

(1.8.2)

In the case when \(m = 1\), it reduces to the ordinary shift operator, while for \(m = 2\) it can be identified with the operatorial version of the Gauss transform

\[
a^\lambda \left( \frac{\partial}{\partial x} \right) f(x) = f(x + \lambda \ln(a))
\]  

(1.8.3)

Making use of the following identity, we have

\[
e^\varphi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2\varphi \xi} d\xi,
\]  

(1.8.4)

we find

\[
a^\lambda \left( \frac{\partial^2}{\partial x^2} \right) f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\xi^2 + 2\sqrt{\pi \lambda \ln(a)} \xi \frac{\partial}{\partial x} \right) f(\xi) d\xi
\]

\[
= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} f(x + 2\xi \sqrt{\lambda \ln(a)}) d\xi
\]

or

\[
a^\lambda \left( \frac{\partial^2}{\partial x^2} \right) f(x) = \frac{1}{2\sqrt{\pi \lambda \ln(a)}} \int_{-\infty}^{\infty} e^{\frac{(x-x_0)^2}{4\sqrt{\lambda \ln(a)}}} f(\xi) d\xi.
\]  

(1.8.5)

after a suitable change of variables.

Both the eqs. (1.8.3) and (1.8.5) are solution of the partial differential equation:

\[
\begin{align*}
\frac{\partial}{\partial \lambda} F(x, \lambda \ln(a)) &= \ln(a) \left( \frac{\partial}{\partial x} \right)^m F(x, \lambda \ln(a)), \\
F(x, 0) &= f(x), \quad m = 1, 2.
\end{align*}
\]  

(A)
In case when $m > 2$, the exponential operator $\hat{A}_m = a^{\lambda(\frac{\partial}{\partial x})^m}$ provides formal solution for the generalized heat equation. It does not seem possible to associate it to any transformation of the Gauss type. We must, however, emphasize that the Hermite-Kampé de Fériet polynomials [3] of the type

$$H_n^{(m)}(x, y \ln(a)) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{x^{n-2r}(y \ln(a))^r}{(n-2r)!r!} = g_n^{(m)}(x, y \ln(a))$$

(1.8.6)

or equivalently the Gould-Hopper polynomials [[199], p. 76, eq. (1.9)]:

$$g_n^{(m)}(x, y \ln(a)) = \sum_{r=0}^{[\frac{n}{2}]} \frac{n!}{r!(n-2r)!} x^{n-2r}(y \ln(a))^r$$

These polynomials are a solution of

$$\frac{\partial}{\partial \lambda} F(x, \lambda \ln(a)) = \ln(a) \left( \frac{\partial}{\partial x} \right)^m F(x, \lambda \ln(a)),$$

$$F(x, 0) = x^n.$$ (A')

or in other words[136] and [157]

$$a^{\lambda(\frac{\partial}{\partial x})^m} x^n = H_n^{(m)}(x, y \ln(a))$$

(1.8.7)

This last result is particularly important, since it allows the conclusion that if $f(x)$ is an analytic function defined by the series expansion

$$f(x) = \sum_n c_n x^n$$

(1.8.8)

then, by Taylor Theorem, we write

$$a^{\lambda(\frac{\partial}{\partial x})^m} f(x) = \sum_n c_n H_n^{(m)}(x, y \ln(a))$$

(1.8.9)
The polynomials $H_n^{(m)}(x, y \ln(a))$ will be said to be the polynomials of index $n$ and order $m$.

**Particular case:** The replacement of $a$ with $e$ into the equations of this section give raise to the eqs. given in the first section of Dattoli et al. [39].

### 1.9 Generalized Operational Methods, Fractional Operators and Special Polynomials

In 2003, Dattoli [37] used operators on the natural base $e$ for determining fractional operators, integral transforms and new family of special polynomials. For determining the new family of polynomials, we introduce and use operators on the base $a$ ($a > 0, \neq 1$).

It is well known that Hermite and Laguerre polynomials are defined through operational identities (see [33]) i.e. through the exponential operators defined on natural base $e$ [37].

We define Hermite polynomial through the operational identities, the exponential operators defined on the base $a$ ($a > 0, \neq 1$), as follows:

\[
a^y \frac{\partial^2}{\partial x^2} x^n = e^{y \ln(a) \frac{\partial^2}{\partial x^2}} x^n
\]

\[
= \sum_{r=0}^{\infty} \frac{(\ln(a) y \frac{\partial^2}{\partial x^2})^r}{r!} x^n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(y \ln(a))^r n! x^{n-2r}}{(n-2r)! r!}
\]

or

\[
a^y \frac{\partial^2}{\partial x^2} x^n = H_n(x, \ln(a) y).
\] (1.9.1)

Similarly, we define Laguerre polynomial through the operational identities, the exponential operators defined on the base $a$ ($a > 0, \neq 1$), given below:
Chapter 1: Introduction

\[ a^{-y \frac{\partial}{\partial x} x} \left[ \frac{(-1)^n}{n!} x^n \right] = e^{-y \ln(a) \frac{\partial}{\partial x} x} \left[ \frac{(-1)^n}{n!} x^n \right] \]

\[ = \sum_{r=0}^{\infty} \left( -y \ln(a) \frac{\partial}{\partial x} x \right)^r \left[ \frac{(-1)^n}{n!} x^n \right] = \sum_{r=0}^{\infty} \left( -y \ln(a) \right)^r \left( \frac{\partial}{\partial x} x \right)^r \frac{(-1)^n}{r!(n-r)!} \]

\[ = \sum_{r=0}^{n} \frac{(-1)^{n+r} (y \ln(a))^r n! x^{n-r}}{r! (n-r)!^2} = n! \sum_{r=0}^{n} \frac{(-1)^r (y \ln(a))^{n-r} x^r}{(n-r)! (r!)^2} \]

or

\[ a^{-y \frac{\partial}{\partial x} x} \left[ \frac{(-1)^n}{n!} x^n \right] = L_n(x, y \ln(a)). \quad (1.9.2) \]

which will play crucial role in applications [208, 209].

The polynomials (1.9.1) and (1.9.2) are generalized forms of Hermite and
Laguerre polynomials and are linked to the ordinary case by

\[ (-i)^n (y \ln(a))^\frac{3}{2} H_n \left( \frac{ix}{2 \sqrt{y \ln(a)}} \right) = (-i)^n (y \ln(a))^\frac{3}{2} \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r n! \left( \frac{2x}{2 \sqrt{y \ln(a)}} \right)^{n-2r}}{r!(n-2r)!} \]

\[ = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{2n-r} (y \ln(a))^r x^{n-2r}}{r!(n-2r)!} = n! \sum_{r=0}^{\frac{n}{2}} \frac{(y \ln(a))^r x^{n-2r}}{r!(n-2r)!} \]

or

\[ (-i)^n (y \ln(a))^\frac{3}{2} H_n \left( \frac{ix}{2 \sqrt{y \ln(a)}} \right) = H_n(x, y \ln(a)). \quad (1.9.3) \]

and

\[ (y \ln(a))^n L_n \left( \frac{x}{y \ln(a)} \right) = (y \ln(a))^n \sum_{r=0}^{n} \frac{(-1)^r n! \left( \frac{x}{y \ln(a)} \right)^r}{(r!)^2(n-r)!} \]
\[ = \sum_{r=0}^{n} \frac{(-1)^r n! x^r (y \ln(a))^{n-r}}{(r!)^2 (n-r)!} \]

or

\[ (y \ln(a))^n L_n \left( \frac{x}{y \ln(a)} \right) = L_n(x, y \ln(a)). \quad (1.9.4) \]

The use of the following identity [209]

\[ a^{B^2} = e^{B \ln(a)} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2 + 2B \sqrt{\ln(a)} \xi) d\xi. \quad (1.9.5) \]

enables us to concentrate on the eq. (1.9.1), in particular on the generalized exponential operator, which, according to the standard procedure [27], can be written in the following form

\[ a^{\partial^2 \over \partial x^2} = e^{y \ln(a) \partial^2 \over \partial x^2} \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\xi^2 + 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) d\xi. \quad (1.9.6) \]

The use of the above identity and of the following fact

\[ a^{\lambda \partial} f(x) = f(x + \lambda \ln(a)) \quad (1.9.7) \]

allows to conclude that the polynomials \( H_n(x, y \ln(a)) \) satisfy the integral representation

\[ H_n(x, y \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\xi^2 + 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) x^n d\xi \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2) \exp \left( 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) x^n d\xi \]

or
Chapter 1: Introduction

\[ H_n(x, y \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2)[x + 2\xi \sqrt{y \ln(a)}]^n d\xi, \]  

(1.9.8)

The above example shows
(a) it is possible to define polynomials by means of an operational identity,
(b) such operational identity can in turn be used to derive integral representation.

**Particular case:** The substitution \( a = e \), into the eqs. (1.9.1), (1.9.2), (1.9.3), (1.9.4), (1.9.6), (1.9.7) and (1.9.8) reduce to the results (1), (2), (3), (4), (5) and (6) due to Dattoli [37].

Methods employing the combined use of generalized exponential operators and integral transforms provide a powerful tool for the solution of P.D.E. of evolution type. An appropriate example follows from the equation associated with the Black-Scholes financial model [210]

\[ \frac{1}{(\ln(a))^2} \frac{\partial}{\partial \tau} A = S^2 \frac{\partial^2}{\partial S^2} A + \lambda S \frac{\partial}{\partial S} A - \lambda A \]

\[ A(S, 0) = f(S) \]

which can be rewritten as

\[ \frac{1}{(\ln(a))^2} \frac{\partial}{\partial \tau} A = \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 A - \left( \frac{\lambda + 1}{2} \right)^2 A \]

or

\[ \frac{\partial}{\partial \tau} A - \left[ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right] (\ln(a))^2 A = 0 \]

(1.9.10)

which is a linear differential equation in \( A \).

whose **Integrating Factor** is

\[ \exp \left[ - \left\{ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right\} (\ln(a))^2 \tau \right] \]
and which admits the formal solution

\[ A(S, \tau(\ln(a))^2) = \exp \left[ \left\{ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right\} (\ln(a))^2 \tau \right] f(S) \]

\[ = \exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 (\ln(a))^2 \tau \right] \times \exp \left[ \left\{ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 \right\} (\ln(a))^2 \tau \right] f(S) \]

(1.9.11)

\[ = \exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 \tau(\ln(a))^2 \right] \frac{\sqrt{\pi}}{\int_{-\infty}^{\infty} \exp \left[ -\xi^2 + 2 \left( \frac{\lambda - 1}{2} \right) \sqrt{\tau} \ln(a) \xi \right] f(S) d\xi, \]

or

\[ A(S, \tau(\ln(a))^2) = \frac{\exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 \tau(\ln(a))^2 \right]}{\sqrt{\pi}} \]

\times \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + (\lambda - 1)\xi \sqrt{\tau} \ln(a) \right] \exp \left( 2\sqrt{\tau} \ln(a) S \frac{\partial}{\partial S} \right) f(S) d\xi, \]  

(1.9.12)

by the dilatation operator, we have

\[ a^{\lambda x} \frac{\partial}{\partial x} f(x) = \exp \left( \lambda \ln(a) x \frac{\partial}{\partial x} \right) = f(xa^{\lambda}) \]  

(1.9.13)

from the eqs. (1.9.12) and (1.9.13), we have

\[ A(S, \tau(\ln(a))^2) = \frac{\exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 \tau(\ln(a))^2 \right]}{\sqrt{\pi}} \]

\times \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + (\lambda - 1)\xi \sqrt{\tau} \ln(a) \right] f(\exp(2\sqrt{\tau} \ln(a)) S) d\xi \]

(1.9.14)
or

\[ A(S, \tau(\ln(a))^2) = \exp \left[ - \left( \frac{\lambda+1}{2} \right)^2 \tau(\ln(a))^2 \right] \frac{\sqrt{\pi}}{\sqrt{\pi}} \]

\[ \times \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + (\lambda - 1)\xi \sqrt{\tau} \ln(a) \right] f(a^{2\xi} \sqrt{S}) d\xi. \tag{1.9.15} \]

This result shows that methods employing operational techniques can be used in fairly wide context and allow noticeable flexibility.

In the last chapter we introduce new families of special polynomials starting from a suitable definition. It shall also be shown that the concept we develop is useful in different variety including the theory of fractional derivatives.

**Particular case:** When we put \( a = e \), into the eqs. (1.9.9), (1.9.10), (1.9.11), (1.9.13) and (1.9.15) reduce to the results (7), (8), (9), (10) and (11) due to Dattoli [37].
Publications

(i) Khan, M. A.; Asif M.; *Shift operators on the base $a(a > 0, \neq 1)$ and Monomial Type Functions*, International Transactions in Mathematical Sciences and Computer, 2, No. 2, (2009), 453-462.

(ii) Khan, M. A.; Asif M.; *Generalized exponential operators and difference equations*, Accepted for publication in Italian Journal of Pure and Applied Mathematics no. 30.

(iii) Khan, M. A.; Asif M.; *Shift operators on the base $a(a > 0, \neq 1)$ and pseudo-polynomials of fractional order*, Accepted for publication in International Journal of Mathematical Analysis.

(iv) Khan, M. A.; Asif M.; *Generalized operational methods, fractional operators and special polynomials*, Communicated for publication.

(v) Khan, M. A.; Asif, M.; *q-Analogue of exponential operators and difference equations*, Communicated for publication.

(vi) Khan, M. A.; Asif, M.; *q-Analogue shift operators and pseudo-polynomials of fractional order*, Communicated for publication.

(vii) Khan, M. A.; Asif, M.; *q-Analogue of the operational methods, fractional operators and special polynomials*, Communicated for publication.
(viii) Khan, M. A.; Asif, M.; *Generalized q-shift operators and monomial type functions*, Communicated for publication.

(ix) Khan, M. A.; Asif, M.; *A note on generating function of q-Gottlieb polynomials*, Communicated for publication.

(x) Khan, M. A. and Asif, M.; *Jacobi type and Gegenbauer type generalization of certain polynomials*, Communicated for publication.

(xi) Khan, M. A. and Asif, M.; *On certain generating functions of q-analogues of Jacobi type and Gegenbauer type generalized polynomials*, Communicated for publication.
Chapter 2

Jacobi Type And Gegenbauer Type Generalization Of Certain Polynomials

ABSTRACT: This chapter deals with the Jacobi type and Gegenbauer type generalizations of certain polynomials and their generating functions. Relationships among those generalized polynomials have also been indicated.

2.1 Introduction

In 1947, Sister Celine (Fasenmyer[48]) obtained some basic formal properties of the hypergeometric polynomials. Sister Celine’s Polynomials are defined by the following generating relation

\[
(1 - t)^{-1} r,F_s \left[ \begin{array}{c} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} \right] - \frac{4x}{(1-t)^2} = \sum_{n=0}^{\infty} f_n \left[ \begin{array}{c} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} \right] t^n \]

which yields

\[
f(a_i; b_i; x) \equiv f_n(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; x) = r,F_{s+2} \left[ \begin{array}{c} -n, n + 1, a_1, \ldots, a_r; \\ 1, \frac{1}{2}, b_1, \ldots, b_s; \end{array} x \right] (2.1.2)
\]
(n is non-negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention.

Her polynomials include as special cases of Legendre’s polynomials $P_n(1 - 2x)$, some special Jacobi polynomials, Rice’s polynomials $H_n(\xi, p, \nu)$, Bateman’s $Z_n(x)$, $F_n(z)$ and Pasternak’s $F_n^m(z)$ which is a generalization of Bateman’s polynomials $F_n(z)$.

**The Bateman’s Polynomials**

Bateman defined the following polynomials (see [191], [199])

$$F_n(z) = 3F_2 \left[ \begin{array}{c}
-n, n + 1, \frac{1}{2}(1 + z); \\
1, 1;
\end{array} \right]$$ (2.1.3)

Bateman ([7], [8]) obtained the following generating functions

$$\sum_{n=0}^{\infty} F_n(z)t^n = \frac{1}{1 - t} 2F_1 \left[ \begin{array}{c}
\frac{1}{2}, \frac{1}{2} + \frac{1}{2}z; \\
1;
\end{array} \right]$$ (2.1.4)

$$\sum_{n=0}^{\infty} (F_n(z-2) - F_n(z))t^n = \frac{2t}{(1 - t)^3} 2F_1 \left[ \begin{array}{c}
\frac{3}{2}, \frac{1}{2} + \frac{1}{2}z; \\
2, \frac{-4t}{(1 - t)^3}
\end{array} \right]$$ (2.1.5)

**The Pasternack’s Polynomials**

The generalization of the Bateman’s Polynomial due to Pasternack is given below:

$$F_n^m(z) = 3F_2 \left[ \begin{array}{c}
-n, n + 1, \frac{1}{2}(z + m + 1); \\
1, m + 1;
\end{array} \right]$$ (2.1.6)
which is a generalization of Bateman’s polynomials $F_n(z)$. Generating function of generalization of Pasternack’s polynomials is given below:

$$
\sum_{n=0}^{\infty} F_n^m(z)t^n = \frac{1}{(1-t)} 2F_1\left[\frac{1}{2}, \frac{1}{2}(z+m+1); \frac{-4t}{(1-t)^2}; m+1\right] (2.1.7)
$$

In 1936, Bateman [191] was interested in constructing inverse Laplace transforms. For this purpose he introduced the following polynomial

$$
Z_n(x) = \frac{n!}{1-t} 2F_2\left[-n, n+1; 1, 1; x\right]. \quad (2.1.8)
$$

Rainville (see [191], p. 137, Theorem 48) writes the generating function as follows:

$$
\sum_{n=0}^{\infty} Z_n(x)t^n = \frac{1}{1-t} 2F_1\left[\frac{1}{2}, n+1; \frac{-4xt}{(1-t)^2}; 1\right]. \quad (2.1.9)
$$

In 1939, Pasternack [191] obtained the following generating function of Bateman’s polynomials:

$$
\sum_{n=0}^{\infty} F_m(-2n-1)\frac{(-t)^n}{n!} = e^{-t}Z_m(t) \quad (2.1.10)
$$

$$
\sum_{n=0}^{\infty} F_n(-2m-1)t^n = (1-t)^{-m-1}(1+t)^m P_m\left(\frac{1+t^2}{1-t^2}\right). \quad (2.1.11)
$$

**The Rice’s Polynomials**

S. O. Rice made a considerable study of the polynomials defined by

$$
H_n(\xi, p, \nu) = 3F_2\left[-n, n+1, \xi; 1, p; \nu\right]. \quad (2.1.12)
$$

The generalized Rice’s Polynomial due to Khandekar [178] is given below:
\[ \frac{n!}{(1 + \alpha)_n} H_n^{(\alpha, \beta)}(\xi, p, \nu) = \binom{-n}{1 + \alpha, p; \nu} \]  
\[ (2.1.13) \]

Generating function of the generalized Rice polynomial due to Khandekar [178] is given below:

\[ \sum_{n=0}^{\infty} \frac{(2\alpha + 2k)^n}{(1 + \alpha)_n} H_n^{(\alpha, \beta)}(\xi, p, \nu) t^n \]
\[ = (1 - t)^{-\alpha - \beta - 1} \binom{\Delta(2; \alpha + \beta + 1), \xi; 1 + \alpha, p; \nu}{1 - \frac{4\nu t}{(1 - t)^2}}. \]  
\[ (2.1.14) \]

**The Hahn Polynomials**

Hahn polynomial is defined as

\[ Q_n(x; \alpha, \beta, N) = \binom{-n, n + \alpha + \beta, -x; 1}{1 + \alpha, -N} \]
\[ \alpha, \beta > -1, \quad n, x = 0, 1, \ldots, N. \]  
\[ (2.1.15) \]

The following generating functions are satisfied by the Hahn polynomial (2.1.15):

\[ \sum_{n=0}^{\infty} \frac{(-N)_n}{(\beta + 1)_n n!} Q_n(x; \alpha, \beta, N) t^n = \binom{-x; -t}{\alpha + 1; \beta + 1; t} \]
\[ x = 0, 1, \ldots, N, \]  
\[ (2.1.16) \]

and
Chapter 2: Jacobi Type and Gegenbauer Type Generalization.

Motivated by the Jacobi type generalization of the Rice’s polynomials obtained by Khandekar [178], we aim here to obtain Jacobi type generalization of the polynomials mentioned in the first section of this chapter.

2.2 Jacobi Type Generalization of Certain Polynomials and Their Generating Functions

Before obtaining the Gegenbauer type generalization of the polynomials, we shall first discuss the Jacobi type generalization of the polynomials. The Gegenbauer type generalizations of the polynomials shall be discussed in the next section of this chapter.

The Jacobi type generalization of Sister Celine’s polynomial is given below:

2.2.1 The Sister Celine’s Polynomial

Generalized Sister Celine’s Polynomials are defined by means of the following generating relation

\[
(1 - t)^{-c} \sum_{n=0}^{\infty} \binom{-c}{n} \frac{t^n}{n!} x^n = \sum_{n=0}^{\infty} \binom{-c}{n} \frac{t^n}{n!} \left[ 1 + \alpha, \frac{1}{2} + \alpha, b_1, \ldots, b_s; -\frac{4\pi t}{(1-\alpha)^2} \right]_n
\]

\[
= \sum_{n=0}^{\infty} f_n \left[ 1 + \alpha, \frac{1}{2} + \alpha, b_1, \ldots, b_s; \right]_n x^n
\]

(2.2.1)
which produces the following relation
\[
\begin{align*}
\binom{c}{n} \sum_{n=0}^{\infty} & \frac{n!}{n!} \binom{c}{n} \binom{c}{n} x^n \\
&= (1-t)^{-c} \sum_{k=0}^{\infty} \binom{c}{k} \frac{1+c}{2} \binom{1+c}{k} \binom{a_1}{k} \cdots \binom{a_r}{k} \\
&= (1-t)^{-c} \sum_{k=0}^{\infty} \binom{c}{k} \frac{1+c}{2} \binom{1+c}{k} \binom{a_1}{k} \cdots \binom{a_r}{k} (1-t)^{2k}
\end{align*}
\]

which is the required eq. (2.2.1).

Case (i) For \( \alpha = 0 \), it reduces to

\[
\binom{c}{n} \sum_{n=0}^{\infty} \frac{n!}{n!} \binom{c}{n} \binom{c}{n} x^n
\]
Chapter 2: Jacobi Type and Gegenbauer Type Generalization....

\[(1 - t)^{-c} \binom{\frac{c}{2} + \frac{1 + c}{2}, a_1, \ldots, a_r}{1, \frac{1}{2}, b_1, \ldots, b_s; \frac{4\pi t}{(1-t)^2}} \]

\[= \sum_{n=0}^{\infty} f_n \begin{bmatrix} \frac{c}{2} + \frac{1 + c}{2}, a_1, \ldots, a_r; \\ 1, \frac{1}{2}, b_1, \ldots, b_s; \\ x \end{bmatrix} t^n \quad (2.2.3)\]

Case (ii) For \(\alpha = 0\) and \(c = 1\), it reduces to original Sister Celine Polynomial (see [191] eq.(1) pp.290 of Rainville) and eq. (2.1.1).

Case (iii) For \(c = 1 + \alpha + \beta\), it gives Jacobi type generalization of Sister Celine's polynomial

\[= \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)n}{n!} \binom{1 + \alpha + \beta}{1 + \alpha, \frac{1}{2} + \alpha, b_1, \ldots, b_s; \frac{4\pi t}{(1-t)^2}} \]

\[= \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! (n + 1 + \alpha + \beta)(a_1)_k \ldots (a_r)_k x^k}{(1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]

\[= \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! (n + 1 + \alpha + \beta)(a_1)_k \ldots (a_r)_k x^k}{(n - k)! (1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1 + \alpha + \beta)n! (a_1)_k \ldots (a_r)_k (-x)^k}{(n - k)! (1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]

Proof Substituting \(c = 1 + \alpha + \beta\), in equation (2.2.2), we have

\[= \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! (n + 1 + \alpha + \beta)(a_1)_k \ldots (a_r)_k x^k}{(1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]

\[= \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! (n + 1 + \alpha + \beta)(a_1)_k \ldots (a_r)_k x^k}{(n - k)! (1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1 + \alpha + \beta)n! (a_1)_k \ldots (a_r)_k (-x)^k}{(n - k)! (1 + \alpha)_k(1 + \alpha + \beta)(b_1)_k \ldots (b_s)_k k!} t^n \]
Chapter 2: Jacobi Type and Gegenbauer Type Generalization

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 + \alpha + \beta)_{n+2k}(a_1)_k \cdots (a_r)_k (-xt)^k}{k!n!(1 + \alpha)_k(\frac{1}{2} + \alpha)_k(b_1)_k \cdots (b_s)_k} t^{n+k} \]

\[ = \sum_{k=0}^{\infty} \frac{(1 + \alpha + \beta)_{2k}(a_1)_k \cdots (a_r)_k (-xt)^k}{k!(1 + \alpha)_k(\frac{1}{2} + \alpha)_k(b_1)_k \cdots (b_s)_k} \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta + 2k)_n}{n!} t^n \]

\[ = (1 - t)^{-1-\alpha-\beta} \sum_{k=0}^{\infty} \frac{\left(\frac{1+\alpha+\beta}{2}\right)_k \left(\frac{2+\alpha+\beta}{2}\right)_k (a_1)_k \cdots (a_r)_k (-4xt)^k}{k!(1 + \alpha)_k(\frac{1}{2} + \alpha)_k(b_1)_k \cdots (b_s)_k} (1 - t)^{2k} \]

\[ = \frac{1}{(1 - t)^{1+\alpha+\beta}} \, 2^{r+s} F_{2+s} \left[ \begin{array}{c} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}, a_1, \ldots, a_r; \\ 1 + \alpha, \frac{1}{2} + \alpha, b_1, \ldots, b_s; \end{array} \right] x \left( \begin{array}{c} \frac{4xt}{(1-t)^2} \end{array} \right) \]

For \( \alpha = \beta \), we get the following ultraspherical type generalization of Sister Celine’s polynomials

\[ \sum_{n=0}^{\infty} \frac{(1 + 2\alpha)_n}{n!} \, 2^{r+s} F_{2+s} \left[ \begin{array}{c} -n, n + 2\alpha + 1, a_1, \ldots, a_r; \\ 1 + \alpha, \frac{1}{2} + \alpha, b_1, \ldots, b_s; \end{array} \right] x \left( \begin{array}{c} t^n \end{array} \right) \]

\[ = \frac{1}{(1 - t)^{1+2\alpha}} \, 2^s F_s \left[ \begin{array}{c} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} \right] \left( \begin{array}{c} \frac{4xt}{(1-t)^2} \end{array} \right) \] (2.2.5)

**Case(iv)** For \( c = 1 + \alpha + \beta \) and \( r = s = 0 \), we have

\[ \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta)_n}{n!} \, 2 F_2 \left[ \begin{array}{c} -n, n + 1 + \alpha + \beta; \\ 1 + \alpha, \frac{1}{2} + \alpha; \end{array} \right] x \left( \begin{array}{c} t^n \end{array} \right) \]

\[ = \frac{1}{(1 - t)^{1+\alpha+\beta}} \, 2 F_2 \left[ \begin{array}{c} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ 1 + \alpha, \frac{1}{2} + \alpha; \end{array} \right] \left( \begin{array}{c} \frac{4xt}{(1-t)^2} \end{array} \right) \] (2.2.6)
Proof

\[ \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)n}{n!} \binom{-n, n+1+\alpha+\beta}{1+\alpha, \frac{1}{2}+\alpha} x^n = \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n+1+\alpha+\beta)_k x^k}{(1+\alpha)_k(1/2+\alpha)_k k!} t^n \]

\[ = \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)n}{n!} \sum_{k=0}^{n} \frac{(-1)^kn! (n+1+\alpha+\beta)_k x^k}{(n-k)! (1+\alpha)_k(k/2+\alpha)_k k!} t^n \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)n+2k}{n!(1+\alpha)_k(1/2+\alpha)_k} \frac{(-xt)^k}{k!} t^n \]

\[ = \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)2k}{(1+\alpha)_k(1/2+\alpha)_k k!} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+2k)_n}{n!} t^n \]

\[ = (1-t)^{-1-\alpha-\beta} \sum_{k=0}^{\infty} \frac{\binom{1+\alpha+\beta}{1/2}_k \binom{2+\alpha+\beta}{1/2}_k}{(1+\alpha)_k(1/2+\alpha)_k k!} \frac{(-4xt)^k}{(1-t)^{2k}} \]

\[ = \frac{1}{(1-t)^{1+\alpha+\beta}} 2F_2 \left[ \begin{array}{c} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2} \\ 1+\alpha, \frac{1}{2}+\alpha \end{array} \right] \frac{4xt}{(1-t)^2} \]

Case (v) when \( \alpha = \beta = r = s = 0 \), we have the following form of Sister Celine's Polynomial

\[ \sum_{n=0}^{\infty} 2F_2 \left[ \begin{array}{c} -n, n+1; \\ 1, \frac{1}{2} \end{array} \right] x^n = \frac{1}{1-t} \exp \left[ -\frac{4xt}{(1-t)^2} \right] \quad (2.2.7) \]

2.2.2 The Bateman's Polynomials

In 1999, M. A. Khan and A. K. Shukla [102] defined the Jacobi type generalization of the Bateman's polynomials as follows:
Chapter 2: Jacobi Type and Gegenbauer Type Generalization...

\[ F_{n}^{(\alpha,\beta)}(p, z) = \binom{-n, n + \alpha + \beta + 1, 1/2(1 + z)}{1 + \alpha, \beta}. \]  \hspace{1cm} (2.2.8)

We define the generating function for the Jacobi type generalization of Bateman’s polynomials defined by Khan and Shukla \cite{102}.

\[
\sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} F_{n}^{(\alpha,\beta)}(p, z) t^{n} = \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=0}^{n} \frac{(-n)_{k}(n + \alpha + \beta + 1)_{k}(1/2(1 + z))_{k} t^{n}}{(1 + \alpha)_{k}(p)_{k}k!}
\]

\[ = \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=0}^{n} \frac{(-1)^{k}(1)_{n+\alpha+\beta+k} \left(\frac{1}{2}(1 + z)\right)_{k} t^{n}}{(n - k)! (1 + \alpha)_{k}(p)_{k}k!}
\]

\[ = \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \frac{\left(\frac{1}{2}(1 + z)\right)_{k} t^{n+2k}}{n! (1 + \alpha)_{k}(p)_{k}k!}
\]

\[ = \sum_{k=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \frac{\left(\frac{1}{2}(1 + z)\right)_{k} \left(\frac{1}{2}(1 + z)\right)_{k} (-t)^{k}}{(1 + \alpha)_{k}(p)_{k}k!}
\]

\[ = \frac{1}{(1 - t)^{1 + \alpha + \beta}} \sum_{k=0}^{\infty} \frac{2^{2k} \left(\frac{1 + \alpha + \beta}{2}\right)_{k} \left(\frac{2 + \alpha + \beta}{2}\right)_{k} \left(\frac{1}{2}(1 + z)\right)_{k} (-t)^{k}}{(1 + \alpha)_{k}(p)_{k}k!}
\]

\[ = \frac{1}{(1 - t)^{1 + \alpha + \beta}} 3F_{2} \left[ \begin{array}{c} \frac{1 + \alpha + \beta}{2}, \frac{2 + \alpha + \beta}{2}, 1/2(1 + z); \\ 1 + \alpha, p; \end{array} \right] - \frac{4t}{(1 - t)^{2}}. \]  \hspace{1cm} (2.2.9)

Another kind of generating function can also be defined for the Jacobi type generalization of the Bateman’s polynomials given below:

\[
\sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} [F_{n}^{(\alpha,\beta)}(p, z - 2) - F_{n}^{(\alpha,\beta)}(p, z)] t^{n}
\]
\[ \begin{align*}
&= \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=0}^{n} \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(1 + \alpha)_k (p)_k k!} \left\{ \left( \frac{1}{2} (1 + z) \right)_k - \left( \frac{1}{2} (1 + z) \right)_{k-1} \right\} t^n \\
&= \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=1}^{n} \frac{(-1)^k (1)_n (n + \alpha + \beta + 1)_k}{(n - k)! (1 + \alpha)_k (p)_k k!} \left\{ (-k) \left( \frac{1}{2} (1 + z) \right)_{k-1} \right\} t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1} (1)_{n+\alpha+\beta+k}}{(n-k)!} \frac{\left( \frac{1}{2} (1 + z) \right)_{k-1} t^n}{(1)_{\alpha+\beta}(1 + \alpha)_k (p)_{k} (k - 1)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^{k+2} (1)_{n+\alpha+\beta+k+1}}{(n-k-1)!} \frac{\left( \frac{1}{2} (1 + z) \right)_k t^{n-1}}{(1)_{\alpha+\beta}(1 + \alpha)_{k+1} (p)_{k+1} k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (1)_{n+\alpha+\beta+2k}}{(n-2)!} \frac{\left( \frac{1}{2} (1 + z) \right)_k t^n}{(1)_{\alpha+\beta}(1 + \alpha)_{k+1} (p)_{k+1} k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (1)_{n+\alpha+\beta+2k+2}}{n!} \frac{\left( \frac{1}{2} (1 + z) \right)_k t^{n+k}}{(1)_{\alpha+\beta}(1 + \alpha)_{k+1} (p)_{k+1} n! k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1 + \alpha + \beta)_2 (3 + \alpha + \beta)_{n+2k} (1/2 (1 + z))_k (-t)^k t^n}{(1 + \alpha)_{k+1} (p)_{k+1} n! k!} \\
&= (1 + \alpha + \beta)_2 \sum_{k=0}^{\infty} \frac{(3 + \alpha + \beta)_{2k} (1/2 (1 + z))_k (-t)^k t^n}{(1 + \alpha)_{k+1} (p)_{k+1} n! k!} \sum_{n=0}^{\infty} \frac{(3 + \alpha + \beta + 2k) t^n}{n!} \\
&= (1 + \alpha + \beta)_2 \sum_{k=0}^{\infty} \frac{2^{2k} (3 + \alpha + \beta)_k (4 + \alpha + \beta)_k (1/2 (1 + z))_k (-t)^k}{(1 + \alpha)_{2k} (2 + \alpha)_k (p)_{2k} (p + 1) k!} \frac{1}{(1 - t)^{2k}} \\
&= \frac{(1 + \alpha + \beta)_2}{(1 - t)^{3 + \alpha + \beta} (1 + \alpha)_p} \sum_{k=0}^{\infty} \frac{2^{2k} (3 + \alpha + \beta)_k (4 + \alpha + \beta)_k (1/2 (1 + z))_k (-t)^k}{(1 + \alpha)_{2k} (2 + \alpha)_k (p)_{2k} (p + 1) k!} \left[ \frac{3 + \alpha + \beta}{2}, \frac{4 + \alpha + \beta}{2}, \frac{1}{2} (1 + z) ; -\frac{4t}{(1 - t)^2} \right] \right] \left[ 2 + \alpha, p + 1 \right] \\
&= (2.2.10)
\]
For $\alpha = \beta$, we get ultraspherical type generalization of Bateman’s polynomials

$$F^{(\alpha,\alpha)}_n(p, z) = \left[ \begin{array}{c} -n, \ n + 2\alpha + 1, \ \frac{1}{2}(1 + z); \\ 1 + \alpha, \ p; \end{array} \right] (2.2.11)$$

Generating function for the ultraspherical generalization of Bateman’s polynomials is given below:

$$\sum_{n=0}^{\infty} \frac{(1+n)^{2\alpha}}{(1)^{2\alpha}} \left\{ F^{(\alpha,\alpha)}_n(p, z-2) - F^{(\alpha,\alpha)}_n(p, z) \right\} t^n$$

$$= \left( \frac{1 + 2\alpha}{1-t} \right)^{3+2\alpha(1+\alpha)} \left[ \begin{array}{c} \frac{3+2\alpha}{2}, \ \frac{1}{2}(1+z); \\ p+1, \end{array} \right] (2.2.12)$$

### 2.2.3 The Pasternack’s Polynomial

Khan and Shukla [102] obtained Jacobi type generalization of Pasternack’s generalized Bateman’s polynomial $F^{m}_n(z)$, given below:

$$F^{(\alpha,\beta)}_{n,m}(z) = \left[ \begin{array}{c} -n, \ n + \alpha + \beta + 1, \ \frac{1}{2}(1 + z + m); \\ 1 + \alpha, \ 1 + m; \end{array} \right] (2.2.13)$$

We derive here the generating function of the Jacobi type generalization of Pasternack’s generalized Bateman’s polynomials obtained by Khan and Shukla [102] given below:

$$\sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} F^{(\alpha,\beta)}_{n,m}(z)t^n = \sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(-1)^k(n + \alpha + \beta + 1)k(\frac{1}{2}(1 + z + m))_k t^n}{(1 + \alpha)_k(1 + m)_k k!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+n)_{\alpha+\beta}}{(1)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(-1)^k(1)_n (n + \alpha + \beta + 1)k(\frac{1}{2}(1 + z + m))_k t^n}{(n - k)! (1 + \alpha)_k(1 + m)_k k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(1)_n(\alpha+\beta+k) (\frac{1}{2}(1 + z + m))_k t^n}{(n - k)! (1)_{\alpha+\beta}(1 + \alpha)_k(1 + m)_k k!}$$
Chapter 2: Jacobi Type and Gegenbauer Type Generalization

\[
\begin{align*}
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1)_{n+\alpha+\beta+2k}}{n!} \frac{\left(\frac{1}{2}(1+z+m)\right)_k t^{n+k}}{(1)_{\alpha+\beta}(1+\alpha)_k(1+m)_k k!} &= \\
&= \sum_{k=0}^{\infty} \frac{(1)_{\alpha+\beta}(1+\alpha+\beta)2k(1/2(1+z+m))_k (-t)^k}{(1)_{\alpha+\beta}(1+\alpha)_k(1+m)_k k!} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+2k)_nt^n}{n!} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{2^k (1+\alpha+\beta)_{k/2} (1+\alpha+\beta+2k)_{k/2} (1/2(1+z+m))_k (-t)^k}{(1+\alpha)_k(1+m)_k k!} \frac{1}{(1-t)^{2k}} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}} 3F_2 \left[ \begin{array}{c}
\frac{1+\alpha+\beta}{2}, \frac{1+\alpha+\beta+2\alpha}{2}, \frac{1}{2}(1+z+m); \\
1+\alpha, 1+m;
\end{array} \right] - \frac{4t}{(1-t)^2} \right]. \tag{2.2.14}
\end{align*}
\]

Substitution of \( \alpha = \beta \) in the Jacobi type generalization of Pasternack's generalized Bateman's polynomial gives us its ultraspheriacal type generalization

\[
F_{n,m}^{(\alpha,\alpha)}(z) = 3F_2 \left[ \begin{array}{c}
-n, n + 2\alpha + 1, \frac{1}{2}(1+z+m); \\
1+\alpha, 1+m;
\end{array} \right]. \tag{2.2.15}
\]

The following is the generating function of the ultraspherical type generalized of the polynomial (2.2.15)

\[
\sum_{n=0}^{\infty} \frac{(1+n)_{2\alpha}}{(1)_{2\alpha}} F_{n,m}^{(\alpha,\alpha)}(z) t^n \\
= \frac{1}{(1-t)^{1+2\alpha}} 2F_1 \left[ \begin{array}{c}
\frac{1+2\alpha}{2}, \frac{1}{2}(1+z+m); \\
1+m;
\end{array} \right] - \frac{4t}{(1-t)^2} \right]. \tag{2.2.16}
\]
2.2.4 The Bateman’s Polynomial $Z_n(x)$

Jacobi type generalization of Bateman’s polynomials $Z_n(x)$ and was considered to be new in the last decade. For $\alpha = \beta = \nu - \frac{1}{2}$, it reduces to Gegenbauer generalization of the Bateman’s polynomials and also for $\alpha = \beta = 0$, reduces to Bateman’s polynomial. Khan and Shukla [102] adopted the symbol $Z_n^{(\alpha, \beta)}(b, x)$ to denote the Jacobi type generalization of Bateman’s polynomial $Z_n(x)$.

$$Z_n^{(\alpha, \beta)}(b, x) = {}_2F_2 \left[ \begin{array}{c} -n, n + \alpha + \beta + 1; \\ 1 + \alpha, b + 1; \end{array} \right] x.$$  \hspace{1cm} (2.2.17)

We determine the following generating function for the polynomial $Z_n^{(\alpha, \beta)}(b, x)$:

$$\sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} Z_n^{(\alpha, \beta)}(b, x) t^n = \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=0}^{n} \frac{(-n)_k (n + \alpha + \beta + 1)_k x^k t^n}{(1 + \alpha)_k (b + 1)_k k!}$$

$$= \sum_{n=0}^{\infty} \frac{(1 + n)_{\alpha + \beta}}{(1)_{\alpha + \beta}} \sum_{k=0}^{n} \frac{(-1)^k (1)_{n + \alpha + \beta + k}}{(n - k)!} \frac{(n + \alpha + \beta + 1)_k x^k t^n}{(1 + \alpha)_k (b + 1)_k k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (1)_{n + \alpha + \beta + 2k}}{n!} \frac{x^k t^n}{(1 + \alpha)_k (b + 1)_k k!}$$

$$= \sum_{k=0}^{\infty} \frac{(1 + \alpha + \beta + 2k)_k (1 + \alpha + \beta)_k (2 + 2k)_k (-xt)^k}{(1 + \alpha)_k (b + 1)_k k!} \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta + 2k)_k (1 + \alpha + \beta)_k (2 + 2k)_k (-xt)^k}{n!} t^n$$

$$= \frac{1}{(1 - t)^{1 + \alpha + \beta}} \sum_{k=0}^{\infty} \frac{2^{2k} (1 + \alpha + \beta)_k (2 + 2k)_k (1 - xt)^k}{(1 + \alpha)_k (b + 1)_k k!} \frac{1}{(1 - t)^{2k}}$$

$$= \frac{1}{(1 - t)^{1 + \alpha + \beta}} {}_2F_2 \left[ \begin{array}{c} \frac{1 + \alpha + \beta}{2}, \frac{2 + \alpha + \beta}{2}; \\ 1 + \alpha, b + 1; \end{array} \right] \frac{-4xt}{(1-t)^2}.$$  \hspace{1cm} (2.2.18)
The generating function given below establishes a relation between $F_{m,n}^{(\alpha,\beta)}$ and $Z_n^{(\alpha,\beta)}(m, t)$:

$$F_{m,n}^{(\alpha,\beta)}(z) = {}_3F_2\left[\begin{array}{c}
-m, \ m + \alpha + \beta + 1, \ \frac{1}{2}(1 + z + n); \\
1 + \alpha, \ 1 + m;
\end{array} \bigg| 1 \right]$$

$$\sum_{n=0}^{\infty} F_{m,n}^{(\alpha,\beta)}(-2n-1) \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-m)_k(m + \alpha + \beta + 1)_k(-1)^k(1)_n(-t)^n}{(1 + \alpha)_k(1 + m)_k k! n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-m)_k(m + \alpha + \beta + 1)_k(-1)^k(1)_n(-t)^n}{(1 + \alpha)_k(1 + m)_k k! n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{k=0}^{\infty} \frac{(-m)_k(m + \alpha + \beta + 1)_k t^k}{(1 + \alpha)_k(1 + m)_k k!}$$

$$= e^{-t} {}_2F_2\left[\begin{array}{c}
-m, \ m + \alpha + \beta + 1; \\
1 + \alpha, \ 1 + m;
\end{array} \bigg| t \right] = e^{-t} Z_m^{(\alpha,\beta)}(m, t)$$

For $\alpha = \beta$, we get the ultraspherical type generalization of it, as given below:

$$F_{m,n}^{(\alpha,\alpha)}(z) = {}_3F_2\left[\begin{array}{c}
-m, \ m + 2\alpha + 1, \ \frac{1}{2}(1 + z + n); \\
1 + \alpha, \ 1 + m;
\end{array} \bigg| 1 \right]$$

$$\sum_{n=0}^{\infty} F_{m,n}^{(\alpha,\alpha)}(-2n-1) \frac{(-t)^n}{n!}$$
\[
= e^{-t} \binom{2F2}{-m, m + 2\alpha + 1; \ 1 + \alpha, 1 + m; \ t} = e^{-t} Z_{m}^{(\alpha, \alpha)}(m, t) \quad (2.2.22)
\]

In particular, for \(\alpha = \beta\), we obtain the ultraspherical generalization of Bateman's polynomial due to Khan and Shukla [102]

\[
Z_{n}^{(\alpha, \alpha)}(b, x) = \binom{2F2}{-n, n + 2\alpha + 1; \ 1 + \alpha, b + 1; \ x}. \quad (2.2.23)
\]

Generating function for the ultraspherical generalized Bateman's polynomials is given below:

\[
\sum_{n=0}^{\infty} \frac{(1 + n)^{2\alpha}}{(1)_{2\alpha}} Z_{n}^{(\alpha, \alpha)}(b, x) t^n = \frac{1}{(1 - t)^{1 + 2\alpha}} \binom{1 + 2\alpha}{b + 1; \ -\frac{4xt}{(1-t)^2}} \quad (2.2.24)
\]

Substitution of \(\alpha = \beta\) reduces Khandekar's polynomial to ultraspherical type generalization of Rice's polynomials, therefore, we have

\[
\frac{n!}{(1 + \alpha)_n} H_{n}^{(\alpha, \alpha)}(\xi, p, \nu) = \binom{3F2}{-n, n + 2\alpha + 1, \xi; \ 1 + \alpha, p; \ \nu}. \quad (2.2.25)
\]

Generating functions for the ultraspherical type generalized Rice's polynomial is given below:

\[
\sum_{n=0}^{\infty} \frac{(2\alpha + 1)_n}{(1 + \alpha)_n} \frac{H_{n}^{(\alpha, \alpha)}(\xi, p, \nu)}{t^n}
\]

\[
= \sum_{n=0}^{\infty} \frac{(2\alpha + 1)_n (1 + \alpha)_n}{(1 + \alpha)_n} \sum_{k=0}^{n} \frac{(-n)_k (n + 2\alpha + 1)_k (\xi)_k \nu^k t^n}{(1 + \alpha)_k (p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2\alpha + 1)_n (n + 2\alpha + 1)_k (\xi)_k \nu^k t^n}{(n - k)! (1 + \alpha)_k (p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2\alpha + 1)_n (\xi)_k \nu^k t^n}{(n - k)! (1 + \alpha)_k (p)_k k!}
\]
Chapter 2: Jacobi Type and Gegenbauer Type Generalization....

\[\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\alpha + 1)_n 2k(\xi)_k (-\nu t)^k t^n}{n!(1 + \alpha)_k(p)_k k!} = \sum_{k=0}^{\infty} \frac{(2\alpha + 1)_k(\xi)_k (-\nu t)^k}{(1 + \alpha)_k(p)_k k!} \sum_{n=0}^{\infty} \frac{(2\alpha + 1 + 2k)t^n}{n!}\]

\[= \sum_{k=0}^{\infty} \frac{2^{2k}(1/2 + \alpha)_k(1 + \alpha)_k(\xi)_k (-\nu t)^k}{(1 + \alpha)_k(p)_k k!} \frac{1}{(1 - t)^{2\alpha + 2}}\]

\[= \frac{1}{(1 - t)^{2\alpha + 1}} \sum_{k=0}^{\infty} \frac{(1/2 + \alpha)_k(\xi)_k (-4\nu t)^k}{(p)_k k!(1 - t)^{2k}}\]

\[= \frac{1}{(1 - t)^{2\alpha + 1}} \binom{1/2 + \alpha}{p} \binom{-4\nu t}{(1 - t)^2} . \]

Ultraspherical type generalization of the Hahn polynomial is given below:

\[Q_n(x; \alpha, \alpha, N) = \binom{1 + \alpha}{p} \binom{-N}{N} . \]

Polynomials given by (2.2.27), satisfy the following generating functions

\[\sum_{n=0}^{\infty} \frac{(1 + 2\alpha)_n}{n!} Q_n(x; \alpha, \alpha, N)t^n = \frac{1}{(1 - t)^{2\alpha + 1}} \binom{1/2 + \alpha}{p} \binom{-N}{N} . \]

2.3 Gegenbauer Type Generalization of Certain Polynomials and Their Generating Functions

In this section we determine Gegenbauer type generalization of the polynomials mentioned in the introduction section of this chapter.
2.3.1 The Sister Celine's Polynomial

Let us define Gegenbauer type generalization of the Sister Celine’s polynomials is given below:

\[ f^\nu_n(a_1, \cdots, a_r; b_1, \cdots, b_s; x) \]

\[
= \frac{(2\nu)^n}{n!} 2_{r+2} F_{2+s} \left[ \begin{array}{c} -n, n + 2\nu, a_1, \cdots, a_r; \\ \frac{1}{2} + \nu, b_1, \cdots, b_s; \\ x \end{array} \right].
\]  \hspace{1cm} (2.3.1)

Generating function of the Gegenbauer type generalization of the Sister Celine’s polynomial is given below:

\[
\sum_{n=0}^{\infty} \frac{(2\nu)^n}{n!} 2_{r+2} F_{2+s} \left[ \begin{array}{c} -n, n + 2\nu, a_1, \cdots, a_r; \\ \frac{1}{2} + \nu, b_1, \cdots, b_s; \\ x \end{array} \right] t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(2\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)^{n+k} (a_1)_k \cdots (a_r)_k \cdot x^k t^n}{(n-k)!(\frac{1}{2} + \nu)_k (b_1)_k \cdots (b_s)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)^{n+2k} (a_1)_k \cdots (a_r)_k \cdot x^{k+n+2k}}{k!(\frac{1}{2} + \nu)_k (b_1)_k \cdots (b_s)_k n!}
\]

\[
= \sum_{k=0}^{\infty} \frac{2^{2k} (a_1)_k \cdots (a_r)_k (\frac{-xt}{2})^k}{k!(b_1)_k \cdots (b_s)_k} \sum_{n=0}^{\infty} \frac{(2\nu + 2k)_n t^n}{n!}
\]

\[
= \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k (\frac{-4xt}{(1-t)^2})^k}{k!(b_1)_k \cdots (b_s)_k}
\]

\[
= \frac{1}{(1-t)^{2\nu}} \, r_{F_s} \left[ \begin{array}{c} a_1, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} \right] \frac{-4xt}{(1-t)^2}
\]  \hspace{1cm} (2.3.2)
2.3.2 The Bateman’s Polynomial

The Gegenbauer type generalization of the Bateman’s polynomial \( F_n(z) \) is given below:

\[
F_n^{(\nu)}(p, z) = 3F_2 \left[ \begin{array}{c}
-n, n + 2\nu, \frac{1}{2}(1 + z) \\
\nu + \frac{1}{2}, p
\end{array} \right]. (2.3.3)
\]

One of the generating functions of the polynomials \((2.3.3)\) is determined below:

\[
\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} F_n^{(\nu)}(p, z) t^n = \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (n + 2\nu)_k (\frac{1}{2}(1 + z))_k t^n}{(\nu + \frac{1}{2})_k (p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n! (n + 2\nu)_k (\frac{1}{2}(1 + z))_k t^n}{(n - k)! (\nu + \frac{1}{2})_k (p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2\nu)_{n+k} (\frac{1}{2}(1 + z))_k t^n}{n! (\nu + \frac{1}{2})_k (p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)_{n+2k} (\frac{1}{2}(1 + z))_k t^{n+k}}{n! (\nu + \frac{1}{2})_k (p)_k k!}
\]

\[
= \frac{1}{(1 - t)^{2\nu}} \sum_{k=0}^{\infty} \frac{2^{2k}(\nu)_k (\nu + \frac{1}{2})_k (\frac{1}{2}(1 + z))_k (-t)^k}{(\nu + \frac{1}{2})_k (p)_k k!} \frac{1}{(1 - t)^{2k}}
\]

\[
= \frac{1}{(1 - t)^{2\nu}} 2F_1 \left[ \begin{array}{c}
\nu, \frac{1}{2}(1 + z) \\
p
\end{array} \right] - \frac{4t}{(1-t)^2} \right]. (2.3.4)
\]

Another generating function of the polynomial \((2.3.3)\) is obtained below:

\[
\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \{F_n^{(\nu)}(p, z - 2) - F_n^{(\nu)}(p, z)\} t^n
\]
Chapter 2: Jacobi Type and Gegenbauer Type Generalization.

\[
= \sum_{n=0}^{\infty} \frac{(2\nu)^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n + 2\nu)_k}{(\nu + \frac{1}{2})_k(p)_k k!} \left\{ \left( \frac{1}{2}(-1 + z) \right)_k - \left( \frac{1}{2}(1 + z) \right)_k \right\} t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=1}^{n} \frac{(-1)^k n! (n + 2\nu)_k}{(n - k)! (\nu + \frac{1}{2})_k(p)_k k!} \left\{ (-k) \left( \frac{1}{2}(1 + z) \right)_{k-1} \right\} t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(n-k)! (\nu + \frac{1}{2})_k(p)_k(k-1)!} \left\{ \left( \frac{1}{2}(1 + z) \right)_{k-1} \right\} t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+2}(2\nu)_{n+k}}{(n-k-1)! (\nu + \frac{3}{2})_{k+1}(p)_{k+1}k!} (\frac{1}{2}(1 + z))_{k} t^{n-k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(2\nu)_{n+2k-1}}{(n-2)! (\nu + \frac{3}{2})_{k+1}(p)_{k+1}k!} (\frac{1}{2}(1 + z))_{k} t^{n+k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(2\nu)_{n+2k}}{n! (\nu + \frac{3}{2})_{k+1}(p)_{k+1}k!} (\frac{1}{2}(1 + z))_{k} t^{n+k}
\]

\[
= \sum_{k=0}^{\infty} \frac{(2\nu)(2\nu + 1)_{2k}(\frac{1}{2}(1 + z))_{k}(-t)^k}{(\nu + \frac{3}{2})_{k+1}(p)_{k+1}k!} \sum_{n=0}^{\infty} \frac{(2\nu + 2k + 1)_n t^n}{n!}
\]

\[
= \frac{(2\nu)}{(1 - t)^{1+2\nu}} \sum_{k=0}^{\infty} \frac{2^{2k}(\nu + \frac{1}{2})_k(\nu + 1)_k(\frac{1}{2}(1 + z))_k(-t)^k}{(\nu + \frac{3}{2})_{k+1}(p)_{k+1}k!} \frac{1}{(1 - t)^{2k}}
\]

\[
= \frac{(2\nu)}{(1 - t)^{1+2\nu}(\nu + \frac{1}{2})(p)} 3F_2 \left[ \begin{array}{c} \nu + \frac{1}{2}, \nu + 1, \frac{1}{2}(1 + z); \\ \nu + \frac{3}{2}, p + 1; \end{array} \right| \frac{-4t}{(1-t)^2} \right].
\]

Another Gegenbauer Type generalization of Bateman's Polynomials \( Z_n(b, x) \) is given below:

\[
Z_n^{(\nu)}(b, x) = 2F_2 \left[ \begin{array}{c} -n, n + 2\nu; \\ \nu + \frac{1}{2}, 1 + b; \end{array} \right| \frac{1}{x} \right].
\]
Chapter 2: Jacobi Type and Gegenbauer Type Generalization

Gegenbauer type generalized Bateman’s polynomial (2.3.6) satisfies the following generating function

$$
\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} G_n^{(\nu)}(b, x)t^n = \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n + 2\nu)_k x^k t^n}{(\nu + \frac{1}{2})_k(1 + b)_k k!}
$$

$$
= \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n - k)!} \frac{(n + 2\nu)_k x^k t^n}{(\nu + \frac{1}{2})_k(1 + b)_k k!}
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(2\nu)_{n+k}}{n!} \frac{x^k t^{n+k}}{(\nu + \frac{1}{2})_k(1 + b)_k k!}
$$

$$
= \sum_{n=0}^{\infty} \frac{(2\nu)_{2k}(-xt)^k}{(\nu + \frac{1}{2})_k(1 + b)_k k!} \sum_{k=0}^{\infty} \frac{(2\nu + 2k)_n t^n}{n!}
$$

$$
= \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{2^{2k}(\nu + \frac{1}{2})_k(-xt)^k}{(\nu + \frac{1}{2})_k(1 + b)_k k!} \frac{1}{(1-t)^{2k}}
$$

$$
= \frac{1}{(1-t)^{2\nu}} \left[ \frac{\nu;}{1 + b; - \frac{4t}{(1-t)^2}} \right]. \quad (2.3.7)
$$

2.3.3 The Pasternack’s Polynomial

We define here the Gegenbauer Type Generalization of Pasternack’s generalization of the Bateman’s Polynomial $F^{(m)}_{\nu}(p, z)$ as given below:

$$
F^{(\nu)}_{n,m}(p, z) = _3F_2 \left[ \begin{array}{c}
-n, \ n + 2\nu, \ \frac{1}{2}(z + m + 1); \\
\nu + \frac{1}{2}, \ p;
\end{array} \right]. \quad (2.3.8)
$$

For the above polynomial (2.3.8), we easily obtain the following generating function relation

$$
\sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} F^{(\nu)}_{n,m}(p, z)t^n = \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n + 2\nu)_k(\frac{1}{2}(z + m + 1))_k t^n}{(\nu + \frac{1}{2})_k(p)_k k!}
$$
Chapter 2: Jacobi Type and Gegenbauer Type Generalization

Another generating function for the polynomial (2.3.6) is obtained as given below:

$$F_{m,n}^{(\nu)}(z) = \, 3 \, _2F_1 \left[ \begin{array}{c} -m, \, m + 2\nu, \, \frac{1}{2}(1 + z + n); \\ \nu + \frac{1}{2}, \, 1 + m; \end{array} \right]$$

(2.3.10)

$$\sum_{n=0}^{\infty} F_{m,n}^{(\nu)}(-2n - 1) \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-m)_k(m + 2\nu)_k(-n)_k(-t)^n}{(\nu + \frac{1}{2})_k(1 + m)_k k! n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-m)_k(m + 2\nu)_k (-1)_k(n)_k(-t)^n}{(\nu + \frac{1}{2})_k(1 + m)_k k! (n - k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-m)_k(m + 2\nu)_k (-1)^{n+k}(-t)^n}{(\nu + \frac{1}{2})_k(1 + m)_k k! n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{k=0}^{\infty} \frac{(-m)_k(m + 2\nu)_k t^k}{(\nu + \frac{1}{2})_k(1 + m)_k k!}$$
\[ e^{-t} \binom{2}{2} \begin{array}{c} -m, m + 2\nu; \\ \nu + \frac{1}{2}, 1 + m; \end{array} t \right] = e^{-t} Z^{(\nu)}_m(m, t). \quad (2.3.11) \]

### 2.3.4 The Rice’s Polynomials

With the steps taken by Khandekar [178] in the definition of Jacobi type generalization of Rice’s polynomials, we define Gegenbauer generalization of Rice’s polynomials as follows:

\[ \frac{n!}{(\nu + \frac{1}{2})^n} H^{(\nu)}_n(\xi, p, \nu) = \binom{3}{2} \begin{array}{c} -n, n + 2\nu, \xi; \\ \nu + \frac{1}{2}, p; \end{array} \nu \right]. \quad (2.3.12) \]

We now find the generating function of Gegenbauer type generalization of the Rice’s polynomial as follows:

\[
\sum_{n=0}^{\infty} \frac{(2\nu)_n}{(\nu + \frac{1}{2})^n} H^{(\nu)}_n(\xi, p, \nu)t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(2\nu)_n}{(\nu + \frac{1}{2})^n} \frac{(\nu + \frac{1}{2})^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n + 2\nu)_k(\xi)_k\nu^k t^n}{(\nu + \frac{1}{2})_k(p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (2\nu)_n(n + 2\nu)_k(\xi)_k\nu^k t^n}{(n - k)! (\nu + \frac{1}{2})_k(p)_k k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (2\nu)_{n+k}(\xi)_k(-\nu t)^k t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\nu)_{n+2k}(\xi)_k(-\nu t)^k t^n}{n!(\nu + \frac{1}{2})_k(p)_k k!}
\]

\[
= \sum_{k=0}^{\infty} \frac{(2\nu)_{2k}(\xi)_k(-\nu t)^k}{(\nu + \frac{1}{2})_k(p)_k k!} \sum_{n=0}^{\infty} \frac{(2\nu + 2k)t^n}{n!}
\]

\[
= \sum_{k=0}^{\infty} \frac{2^{2k}(\nu)_{k}(\nu + \frac{1}{2})_k(\xi)_k(-\nu t)^k}{(\nu + \frac{1}{2})_k(p)_k k!} \frac{1}{(1 - t)^{2\nu + 2k}}
\]
\[ \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(\nu)_k(\xi)_k(-4\nu t)^k}{(p)_k k! (1-t)^{2k}} \]

\[ = \frac{1}{(1-t)^{2\nu}} {}_2F_1 \left[ \frac{\nu}{p}; -\frac{4\nu t}{(1-t)^2} \right] \]  \quad (2.3.13)

### 2.3.5 The Hahn Polynomials

We define the Gegenbauer type generalization of Hahn’s polynomial as given below:

\[ Q_n^{(\nu)}(x; N) = {}_3F_2 \left[ \begin{array}{c} -n, n + 2\nu, -x; \\ \frac{1}{2} + \nu, -N; \end{array} 1 \right] . \]  \quad (2.3.14)

The following generating function is satisfied by the Hahn’s Polynomial:

\[ \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} Q_n^{(\nu)}(x; N) t^n = \sum_{n=0}^{\infty} \frac{(2\nu)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n + 2\nu)_k(-x)_k t^n}{(\frac{1}{2} + \nu)_k(-N)_k k!} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(2\nu)_{n+k}(-x)_k t^n}{(n-k)!(\frac{1}{2} + \nu)_k(-N)_k k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(2\nu)_{n+2k}(-x)_k t^{n+k}}{n!(\frac{1}{2} + \nu)_k(-N)_k k!} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k(2\nu)_{2k}(-x)_k t^k}{(\frac{1}{2} + \nu)_k(-N)_k k!} \sum_{n=0}^{\infty} \frac{(2\nu + 2k)_n t^n}{n!} \]

\[ = \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{2^{2k}(\nu)_{\frac{1}{2} + \nu}_k(-x)_k(-t)^k}{(\frac{1}{2} + \nu)_k(-N)_k k!(1-t)^{2k}} \]

\[ = \frac{1}{(1-t)^{2\nu}} {}_2F_1 \left[ \frac{\nu}{-x}; -\frac{4t}{(1-t)^2} \right] . \]  \quad (2.3.15)
Chapter 3

On Some Generating Functions
Of $q$-Analogues Of Jacobi Type
And Gegenbauer Type
Generalized Polynomials

ABSTRACT: The aim of the present paper is to determine $q$-analogue of the Jacobi type and Gegenbauer type generalizations of certain polynomials. Moreover, $q$-analogue of their generating functions have also been established.

3.1 Introduction

In the chapter 2, motivated by the work done by Khandekar [178] in the Jacobi type generalization of the Rice's polynomials, we have studied Jacobi type and Gegenbauer type generalization of certain polynomials and their generating functions.

The present chapter deals with $q$-analogues of the polynomials studied in the chapter 2. Thus, this chapter has been divided into two broad sections. The section 3.2 of this chapter deals $q$-analogues of the Jacobi type generalized polynomials and their generating functions relation while section 3.3
and their generating functions treats $q$-analogues of the Gegenbaur type generalized polynomials.

3.2 The $q$-Analogue of The Jacobi Type Generalization of Certain Polynomials and Their Generating Functions

In view of the definitions, identities available in $q$-analysis [50], we begin with the $q$-analogue of the Jacobi type generalization of the Sister Celine’s polynomial, given below:

3.2.1 The Sister Celine’s Polynomial

The $q$-analogue of the Sister Celine’s polynomials are defined by means of the following generating functions

\[
\frac{1}{(1-t)^c} \phi_{8+s} \left[ \begin{array}{c}
0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, \ldots, q^r
\end{array} \right]
\]

\[
\phi_{4+s} \left[ \begin{array}{c}
q^{1+\alpha}, -q^{1+\alpha}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, (q^c)^{1/2}, -(q^c)^{1/2}, (q^{c+1})^{1/2}, -(q^{c+1})^{1/2}, q^{h_1}, \ldots, q^{h_s}
\end{array} \right]
\]

which produces the following relation

\[
\varphi_n(q^{(a_1)}, \ldots, q^{(a_r)}, q^{(b_1)}, \ldots, q^{(b_s)}; x)
\]

\[
\frac{(q^c; q)_n}{(q; q)_n} \phi_{4+s} \left[ \begin{array}{c}
q^{-n}, q^{n+c}, q^{a_1}, \ldots, q^{a_r}
\end{array} \right] q, q^n x
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} \phi_{4+s} \left[ \begin{array}{c}
q^{1+\alpha}, -q^{1+\alpha}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, q^{b_1}, \ldots, q^{b_s}
\end{array} \right] q, q^n x
\]

(3.2.1)

which produces the following relation

\[
\varphi_n(q^{(a_1)}, \ldots, q^{(a_r)}; q^{(b_1)}, \ldots, q^{(b_s)}; x)
\]

\[
= \frac{(q^c; q)_n}{(q; q)_n} \phi_{4+s} \left[ \begin{array}{c}
q^{-n}, q^{n+c}, q^{a_1}, \ldots, q^{a_r}
\end{array} \right] q, q^n x
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} \phi_{4+s} \left[ \begin{array}{c}
q^{1+\alpha}, -q^{1+\alpha}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, q^{b_1}, \ldots, q^{b_s}
\end{array} \right] q, q^n x
\]

(3.2.2)
Proof Let us rewrite eq. (3.2.2), we have

\[
\sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+c}; q)_k(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^k(\frac{k}{2})^{s+3-r} q^{nk} x^k t^n)}{(q; q)_n-k(q^{1+2a}; q)_2k(q^{b_1}; q)_k \ldots (q^{b_r}; q)_k(q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^c; q)_{n-k}(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^k(\frac{k}{2})^{s+4-r} x^k t^n)}{(q; q)_n-k(q^{1+2a}; q)_2k(q^{b_1}; q)_k \ldots (q^{b_r}; q)_k(q; q)_k}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^c; q)_{2k}(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^k(\frac{k}{2})^{s+4-r} (xt)^k t^n)}{(q^{1+2a}; q)_{2k}(q^{b_1}; q)_k \ldots (q^{b_r}; q)_k(q; q)_k} \sum_{n=0}^{\infty} \frac{(q^{c+2k}; q)_n t^n}{(q; q)_n}
\]

\[
= \frac{1}{(1-t)} \sum_{k=0}^{\infty} \frac{(q^c; q)_{2k}(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^k(\frac{k}{2})^{s+4-r} (xt)^k t^n)}{(q^{1+2a}; q)_{2k}(q^{b_1}; q)_k \ldots (q^{b_r}; q)_k(q; q)_k}
\]

which is the required eq. (3.2.1)

Case (i) For \(\alpha = 0\), it reduces to the following generating function

\[
\frac{1}{(1-t)} 5^r \phi^{8+s} \left[ a, q^c, -q \frac{c}{2}, q^c, q^{c+1}, \ldots, q^{s+c+1}, q^{s+1}, \ldots, q^{s+r} \right]_{q, xt}
\]

\[
\frac{1}{(1-t)} 5^r \phi^{8+s} \left[ a, q^c, -q \frac{c}{2}, q^c, q^{c+1}, \ldots, q^{s+c+1}, q^{s+1}, \ldots, q^{s+r} \right]_{q, xt}
\]
Chapter 3: On Some Generating Functions of \( q \)-Analogues of....

\[
\sum_{n=0}^{\infty} \frac{(q^n; q)_n}{(q; q)_n} \phi_{4+r}^{2+r} \begin{bmatrix}
q^{-n}, q^{n+1}, q^{a_1}, \ldots, q^{a_r} \\
q^{b_1}, -q^{b_2}, q, -q, q^{b_3}, \ldots, q^{b_s}
\end{bmatrix} \psi ; q, q^n x \right) t^n
\]

**Case (ii)** For \( \alpha = 0 \), and \( c = 1 \), it reduces to the following \( q \)-analogue of the generating function of Sister Celine's polynomial

\[
\frac{1}{1-t^{1+r+4}} \begin{bmatrix}
0, q^{a_1}, \ldots, q^{a_r} \\
(qt)^{\frac{1}{2}}, -(qt)^{\frac{1}{2}}, (qt^2)^{\frac{1}{2}}, -(qt^2)^{\frac{1}{2}}, q^{b_1}, \ldots, q^{b_s}
\end{bmatrix}
\]

\[
= \sum_{n=0}^{\infty} r+2\phi_{8+4} \begin{bmatrix}
q^{-n}, q^{n+1}, q^{a_1}, \ldots, q^{a_r} \\
q, -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{b_1}, \ldots, q^{b_s}
\end{bmatrix} \psi ; q, q^n x \right) t^n
\]

**Proof**

\[
\sum_{n=0}^{\infty} r+2\phi_{8+4} \begin{bmatrix}
q^{-n}, q^{n+1}, q^{a_1}, \ldots, q^{a_r} \\
q, -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{b_1}, \ldots, q^{b_s}
\end{bmatrix} \psi ; q, q^n x \right) t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n(-1)^kq\left(\frac{k}{2}\right)^{-nk}(q^{n+1}; q)_k(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k(-1)^kq\left(\frac{k}{2}\right)^{(s-r+3)k} q^{nk} x^k t^n}{(q; q)_k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)n(-1)^kq\left(\frac{k}{2}\right)^{-nk}(q^{n+1}; q)_k(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k(-1)^kq\left(\frac{k}{2}\right)^{(s-r+4)k} x^k t^n}{(q; q)_n-k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)n-k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}{(q; q)_n-k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)n-k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}{(q; q)_n-k(q; q)_k \cdots (q^{b_s}; q)_k(q; q)_k}
\]
Chapter 3: On Some Generating Functions of $q$-Analogues of....  

\[
\begin{align*}
&= \sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^\left(\begin{array}{c} k \\ \frac{k}{2} \end{array}\right) (x t)^k]}{(q^{b_1};q)_k \ldots (q^{b_s};q)_k (q; q)_k} \\
&= \sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^\left(\begin{array}{c} k \\ \frac{k}{2} \end{array}\right) (x t)^k]}{(q^{b_1};q)_k \ldots (q^{b_s};q)_k (q; q)_k} (1 - q^{2k+1}) \infty \frac{1}{(1 - t) \infty}
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{1 - t} \sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^\left(\begin{array}{c} k \\ \frac{k}{2} \end{array}\right) (x t)^k]}{(q^{b_1};q)_k \ldots (q^{b_s};q)_k (q; q)_k}
\end{align*}
\]

\[
\frac{1}{1 - t} 1 + r \phi_{4 + s} \left[ \begin{array}{c}
0, q^{a_1}, \ldots, q^{a_r} \\
(q t)^{\frac{1}{2}}, -(q t)^{\frac{1}{2}}, (q^2 t)^{\frac{1}{2}}, -(q^2 t)^{\frac{1}{2}}, q^{b_1}, \ldots, q^{b_s}
\end{array} \right] ; q, x t
\]

it was to be proved.

**Case (iii)** For $c = 1 + \alpha + \beta$, it gives the $q$-analogue of the Jacobi type generalization of Sister Celine’s polynomials.

\[
\frac{1}{(1 - t)^{1+\alpha+\beta}}
\]

\[
\times 5 + r \phi_{8 + s} \left[ \begin{array}{c}
o, q^{\frac{1}{2}(1+\alpha+\beta)}, -q^{\frac{1}{2}(1+\alpha+\beta)}, q^{\frac{1}{2}(2+\alpha+\beta)}, -q^{\frac{1}{2}(2+\alpha+\beta)}, q^{a_1}, \ldots, q^{a_r} \\
q^{\frac{1}{2}\alpha}, -q^{\frac{1}{2}\alpha}, q^{\frac{1}{2}\alpha}, -q^{\frac{1}{2}\alpha}, (q^{1+\alpha+\beta})^{1/2}, -(q^{1+\alpha+\beta})^{1/2}, (q^{2+\alpha+\beta})^{1/2}, -(q^{2+\alpha+\beta})^{1/2}, \ldots, q^{b_s}
\end{array} \right] ; q, x t
\]

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_n (q; q)_n}{2 + r \phi_{4 + s}} \left[ \begin{array}{c}
q^{-n}, q^{n+1+\alpha+\beta}, q^{a_1}, \ldots, q^{a_r} \\
q^{1+\alpha}, -q^{1+\alpha}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, q^{b_1}, \ldots, q^{b_s}
\end{array} \right] t^n
\end{align*}
\]

\[(3.2.5)\]

**Proof** Substituting $c = 1 + \alpha + \beta$ in the eq. (3.2.2), we have

\[
\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_n}{(q; q)_n} 2 + r \phi_{4 + s} \left[ \begin{array}{c}
q^{-n}, q^{n+1+\alpha+\beta}, q^{a_1}, \ldots, q^{a_r} \\
q^{1+\alpha}, -q^{1+\alpha}, q^{\frac{1}{2}+\alpha}, -q^{\frac{1}{2}+\alpha}, q^{b_1}, \ldots, q^{b_s}
\end{array} \right] t^n
\]
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}, q)_n}{(q; q)_n} \\
&\times \sum_{k=0}^{n} \frac{(q^{-n}, q)_k(q^{n+1+\alpha+\beta}, q)_k(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k((-1)^k q^{k \choose 2})_{s+3-r} q^{nk} x^k t^n}{(q^{1+\alpha}; q)_k(-q^{1+\alpha}; q)_k(q^{\frac{1}{2}+\alpha}; q)_k(-q^{\frac{1}{2}+\alpha}; q)_k(q^{b_1}; q)_k \cdots (q^{b_r}; q)_k(q; q)_k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{1+\alpha+\beta}, q)_n+k(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k((-1)^k q^{k \choose 2})_{s+4-r} x^k t^n}{(q; q)_{n-k}(q^{1+2\alpha}; q)_{2k}(q^{b_1}; q)_k \cdots (q^{b_r}; q)_k(q; q)_k} \\
&= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}, q)_{2k}(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k((-1)^k q^{k \choose 2})_{s+4-r} (xt)^k}{(q^{1+2\alpha}; q)_{2k}(q^{b_1}; q)_k \cdots (q^{b_r}; q)_k(q; q)_k} \\
&= \sum_{k=0}^{n} \frac{(q^{1+\alpha+\beta}, q)_{2k}(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k((-1)^k q^{k \choose 2})_{s+4-r} (xt)^k}{(q^{1+2\alpha}; q)_{2k}(q^{b_1}; q)_k \cdots (q^{b_r}; q)_k(q; q)_k} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}, q)_{2k}(q^{a_1}; q)_k \cdots (q^{a_r}; q)_k((-1)^k q^{k \choose 2})_{s+4-r} (xt)^k}{(q^{1+2\alpha}; q)_{2k}(q^{b_1}; q)_k \cdots (q^{b_r}; q)_k(q; q)_k(q^{1+\alpha+\beta} t; q)_{2k}} \\
&= \frac{1}{(1-t)^{1+\alpha+\beta}} \\
&\times 5+\varphi_{8+s}
\begin{bmatrix}
0, q^{\frac{1}{2}(1+\alpha+\beta)}, -q^{\frac{1}{2}(1+\alpha+\beta)}, q^{\frac{1}{2}(2+\alpha+\beta)}, -q^{\frac{1}{2}(2+\alpha+\beta)}, q^{a_1}, \ldots, q^{a_r}, \\
q^{1+a}, -q^{1+a}, q^{\frac{1}{2}+a}, -q^{\frac{1}{2}+a}, (q^{1+a+\beta}/2), -(q^{1+a+\beta}/2), (q^{2+a+\beta}/2), -(q^{2+a+\beta}/2), \ldots, q^{b_r}
\end{bmatrix}
\end{align*}
this completes the proof of the eq. (3.2.5).

For $\alpha = \beta$, we get the following $q$-analogue of the Ultraspherical generalization of the Sister Celine's polynomials

$$\sum_{n=0}^{\infty} \frac{(q^{1+2\alpha}; q)_n}{(q; q)_n} 2^{r+1} \phi_{4+s} \left[ \begin{array}{c} q^{-n}, q^{n+1+2\alpha}, q^{a_1}, \ldots, q^{a_r} \\ q^{1+\alpha}, -q^{1+\alpha}, q^{1+2\alpha}, -q^{1+2\alpha}, q^{b_1}, \ldots, q^{b_s} \end{array} ; q, q^n x \right] t^n$$

$$= \frac{1}{(1 - t)^{1+2\alpha}}$$

$$\times \prod_{s} \phi_{4+s} \left[ \begin{array}{c} q^{n+1}, \ldots, q^{n+r} \\ q^{1+2\alpha} \end{array} ; q \right]$$

Proof

$$\sum_{n=0}^{\infty} \frac{(q^{1+2\alpha}; q)_n}{(q; q)_n} 2^{r+1} \phi_{4+s} \left[ \begin{array}{c} q^{-n}, q^{n+1+2\alpha}, q^{a_1}, \ldots, q^{a_r} \\ q^{1+\alpha}, -q^{1+\alpha}, q^{1+2\alpha}, -q^{1+2\alpha}, q^{b_1}, \ldots, q^{b_s} \end{array} ; q, q^n x \right] t^n$$

$$= \sum_{n=0}^{\infty} \frac{(q^{1+2\alpha}; q)_n}{(q; q)_n}$$

$$\times \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+1+2\alpha}; q)_k(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^{k(1+2\alpha)} )_{s+3-r} g_{n-k}^{k^2} t^n}{(q^{1+\alpha}; q)_k(-q^{1+\alpha}; q)_k(q^{1+2\alpha}; q)_k(-q^{1+2\alpha}; q)_k(q^{b_1}; q)_k \ldots (q^{b_s}; q)_k(q; q)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{1+2\alpha}; q)_{n+k}(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^{k(1+2\alpha)} )_{s+4-r} g_{n+k}^{k^2} t^n}{(q; q)_{n-k}(q^{1+2\alpha}; q)_2(q^{b_1}; q)_k \ldots (q^{b_s}; q)_k(q; q)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{1+2\alpha}; q)_{n+2k}(q^{a_1}; q)_k \ldots (q^{a_r}; q)_k((-1)^k q^{k(1+2\alpha)} )_{s+4-r} g_{n+2k}^{k^2} t^n}{(q; q)(q^{1+2\alpha}; q)_2(q^{b_1}; q)_k \ldots (q^{b_s}; q)_k(q; q)_k}$$
Chapter 3: On Some Generating Functions of $q$-Analogues of...

\[
\sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^{k \left( \frac{1}{2} \right)}]^{s+r-1} (xt)^k}{(g^{b_1};q)_k \ldots (g^{b_s};q)_k (q;q)_k} \sum_{n=0}^{\infty} \frac{(q^{1+2s+2k};q)_n t^n}{(q;q)_n}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^{k \left( \frac{1}{2} \right)}]^{s+r-1} (xt)^k (1 - q^{1+2s+2k} t)_{\infty}}{(1 - t)_{\infty}}
\]

\[
= \frac{1}{(1 - t)_{1+2\alpha}} \sum_{k=0}^{\infty} \frac{(q^{a_1};q)_k \ldots (q^{a_r};q)_k [(-1)^k q^{k \left( \frac{1}{2} \right)}]^{s+r-1} (xt)^k}{(g^{b_1};q)_k \ldots (g^{b_s};q)_k (q;g)_k (q^{1+2s} t; q)_{2k}}
\]

\[
= \frac{1}{(1 - t)_{1+2\alpha}}
\]

\[
= 1^{1+\alpha+\beta} \left[ \frac{0, q^{a_1}, \ldots, q^{a_r}}{(q^{1+2\alpha+\beta})^{1/2}, -q^{1+2\alpha+\beta} \ldots, -q^{1+2\alpha+\beta} \ldots, (q^{2+2\alpha+\beta})^{1/2}, -q^{2+2\alpha+\beta} \ldots, -q^{2+2\alpha+\beta} \ldots, (q^{1+2\alpha+\beta})^{1/2}, -q^{1+2\alpha+\beta} \ldots, -q^{1+2\alpha+\beta} \ldots} \right]_{q;xt}
\]

**Case (iv)** For $c = 1 + \alpha + \beta$ and $r = s = 0$, we have

\[
\frac{1}{(1 - t)_{1+\alpha+\beta}}
\]

\[
\times \frac{1}{5+r} \phi_{8+s} \left[ \frac{0, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}, q^{1+\alpha+\beta}}{(q^{1+\alpha+\beta})^{1/2}, -q^{1+\alpha+\beta} \ldots, -q^{1+\alpha+\beta} \ldots, (q^{2+\alpha+\beta})^{1/2}, -q^{2+\alpha+\beta} \ldots, -q^{2+\alpha+\beta} \ldots, (q^{1+\alpha+\beta})^{1/2}, -q^{1+\alpha+\beta} \ldots, -q^{1+\alpha+\beta} \ldots} \right]_{q;xt}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_n}{(q;q)_n} 2^{r+\phi_{4+s}} \left[ \frac{q^{-n}, q^{n+1+\alpha+\beta}}{q^{1+\alpha}, -q^{1+\alpha}, q^{1+\alpha}, -q^{1+\alpha}, q^{1+\alpha}, -q^{1+\alpha}, q^{1+\alpha}, -q^{1+\alpha}, q^{1+\alpha}} \right]_{t^n}^{q; q^nx}
\]

\[
(3.2.7)
\]

**Case (v)** When $\alpha = \beta = r = s = 0$, we get the following form of the $q$-analogue of the Jacobi type generalization of Sister Celine’s polynomial.
\[
\frac{1}{1-t} 3\phi_4 \left[ \begin{array}{c}
0, -q^{\frac{1}{3}}, -q \\
(qt)^{\frac{1}{3}}, -(qt)^{\frac{1}{3}}, (q^2t)^{\frac{1}{3}}, -(q^2t)^{\frac{1}{3}} \\
0, -q, xt
\end{array} \right] \\
= \sum_{n=0}^{\infty} 2\phi_2 \left[ \begin{array}{c}
q^{-n}, q^{n+1} \\
q, q^{\frac{1}{2}} \\
q, q^n x
\end{array} \right] t^n
\]

Proof

\[
\sum_{n=0}^{\infty} 2\phi_2 \left[ \begin{array}{c}
q^{-n}, q^{n+1} \\
q, q^{\frac{1}{2}} \\
q, q^n x
\end{array} \right] t^n
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (q^{-n}; q)_k (q^{n+1}; q)_k (-1)^k q \frac{k}{2} q^{nk} x^k t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (q; q)_n (q^{n+1}; q)_k q^{\frac{k}{2}} x^k t^n
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n (q^{n+1}; q)_k 2^{\frac{k}{2}} (x t)^k t^n}{(q; q)_n (q^{\frac{1}{2}}; q)_k (q; q)_k}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q; q)_k 2^{\frac{k}{2}} (x t)^k}{(q; q)_k (1 - t)_1 t^{1+2k}} \sum_{n=0}^{\infty} \frac{(q^{2k+1}; q)_n t^n}{(q; q)_n}
= \sum_{k=0}^{\infty} \frac{-(-q; q)_k (q; q)_k 2^{\frac{k}{2}} (x t)^k}{(q; q)_k (1 - t)_1 t^{1+2k}}
\]

\[
= \frac{1}{1-t} 3\phi_4 \left[ \begin{array}{c}
0, -q^{\frac{1}{3}}, -q \\
(qt)^{\frac{1}{3}}, -(qt)^{\frac{1}{3}}, (q^2t)^{\frac{1}{3}}, -(q^2t)^{\frac{1}{3}} \\
0, -q, xt
\end{array} \right]
\]
3.2.2 The Bateman’s Polynomials

In our research paper [167], we have defined Jacobi type generalization of Bateman’s polynomial [191]. Here we define its q-analogue as follows:

\[ F_{n,q}^{(\alpha,\beta)}(p, z) = \begin{pmatrix} q^{-n}, q^{n+\alpha+\beta+1}, q^\frac{1}{2}(1+z) \\ q^{1+\alpha}, q^p \end{pmatrix} 3\phi_2 \left[ q^{-n}, q^{n+\alpha+\beta+1}, q^\frac{1}{2}(1+z) ; q, q^n q \right] \]

and this satisfies the following generating function

\[
\sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} F_{n,q}^{(\alpha,\beta)}(p, z) t^n = \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} 3\phi_2 \left[ q^{-n}, q^{n+\alpha+\beta+1}, q^\frac{1}{2}(1+z) ; q, q^n q \right] t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_{k}(q^{n+\alpha+\beta+1}; q)_{k}(q^\frac{1}{2}(1+z); q)_{k}q^k q^{nk} t^n}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_{n+\alpha+\beta}(q^{n+\alpha+\beta+1}; q)_{k}(q^\frac{1}{2}(1+z); q)_{k}((-1)^k q^\frac{k}{2})}{(q; q)_{\alpha+\beta}(q; q)_{n+k}(q^{1+\alpha}; q)_{k}(q; q)_{k}(q^p; q)_{k}} (qt)^k t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_{n+\alpha+\beta+2k}(q^\frac{1}{2}(1+z); q)_{k}((-1)^k q^\frac{k}{2})}{(q; q)_{\alpha+\beta}(q; q)_{n}(q^{1+\alpha}; q)_{k}(q; q)_{k}(q^p; q)_{k}} (qt)^k t^n
\]

\[
= \sum_{k=0}^{\infty} \frac{(q; q)_{\alpha+\beta+2k}(q^\frac{1}{2}(1+z); q)_{k}((-1)^k q^\frac{k}{2})}{(q; q)_{\alpha+\beta}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}} \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta+2k}; q)_{n} t^n}{(q; q)_{n}}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^\frac{1}{2}(1+z); q)_{k}((-1)^k q^\frac{k}{2})}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}} (qt)^k \frac{(1 - q^{1+\alpha+\beta+2k})_{\infty}}{(1 - t)_{\infty}}
\]
The $q$-analogue of the ultraspherical type generalization of Bateman’s polynomials is given by

\[
F^{(\alpha,\alpha)}_{n,q}(p, z) = 3\phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+2\alpha+1}, \frac{1}{2}(1+z) \\ q^{1+\alpha}, q^p \end{array} \right] q^k q^{n_k} \quad (q, q) \quad (3.2.11)
\]

and its generating function is determined below:

\[
\sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{2\alpha}}{(q; q)_{2\alpha}} F^{(\alpha,\alpha)}_{n,q}(p, z)t^n = \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{2\alpha}}{(q; q)_{2\alpha}} 3\phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+2\alpha+1}, \frac{1}{2}(1+z) \\ q^{1+\alpha}, q^p \end{array} \right] t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{2\alpha}}{(q; q)_{2\alpha}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_{k}(q^{n+2\alpha+1}; q)_{k}(q^{\frac{1}{2}(1+z)}; q)_{k}q^{k}q^{n_k}t^n}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{1+n}; q)_{2\alpha}}{(q; q)_{2\alpha}} (q^{1+\alpha}; q)_{k}((q^{-n}; q)_{k}(q^{n+2\alpha+1}; q)_{k}(q^{\frac{1}{2}(1+z)}; q)_{k}(-1)^k q^{\frac{k}{2}}t^n}{(q; q)_{n-k}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{1+n}; q)_{2\alpha}}{(q; q)_{2\alpha}} (q^{1+\alpha}; q)_{k}((q^{-n}; q)_{k}(q^{n+2\alpha+1}; q)_{k}(q^{\frac{1}{2}(1+z)}; q)_{k}(-1)^k q^{\frac{k}{2}}t^n}{(q; q)_{n-k}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^p; q)_{k}}
\]
Chapter 3: On Some Generating Functions of q-Analogues of...

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q; q)_{n+2\alpha+2k}(q^{\frac{1}{2}(1+z)}; q)_k[(-1)^k q^{\frac{k}{2}}]}{(q; q)_{2\alpha}(q; q)_k(q^{1+\alpha}; q)_k(q; q)_k(q^p; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(q; q)_{2\alpha+2k}(q^{\frac{1}{2}(1+z)}; q)_k[(-1)^k q^{\frac{k}{2}}]}{(q; q)_{2\alpha} q^{1+\alpha} q^k(q; q)_{2k}} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{1+2\alpha}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k[(-1)^k q^{\frac{k}{2}}]}{(q; q)_k(q^{1+\alpha}; q)_k(q^p; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{1+2\alpha}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k[(-1)^k q^{\frac{k}{2}}]}{(q; q)_k(q^{1+\alpha}; q)_k(q^p; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{1+2\alpha}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k[(-1)^k q^{\frac{k}{2}}]}{(q; q)_k(q^{1+\alpha}; q)_k(q^p; q)_k} \]

The \( q \)-analogue of the Jacobi type generalized Bateman’s Polynomial \( Z^{(\alpha, \beta)}_{n,q}(b, x) \) is given by

\[ Z^{(\alpha, \beta)}_{n,q}(b, x) = \phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{1+\alpha}, q^{b+1} \end{array} ; q, q^n x \right] \]

and the its generating function is given below:

\[ \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta} Z^{(\alpha, \beta)}_{n,q}(b, x)t^n}{(q; q)_{\alpha+\beta}} = \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{1+\alpha}, q^{b+1} \end{array} ; q, q^n x \right] t^n \]
\[ \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_{k}(q^{n+\alpha+\beta+1}; q)_{k}[(1-q^{k})^{\frac{k}{2}}]_{2}q^{nk}x^{k}t^{n}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{b+1}; q)_{k}} = \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(q; q)_{n}(q^{n+\alpha+\beta+1}; q)_{k}[(1-q^{k})^{\frac{k}{2}}]_{2}q^{nk}x^{k}t^{n}}{(q; q)_{n-k}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{b+1}; q)_{k}} = \sum_{n=0}^{\infty} \frac{(q; q)_{\alpha+\beta+n+2k}[(1-q^{k})^{\frac{k}{2}}]_{2}(xt)^{k}t^{n}}{(q; q)_{\alpha+\beta}(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{b+1}; q)_{k}} = \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}[(1-q^{k})^{\frac{k}{2}}]_{2}(xt)^{k}t^{n}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{b+1}; q)_{k}} = \frac{1}{(1-t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}[(1-q^{k})^{\frac{k}{2}}]_{2}(xt)^{k}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{b+1}; q)_{k}(q^{1+\alpha+\beta}; q)_{2k}} = \frac{1}{(1-t)_{1+\alpha+\beta}} \times 5\phi_{6} \left[ 0, q^{\frac{1}{2}(1+\alpha+\beta)}, -q^{\frac{1}{2}(1+\alpha+\beta)}, q^{\frac{1}{2}(2+\alpha+\beta)}, -q^{\frac{1}{2}(2+\alpha+\beta)} \right] \left[ q^{1+\alpha}, q^{1+b}, (q^{1+\alpha+\beta}t)^{\frac{1}{2}}, -(q^{1+\alpha+\beta}t)^{\frac{1}{2}}, (q^{2+\alpha+\beta}t)^{\frac{1}{2}}, -(q^{2+\alpha+\beta}t)^{\frac{1}{2}} \right] ; q, xt \]

which is the required \( q \)-analogue of the generating function for the polynomial (3.2.13).
3.2.3 The Pasternack’s Polynomials

The \( q \)-analogue of the Jacobi type generalized Pasternack’s polynomial (1.18) is defined below:

\[
F_{n,q}^{(\alpha,\beta)}(p, z) = \phi_2 \left[ q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}(1+z)} \right]^{\frac{1}{2}} \left[ q^{1+\alpha}, q^{m+1} \right]^{\frac{1}{2}} ; q, q^n \]

(3.2.15)

and its generating function is derived as follows:

\[
\sum_{n=0}^{\infty} \frac{(q_1^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} F_{n,q}^{(\alpha,\beta)}(p, z) t^n = \sum_{n=0}^{\infty} \frac{(q_1^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \phi_2 \left[ q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}(1+z)} \right]^{\frac{1}{2}} \left[ q^{1+\alpha}, q^{m+1} \right]^{\frac{1}{2}} ; q, q^n \]

\[
= \sum_{n=0}^{\infty} \frac{(q_1^{1+n}; q)_{\alpha+\beta}}{(q; q)_{\alpha+\beta}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+\alpha+\beta+1}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k q^k q^n t^n}{(q; q)_k(q^{1+\alpha}; q)_k(q^{m+1}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n(q^{n+\alpha+\beta+1}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_n(q^{1+\alpha}; q)_k(q^{m+1}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n(q^{n+\alpha+\beta+1}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_n(q^{1+\alpha}; q)_k(q^{m+1}; q)_k} t^n
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_2(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q^{1+\alpha}; q)_k(q^{m+1}; q)_k} t^n
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_2(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q^{1+\alpha}; q)_k(q^{m+1}; q)_k} t^n
\]

\[
= \frac{1}{(1-t)^{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_2(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q^{1+\alpha}; q)_k(q^{m+1}; q)_k} t^{2k}
\]
Chapter 3: On Some Generating Functions of q-Analogues of....

\[ 3.2.16 \]

3.2.4 The Rice’s Polynomials

We define the q-analogue of the Rice’s Polynomials due to Khandekar [178]

\[ (q^{1+a+\beta}; q)_{n} H^{(\alpha, \beta)}_{n, q}(\xi, p, \nu). \]  

The polynomial (3.2.17) satisfies the following generating function

\[
\sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n}}{(q^{1+a}; q)_{n}} H^{(\alpha, \beta)}_{n, q}(\xi, p, \nu) t^{n}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n} (q^{1+a}; q)_{n}}{(q^{1+a}; q)_{n} (q^{1+a}; q)_{n}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_{k} (q^{n+1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{k} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n}}{(q^{1+a}; q)_{n}} \sum_{k=0}^{n} \frac{(q^{-n}; q)_{k} (q^{n+1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{k} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n}}{(q; q)_{n}} \sum_{k=0}^{n} \frac{(q^{1+a+\beta}; q)_{n} (-1)^{n} q^{\frac{k}{2}} (q^{n+1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{n-k} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n} (q^{1+a}; q)_{n} (-1)^{n} q^{\frac{k}{2}} (q^{n+1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{n-k} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{n+2k} (q^{\xi}; q)_{k} (-1)^{k} q^{\frac{k}{2}} (q^{1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{n} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{2k} (q^{\xi}; q)_{k} (-1)^{k} q^{\frac{k}{2}} (q^{1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(q; q)_{k} (q^{1+a}; q)_{k} (q^{p}; q)_{k}}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+a+\beta}; q)_{2k} (q^{\xi}; q)_{k} (-1)^{k} q^{\frac{k}{2}} (q^{1+a+\beta}; q)_{k} (q^{\xi}; q)_{k} q^{nk} k^{n}}{(1-t q^{1+a+\beta+2kt})_{\infty}}
\]
Chapter 3: On Some Generating Functions of \(q\)-Analogues of...

\[\sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - t)^{k}} = \frac{1}{(1 - t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - q^{1+\alpha+\beta}t)}
\]

\[= \frac{1}{(1 - t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - q^{1+\alpha+\beta}t)}
\]

\[= \frac{1}{(1 - t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - q^{1+\alpha+\beta}t)}
\]

\[= \frac{1}{(1 - t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - q^{1+\alpha+\beta}t)}
\]

\[= \frac{1}{(1 - t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{2k}(q^{2}; q)_{k}[-1]^{k}q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)}}{(q; q)_{k}(q^{1+\alpha}; q)_{k}(q^{\alpha}; q)_{k}(1 - q^{1+\alpha+\beta}t)}
\]

\(\text{eq.(3.2.18) completes the proof of the generating function of the polynomial (3.2.17).}
\]

\subsection*{3.2.5 The Hahn’s Polynomials}

We define the \(q\)-analogue of the Hahn’s Polynomials as follows:

\[Q_{n, q, N}(x; \alpha, \beta, N) = \phi_{2} \begin{bmatrix} q^{-n}, q^{n+1+\alpha+\beta}, q^{\alpha} \\ q^{\alpha}, q^{-N} \end{bmatrix} \]

\[\phi_{2} \begin{bmatrix} q^{\alpha}, q^{-N} \end{bmatrix} \]

the above polynomial (3.2.19), satisfies the following generating function

\[\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[= \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]

\[\sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta}; q)_{n}Q_{n, q, N}(x; \alpha, \beta, N)t^{n}}{(q; q)_{n}}
\]
Chapter 3: On Some Generating Functions of q-Analogues of...

\[
\begin{align*}
&= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_{2k}(q^{-x};q)_k[(-1)^kq^k]{\binom{k}{2}}}{(q;q)_k(q^{1+\alpha};q)_k(q^{-N};q)_k}(qt)^k \sum_{n=0}^{\infty} \frac{(q^{1+\alpha+\beta+2k};q)_{n}}{(q;q)_n} \\
&= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_{2k}(q^{-x};q)_k[(-1)^kq^k]{\binom{k}{2}}}{(q;q)_k(q^{1+\alpha};q)_k(q^{-N};q)_k}(qt)^k \frac{1}{(1-t)_\infty} (1 - q^{1+\alpha+\beta+2kt})_{\infty}
\end{align*}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_{2k}(q^{-x};q)_k[(-1)^kq^k]{\binom{k}{2}}}{(q;q)_k(q^{1+\alpha};q)_k(q^{-N};q)_k}(qt)^k \frac{1}{(1-t)_{1+\alpha+\beta}(1 - q^{1+\alpha+\beta}t)_{2k}}
\]

\[
= \frac{1}{(1-t)_{1+\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{1+\alpha+\beta};q)_{2k}(q^{-x};q)_k[(-1)^kq^k]{\binom{k}{2}}}{(q;q)_k(q^{1+\alpha};q)_k(q^{-N};q)_k(q^{1+\alpha+\beta}t;q)_{2k}}
\]

\[
= \frac{1}{(1-t)_{1+\alpha+\beta}} \phi_0 \left[ \begin{array}{c}
0, -x, \frac{1}{2}(1+\alpha+\beta), -\frac{1}{2}(1+\alpha+\beta), \frac{1}{2}(1+\alpha+\beta), -\frac{1}{2}(1+\alpha+\beta) \\
q^{-N}, q^x, (q^{1+\alpha+\beta}t)^{\frac{1}{2}}, -(q^{1+\alpha+\beta}t)^{\frac{1}{2}}, (q^{1+\alpha+\beta}t)^{\frac{1}{2}}, -(q^{1+\alpha+\beta}t)^{\frac{1}{2}} \end{array} \right] ; q, qt
\]

(3.2.20)

\]

this completes the proof of the generating function of q-Jacobi type generalized Hahn polynomial (3.2.19).

3.3 The q-Analogue of the Gegenbauer Type Generalization of the Certain Polynomials and Their Generating Functions

The main objective of this section is to derive q-analogues of those polynomials and their generating function relations whose Gegenbauer type generalization were obtained in the chapter 2.

3.3.1 The Sister Celine’s Polynomials

To begin with we define q-analogue of Gegenbauer type generalized Sister Celine’s polynomial as given below:

\[
\phi_n^{(\nu)}(q^{(a_1)}, \ldots, q^{(a_r)}, q^{(b_1)}, \ldots, q^{(b_s)}; x)
\]
Chapter 3: On Some Generating Functions of $q$-Analogues of...

\[ \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{n=0}^{\infty} \frac{(q^n;q^n)_{2n+r}}{(q^{2r};q)_n (q^{2r};q)^{n+r}} \left[ \begin{array}{c} q^{-n}, q^{n+2r}, q^{n_1}, \ldots, q^{n_r} \\ q^{n+r}, q^{n_1}, \ldots, q^{n_r} \end{array} \right] (q, q^n x) \]

which produces the following generating function

\[ \frac{1}{(1 - t)^{2r}} \]

\[ \times 1 + r \frac{\phi_4 + s}{(q^{2r};q)_n} \left[ \begin{array}{c} 0, q^{n_1}, \ldots, q^{n_r} \\ (q^{2r}t)^{1/2}, -(q^{2r}t)^{1/2}, (q^{2+2r}t)^{1/2}, -(q^{2+2r}t)^{1/2}, q^{h_1}, \ldots, q^{h_r} \end{array} \right] (q, xt) \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{n};q)_k (q^{n+2r};q)_k (q^{n_1};q)_k \cdots (q^{n_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} t^n}{(q^{2r};q)_k (q^{2r};q)_{2k} (q^{h_1};q)_k \cdots (q^{h_r};q)_k (q; q)_{k}^n} \]

Proof To derive the generating function of the polynomial (3.3.1), we proceed as follows:

\[ \sum_{n=0}^{\infty} \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{n};q)_k (q^{n+2r};q)_k (q^{n_1};q)_k \cdots (q^{n_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} t^n}{(q^{2r};q)_k (q^{2r};q)_{2k} (q^{h_1};q)_k \cdots (q^{h_r};q)_k (q; q)_{k}^n} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r}t)^{1/2}, -(q^{2r}t)^{1/2}, (q^{2+2r}t)^{1/2}, -(q^{2+2r}t)^{1/2}, q^{h_1}, \ldots, q^{h_r}}{(q; q)_{k}^n} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r}t)^{1/2}, -(q^{2r}t)^{1/2}, (q^{2+2r}t)^{1/2}, -(q^{2+2r}t)^{1/2}, q^{h_1}, \ldots, q^{h_r}}{(q; q)_{k}^n} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{n};q)_k (q^{n+2r};q)_k (q^{n_1};q)_k \cdots (q^{n_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k (q^{2r};q)_{2k} (q^{h_1};q)_k \cdots (q^{h_r};q)_k (q; q)_{k}^n} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2r};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{n};q)_k (q^{n+2r};q)_k (q^{n_1};q)_k \cdots (q^{n_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k (q^{2r};q)_{2k} (q^{h_1};q)_k \cdots (q^{h_r};q)_k (q; q)_{k}^n} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{a};q)_k \cdots (q^{a_r};q)_k (-1)^k q^{k} \left( \frac{k}{2} \right)^{s+4-r} (xt)^k t^n}{(q^{2r};q)_k \cdots (q^{h_r};q)_k (q; q)_k} \]
3.3.2 The Bateman’s Polynomials

The $q$-Gegenbauer type generalized Bateman’s polynomial is defined below:

\[ F_{n,q}^{(\nu)}(p, z) = 3\phi_2 \begin{bmatrix} q^{-n}, q^{n+2\nu}, q^{\frac{1}{2}(1+z)} \\ q^{\nu+rac{3}{2}}, q^n \end{bmatrix} \quad (3.3.3) \]

and its generating function

\[ \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} F_{n,q}^{(\nu)}(p, z) t^n = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+2\nu}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k q^{nk} q^k t^n}{(q; q)_k(q^{\nu+rac{3}{2}}; q)_k(q^{\nu}; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+k}(q^{\frac{1}{2}(1+z)}; q)_k((-1)^k q^{\frac{k}{2}})}{(q; q)_{n-k}(q^{\nu+rac{1}{2}}; q)_k(q^{\nu}; q)_k} t^n \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+k}(q^{\frac{1}{2}(1+z)}; q)_k((-1)^k q^{\frac{k}{2}})}{(q; q)_{n-k}(q^{\nu+rac{1}{2}}; q)_k(q^{\nu}; q)_k} t^n \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+2k}(q^{\frac{1}{2}(1+z)}; q)_k((-1)^k q^{\frac{k}{2}})}{(q; q)_{n}(q^{\nu+rac{1}{2}}; q)_k(q^{\nu}; q)_k} \sum_{n=0}^{\infty} \frac{(q^{2\nu+2k}; q)_n t^n}{(q; q)_n} \]
Chapter 3: On Some Generating Functions of q-Analogues of...

\[
\sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k(-1)^kq^{k\left(\frac{k}{2}\right)}(qt)^k}{(q; q)_k(q^\nu+\frac{1}{2}; q)_k(q^\nu+\frac{1}{2}; q)_k} \frac{1}{(1-t)^{2\nu}(1-q^{2\nu}t)_{2k}}
\]

Next, let us obtain the q-analogue of the Bateman's polynomial \(Z_n^{(\nu)}(b, x)\) as given below:

\[
Z_n^{(\nu)}(b, x) = \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k(-1)^kq^{k\left(\frac{k}{2}\right)}(qt)^k}{(q; q)_k(q^\nu+\frac{1}{2}; q)_k(q^\nu+\frac{1}{2}; q)_k} \frac{1}{(1-t)^{2\nu}(1-q^{2\nu}t)_{2k}}
\]

which satisfies the following generating functions

\[
\sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} Z_n^{(\nu)}(p, x) t^n = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \left[ q^{-n}, q^{n+2\nu} \right]_2 \left[ q^{\nu+\frac{1}{2}}, q^{(b+1)} \right] q^{nq} t^n
\]

Next, let us obtain the q-analogue of the Bateman's polynomial \(Z_n^{(\nu)}(b, x)\) as given below:

\[
Z_n^{(\nu)}(b, x) = \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k}(q^{\frac{1}{2}(1+z)}; q)_k(-1)^kq^{k\left(\frac{k}{2}\right)}(qt)^k}{(q; q)_k(q^\nu+\frac{1}{2}; q)_k(q^\nu+\frac{1}{2}; q)_k} \frac{1}{(1-t)^{2\nu}(1-q^{2\nu}t)_{2k}}
\]
\[ \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k[(1-t)^k]}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{1+b}; q)_k} \sum_{n=0}^{\infty} \frac{(q^{2\nu+2k}; q)_n t^n}{(q; q)_n} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k[(1-t)^k]}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{1+b}; q)_k} (1 - q^{2\nu+2k} t)^{\infty} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k[(1-t)^k]}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{1+b}; q)_k} \frac{1}{(1-t)^{2\nu}(1-q^{2\nu})} \]

\[ = \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k[(1-t)^k]}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{1+b}; q)_k(q^{2\nu} t; q)_k} \]

\[ = \frac{1}{(1-t)^{2\nu}} \phi^5 \begin{bmatrix} 0, q^n, -q^n, -q^{\nu+\frac{1}{2}} \\
(q^{1+b}), (q^{2\nu} t)^{\frac{1}{2}}, -(q^{2\nu} t)^{\frac{1}{2}}, (q^{2\nu+1} t)^{\frac{1}{2}}, -(q^{2\nu+1} t)^{\frac{1}{2}} \end{bmatrix} ; q, q^t \] (3.3.6)

### 3.3.3 The Pasternack’s Polynomials

The $q$-Gegenbauer type generalized Pasternack’s polynomial is defined as

\[ F_{n,m,q}(z) = \phi_2 \begin{bmatrix} q^{-n}, q^{n+2\nu}, q^{\frac{1}{2}(z+m+1)} \\
q^{\nu+\frac{1}{2}}, q^{m+1} \end{bmatrix} ; q, q^n q \] (3.3.7)

the polynomial (3.3.7) satisfies the following generating function

\[ \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} F_{n,m,q}(z) t^n = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \phi_2 \begin{bmatrix} q^{-n}, q^{n+2\nu}, q^{\frac{1}{2}(z+m+1)} \\
q^{\nu+\frac{1}{2}}, q^{m+1} \end{bmatrix} ; q, q^n q \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+2\nu}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k q^{nk} q^k t^n}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{m+1}; q)_k} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(-1)^k q^{k\frac{1}{2}}}{(q; q)_{n-k}(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{m+1}; q)_k} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+k}(q^{\frac{1}{2}(z+m+1)}; q)_k(-1)^k q^{k\frac{1}{2}}}{(q; q)_{n-k}(q^{\nu+\frac{1}{2}}; q)_k(q; q)_k(q^{m+1}; q)_k} \]
\[\sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k} (q^{\frac{1}{2}(z+m+1)}; q)_k [(-1)^k q^2 \binom{k}{2}] (qt)^k}{(q; q)_{2k} (q^{\nu+\frac{1}{2}}; q)_k (q^{m+1}; q)_k} = \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k} (q^{\frac{1}{2}(z+m+1)}; q)_k [(-1)^k q^2 \binom{k}{2}] (qt)^k}{(q; q)_{2k} (q^{\nu+\frac{1}{2}}; q)_k (q^{m+1}; q)_k} \frac{1}{(1-t)^{2 \nu}}\]

which was to be proved. For another \(q\)-analogue of the generating function for the Pasternack's polynomials, we have

\[\sum_{n=0}^{\infty} F_{m,n,q}^\nu (-2n - 1) \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-m}; q)_k (q^{m+2\nu}; q)_k (q^{-n}; q)_k q^m q^k (-t)^n}{(q^{\nu+\frac{1}{2}}; q)_k (q^{1+m}; q)_k (q; q)_k (q; q)_n} \]

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-m}; q)_k (q^{m+2\nu}; q)_k [(-1)^k q^{\frac{k}{2}}] q^k (-t)^n}{(q^{\nu+\frac{1}{2}}; q)_k (q^{1+m}; q)_k (q; q)_k (q; q)_{n-k}} \]

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-m}; q)_k (q^{m+2\nu}; q)_k [(-1)^k q^{\frac{k}{2}}] (-qt)^k (-t)^n}{(q^{\nu+\frac{1}{2}}; q)_k (q^{1+m}; q)_k (q; q)_k (q; q)_n} \]

\[= (q^{m+2\nu}; q)_n \sum_{k=0}^{n} \frac{(-t)^n}{(q; q)_n} \frac{(q^{-m}; q)_k (q^{m+2\nu}; q)_k [(-1)^k q^{\frac{k}{2}}] (-qt)^k}{(q^{\nu+\frac{1}{2}}; q)_k (q^{1+m}; q)_k (q; q)_k (q; q)_k} \]

\[= e_q(-t) \frac{\varphi_2}{2} \left[ \frac{q^{-m}, q^{m+2\nu}}{q^{\nu+\frac{1}{2}}, q^{1+m}} ; q, -qt \right] = e_q(-t) \varphi_2^{(\nu)}(m, -qt) \quad (3.3.9)\]
Chapter 3: On Some Generating Functions of q-Analogues of...

The $q$-anologue of the Gegenbauer type generalization of Pasternack's generalized Bateman's polynomial is given below:

\[
F_{n,m,q}^{(\nu)}(p,z) = 3\phi_2 \left[ \begin{array}{c}
q^{-n}, q^{n+2\nu}, q^{\frac{1}{2}(z+m+1)} \\
q^{\nu+\frac{1}{2}}, q^{p}
\end{array} \right] ; q, q^nq
\] (3.3.10)

The polynomial (3.3.10) satisfies the following generating function

\[
\sum_{n=0}^{\infty} \frac{(q^{2\nu};q)_n}{(q; q)_n} F_{n,m,q}^{(\nu)}(p, z)t^n = \frac{1}{(1-t)^{2\nu}} 5\phi_3 \left[ \begin{array}{c}
0, q^{\nu}, -q^{\nu}, -q^{\nu+\frac{1}{2}}, q^{\frac{1}{2}(z+m+1)} \\
q^{p}, (q^{2\nu+1}t)^{\frac{1}{2}}, -(q^{2\nu+1}t)^{\frac{1}{2}}, (q^{2\nu+1}t)^{\frac{1}{2}}, -(q^{2\nu+1}t)^{\frac{1}{2}}
\end{array} \right] ; q, qt
\] (3.3.11)

### 3.3.4 The Rice’s Polynomials

We define the $q$-anologue of the Gegenbauer type generalized Rice's polynomials as follows:

\[
3\phi_2 \left[ \begin{array}{c}
q^{-n}, q^{n+2\nu}, q^{\xi} \\
q^{\nu+\frac{1}{2}}, q^{p}
\end{array} \right] = \frac{(q; q)_n}{(q^{\nu+\frac{1}{2}}; q)_n} H_{n,q}^{(\nu)}(\xi, p, \nu)
\] (3.3.12)

and its generating function is given below:

\[
\sum_{n=0}^{\infty} \frac{(q^{2\nu};q)_n}{(q^{\nu+\frac{1}{2}}; q)_n} H_{n,q}^{(\nu)}(\xi, p, \nu)t^n = \sum_{n=0}^{\infty} \frac{(q^{2\nu};q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+2\nu}; q)_k(q^{\xi}; q)_kq^{nk}\nu^kt^n}{(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{\nu}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{2\nu};q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q; q)_n[(\nu^k\xi)\left(\frac{k}{2}\right)](q^{n+2\nu}; q)_k(q^{\xi}; q)_kq^{nk}\nu^kt^n}{(q; q)_n-k(q; q)_k(q^{\nu+\frac{1}{2}}; q)_k(q^{\nu}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu};q)_{n+k}(q^{\xi}; q)_k[(\nu^k\xi)\left(\frac{k}{2}\right)](q^{2\nu}; q)_k(q^{\xi}; q)_kq^{nk}\nu^kt^n}{(q; q)_{n-k}(q^{\nu+\frac{1}{2}}; q)_k(q^{\nu}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu};q)_{n+2k}(q^{\xi}; q)_k[(\nu^k\xi)\left(\frac{k}{2}\right)](q^{2\nu}; q)_k(q^{\xi}; q)_kq^{nk}\nu^kt^n}{(q; q)_n(q^{\nu+\frac{1}{2}}; q)_k(q; q)_k(q^{\nu}; q)_k}
\]
Chapter 3: On Some Generating Functions of \( q \)-Analogues of....

\[
\sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k(q^\xi; q)_k}{(q; q)_k(q^{\nu + \frac{3}{2}}; q)_k(q^\eta; q)_k} \left(\frac{q}{q}\right)^k \sum_{n=0}^{\infty} \frac{(q^{2\nu + 2k}; q)_n t^n}{(q; q)_n}
\]

\[
= \frac{1}{(1-t)^{2\nu}} \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_{2k}(q^\xi; q)_k((-1)^kq^{\frac{k}{2}})(\nu t)^k}{(q; q)_k(q^{\nu + \frac{3}{2}}; q)_k(q^\eta; q)_k(q^{2\nu t}; q)_{2k}}
\]

\[
= \frac{1}{(1-t)^{2\nu}} \left[ \begin{array}{c}
0, q^\xi, q^\eta, -q^\nu, -q^{\nu + \frac{1}{2}} \\
q^\nu, (q^{2\nu t})^\frac{1}{2}, -(q^{2\nu t})^\frac{1}{2}, (q^{2\nu + 1}t)^\frac{1}{2}, -(q^{2\nu + 1}t)^{\frac{1}{2}}
\end{array} \right]_{q, \nu t}
\]

(3.3.13)

### 3.3.5 The Hahn’s Polynomials

We define the \( q \)-analogue of the Gegenbaur type generalized Hahn’s polynomials as follows:

\[
Q_{n, \nu}(x; \nu, N) = \phi_2 \left[ \begin{array}{c}
q^{-n}, q^{n+2\nu}, q^{-x} \\
q^{\nu + \frac{1}{2}}, q^{-N}
\end{array} \right]_{q, q^n q}
\]

(3.3.14)

and its generating function

\[
\sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} Q_{n, \nu}(x; \nu, N)t^n = \sum_{n=0}^{\infty} \frac{(q^{2\nu}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, q)_k(q^{n+2\nu}, q)_k(q^{-x}, q)_k(q^{-N}, q)_k}{(q; q)_k(q^{\nu + \frac{3}{2}}; q)_k(q^{N}, q)_k(q^{2\nu t}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+k}(q^{-x}, q)_k((-1)^kq^{\frac{k}{2}})}{(q; q)_{n-k}(q^{\nu + \frac{3}{2}}; q)_k(q; q)_k(q^{N}, q)_k(q^{2\nu t}; q)_k} k^t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{2\nu}; q)_{n+2k}(q^{-x}, q)_k((-1)^kq^{\frac{k}{2}})}{(q; q)_{n+k}(q^{\nu + \frac{3}{2}}; q)_k(q; q)_k(q^{N}, q)_k(q^{2\nu t}; q)_k} k^t^n
\]
Chapter 3: On Some Generating Functions of $q$-Analogues of....

\[ \sum_{k=0}^{\infty} \frac{(q^{2\nu}; q)_k (q^{-x}; q)_k (1 - t)^2}{(q; q)_k (q^{\nu+\frac{1}{2}}; q)_k (q^{-N}; q)_k} = \frac{\phi_5}{(1 - t)^{2\nu}} \]

\[ \left[ 0, \frac{q^{-x}}, \frac{q^\nu}{q}, -\frac{q^\nu}{q}, -\frac{q^{\nu+\frac{1}{2}}}{q} \right] \]

\[ \left[ q^{-N}, \frac{q^{2\nu}t^{\frac{1}{2}}}{q}, \frac{q^{2\nu+1}t^{\frac{1}{2}}}{q}, \frac{q^{2\nu+2}t^{\frac{1}{2}}}{q} \right] \]

(3.3.15)

this completes the proof of the $q$-analogue of the the Gegenbauer type generalized Hahn polynomials.
Chapter 4

On Some Generating Functions Of Certain $q$-Polynomials

ABSTRACT: The present chapter deals with the $q$-analogues of certain polynomials and their generating functions.

4.1 Introduction

In the present chapter a study of the $q$-analogue of the Bateman’s polynomials $F_n(z)$, $Z_n(x)$, Pasternack’s generalization of Bateman’s polynomial $F^m_n(z)$, Shively’s pseudo-lagurre and other polynomials $R_n(a, x)$, Cesaro’s polynomials $g_n^{(p)}(x)$, Gottlieb polynomials $L_n(x; \lambda)$, and generalized Hypergeometric polynomial sets has been made. Further $q$-analogue of their generating functions relations have also been determined.

Apart from the polynomials noted in the introduction section of chapter 2, some more polynomials and their generating functions are also included below whose $q$-analogues shall be studied in the next consecutive sections of this chapter.
4.1.1 Shively’s Pseudo-Laguerre Polynomials

Shively studied the pseudo-Laguerre set

\[ R_n(a, x) = \frac{(a)_{2n}}{n!(a)_n} \ _1F_1(-n; a + n; x) \]  

(4.1.1)

which are related to the proper simple Laguerre polynomial

\[ L_n(x) = \ _1F_1(-n; 1; x) \]  

(4.1.2)

by

\[ R_n(a, x) = \frac{1}{(a - 1)_n} \sum_{k=0}^{n} \frac{(a - 1)_{n+k}L_{n-k}(x)}{k!} \]  

(4.1.3)

Shively obtained Toscano’s other generating relation

\[ \sum_{n=0}^{\infty} \frac{R_n(a, x)t^n}{(\frac{1}{2} + \frac{1}{2}a)_n} = e^{2t} \ _0F_1\left[ \begin{array}{c} -\frac{3}{2} \cr \frac{1}{2} + \frac{1}{2}a; \ t^2 - xt \end{array} \right] \]  

(4.1.4)

4.1.2 Cesàro Polynomials

Cesàro polynomials [199] are defined as

\[ g_n^{(s)}(x) = \binom{s + n}{n} \ _2F_1\left[ \begin{array}{c} -n, 1; \cr -s - n; \ x \end{array} \right] \]  

(4.1.5)

and eq. (4.1.5) satisfies the following generating functions

\[ \sum_{n=0}^{\infty} g_n^{(s)}(x)t^n = (1 - t)^{-s-1}(1 - xt)^{-1} \]  

(4.1.6)

\[ \sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(s)}(x)t^n = (1 - t)^{-s-1-k}(1 - xt)^{-1} g_k^{(s)}\left( \frac{x(1-t)}{1-xt} \right) \]  

(4.1.7)
4.1.3 Gottlieb Polynomials

Gottlieb polynomials are given below: (see [5], [191], [199])

\[
l_n(x; \lambda) = e^{-n\lambda} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{k} \right) (1-e^\lambda)^k
\]

or

\[
l_n(x; \lambda) = e^{-n\lambda} \, _2F_1 \left[ \begin{array}{c} -n, -x; \\ 1 \\
\end{array} \right] (1-e^\lambda)
\]

Above Gottlieb polynomials satisfy the following generating functions (see [191], [199])

\[
\sum_{n=0}^{\infty} l_n(x; \lambda) t^n = (1-t)^{\lambda} (1-te^{-\lambda})^{-x-1}, \quad |t| < 1
\]

\[
\sum_{n=0}^{\infty} l_n(x; \lambda) \frac{e^{n\lambda} t^n}{n!} = \exp(te^{-\frac{\lambda}{2}}) \, _1F_1 \left[ \begin{array}{c} -x; \\ 1 \\
\end{array} \right] -(e^\frac{\lambda}{2} - e^{-\frac{\lambda}{2}}) t
\]

\[
\sum_{n=0}^{\infty} l_n(x; \lambda) t^n = e^t \, _1F_1 \left[ \begin{array}{c} x + 1; \\ 1 \\
\end{array} \right] - (1-e^{-\lambda}) t
\]

\[
\sum_{n=0}^{\infty} \frac{(c)_n}{n!} l_n(x; \lambda) t^n = (1-te^{-\lambda})^{-c} \, _2F_1 \left[ \begin{array}{c} c, -x; \\ 1 \\
\end{array} \right] \frac{(1-e^{-\lambda}) t}{1-te^{-\lambda}}
\]

\[
\sum_{n=0}^{\infty} \binom{n+k}{k} l_{n+k}(x; \lambda) t^n = (1-t)^{x-k} (1-te^{-\lambda})^{-x-1} l_k(x, \alpha), \quad (4.1.14)
\]

where \( \alpha = \log \left[ \frac{e^\lambda - 1}{1-t} \right] \).
4.1.4 Generalized Hypergeometric Polynomial Set

In 1994, S. D. Bajpai and M. S. Arora [4] studied some properties of the generalized hypergeometric polynomial set, given below:

\[ \mathcal{U}_n(\beta; \gamma; x) = x^n \binom{2}{1} \qquad (4.1.15) \]

where \( n \) is a non-negative integer and \( x \) is any non-zero complex variable and \( \beta, \gamma \) are independent of \( n \) for if \( \beta, \gamma \) dependent upon \( n \) then many properties which are valid for \( \beta, \gamma \) independent of \( n \) fail to be valid for \( \beta, \gamma \) dependent upon \( n \).

In 1997, I. K. Khanna and V. Srinivasa Bhagavan [179] derived generating function of generalized hypergeometric polynomial set (4.1.15) in terms of Gottlieb Polynomials, is given below:

\[ \mathcal{U}_n(-x; 1; (1 - e^\lambda)^{-1}) = (e^{-\lambda} - 1)^{-n} l_n(x; \lambda) \quad (4.1.16) \]

Recently, M. A. Khan and M. Akhlaq [137] defined two variable and three variable analogues of the Gottlieb Polynomials. Of which, the two variable analogue of Gottlieb polynomials is given below:

\[ l_n(x, y; \lambda, \mu) = e^{-n(\lambda+\mu)} F \left[ \begin{array}{c} -n : -x; -y; \\ 1 : -; -; \end{array} \right] 1 - e^\lambda, 1 - e^\mu \quad (4.1.17) \]

or in other words,

\[ l_n(x, y; \lambda, \mu) = e^{-n(\lambda+\mu)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_r (-x)_r (-y)_s (1 - e^\lambda)^r (1 - e^\mu)^s}{r! s! (1)_{r+s}} \quad (4.1.18) \]

and the three variable analogue as follows [137]:
Chapter 4: On Some Generating Functions of Certain q-Polynomials

\[ l_n(x, y, z; \lambda, \mu, \eta) = \]

\[ e^{-n(\lambda+\mu+\eta)} F \left[ \begin{array}{c} -n \:::\; -x; -y; -z; \\ 1 \:::\; -x; -y; -z; \\ \end{array} \left| \begin{array}{c} 1 - e^\lambda, 1 - e^\mu, 1 - e^\eta \end{array} \right| \right] \]

or in other words,

\[ l_n(x, y, z; \lambda, \mu, \eta) = \]

\[ e^{-n(\lambda+\mu+\eta)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{r-s} \frac{(-n)_r (-x)_r (-y)_s (-z)_k (1 - e^\lambda)^r (1 - e^\mu)^s (1 - e^\eta)^k}{r! s! k! (1)_r s+k} \]

(4.1.20)

They [137] obtained the following generating functions similar to (4.1.14)

\[ \sum_{n=0}^{\infty} l_n(x, y; \lambda, \mu) t^n = (1 - te^\mu)^x (1 - te^\lambda)^y (1 - te^{-\lambda-\mu})^{x-y-1}, \quad |t| < 1 \]

(4.1.21)

\[ \sum_{n=0}^{\infty} (c)_n l_n(x, y; \lambda, \mu) \frac{t^n}{n!} = \]

\[ (1 - te^{-(\lambda+\mu)-c}) F \left[ \begin{array}{c} c \:::\; -x, -y; \\ 1 \:::\; -x, -y; \\ \end{array} \left| \begin{array}{c} \frac{t(e^\lambda-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}}, \frac{t(e^\mu-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}} \end{array} \right| \right] \]

(4.1.22)

and for result (4.1.18) as

\[ \sum_{n=0}^{\infty} l_n(x, y, z; \lambda, \mu, \eta) t^n = \]
Motivated by the above works, we investigate here generating functions of certain $q$-polynomials. For this we require the definitions and notations of the $q$-theory (see [50] and Chapter 1).

### 4.2 $q$-Analogue of Certain Polynomials

We define the $q$-analogs of the above mentioned polynomials and their generating functions as follows:

#### 4.2.1 $q$-Bateman’s Polynomial

We define the $q$-analogue of the Bateman’s polynomial (2.1.3) in the following manner:

\[
F_{n,q}(z) = \, _3\phi_2\left[\begin{array}{c}
q^{-n}, q^{n+1}, q^{\frac{1}{2}(1+z)} \\
q, q^n
\end{array}\right]_q.
\]

\[
Z_{n,q}(x) = \, _2\phi_2\left[\begin{array}{c}
q^{-n}, q^{n+1} \\
q, q^n x
\end{array}\right]_q.
\]

Another $q$-analogue of the Bateman’s polynomial is given below:
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

\[ F_{m,q}(-2n - 1) = 3\phi_2 \left[ q^{-n}, q^{n+1}, q^{-n} \middle| q, q^n \right] . \]  \hspace{1cm} (4.2.3)

### 4.2.2 $q$-Pasternack’s Polynomials

The $q$-analogue of the Pasternack’s generalization of Bateman’s polynomial is given below:

\[ F^{m}_{n,q}(z) = 3\phi_2 \left[ q^{-n}, q^{n+1}, q^{\frac{1}{2}(z+m+1)} \middle| q, q^{m+1} \right] . \]  \hspace{1cm} (4.2.4)

### 4.2.3 $q$-Shively’s Pseudo-Laguerre Polynomials

The $q$-analogue of (4.1.5) is given below:

\[ R_{n,q}(q^{a}; x) = \frac{(q^{a}; q)_{2n}}{(q^{a}; q)_{n}(q; q)_{n}} 1\phi_1 \left[ q^{-n} \middle| q, q^{n}x \right] . \]  \hspace{1cm} (4.2.5)

### 4.2.4 $q$-Cesàro Polynomial

The $q$-analogue of the Cesàro’s polynomial is defined below:

\[ g^{(s)}_{n,q}(x) = \frac{(q^{1+s}; q)_{n}}{(q; q)_{n}} 2\phi_1 \left[ q^{-n}, q \middle| q^{-s-n}, x \right] . \]  \hspace{1cm} (4.2.6)

### 4.2.5 $q$-Gottlieb Polynomials

We define the following $q$-Gottlieb polynomials

\[ l_{n,q}(x; \lambda) = \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} x \\ k \end{array} \right]_q \lambda^{k}\left(1 - e_q(\lambda)\right)^k \]  \hspace{1cm} (4.2.7)
\[ = \left\{ E_q(-\lambda) \right\}^n \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k} \frac{(q; q)_x}{(q; q)_{k-x}} q^{(k-1)-xk} (1 - e_q(\lambda))^k \]

using the following identity (see [50])

\[ (q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\left( \begin{array}{c} k \\ 2 \end{array} \right) - nk} , \]

we have

\[ = \left\{ E_q(-\lambda) \right\}^n \sum_{k=0}^{n} (-1)^k (q^{-n}; q)_k q^{\left( \begin{array}{c} k \\ 2 \end{array} \right) + nk} (q^{-x}; q)_k q^{\left( \begin{array}{c} k \\ 2 \end{array} \right) + xk} q^{k(k-1)-xk} (1 - e_q(\lambda))^k \]

or

\[ l_n(x; \lambda) = \left\{ E_q(-\lambda) \right\}^n \, _2\phi_1 \left[ \begin{array}{c} q^{-n}, \ q^{-x} \\ q^n(1 - e_q(\lambda)) \end{array} ; q \right] . \hspace{1cm} (4.2.8) \]

4.2.6 \( q \)-Generalized Hypergeometric Polynomial Set

The \( q \)-analogue of the generalized hypergeometric polynomial set (4.1.15), the hypergeometric polynomial set derived by S.D. Bajpai and M. S. Arora [4], is given below:

\[ U_n(\beta; \gamma; q, x) = x^n \, _2\phi_1 \left[ \begin{array}{c} q^{-n}, \ q^\beta \\ q^\gamma \end{array} ; q, \frac{1}{x} \right] . \hspace{1cm} (4.2.9) \]

4.3 Generating Functions of Certain \( q \)-Polynomials

The following are the generating functions of the above mentioned \( q \)-polynomials
4.3.1 Generating Functions of \( q \)-Bateman Polynomials

The \( q \)-Bateman polynomial (4.2.1) satisfies the following generating functions

\[
\sum_{n=0}^{\infty} F_{n,q}(z)t^n = \frac{1}{1-t} 5\phi_5 \left[ \begin{array}{c}
-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{1}{2}(1+z)}, 0 \\
q, qt, -qt, (qt)^{\frac{1}{2}}, -(qt)^{\frac{1}{2}}
\end{array} \right]; q, t,
\]

and

\[
\sum_{n=0}^{\infty} [F_{n,q}(z - 2) - F_{n,q}(z)]t^n
\]

\[
= q^{\frac{1}{2}}(1 + q)t \frac{t}{(1-t)^3} 5\phi_5 \left[ \begin{array}{c}
q^{\frac{1}{2}}, -q^{\frac{3}{2}}, -q^2, q^{\frac{1}{2}(1+z)}, 0 \\
q^2, q^3t, -q^3t, (q^3t)^{\frac{1}{2}}, -(q^3t)^{\frac{1}{2}}
\end{array} \right]; q, qt.
\]

4.3.2 \( q \)-Pasternack Polynomials’ Generating Functions

In (1939), Pasternack derived the following generating function for the Bateman Polynomials as follows:

\[
\sum_{n=0}^{\infty} F_{m,q}(-2n - 1) \frac{(-t)^n}{(q;q)_n} = e_q(-t)Z_{m,q}(-tq^{-n}).
\]

Generating Functions of generalization of \( q \)-Pasternack Polynomials

\[
\sum_{n=0}^{\infty} F_{n,q}^m(z)t^n = \frac{1}{1-t} 5\phi_5 \left[ \begin{array}{c}
-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{1}{2}(z+m+1)}, 0 \\
(qt)^{\frac{1}{2}}, -(qt)^{\frac{1}{2}}, (q^2t)^{\frac{1}{2}}, -(q^2t)^{\frac{1}{2}}, q^{m+1}
\end{array} \right]; q, t.
\]
4.3.3 Generating Function of Pseudo-Laguerre Set

The generating function for the Toscano’s other generating relation obtained by Shively[191] whose \( q \)-analogue is given below:

\[
\sum_{n=0}^{\infty} R_{n,q}(q^n; q^x) t^n = \phi_1 \left[ - ; q, x t \right] \phi_2 \left[ q^{x+k}, -q^{x+k}, q^{x+\frac{1}{2}+k} ; q, t \right].
\]

(4.3.6)

4.3.4 Generating Functions of \( q \)-Cesàro’s Polynomials

The \( q \)-Cesàro polynomials satisfy the following generating functions

\[
\sum_{n=0}^{\infty} g_{n,q}(x) t^n = \frac{(1-q^x t)^{-1}}{(1-t)^{s+1}}.
\]

(4.3.7)

4.3.5 Generating Functions of \( q \)-Gottlieb Polynomials

The following generating functions hold for \( q \)-Gottlieb Polynomials (4.3.7)

\[
\sum_{n=0}^{\infty} l_{n,q}(x; \lambda) t^n = (1 - t E_q(-\lambda))^{-1} \phi_1 \left[ q^{-x} ; q, -(1 - E_q(-\lambda)) t \right].
\]

(4.3.8)

\[
\sum_{n=0}^{\infty} l_{n,q}(x; \lambda) \frac{t^n}{(q; q)_n} = \phi_1 \left[ q^{-x} ; q, -(1 - E_q(-\lambda)) t \right].
\]

(4.3.9)

\[
\sum_{n=0}^{\infty} \frac{\{e_q(\lambda)\}^{\frac{n}{2}}}{(q; q)_n} l_{n,q}(x; \lambda) t^n
\]
Chapter 4: On Some Generating Functions of Certain q-Polynomials

\[ e_q(t \{E_q(-\lambda)\}^{1/2}) \phi_1 \left[ \begin{array}{c} q^{-x} \\ \phi \end{array} ; q, -\left( \{e_q(\lambda)\}^{-1/2} - \{e_q(\lambda)\}^{1/2} \right) t \right]. \quad (4.3.10) \]

\[
\sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} l_{n,q}(x; \lambda)t^n
\]

\[ = \frac{(1 - tq^c E_q(-\lambda))_{\infty}}{(1 - t E_q(-\lambda))_{\infty}} 2\phi_2 \left[ \begin{array}{c} q^c, q^{-x} \\ \phi \end{array} ; q, -(1 - E_q(-\lambda))t \right]. \quad (4.3.11) \]

### 4.3.6 Generating Function of q-Generalized Hypergeometric Polynomial Set

The generating function for the q-analogue of the generalized hypergeometric polynomial set (4.1.15), the hypergeometric polynomial set derived by S.D. Bajpai and M. S. Arora [4], is given below:

\[ U_{n,q} \left(-x; q; q, \frac{1}{1 - e_q(\lambda)} \right) = (E_q(-\lambda) - 1)^{-n} l_{n,q}(x; \lambda). \quad (4.3.12) \]

### Proofs of the Generating Functions Mentioned in the Section 4.3

Let us prove one by one the generating functions from (4.3.1) to (4.3.12).

**Proof** let us determined the generating function (4.3.1), which is satisfied by the q-Bateman polynomials

\[
\sum_{n=0}^{\infty} F_{n,q}(z)t^n = \sum_{n=0}^{\infty} 3\phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+1}, q^{1+[1+z]} \\ \phi \end{array} ; q, q^n \right] t^n
\]
by making use of the identity eq. (1.3.65), we have

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k \left[ (-1)^k q^k \binom{k}{2} \right]^{2+1-3}}{(q; q)_k(q; q)_k(q; q)_k} q^{nk} t^n
$$

(4.3.13)

By making use of the following identity (see [50]; ex-1.1(iv) p. 24)

$$
(a; q)_n = (a; q)_n (-a; q)_n ((aq)^{\frac{1}{2}}; q)_n ((aq)^{-\frac{1}{2}}; q)_n,
$$

(4.3.15)

we have

$$
\sum_{k=0}^{\infty} \frac{(-q; q)_k(q^{\frac{1}{2}}; q)_k(-q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k(-1)^k q^k \binom{k}{2} t^k}{(q; q)_k(q; q)_k} \frac{(q^{2k+1}; q)_n t^n}{(q; q)_n}
\sum_{n=0}^{\infty} \frac{(q^{2k+1}; q)_n t^n}{(q; q)_n}
$$

$$
\sum_{k=0}^{\infty} \frac{(-q; q)_k(q^{\frac{1}{2}}; q)_k(-q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k(-1)^k q^k \binom{k}{2} t^k (1 - q^{2k+1}t)_\infty}{(q; q)_k(q; q)_k}
\frac{1}{(1 - t)_\infty}
$$
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

104

\[ E^f_{c=0}(z^q: q)k = \frac{1}{(1 - t)^{1+2k}} \]

\[ = \sum_{k=0}^{\infty} \frac{(-q; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k(-1)^k q^{k \frac{k}{2}} t^k}{(q; q)_k(q; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(-q; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k(-1)^k q^{k \frac{k}{2}} t^k}{(q; q)_k(q; q)_k} \]

\[ = \frac{1}{1 - t} \sum_{k=0}^{\infty} \frac{(-q; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}}; q)_k(q^{\frac{1}{2}(1+z)}; q)_k(-1)^k q^{k \frac{k}{2}} t^k}{(q; q)_k(q; q)_k} \]

\[ \phi_5 \left[ \begin{array}{c}
-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{1}{2}(1+z)}, 0 \\
q, qt, -qt, (qt)^{\frac{1}{2}}, -(qt)^{\frac{1}{2}}
\end{array} \right] 
\]

which was to be proved.

Proof The generating function (4.3.2) is determined as follows:

\[ \sum_{n=0}^{\infty} \left[ F_{n,q}(z - 2) - F_{n,q}(z) \right] t^n = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(z-1)}; q)_k q^{nk}}{(q; q)_k(q; q)_k(q; q)_k} \right] t^n 
- \sum_{k=0}^{n} \left[ \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(z+1)}; q)_k q^{nk}}{(q; q)_k(q; q)_k(q; q)_k} \right] t^n \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(z-1)}; q)_k q^{nk} \left\{ (q^{\frac{1}{2}(z-1)}; q)_k - (q^{\frac{1}{2}(z+1)}; q)_k \right\} t^n}{(q; q)_k(q; q)_k(q; q)_k} 
\]

\[ = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(z-1)}; q)_k q^{nk} t^n}{(q; q)_k(q; q)_k(q; q)_k} (1 - q^k)(q^{\frac{1}{2}(z+1)}; q)_k^{-1} q^{nk} t^n \]

(4.3.16)

from eqs. (1.3.65) and (4.3.16), we have
\[ \sum_{n=0}^{\infty} \sum_{k=1}^{n} (-1)^{k} (q; q)_{n+k} \binom{k}{2} (-q^{2}z^{-1})(q^{2}(z+1); q)_{k-1} q^{nk} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=1}^{n} (-1)^{k} (q; q)_{n+k} \binom{k}{2} (-q^{2}z^{-1})(q^{2}(z+1); q)_{k-1} q^{nk} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=1}^{n} (-1)^{k+2} (q; q)_{n+2k+1} \binom{k+1}{2} (q^{2}(z+1); q)_{k} q^{nk} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=1}^{n} (-1)^{k} (q; q)_{n+2k+1} \binom{k+1}{2} (q^{2}(z+1); q)_{k} q^{nk} \]

\[ = q^{2z} t \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q; q)_{2} q^{2}; q)_{n+2k} (q^{2}(z+1); q)_{k} (-1)^{k} \binom{k}{2} (qt)^{k} q^{nk}}{(1-q^{2})^{2} q^{2}(q^{2}; q)_{k}(q; q)_{k} q^{n}} \]

\[ = q^{2z} (1+q) t \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{3}; q)_{2k} (q^{3+2k}; q)_{n} (q^{2}(z+1); q)_{k} (-1)^{k} \binom{k}{2} (qt)^{k} q^{nk}}{(q^{2}; q)_{k}(q; q)_{k} q^{n}} \]

\[ = q^{2z} (1+q) t \sum_{k=0}^{\infty} \frac{(q^{3}; q)_{k} (-q^{3}; q)_{k} (q^{2}; q)_{k} (q^{2}(z+1); q)_{k} (-1)^{k} \binom{k}{2} (qt)^{k} q^{nk}}{(q^{2}; q)_{k}(q; q)_{k} \sum_{n=0}^{\infty} (q^{3+2k}; q)_{n} t^{n}} \]
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

106

\[
q^2(1+q)t \sum_{k=0}^{\infty} \frac{(q^3; q)_k(-q^3; q)_k(-q^2; q)_k(q^{1/2}; q)_k(-1)^k q^{k/2}}{(q^2; q)_k(q; q)_k} (qt)^k (1 - q^{3+2k})_t (1 - t)_\infty
\]

\[
= q^2(1+q)t \sum_{k=0}^{\infty} \frac{(q^3; q)_k(-q^3; q)_k(-q^2; q)_k(q^{1/2}; q)_k(-1)^k q^{k/2}}{(q^2; q)_k(q; q)_k} (qt)^k \frac{1}{(1 - t)_{3+2k}}
\]

\[
= \frac{q^2(1+q)t}{(1 - t)_3} \sum_{k=0}^{\infty} \frac{(q^3; q)_k(-q^3; q)_k(-q^2; q)_k(q^{1/2}; q)_k(-1)^k q^{k/2}}{(q^2; q)_k(q; q)_k(q^3t; q)_{2k}} (qt)^k
\]

\[
= \frac{q^2(1+q)t}{(1 - t)_3} \sum_{k=0}^{\infty} \frac{(q^3; q)_k(-q^3; q)_k(-q^2; q)_k(q^{1/2}; q)_k(-1)^k q^{k/2}}{(q^2; q)_k(q; q)_k(q^3t; q)_{2k}} (qt)^k
\]

\[
= \frac{q^2(1+q)t}{(1 - t)_3} \sum_{k=0}^{\infty} \frac{(q^3; q)_k(-q^3; q)_k(-q^2; q)_k(q^{1/2}; q)_k(-1)^k q^{k/2}}{(q^2; q)_k(q; q)_k(q^3t; q)_{2k}} (qt)^k
\]

which was to be proved.

**Proof** We determine the generating function (4.3.3) as given below:

\[
\sum_{n=0}^{\infty} Z_{n,q}(x) t^n = \sum_{n=0}^{\infty} 2\phi_2 \left[ \begin{array}{c} q^{-n}, q^{n+1} \\ q, q^n \end{array} \right] t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(-1)^k q^{k/2}}{(q; q)_k(q; q)_k} \frac{x^k q^{nk} t^n}{(q; q)_k} \tag{4.3.20}
\]

from eqs. (1.3.65), (4.3.20), we get

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n(-1)^k q^{k/2} (q^{n+1}; q)_k(-1)^k q^{k/2}}{(q; q)_n-k(q; q)_k(q; q)_k} \frac{x^k q^{nk} t^n}{(q; q)_k}
\]
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_{n+k} (\frac{k}{2})!^2}{(q; q)_{n-k} (q; q)_{k} (q; q)_{k}} x^{k} t^{n} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_{n+2k} (\frac{k}{2})!^2}{(q; q)_{n} (q; q)_{k} (q; q)_{k}} x^{k} t^{n+k} \]

\[ = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q; q)_{2k} (\frac{k}{2})!^2}{(q; q)_{k} (q; q)_{k}} x^{k} t^{n+k} \]

using identity (4.3.15), we obtain

\[ = \sum_{k=0}^{\infty} \frac{(-q; q)_{k} (q^{\frac{1}{2}}; q)_{k} (-q^{\frac{1}{2}}; q)_{k} (\frac{k}{2})!^2 (xt)^{k} (1 - q^{2k+1} t)_{\infty}}{(1 - t)_{\infty}} \]

\[ = \sum_{k=0}^{\infty} \frac{(-q; q)_{k} (q^{\frac{1}{2}}; q)_{k} (-q^{\frac{1}{2}}; q)_{k} (\frac{k}{2})!^2 (xt)^{k}}{(q; q)_{k} (gt; q)_{k}} \]

\[ = \frac{1}{1 - t} \sum_{k=0}^{\infty} \frac{(-q; q)_{k} (q^{\frac{1}{2}}; q)_{k} (-q^{\frac{1}{2}}; q)_{k} (\frac{k}{2})!^2 (xt)^{k}}{(q; q)_{k} (gt; q)_{k}} \]

\[ = \frac{1}{1 - t} \frac{4 \phi_{5} \left[ -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, 0 \right]}{q, qt, -qt, (qt)^{\frac{1}{2}}, -(qt)^{\frac{1}{2}}; q, xt} \]

it was to be proved.

**Proof** The generating function (4.3.4) is determined as given below:

\[ \sum_{n=0}^{\infty} F_{m, q} (-2n - 1) (\frac{-t}{q})^{n} = \sum_{n=0}^{\infty} 3 \phi_{2} \left[ q^{-m}, q^{m+1}, q^{-n} \right] \frac{(t)^{n}}{(q; q)_{n}} \]

\[ \frac{q, q^{n}}{q, q^{n}} \left( -t \right) \]
from eq. (1.3.65) and (4.3.22), we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-m}; q)_k(q^{m+1}; q)_k(q^{-n}; q)_kq^{nk}(-t)^n}{(q; q)_k(q; q)_k} = e_q(-t)\ _2\phi_2 \left[ \begin{array}{c} q^{-m}, q^{m+1} \\ q, q \end{array} \right] _{q, -t}
\]

which is the required result.

**Proof** The proof of the generating function (4.3.5) of \( q \)-analogue of Paster-nack's generalization of Bateman's Polynomial, is given below:

\[
\sum_{n=0}^{\infty} F_{m,n}^q (z) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+1}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k q^{nk} t^n}{(q; q)_k(q; q)_k(q^{m+1}; q)_k}
\]

with the help of identity (1.3.65), we write

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_n(-1)^k q^{k/2}(q^{n+1}; q)_k(q^{\frac{1}{2}(z+m+1)}; q)_k q^{nk} t^n}{(q; q)_n-k(q; q)_k(q^{m+1}; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (q; q)_{n+k}(q^{\frac{1}{2}(z+m+1)}; q)_k[(-1)^k q^{k/2}] t^n
\]
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q; q)_{n+k}(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_n(q; q)_k(q^{m+1}; q)_k} (-1)^k q^{\binom{k}{2}} t^n$$

$$= \sum_{k=0}^{\infty} \frac{(q; q)_{2k}(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q; q)_k(q^{m+1}; q)_k} (-1)^k q^{\binom{k}{2}} t^k \frac{(1 - q^{1+2k} t)_{\infty}}{(1 - t)_{\infty}}$$

$$= \sum_{k=0}^{\infty} \frac{(q; q)_{2k}(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q; q)_k(q^{m+1}; q)_k} (-1)^k q^{\binom{k}{2}} t^k \frac{(1 - q^{1+2k} t)_{\infty}}{(1 - t)_{\infty}}$$

$$= \sum_{k=0}^{\infty} \frac{(q; q)_{2k}(q^{\frac{1}{2}(z+m+1)}; q)_k}{(q; q)_k(q; q)_k(q^{m+1}; q)_k} (-1)^k q^{\binom{k}{2}} t^k \frac{1}{(1 - t)(1 - qt)_{2k}}$$

$$= \frac{1}{1 - t} \sum_{k=0}^{\infty} \frac{(q; q)_{2k}(q^{\frac{1}{2}(z+m+1)}; q)_k(-1)^k q^{\binom{k}{2}} t^k}{(q; q)_k(q; q)_k(q^{m+1}; q)_k}$$

$$\frac{1}{1 - t} \Phi_5\left[-q, \frac{q}{2}, \frac{q}{2}, q^{\frac{1}{2}(z+m+1)}, 0 \left(qt, \frac{1}{2}, -(qt)^{\frac{1}{2}}, (qt^2)^{\frac{1}{2}}, -(qt^3)^{\frac{1}{2}}, q^{m+1}\right)\right]$$

Proof: $q$-analogue of the pseudo-Laguerre Set (4.3.6) obtained by Shively obtained Toscano's other generating relation is given below:

$$\sum_{n=0}^{\infty} \frac{R_{n,q}(q^a, x)t^n}{(q^{\frac{1}{2}+\frac{a}{2}}, q)_n}$$
Chapter 4: On Some Generating Functions of Certain \( q \)-Polynomials

\[ 110 \]

\[ \sum_{n=0}^{\infty} \frac{(q^n; q)_{2n}}{(q^{n+1}; q)_n(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k[(-1)^k q \binom{k}{2}]_x^k q^{kn} t^n}{(q^{a+n}; q)_n(q; q)_k} \]

\[ 4.3.24 \]

\[ = \sum_{n=0}^{\infty} \frac{(q^n; q)_n(-q^{\frac{n}{2}}; q)_n(-q^{\frac{n+1}{2}}; q)_n}{(q^{a}; q)_n(q; q)_n} \sum_{k=0}^{n} \frac{(-1)^k q \binom{k}{2} x^k q^{kn} t^n}{(q^{a+n}; q)_n(q; q)_k} \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^n; q)_k(-q^{\frac{n}{2}}; q)_k(-q^{\frac{n+1}{2}}; q)_k q^2 \binom{k}{2} (xt)^k}{(q^{a}; q)_{n+2k}(q; q)_k(q^n; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{\frac{n}{2}}; q)_k(-q^{\frac{n}{2}}; q)_k(-q^{\frac{n+1}{2}}; q)_k q^2 \binom{k}{2} (xt)^k}{(q^{a}; q)_{n+2k}(q; q)_k(q^n; q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k q \binom{k}{2} (xt)^k}{(q^{\frac{n}{2}}+\frac{1}{2}; q)_k(q^{a}; q)_k(q; q)_k} \sum_{n=0}^{\infty} \frac{(q^n; q)_n(-q^{\frac{n}{2}}+k; q)_n(-q^{\frac{n+1}{2}}; q)_n t^n}{(q^{a+2k}; q)_n(q; q)_n} \]

or

\[ \sum_{n=0}^{\infty} \frac{R_{n,q}(q^n; x) t^n}{(q^{\frac{n}{2}}+\frac{1}{2}; q)_n} = \phi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} q^{\frac{n}{2}} ; q, xt \]

\[ 3 \phi_2 \begin{bmatrix} q^{\frac{n}{2}}+k, -q^{\frac{n}{2}}+k, 2^{\frac{n+1}{2}}+k \\ 0, q^{a+2k} \end{bmatrix} ; q, t \]
this was the required result.

**Proof** The proof of the generating function (4.3.7), satisfied by the \( q \)-Cesàro polynomials, is given below:

\[
\sum_{n=0}^{\infty} g_{n,q}^{(s)}(x) t^n = \sum_{n=0}^{\infty} \left( \frac{q^{1+s}; q}{q; q} \right)_n \sum_{k=0}^{n} \left( \frac{q^{-n}; q}{q; q} \right)_k (q; q)_k x^k t^n
\]

using the identity (1.3.65), we have

\[
= \sum_{n=0}^{\infty} \left( \frac{q^{1+s}; q}{q; q} \right)_n \sum_{n=0}^{\infty} \left( \frac{q^{-n}; q}{q; q} \right)_k (q; q)_k x^k t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{q^{1+s}; q}{q; q} \right)_n \left( \frac{q^{-n}; q}{q; q} \right)_k (q; q)_k x^k t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{1+s}; q)_n (q^{1+s}; q)_k}{(q; q)_n (q; q)_k} x^k t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(1-q^{1+s})_n}{(1-t)_n} \sum_{k=0}^{n} \frac{(q; q)_k t^k}{(1-q^s t)_k}
\]

\[
= \frac{(1-q^{1+s})_\infty}{(1-t)_\infty} \frac{(1-q^{1+s} t)_\infty}{(1-q^s t)_\infty} = \frac{(1-q^s t)^{-1}}{(1-t)^{s+1}}
\]

it was to be proved.

**proof** The proof of the generating function (4.3.8) for Gottlieb polynomial is given below:
\[ \sum_{n=0}^{\infty} l_{n,q}(x; \lambda) t^n = \sum_{n=0}^{\infty} \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{-x};q)_k}{(q;q)_k(q;q)_k} q^{nk}(1 - e_q(\lambda))^k \] 

(4.3.26)

from eqs. (1.3.65) and (4.3.26), we have

\[ \sum_{n=0}^{\infty} \{E_q(-\lambda)\}^n t^n \sum_{k=0}^{n} \frac{(q;q)_n(-1)^k(q^{-x};q)_k}{(q;q)_n(q;q)_k(q;q)_k} q^{nk}(1 - e_q(\lambda))^k \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n+k} \frac{(q;q)_{n+k}(q^{-1})^k(q^{-x};q)_k}{(q;q)_n(q;q)_k(q;q)_k} q^{nk}(1 - e_q(\lambda))^k \] 

(4.3.27)

now making use of the following identity (see[50]; eq.(1.2.33), p.(6)], eq. (1.3.62))

\[ (a; q)_{n+k} = (a; q)_n(aq^n; q)_k \] 

(4.3.28)

substituting \( a = q \), in the (1.3.62) identity and then placing the outcome in the eq. (4.3.27), we have

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{-1};q)_n(q^{-x};q)_k(-1)^k q^{k} \{tE_q(-\lambda)\}^k(1 - e_q(\lambda))^k \{tE_q(-\lambda)\}^n}{(q;q)_n(q;q)_k} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{k}}{(q;q)_k} \{E_q(-\lambda) - 1\}^{k} t^{k} \sum_{n=0}^{\infty} \frac{(q^{k+1};q)_n \{tE_q(-\lambda)\}^n}{(q;q)_n} \]

\[ = \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{k}}{(q;q)_k} \{E_q(-\lambda) - 1\}^{k} t^{k} \frac{[1 - tE_q^{k+1}\{E_q(-\lambda)\}]_\infty}{[1 - tE_q(-\lambda)]_\infty} \]
\[
\sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^\left(\frac{k}{2}\right) (E_q(-\lambda) - 1)^k t^k}{(q;q)_k} \frac{1}{[1 - t E_q(-\lambda)]_{k+1}}
\]

\[
= [1 - t E_q(-\lambda)]^{-1} \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^\left(\frac{k}{2}\right) (E_q(-\lambda) - 1)^k t^k}{(q;q)_k} \frac{1}{[1 - t E_q(-\lambda)]_k}
\]

or

\[
\sum_{n=0}^{\infty} l_{n;q}(x;\lambda) t^n = (1 - t E_q(-\lambda))^{-1} _1\phi_1 \left[ \begin{array}{c} q^{-x} \\ qt E_q(-\lambda) \end{array} ; q, -(1 - E_q(-\lambda)) t \right]
\]

which was to be proved.

**Proof** The proof of the generating function (4.3.9) is given below:

\[
\sum_{n=0}^{\infty} l_{n;q}(x;\lambda) \frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} \left\{ E_q(-\lambda) \right\}^n \frac{t^n}{(q;q)_n} \frac{1}{2} \phi_1 \left[ \begin{array}{c} q^{-n}, q^{-x} \\ q^n (1 - e_q(\lambda)) \end{array} ; q, q^n (1 - e_q(\lambda)) \right]
\]

\[
= \sum_{n=0}^{\infty} \left\{ E_q(-\lambda) \right\}^n \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k q^{nk} (1 - e_q(\lambda))^k}{(q;q)_k (q;q)_k} (4.3.29)
\]

using (1.3.65) and (4.3.29), we have

\[
= \sum_{n=0}^{\infty} \left\{ E_q(-\lambda) \right\}^n \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k q^{nk} q^\left(\frac{k}{2}\right)}{(q;q)_n (q;q)_k (q;q)_k} (1 - e_q(\lambda))^k
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(t E_q(-\lambda))^{n+k} (q^{-x};q)_k q^\left(\frac{k}{2}\right)}{(q;q)_n (q;q)_k (q;q)_k} (1 - e_q(\lambda))^k
\]

\[
= \sum_{n=0}^{\infty} \frac{(t E_q(-\lambda))^{n}}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-x};q)_k (-1)^k q^\left(\frac{k}{2}\right)}{(q;q)_k (q;q)_k} (t(E_q(-\lambda) - 1))^k
\]
or

\[
\sum_{n=0}^{\infty} l_{n,q}(x; \lambda) \frac{t^n}{(q;q)_n} = e_q(tE_q(-\lambda)) \phi_1 \left[ \begin{array}{c} q^{-x} \\ q \\ q, -(1 - E_q(-\lambda))t \end{array} \right]_{1}
\]

which was to be proved.

**Proof** The proof of the generating function (4.3.10) for the Gottlieb polynomial is given below:

\[
\sum_{n=0}^{\infty} \frac{\{e_q(\lambda)\}^{\frac{n}{2}}}{(q; q)_n} l_{n,q}(x; \lambda) t^n = \sum_{n=0}^{\infty} \frac{\{E_q(\lambda)\}^{\frac{n}{2}}}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (q^{-x}; q)_k (1 - e_q(\lambda))^k q^{kn} t^n}{(q; q)_k (q; q)_k}
\]

(4.3.30)

using the identity (1.3.65), we have

\[
= \sum_{n=0}^{\infty} \frac{\{E_q(\lambda)\}^{\frac{n}{2}}}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (q^{-x}; q)_k (1 - e_q(\lambda))^k q^{kn} t^n}{(q; q)_k (q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\{E_q(\lambda)\}^{\frac{n+k}{2}}}{(q; q)_n} \frac{(q^{-x}; q)_k (1 - e_q(\lambda))^k q^{kn+k} t^{n+k}}{(q; q)_n (q; q)_k (q; q)_k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\{E_q(\lambda)\}^{\frac{n+k}{2}}}{(q; q)_n} \frac{(q^{-x}; q)_k (1 - e_q(\lambda))^k q^{kn+k} t^{n+k}}{(q; q)_n (q; q)_k (q; q)_k}
\]

\[
= e_q(t\{E_q(-\lambda)\}^{\frac{1}{2}}) \phi_1 \left[ \begin{array}{c} q^{-x} \\ q \\ q, -\left(\{e_q(\lambda)\}^{-\frac{1}{2}} - \{e_q(\lambda)\}^{\frac{1}{2}}\right) t \end{array} \right]_{1}
\]
which was to be proved.

**Proof** The proof of another generating function (4.3.11) for the Gottlieb polynomial is determined below:

\[
\sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} n \frac{\binom{x}{n}}{\binom{\lambda}{n}} \{E_q(-\lambda)\}^n 2\phi_1 \left[ \begin{array}{c} q^{-n}, q^{x} \\ q \\ q^n(1 - e_q(\lambda)) \end{array} \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^c; q)_n t^n}{(q; q)_n} \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{-x}; q)_k q^{nk} (1 - e_q(\lambda))^k}{(q; q)_{n-k}(q; q)_k (q; q)_k} (q; q)_n (q; q)_k (1 - e_q(\lambda))^k
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^c; q)_{n+k} \{E_q(-\lambda)\}^{n+k} (-1)^k (q^{-x}; q)_k q^{nk} \left( \frac{k}{2} \right)^{-nk} (1 - e_q(\lambda))^k}{(q; q)^n(q; q)_k(q; q)_k}
\]

现在利用以下恒等式（参见[50] ; eq.(1.2.33), p.(6)) 或 eq. (1.3.62)

\[(a; q)_n = (a; q)_n(aq^n; q)_k\]

将 \(a = q^c\) ，在上述恒等式中代入，然后将结果置于等式 (4.3.32) 中，我们有

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^c; q)_k (q^{c+k}; q)_n (q^{-x}; q)_k (-1)^k q^{\left( \frac{k}{2} \right)} \{tE_q(-\lambda)\}^k \{tE_q(-\lambda)\}^n}{(q; q)_n(q; q)_k(q; q)_k (1 - e_q(\lambda))^k}
\]

\[
= \sum_{k=0}^{\infty} \frac{(q^c; q)_k (q^{-x}; q)_k \{E_q(-\lambda) - 1\}_k (-1)^k q^{\left( \frac{k}{2} \right)} \sum_{n=0}^{\infty} \frac{(q^{c+k}; q)_n \{tE_q(-\lambda)\}^n}{(q; q)_n}}{(q; q)_k(q; q)_k (q; q)_k (1 - e_q(\lambda))^k}
\]
Chapter 4: On Some Generating Functions of Certain $q$-Polynomials

$$= \sum_{k=0}^{\infty} \frac{(q^r; q)_k (q^{-x}; q)_k (E_q(-\lambda) - 1)^k}{(q; q)_k (q; q)_k} (-1)^k q \left( \begin{array}{c} k \\ 2 \end{array} \right) \frac{[1 - tq^{c+k} E_q(-\lambda)]_\infty}{[1 - t E_q(-\lambda)]_\infty}$$

$$= \sum_{k=0}^{\infty} \frac{(q^r; q)_k (q^{-x}; q)_k (E_q(-\lambda) - 1)^k}{(q; q)_k (q; q)_k} (-1)^k q \left( \begin{array}{c} k \\ 2 \end{array} \right) \frac{[1 - t q^c E_q(-\lambda)]_\infty [1 - t q^{c+k} E_q(-\lambda)]_\infty}{[1 - t q^c E_q(-\lambda)]_\infty [1 - t E_q(-\lambda)]_\infty}$$

$$= \frac{[1 - t q^c E_q(-\lambda)]_\infty}{[1 - t E_q(-\lambda)]_\infty} \sum_{k=0}^{\infty} \frac{(q^r; q)_k (q^{-x}; q)_k (E_q(-\lambda) - 1)^k}{(q; q)_k [1 - t q^c E_q(-\lambda)]_k (q; q)_k} (-1)^k q \left( \begin{array}{c} k \\ 2 \end{array} \right)$$

or

$$\sum_{n=0}^{\infty} \frac{(q^r; q)_n}{(q; q)_n} l_{nq}(x; \lambda) t^n = \frac{(1 - t q^c E_q(-\lambda))_\infty}{(1 - t E_q(-\lambda))_\infty} 2\phi_2 \left[ \frac{q^r, q^{-x}}{q, t q^c E_q(-\lambda)} ; q, -(1 - E_q(-\lambda)) t \right]$$

which was to be proved.

**Proof** The proof of generating function (4.3.12) of $q$-generalized Hypergeometric set is determined by replacing $\beta$ by $-x$, $\gamma$ by $q$ and $x$ by $\frac{1}{q^r(1-e_q(\lambda))}$, in eq.(4.3.8) we have

$$U_{nq} \left( -x; q; q, \frac{1}{q^n(1-e_q(\lambda))} \right) = (q^n(1-e_q(\lambda)))^{-n} 2\phi_1 \left[ \frac{q^{-n}, q^{-x}}{q} ; q, q^n(1-e_q(\lambda)) \right]$$

$$= \frac{E_q(-\lambda)^n}{(q^n(E_q(-\lambda) - 1))^{n}} 2\phi_1 \left[ \frac{q^{-n}, q^{-x}}{q} ; q, q^n(1-e_q(\lambda)) \right]$$

or

$$U_{nq} \left( -x; q; q, \frac{1}{1-e_q(\lambda)} \right) = (q^n(E_q(-\lambda) - 1))^{-n} l_{nq}(x; \lambda)$$

which is the required result.
Chapter 5

Generalized Exponential Operators And Difference Equations

**ABSTRACT:** The present chapter deals with the generalization of exponential operators used by Dattoli and Levi [34] for translation and diffusive operator which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

### 5.1 Introduction

In 2000, Dattoli and Levi [34] discussed general methods for the solution of difference equations, arising in physical and biological problems. Their technique play crucial role in unifying the generalized families of the difference equations.

The present chapter deals with the generalization of exponential operators used in [34] to operators of the type \( a^{\lambda g(x) \frac{d}{dx}} \), where base \( a \) (\( a > 0, \ a \neq 1 \)) is a
real number. In particular when $a = e$, the operator reduces to the operators used by Dattoli et al. [34].

The action of the generalized exponential operator on a generic function $f(x)$ is defined as

$$a^{\lambda q(x)} \frac{d}{dx} f(x) = e^{(\lambda \ln(a))q(x)} \frac{d}{dx} f(x)$$

$$= f(F^{-1}(\lambda \ln(a) + F(x))). \quad (5.1.1)$$

where $F(x)$ (called the Similarity Factor (S.F.)) denotes the function

$$F(x) = \int^{x} \frac{d\xi}{q(\xi)},$$

and $F^{-1}(\sigma)$ is its inverse.

For $q(x) = 1$, the SF is given by

$$F(x) = \int^{x} d\xi = x, \quad (5.1.2)$$

therefore $F^{-1}(x) = x$, then the operator (5.1.1) reduces to the ordinary translation or shift operator as follows:

$$a^{\lambda} \frac{d}{dx} f(x) = f(F^{-1}(\lambda \ln(a) + x))$$

$$= f(\lambda \ln(a) + x). \quad (5.1.3)$$

Another example of application of the operator (5.1.1), for $q(x) = x$, the SF is given by

$$F(x) = \int^{x} \frac{d\xi}{\xi} = \ln(x), \quad (5.1.4)$$
so that $F^{-1}(x) = e^x$, and hence the operator (5.1.1) reduces to the dilatation operator

$$a^{\lambda \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + \ln(x)))$$

$$= f(e^{\lambda \ln(a)+\ln(x)}) = f(a^\lambda x).$$ \hspace{1cm} (5.1.5)

The ordinary shift operators and their properties play a central role within the context of the theory of difference equations [54]. One can, therefore, suspect that the above generalized exponential operators and the wealth of their properties can be exploited to develop tools which allow the solution of different forms of difference equations.

**5.1(a) Particular case:** The substitution of $a = e$, into the eqs.(5.1.1), (5.1.3) and (5.1.5) reduce to the eqs. (1), (2') and (3) of Dattoli et al. [34].

A simple example of how the exponential operators can help us to solve difference equations may be illuminating. Let us consider the linear dilatation difference equation of the type

$$b_1f(a^2x) + b_2f(ax) + b_3f(x) = 0, \hspace{1cm} (5.1.6)$$

which, according to eq. (5.1.5), eq. (5.1.6) can be written in the following form

$$\left[b_1 x^{2x \frac{d}{dx}} + b_2 x^x \frac{d}{dx} + b_3\right] f(x) = 0. \hspace{1cm} (5.1.7)$$

Suppose $f(x) = R^\ln(x)$, we have

$$a^{\lambda x \frac{d}{dx}} R^\ln(x) = e^{\lambda \ln(a)x \frac{d}{dx}} R^\ln(x),$$

where
\[ q(x) = x, \text{ so that } F(x) = \ln(x) \text{ and } F^{-1}(x) = e^x \]
or
\[ F^{-1}(\lambda \ln(a) + \ln(x)) = e^{\lambda \ln(a) + \ln(x)} = xa^\lambda, \]
therefore,
\[ a^{\lambda x} \frac{d}{dx} R^{\ln x} = R^{\ln(xa^\lambda)} = R^{\lambda \ln(a)} R^{\ln x}. \quad (5.1.8) \]
Hence we can associate with eq. (5.1.7) the characteristic equation
\[ [b_1 R^{2\ln(a)} + b_2 R^{\ln(a)} + b_3] R^{\ln(x)} = 0 \]
or
\[ b_1 R^{2\ln(a)} + b_2 R^{\ln(a)} + b_3 = 0, \quad (5.1.9) \]
whose roots \( R_1^{\ln(a)} \) and \( R_2^{\ln(a)} \) allow to write \( f(x) \) in terms of the following linear combination of independent solutions:
\[ f(x) = c_1 R_1^{\ln(x)} + c_2 R_2^{\ln(x)} = \sum_{\alpha=1}^{2} c_\alpha R_\alpha^{\ln(x)}. \quad (5.1.10) \]
The above example indicates that we can extend well-established methods of solutions of difference equations to other types of equations reducible to ordinary difference equations, after a proper change of variable implicit in eqs. (5.1.1), (5.1.3).

5.1(b) Particular case: The replacement of \( a \) with \( e \) in the eqs. (5.1.6), (5.1.7), (5.1.8) and (5.1.9) give raise to the eqs. (5), (6), (7), and (8) of Dattoli et al. [34].
To give a further example of the flexibility of the formalism associated with exponential operators, let us consider the generalized Heat Equation of the following type

\[
\begin{align*}
\frac{\partial}{\partial \lambda} Q(x, \lambda \ln(a)) &= \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 Q(x, \lambda \ln(a)), \\
Q(x, 0) &= g(x),
\end{align*}
\]

which can formally be solved by rewriting eq (5.1.11) as

\[
\frac{\partial}{\partial \lambda} Q(x, \lambda \ln(a)) - \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 Q(x, \lambda \ln(a)) = 0,
\]

which can formally be solved by considering this as ordinary linear differential equation of order one, whose I.F. is determined as

\[
e^{-\int \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 d\lambda} = e^{-\ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^2 \lambda} = a^{-\lambda \left[ q(x) \frac{\partial}{\partial x} \right]^2},
\]

we can, therefore, find its general solution as

\[
Q(x, \lambda \ln(a)) a^{-\lambda \left[ q(x) \frac{\partial}{\partial x} \right]^2} = C,
\]

where \( C \) is any constant and using the given initial condition, we get

\[
Q(x, 0) = g(x) = C,
\]

and finally, we obtain the solution of the Heat equation (5.1.11) as

\[
Q(x, \lambda \ln(a)) = a^{\lambda \left[ q(x) \frac{\partial}{\partial x} \right]^2} g(x).
\]

The use of the identity

\[
\phi^3 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2k\xi} d\xi,
\]
Chapter 5: Generalized Exponential Operators And Difference Equations

122

replacing \( b^2 \) with \( \lambda \ln(a)q(x)\frac{\partial}{\partial x} \), we have

\[
e^{\lambda \ln(a)q(x)\frac{\partial}{\partial x}^2} = a^{\lambda q(x)\frac{\partial}{\partial x}^2} \]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\sqrt{\lambda \ln(a)}\xi} d\xi, \quad (5.1.13)
\]

with the use of the eq. (5.1.1), finally yields the solution of eq (5.1.11) in the form of an integral transform, which can be viewed as a generalized Gauss transform

\[
a^{\lambda q(x)\frac{\partial}{\partial x}^2} g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi \sqrt{\lambda \ln(a)} + F(x))) d\xi. \quad (5.1.14)
\]

or, in other words, we have

\[
Q(x, \lambda \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi \sqrt{\lambda \ln(a)} + F(x))) d\xi. \quad (5.1.14)'
\]

It is evident that the formalism associated with generalized exponential operators can be exploited in many flexible ways in finding the general solution of a large number of problems. This chapter is devoted to the discussion of methods which provide the solution of the classes of “difference” and generalized “Heat” equations and we shall see that the techniques we propose offer reliable analytical tools and efficient numerical algorithms.

5.1(c) Particular case: To put \( a = e \), in the eqs. (5.1.11), (5.1.12), and (5.1.14) give rise the same forms of the eqs. (10), (11) and (13) respectively of Dattoli et al. [34].
5.2 Generalized Difference Equations

Before discussing the problem in its generality, let us consider the eq. of the following type, as a further example, which reduces to ([34]; p. 655 (14)) when we consider \( a = e \), we have

\[
\sum_{\alpha=0}^{N} b_{\alpha} f(\cos(\alpha \ln(a)) + \sqrt{1 - x^2} \sin(\alpha \ln(a))) = 0, \tag{5.2.1}
\]

which belongs to the families of generalized difference equation. This equation can be obtained by the action of the generalized exponential operator on the function \( f(x) \).

\[
\sum_{\alpha=0}^{N} b_{\alpha} f(\sin^{-1} x \cos(\alpha \ln(a)) + \cos^{-1} \sqrt{1 - x^2} \sin(\alpha \ln(a))) = 0,
\]

or

\[
\sum_{\alpha=0}^{N} b_{\alpha} f(\sin^{-1} x \cos(\alpha \ln(a)) + \cos(\sin^{-1} x) \sin(\alpha \ln(a))) = 0,
\]

or

\[
\sum_{\alpha=0}^{N} b_{\alpha} f(\sin^{-1} x + \alpha \ln(a))) = 0,
\]

or

\[
\sum_{\alpha=0}^{N} b_{\alpha} \alpha^{1-x^2} f(x) = 0.
\]

According to the discussion of the previous section, the use of the exponential operator

\[
\sum_{\alpha=0}^{N} b_{\alpha} \lambda^{\alpha} f(x) = 0,
\]
where

\[ \hat{A} = a^{\sqrt{1-x^2}} \frac{d}{dx} \]  

(5.2.2)

allows to cast (5.2.1) in the operator form

\[ \Psi(\hat{A}) f(x) = 0, \]

where

\[ \Psi(\hat{A}) = \sum_{\alpha=0}^{N} b_{\alpha} \hat{A}^{\alpha}. \]  

(5.2.3)

In this case the SF associated with (5.2.2) is

\[ F(x) = \sin^{-1}(x). \]  

(5.2.4)

Independent solutions of (5.2.1) can therefore be constructed in terms of the function \( R^{\sin^{-1}(x)} \), which satisfies the identity

\[ \hat{A}^{\alpha} R^{\sin^{-1}(x)} = R^{\alpha \ln(a)} R^{\sin^{-1}(x)}, \]  

(5.2.5)

the general solution of (5.2.1) can finally be written as

\[ f(x) = \sum_{\alpha=0}^{N} c_{\alpha} R^{\sin^{-1}(x)}. \]  

(5.2.6)

Similarly, if we consider the following example, we have

\[ \sum_{\alpha=0}^{N} b_{\alpha} f(x \cos(\alpha \ln(a)) - \sqrt{1-x^2} \sin(\alpha \ln(a))) = 0, \]  

(5.2.1)'

which belongs to the families of generalized difference equation. According to the discussion of the previous section, the use of the exponential operator

\[ \hat{A} = a^{-\sqrt{1-x^2}} \frac{d}{dx} \]  

(5.2.2)'}
allows to cast (5.2.1)' in the operator form (5.2.3). In this case the SF associated with (5.2.2)' is

\[ F(x) = \cos^{-1}(x). \] (5.2.4)'

Independent solutions of (5.2.1)' can be therefore constructed in terms of the function \( R_{\cos^{-1}(x)} \), which satisfies the identity

\[ \tilde{A}^\alpha R_{\cos^{-1}(x)} = R_{\alpha \ln(a)} R_{\cos^{-1}(x)}. \] (5.2.5)'

The general solution of (5.2.1)' can finally be written as

\[ f(x) = \sum_{a=0}^{N} c_a R_{\alpha}^{\cos^{-1}(x)}, \] (5.2.6)'

where \( R_{\alpha}^{\ln(a)} \) are the roots of the characteristic equation

\[ \Psi(R_{\ln(a)}) = 0. \] (5.2.7)

From the above discussion it is now clear that, whenever one deals with equations of the type

\[ \sum_{a=0}^{N} b_n f(F^{-1}(\alpha \ln(a) + F(x))) = 0, \] (5.2.8)

one can associate it with the generalized exponential operator

\[ \tilde{A} = a^{q(x) \frac{d}{dx}}, \] (5.2.9)

which allows to cast (5.2.8) in the operator form (5.2.3) and we get the relevant solution in the form

\[ f(x) = \sum_{a=0}^{N} c_a R_{\alpha}^{\int F^{-1}(\ln(a) + F(x))}. \] (5.2.10)
5.2(a) Particular case: when we substitute $a = e$ in the eqs. (5.2.1), (5.2.2), (5.2.3), (5.2.5), (5.2.7), (5.2.8) and (5.2.9) then these equation lead to the eqs. (14), (15), (16), (18), (20), (21) and (22) respectively due to Dattoli et al. [34].

A useful example is given by the equation

$$\sum_{\alpha=0}^{N} b_\alpha f \left( \frac{x}{1-\alpha \ln(a)x} \right) = 0,$$  \hspace{1cm} (5.2.11)

by making use of the shift operator $a^{2x \frac{d}{dx}}$, which allows to cast (5.2.11) in the operator form (5.2.3) i.e.

$$\sum_{\alpha=0}^{N} b_\alpha f \left( \frac{1}{\alpha \ln(a) - \frac{1}{x}} \right) = 0,$$

or

$$\sum_{\alpha=0}^{N} b_\alpha a^{\alpha x \frac{d}{dx}} f(x) = 0,$$

or

$$\sum_{\alpha=0}^{N} b_\alpha \hat{A}^\alpha f(x) = 0,$$

where $\hat{A} = a^{x \frac{d}{dx}}$, $\Psi(\hat{A}) f(x) = 0$ and $\Psi(A) = \sum_{\alpha=0}^{N} b_\alpha (A)\alpha$.

In this case the SF associated with (5.2.11) is $F(x) = -\frac{1}{x}$, Its solution can thus be written as

$$f(x) = \sum_{\alpha=1}^{N} c_\alpha R^{\frac{1}{a \alpha}}.$$  \hspace{1cm} (5.2.12)

The validity of the above solutions is limited to the case in which $\ln(a)$ is not a multiple root of the characteristic equation; this point will be discussed in the concluding section.
5.2(b) Particular case: Replacing $a$ with $e$ in the eq. (5.2.11) reduce to Dattoli et al. ([34]; p.656(24)).

In the tunes of Dattoli et al. ([34]; p. 656(26)), let us introduce the following operational identities:

\[
\begin{align*}
\hat{A}^\pm \alpha b \int_x^0 \frac{df}{\sigma(t)} &= b^\pm \alpha \ln(a) b \int_x^0 \frac{df}{\sigma(t)}, \\
\hat{A}^\pm \alpha \left(b \int_x^0 \frac{df}{\sigma(t)} \phi(x)\right) &= b \int_x^0 \frac{df}{\sigma(t)} \left(b^\ln(a) \hat{A}\right)^\pm \alpha \phi(x) .
\end{align*}
\]

valid for exponential operators of the form (5.2.9).

We note that according to the first of (5.2.13) the non-homogeneous equation

\[
\Psi(\hat{A})f(x) = C b \int_x^0 \frac{df}{\sigma(t)},
\]

where $C$ is a constant and $b^{\ln(a)}$ is not a root of the characteristic equation, admits the particular solution

\[
f(x) = \frac{C b \int_x^0 \frac{df}{\sigma(t)}}{\Psi(b^{\ln(a)})}.
\]

In the slightly more complicated case

\[
\Psi(\hat{A})f(x) = C b \int_x^0 \frac{df}{\sigma(t)} \phi(x),
\]

the second of (5.2.13) yields

\[
f(x) = C b \int_x^0 \frac{df}{\sigma(t)} \frac{1}{\Psi(b^{\ln(a)} \hat{A})} \phi(x).
\]

Further comments shall be discussed in the concluding section.

5.2(c) Particular case: When $a = e$, the eqs. (5.2.14), (5.2.15), (5.2.16) and (5.2.17) convert into the eqs (27), (28), (29) and (30) of Dattoli et al. [34].
5.3 Generalized Shift Operators and Jackson Derivatives

In the previous section we have considered linear equations involving discrete power of the generalized exponential operator. Here we shall discuss examples in which the exponents are not necessarily integers. The introductory example is

\[
\frac{f(a^\lambda x) - f(x)}{\lambda \ln(a)} = g(x),
\]

(5.3.1)

where \( f(x) \) is unknown, \( \lambda \in C \), and \( g(x) \) is an analytical function. The use of the dilatation operator allows to cast eq. (5.2.17) in the form of a the Jackson derivative [58], namely

\[
\frac{a^\lambda \frac{d}{\xi} - 1}{\lambda \ln(a)} f(x) = g(x).
\]

(5.3.2)

The operator on the left hand side can formally be inverted and by writing the differentiation variable in terms of the inverse of the SF we find

\[
f(e^\xi) = \frac{\lambda \ln(a)}{a^\lambda \frac{d}{\xi} - 1} g(e^\xi).
\]

(5.3.3)

The operator on the r.h.s. of (5.3.3) can be expanded as

\[
\frac{\lambda \ln(a)}{a^\lambda \frac{d}{\xi} - 1} = \frac{\lambda \ln(a)}{\lambda \ln(a) \frac{d}{\xi} + \frac{1}{2!} (\lambda \ln(a) \frac{d}{\xi})^2 + \frac{1}{3!} (\lambda \ln(a) \frac{d}{\xi})^3 + \cdots}
\]

\[
= \frac{1}{\frac{d}{\xi} \left[ 1 + \frac{1}{2!} \left( \lambda \ln(a) \frac{d}{\xi} \right)^2 + \frac{1}{3!} \left( \lambda \ln(a) \frac{d}{\xi} \right)^3 + \cdots \right]}
\]

\[
= D_\xi^{-1} \left[ 1 + \left( \frac{1}{2!} \lambda \ln(a) \frac{d}{d\xi} + \frac{1}{3!} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 + \cdots \right) \right]^{-1}
\]
\begin{align*}
= D^{-1}_\xi \left[ 1 - \left( \frac{1}{2!} \lambda \ln(a) \frac{d}{d \xi} + \frac{1}{3!} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^2 + \cdots \right) \\
+ \left( \frac{1}{2!} \lambda \ln(a) \frac{d}{d \xi} + \frac{1}{3!} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^2 + \cdots \right)^2 + \cdots \right] \\
= D^{-1}_\xi \left[ 1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d \xi} + \left( \frac{1}{4} - \frac{1}{6} \right) \left( \lambda \ln(a) \frac{d}{d \xi} \right)^2 - \frac{1}{24} + \frac{1}{8} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^3 + \cdots \right] \\
= D^{-1}_\xi \left[ 1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d \xi} + \frac{1}{12} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^2 - \frac{1}{6} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^3 + \cdots \right] \\
= D^{-1}_\xi \left[ B_0 + B_1 \lambda \ln(a) \frac{d}{d \xi} + \frac{B_2}{2!} \left( \lambda \ln(a) \frac{d}{d \xi} \right)^2 + B_3 \left( \lambda \ln(a) \frac{d}{d \xi} \right)^3 + \cdots \right].
\end{align*}

or

\begin{equation}
\frac{\lambda \ln(a)}{a^{\frac{d}{d \xi}} - 1} = D^{-1}_\xi \sum_{n=0}^{\infty} \frac{B_n}{n!} \left( \lambda \ln(a) \right)^n \left( \frac{d}{d \xi} \right)^n.
\end{equation}

where

\begin{equation}
B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = -1, \cdots
\end{equation}

are Bernoulli numbers (see [191]; p.300(9)) and \( D^{-1}_\xi \) is the inverse of the derivative operator. Since \( g(x) \) has a Taylor expansion \( g(x) = \sum_{m=0}^{\infty} b_m x^m \), we get from eqs. (5.3.3), (5.3.4)

\begin{equation}
f(e^\xi) = D^{-1}_\xi \sum_{n=0}^{\infty} \frac{B_n}{n!} \left( \lambda \ln(a) \right)^n \left( \frac{d}{d \xi} \right)^n \left( \sum_{m=0}^{\infty} b_m e^{m\xi} \right).
\end{equation}
\[ D^{-1}_\xi \left[ B_0 + \frac{B_1}{1!} \lambda \ln(a) \frac{d}{d\xi} + \frac{B_2}{2!} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 + \frac{B_3}{3!} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^3 + \cdots \right] \sum_{m=0}^{\infty} b_m e^{m\xi} \]

\[ = D^{-1}_\xi \left[ 1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d\xi} + \frac{1}{12} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 - \frac{1}{6} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^3 + \cdots \right] \sum_{m=0}^{\infty} b_m e^{m\xi} \]

\[ = D^{-1}_\xi \left[ b_0 + \sum_{m=1}^{\infty} b_m e^{m\xi} - \frac{1}{2} \lambda \ln(a) \sum_{m=0}^{\infty} b_m e^{m\xi} \right. \]

\[ + \frac{1}{12} (\lambda \ln(a))^2 \sum_{m=0}^{\infty} b_m (m)^2 e^{m\xi} - \frac{1}{6} (\lambda \ln(a))^3 \sum_{m=0}^{\infty} b_m (m)^3 e^{m\xi} + \cdots \]

\[ = \left[ b_0 \xi + \sum_{m=1}^{\infty} \frac{b_m}{m} e^{m\xi} - \frac{1}{2} \lambda \ln(a) \sum_{m=0}^{\infty} b_m e^{m\xi} \right. \]

\[ + \frac{1}{12} (\lambda \ln(a))^2 \sum_{m=0}^{\infty} b_m (m)^2 e^{m\xi} - \frac{1}{6} (\lambda \ln(a))^3 \sum_{m=0}^{\infty} b_m (m)^3 e^{m\xi} + \cdots \]

\[ = b_0 \xi + \left( 1 - \frac{\lambda \ln(a)}{2} + \frac{(\lambda \ln(a))^2}{12} - \frac{(\lambda \ln(a))^3}{6} + \cdots \right) b_1 e^\xi \]

\[ + \left( 1 - \frac{2\lambda \ln(a)}{2} + \frac{(2\lambda \ln(a))^2}{12} - \frac{(2\lambda \ln(a))^3}{6} + \cdots \right) \frac{b_2}{2} e^{2\xi} + \cdots \]

\[ = b_0 \xi + b_1 e^\xi \left[ 1 + \left( \frac{\lambda \ln(a)}{2!} + \frac{(\lambda \ln(a))^2}{3!} + \cdots \right) \right]^{-1} \]

\[ + b_2 e^\xi \left[ 1 + \left( \frac{2\lambda \ln(a)}{2!} + \frac{(2\lambda \ln(a))^2}{3!} + \cdots \right) \right]^{-1} + \cdots \]
Chapter 5: Generalized Exponential Operators And Difference Equations

\[
\begin{align*}
&= b_0 e^\xi + \frac{b_1 e^\xi}{1!} + \frac{b_2 e^\xi}{2!} + \cdots \\
&= b_0 e^\xi + \frac{\lambda \ln(a) b_1 e^\xi}{1!} + \frac{\lambda \ln(a) b_2 e^\xi}{2!} + \cdots
\end{align*}
\]

or

\[
f(e^\xi) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) e^{m \xi}}{a^m - 1} + b_0 \xi,
\]

going back to the original variable, we get

\[
f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) x^m}{a^m - 1} + b_0 \ln(x).
\]

The series on the right hand side of eq. \((5.3.6)\) provides the solution of our problem.

Take another example \(g(x) = \sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}\), we find

\[
f(e^\xi) = D_\xi^{-1} \sum_{n=0}^{\infty} \frac{B_n (\lambda \ln(a))^n}{n!} \left( \frac{d}{d\xi} \right)^n \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!}
\]

\[
= D_\xi^{-1} \left[ B_0 + \frac{B_1}{1!} \lambda \ln(a) \frac{d}{d\xi} + \frac{B_2}{2!} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 + \frac{B_3}{3!} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^3 + \cdots \right] \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!}
\]
\[ D_\xi^{-1} \left[ 1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d\xi} + \frac{1}{12} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 - \frac{1}{6} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^3 + \cdots \right] \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \]

\[ = D_\xi^{-1} \left[ \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \right] - \frac{1}{2} \left( \lambda \ln(a) \frac{d}{d\xi} \right) \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \]

\[ + \frac{1}{12} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} - \frac{1}{6} \left( \lambda \ln(a) \frac{d}{d\xi} \right)^3 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \cdots \]

\[ = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)(2m+1)!} e^{(2m+1)\xi} - \frac{1}{2} \lambda \ln(a) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi} \]

\[ + \frac{1}{12} (\lambda \ln(a))^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)!} e^{(2m+1)\xi} - \frac{1}{6} (\lambda \ln(a))^3 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)^2}{(2m+1)!} e^{(2m+1)\xi} \cdots \]

\[ = \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[ 1 - \frac{\lambda \ln(a)(2m+1)}{2} \right. \]

\[ + \frac{(\lambda \ln(a))^2(2m+1)^2}{12} - \frac{(\lambda \ln(a))^3(2m+1)^3}{6} + \cdots \right] \]

\[ = \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[ 1 + \frac{\lambda \ln(a)(2m+1)}{2} \right. \left. + \frac{(\lambda \ln(a))^2(2m+1)^2}{3!} + \cdots \right]^{-1} \]

or

\[ f(e^\xi) = \sum_{m=0}^{\infty} \frac{\lambda \ln(a)}{a^{\lambda(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi} \]
Chapter 5: Generalized Exponential Operators And Difference Equations

\[ f(x) = \sum \frac{\lambda \ln(a)}{a^{(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} x^{2m+1}. \] (5.3.8)

It is essentially the series defining \( g(x) \), provided that \( b_m \) is replaced by \( \frac{b_m \lambda \ln a}{a^{\lambda m} - 1} \). If, e.g., we take \( g(x) = \cos(x) \), we find

\[ f(x) = \sum \frac{\lambda \ln(a)}{a^{(2m)} - 1} \frac{(-1)^m}{(2m)!} x^{2m}, \] (5.3.8')

and for \( g(x) = e^{x^i} \), we get

\[ f(x) = \sum_{m=1}^{\infty} \frac{\lambda \ln(a)}{a^{\lambda m} - 1} \frac{x^{2m}}{m!} + \ln(x). \] (5.3.9)

We can therefore conclude that the primitive of a Jackson derivative can be constructed according to the above-quoted recipe.

This method can also be generalized and the concept of Jackson derivative extended to other forms of exponential operators. In this case we consider equation of the type

\[ \frac{f(x) \cos(\lambda \ln(a)) + \sqrt{1 - x^2} \sin(\lambda \ln(a))}{\lambda \ln(a)} = g(x), \] (5.3.10)

with the assistance of eq. (5.2.2), we write the eq. (5.3.10) as follows:

\[ \frac{a^{\lambda \sqrt{1-x^2}} f(x) - f(x)}{\lambda \ln(a)} = g(x) \]

or

\[ \frac{(a^{\lambda \sqrt{1-x^2}} - 1)}{\lambda \ln(a)} f(x) = g(x), \] (5.3.11)

by assuming \( g(x) \) is an odd function there taking \( x = \sin \xi \), we have

\[ \frac{d}{d\xi} = \frac{d}{dx} \frac{dx}{d\xi} = \cos \xi \frac{d}{dx} \]
Chapter 5: Generalized Exponential Operators And Difference Equations

\[ \frac{a^{\lambda \xi} - 1}{\lambda \ln(a)} f(\sin \xi) = g(\sin \xi), \]

\[ f(\sin \xi) = \frac{\lambda \ln(a)}{a^{\lambda \xi} - 1} g(\sin \xi), \quad (5.3.12) \]

Let us find out the expansion of the first factor of the r.h.s. of the eq. \((5.3.12)\) with the help of the eq. \((5.3.4)\), we have

\[ f(\sin \xi) = D_\xi^{-1} \sum_{n=0}^{\infty} \frac{B_n}{n!}(\lambda \ln(a))^n \left(\frac{d}{d\xi}\right)^n g(\sin \xi), \quad (5.3.13) \]

Since \(g(x)\) is an odd and analytic function, then \(g(\sin \xi)\), can be expanded by Taylor expansion such as \(g(\sin \xi) = \sum_{m=0}^{\infty} b_{2m+1} (\sin \xi)^{2m+1}\), we have from eq. \((5.3.13)\),

\[ f(\sin \xi) = D_\xi^{-1} \sum_{n=0}^{\infty} \frac{B_n}{n!}(\lambda \ln(a))^n \left(\frac{d}{d\xi}\right)^n \sum_{m=0}^{\infty} b_{2m+1} (\sin \xi)^{2m+1} \]

\[ = D_\xi^{-1} \sum_{n=0}^{\infty} \frac{B_n}{n!}(\lambda \ln(a))^n \left(\frac{d}{d\xi}\right)^n \sum_{m=0}^{\infty} b_{2m+1} \frac{e^{i\xi} - e^{-i\xi}}{2i} \]

\[ = D_\xi^{-1} \sum_{n=0}^{\infty} \frac{B_n}{n!}(\lambda \ln(a))^n \left(\frac{d}{d\xi}\right)^n \sum_{m=0}^{\infty} b_{2m+1} \frac{(-i)^{2m+1}}{2^{2m+1}} (-e^{-i\xi})^{2m+1} [1 - e^{2i\xi}]^{2m+1} \]

\[ = D_\xi^{-1} \left[ B_0 + \frac{B_1(\lambda \ln(a))}{1!} \left(\frac{d}{d\xi}\right) + \frac{B_2(\lambda \ln(a))^2}{2!} \left(\frac{d}{d\xi}\right)^2 + \frac{B_3(\lambda \ln(a))^3}{3!} \left(\frac{d}{d\xi}\right)^3 \right] \]

\[ \times \sum_{m=0}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} (-e^{-i\xi})^{2m+1} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^{2m+1-s} e^{[2(2m+1-s)]i\xi} \]
substituting the values of Bernoulli's numbers from eq. (5.3.5), we have

\[
D^{-1}_\xi \left[ 1 - \frac{\lambda \ln(a)}{2} \left( \frac{d}{d\xi} \right) + \frac{(\lambda \ln(a))^2}{12} \left( \frac{d}{d\xi} \right)^2 - \frac{(\lambda \ln(a))^3}{6} \left( \frac{d}{d\xi} \right)^3 \cdots \right] 
\]

\[
\times \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} = \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} D^{-1}_\xi \left[ \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} \right] 
\]

\[
\times \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} + \frac{(\lambda \ln(a))^2}{12} \left( \frac{d}{d\xi} \right)^2 \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} 
\]

\[
= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} D^{-1}_\xi \left[ \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} \right] 
\]

\[
- \frac{(\lambda \ln(a))^2}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s i[2(m-s)+1] e^{i[2(m-s)+1]\xi} 
\]

\[
+ \frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s i[2(m-s)+1] e^{i[2(m-s)+1]\xi} 
\]

\[
= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[ \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} \right] 
\]

\[
- \frac{(\lambda \ln(a))^2}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} 
\]
Chapter 5: Generalized Exponential Operators And Difference Equations

\[ \frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m + 1 \\ s \end{array} \right) (-1)^s [i(2(m-s)+1)] e^{i[2(m-s)+1]} \xi - \ldots \]

\[ = \sum_{m=1}^{\infty} \frac{b_{2m+1}}{2^{2m+1}} \left[ \sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m + 1 \\ s \end{array} \right) \frac{(-1)^s e^{i[2(m-s)+1]} \xi}{i[2(m-s)+1]} \right] \]

\[ = \sum_{m=1}^{\infty} \frac{b_{2m+1}}{2^{2m+1}} \left[ \sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m + 1 \\ s \end{array} \right) \right] \]

\[ \times \frac{\left(1 - \frac{(\lambda \ln(a))^3}{6} \right)[i(2(m-s)+1)]^3 + \ldots}{i[2(m-s)+1]} \left[ 1 + \frac{1}{2![i(2(m-s)+1)] \lambda \ln(a)]} + \frac{1}{3![i(2(m-s)+1)] \lambda \ln(a)]^2 + \ldots} \right] \]

\[ = \sum_{m=1}^{\infty} \frac{b_{2m+1}}{2^{2m+1}} \left[ \sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m + 1 \\ s \end{array} \right) \right] \]

\[ \times \frac{(-1)^s \lambda \ln(a) e^{i \xi}}{[i(2(m-s)+1)] \lambda \ln(a)]} + \frac{1}{2![i(2(m-s)+1)] \lambda \ln(a)]^2 + \frac{1}{3![i(2(m-s)+1)] \lambda \ln(a)]^3 + \ldots} \]

\[ = \sum_{m=1}^{\infty} \frac{b_{2m+1}}{2^{2m+1}} \left[ \sum_{s=0}^{2m+1} \left( \begin{array}{c} 2m + 1 \\ s \end{array} \right) \right] \]

\[ \frac{(-1)^s \lambda \ln(a) e^{i \xi}}{e^{i[2(m-s)+1]} \lambda \ln(a) - 1} \]

or

\[ f(\sin \xi) = \sum_{m=1}^{\infty} \frac{b_{2m+1} \lambda \ln(a)}{2^{2m+1}} (-i)^{2m+1} \]

Chapter 5: Generalized Exponential Operators And Difference Equations

\[ \times \left( \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s \left( \sqrt{1 - \sin \xi^2} + i \sin \xi \right)^{2(m-s)+1} / e^{i[2(m-s)+1]\lambda \ln(a) - 1} \right) \]

finally,

\[ f(x) = \sum_{m=1}^{\infty} b_{2m+1} \lambda \ln(a) \frac{\lambda}{2^{2m+1}} (-i)^{2m+1} \]

\[ \times \left( \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s \left( \sqrt{1 - x^2} + ix \right)^{2(m-s)+1} / a^{i[2(m-s)+1]\lambda - 1} \right) \]  

(5.3.15)

It is interesting to note that, in this case too, the criterion to evaluate the primitive of the Jackson derivative, associated with the operator (5.2.2), can easily be inferred.

Let us note that the procedure we have discussed can also be extended to the cases involving the generalized Gauss transform. In fact the solution of

\[ \frac{a^\lambda \left( \frac{d}{dx} \right)^2 - 1}{\lambda \ln(a)} f(x) = g(x) \]  

(5.3.16)

or in other words, we have

\[ \frac{\lambda \ln(a)}{a^\lambda \left( \frac{d}{dx} \right)^2 - 1} g(x) = f(x), \]  

(5.3.17)

let us suppose \( x = e^\xi \), then \( \frac{d}{d\xi} = \frac{d}{dx} \frac{dx}{d\xi} = e^\xi \frac{d}{dx} = x \frac{d}{dx} \).

Now from eq. (5.3.17), we have

\[ \frac{\lambda \ln(a)}{a^\lambda \left( \frac{d}{dx} \right)^2 - 1} g(e^\xi) = f(e^\xi). \]  

(5.3.18)

Further, after following the steps as we followed in getting the result (5.3.4), we obtain the expansion of first factor of l.h.s. as
Chapter 5: Generalized Exponential Operators And Difference Equations

\begin{equation}
\frac{\lambda \ln(a)}{a^{\lambda(a^2 - 1)}} = D_\xi^{-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} (\lambda \ln(a))^n \left( \frac{d}{d\xi} \right)^{2n}.
\end{equation}

(5.3.19)

Now by the virtue of the analyticity of \( g(x) \), we expand \( g(x) \), by Taylor series, i.e.

\[ f(e^\xi) = D_\xi^{-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} (\lambda \ln(a))^n \left( \frac{d}{d\xi} \right)^{2n} \sum_{m=0}^{\infty} b_m e^{mx}, \]

proceeding of the steps as proceeded in finding the result (5.3.7), we obtain

\[ f(e^\xi) = \sum_{m=1}^{\infty} b_m e^{\lambda \ln(a)m^2} \left( \frac{\lambda \ln(a)}{a^{\lambda(a^2 - 1)}} - 1 \right) e^{mx} + b_0 \xi \]

or in other words, if we take \( b_0 = 0 \), then

\[ f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a)}{a^{\lambda(a^2 - 1)}} x^m. \]

(5.3.20)

Further comments on the results of this section will be discussed in the forthcoming concluding section.

**Particular case:** If we substitute \( a = e \), in the eqs. (5.3.1), (5.3.2), (5.3.3), (5.3.4), (5.3.7), (5.3.8), (5.3.9), (5.3.10), (5.3.15), (5.3.16) and (5.3.20), then we obtain the eqs. (31), (32), (33), (34), (36), (37), (38), (39), (40), (41) and (42) on page numbers 657-658 due to Dattoli et al. [34].

### 5.4 The Remarks

In the previous section we have considered linear difference equations, a (trivial) non-linear example, similar to “Riccati” equation, is given below:

\[ f(ax) - f(x) + b^{n(x)} f(ax) f(x) = 0, \]

(5.4.1)
which can be solved using the auxiliary function \( g(x) = \frac{1}{f(x)} \) and thus getting

\[
\frac{1}{f(x)} - \frac{1}{f(ax)} + b^{\ln(x)} = 0,
\]

or

\[
g(ax) - g(x) = b^{\ln(x)}. \tag{5.4.2}
\]

From the operator (5.1.5), we have

\[
a^x \frac{d}{dx} g(x) - g(x) = b^{\ln(x)}, \tag{5.4.3}
\]

or in other words we write the above eq. (5.4.3) as

\[
g(x) = \frac{b^{\ln(x)}}{a^x \frac{d}{dx} - 1} = - \left[ 1 - a^x \frac{d}{dx} \right]^{-1} b^{\ln(x)}
\]

\[
= - \left[ 1 + a^x \frac{d}{dx} + a^{2x} \frac{d^2}{dx^2} + a^{3x} \frac{d^3}{dx^3} + \ldots \right] b^{\ln(x)}
\]

\[
= - \left[ b^{\ln(x)} + b^{\ln(a)} b^{\ln(x)} + b^{2 \ln(a)} b^{\ln(x)} + b^{3 \ln(a)} b^{\ln(x)} + \ldots \right]
\]

\[
= - \left[ 1 + b^{\ln(a)} + b^{2 \ln(a)} + b^{3 \ln(a)} + \ldots \right] b^{\ln(x)}
\]

\[
= - \left[ \frac{1}{1 - b^{\ln(a)}} \right] b^{\ln(x)} = \frac{b^{\ln(x)}}{b^{\ln(a)} - 1}
\]

thus finding as a particular solution

\[
f(x) = \frac{1}{g(x)} = \frac{b^{\ln(a)} - 1}{b^{\ln(x)}}. \tag{5.4.4}
\]

Moreover, in general, equations of the type
Chapter 5: Generalized Exponential Operators And Difference Equations

\[
\sum_{\alpha=0}^{N} b_{\alpha} f(F^{-1}(\alpha \ln(a) + F(x))) = e[f(x)]^{n}, \quad (5.4.5)
\]

standard perturbation methods can be used. At the lowest order in \( e(f = f_0 + ef_1) \)
we find

\[
\sum_{\alpha=0}^{N} b_{\alpha} f_0(F^{-1}(\alpha \ln(a) + F(x))) = 0 \quad (5.4.6)
\]

and

\[
\sum_{\alpha=0}^{N} b_{\alpha} f_1(F^{-1}(\alpha \ln(a) + F(x))) = R^n \int^{R} \frac{df}{\alpha^{(x)}}, \quad (5.4.7)
\]

where \( R^{\ln(a)} \) is one of the roots of the characteristic equation associated with
(5.4.5). The first-order contribution \( f_1 \) can therefore be evaluated by using
eq. (5.2.15), which should be modified as follows:

\[
f(x) = \frac{C(\int q(x) \frac{df}{\alpha^{(x)}}) \Psi' \left( R^{\ln(a)} \right)}{\Psi' \left( R^{\ln(a)} \right)}, \quad \Psi' \left( R^{\ln(a)} \right) = \left[ \frac{d}{dR} \Psi \left( R^{\ln(a)} \right) \right]_{R=b}, \quad (5.4.8)
\]

if \( R^{\ln(a)} \) is a simple root of the characteristic equation.

5.4(a) Particular case: The replacement of \( a \) with \( e \) in the eqs. (5.4.1),
(5.4.4) and (5.4.8) reduce to the eqs. (43), (45) and (49) of Dattoli et al.
[34].

Let us now go back to the problem of treating exponential operators of the
type

\[
\tilde{A}_{m,\lambda} = a^{\lambda (g(x) \frac{df}{dx})^{m}}. \quad (5.4.9)
\]
We have seen that for \( m = 2 \) and \( \lambda > 0 \) they can be viewed as generalized Gauss transform. Before discussing the problem more deeply, we recall the following important relation [27]:

\[
\begin{align*}
\alpha^{\lambda \left( \frac{d}{dx} \right)^m x^n} &= H_n^{(m)}(x, \lambda \ln(a)), \\
H_n^{(m)}(x, \lambda \ln(a)) &= n! \sum_{r=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \frac{(\lambda \ln(a))^{r} x^{n-mr}}{r!(n-mr)!},
\end{align*}
\]

(5.4.10)

which holds for negative or positive \( \lambda \) and \( H_n^{(m)}(x, \lambda \ln(a)) \) are Kampé de Feriét polynomials, and satisfy the identity

\[
\frac{\partial}{\partial \lambda} H_n^{(m)}(x, \lambda \ln(a)) = \ln(a) \left( \frac{\partial}{\partial x} \right)^m H_n^{(m)}(x, \lambda \ln(a)).
\]

(5.4.11)

According to eq. (5.4.8) we also find

\[
a^{\lambda \left( \frac{d}{dx} \right)^m} g(x) = a^{\lambda \left( \frac{d}{dx} \right)^m} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n H_n^{(m)}(x, \lambda \ln(a)).
\]

(5.4.12)

It is, therefore, easy to realize that

\[
\tilde{\Lambda}_{m, \lambda} x = \sum_{n=0}^{\infty} \phi_n H_n^{(m)}(F(x), \lambda \ln(a)),
\]

(5.4.13)

where we have assumed that the function \( F^{-1} \) can be expanded in power series

\[
F^{-1}(\zeta) = \sum_{n=0}^{\infty} \phi_n \zeta^n.
\]

(5.4.14)

It is clear that eq. (5.4.12) can be further handled to extend the action of the operators (5.4.8) to any function \( g(x) \). It is worth considering the possibility of extending the definition of operators (5.4.8) to the case of not necessarily integer \( m \). In the case of \( m = \frac{1}{2} \) eq. (5.4.9) should be replaced by
Chapter 5: Generalized Exponential Operators And Difference Equations

\[
\left\{
\begin{array}{l}
a^{\lambda\left(\frac{d}{dx}\right)^{\frac{1}{2}}} x^n = H^{(\frac{1}{2})}_n(x, \lambda \ln(a)), \\
H^{(\frac{1}{2})}_n(x, \lambda \ln(a)) = n! \sum_{r=0}^{2n} \frac{(\lambda \ln(a))^r x^{n-r}}{r! \Gamma(n - \frac{r}{2} + 1)}.
\end{array}
\right.
\] (5.4.15)

It is evident that in this case \(H^{(\frac{1}{2})}_n(x, \lambda \ln(a))\) is a relation analogous to (5.4.10) holds, namely

\[
\frac{\partial^2}{\partial \lambda^2} H^{(\frac{1}{2})}_n(x, \lambda \ln(a)) = (\ln(a))^2 \left( \frac{\partial}{\partial x} \right)^2 H^{(\frac{1}{2})}_n(x, \lambda \ln(a)).
\] (5.4.16)

or involving semi derivatives [189]

\[
\frac{\partial}{\partial \lambda} H^{(\frac{1}{2})}_n(x, \lambda \ln(a)) = \ln(a) \left( \frac{\partial}{\partial x} \right)^{\frac{1}{2}} H^{(\frac{1}{2})}_n(x, \lambda \ln(a)).
\] (5.4.16)'

This definition can be extended to any \(m = \frac{1}{p}\) \((p, \text{integer})\).

5.4(b) Particular case: When \(a = e\), the eqs. (5.4.9), (5.4.10), (5.4.11), (5.4.12), (5.4.13), (5.4.15) and (5.4.16) lead to Dattoli et al. [34] eqs. (50), (51), (52), (53), (54), (56) and (57).

The contents of this chapter have been accepted for publication in *Italian Journal of Pure and Applied Mathematics*, Vol. 30.
Chapter 6

Shift Operators On The Base $a \ (a > 0, \neq 1)$, Pseudo-Polynomials And Monomial Type Functions

ABSTRACT: The aim of the present chapter is to introduce and use the generalized exponential shift operators, operators on the base $a \ (a > 0, \neq 1)$, to deal with the families of pseudo-Kampé de Fériet polynomials, which can be viewed as the natural complement for the theory of fractional derivatives and partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of exponential operators, operators on the base $a \ (a > 0, \neq 1)$, providing generalized fractional forms of shift operators.

6.1 Introduction

In what follows, we consider analytic function $f(x)$ so that the corresponding Taylor expansion
converges to corresponding value of $f$ in a suitable neighborhood of $x$.

In 2003, Dattoli et al. [39] discussed the exponential operators, the operators on the natural base $e$.

In the present chapter we generalize the exponential operators [39] on the base $a$ ($a > 0, a \neq 1$), as follows:

$$\hat{A}_m = a^{\lambda \left( \frac{\partial}{\partial x} \right)^m}$$

In the case when $m = 1$, it reduces to the ordinary shift operator, while for $m = 2$ it can be identified with the operatorial version of the Gauss transform

$$a^{\lambda \left( \frac{\partial}{\partial x} \right)^2} f(x) = f(x + \lambda \ln(a))$$

Making use of the following identity, we have

$$e^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2b\xi} d\xi,$$

we find

$$a^{\lambda \left( \frac{\partial}{\partial x} \right)^2} f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + 2\sqrt{\pi \lambda \ln(a)} \xi \frac{\partial}{\partial x} \right] f(\xi) d\xi$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} f(x + 2\xi \sqrt{\lambda \ln(a)}) d\xi$$

or

$$a^{\lambda \left( \frac{\partial}{\partial x} \right)^2} f(x) = \frac{1}{2\sqrt{\pi \lambda \ln(a)}} \int_{-\infty}^{\infty} e^{\frac{(x-\xi)^2}{\lambda \ln(a)}} f(\xi) d\xi.$$ (6.1.4)
Both the eqs. (6.1.3) and (6.1.4) are solution of the partial differential equation:

\[
\begin{cases}
\frac{\partial}{\partial \lambda} F(x, \lambda \ln(a)) = \ln(a) \left( \frac{\partial}{\partial x} \right)^m F(x, \lambda \ln(a)), \\
F(x, 0) = f(x), \quad m = 1, 2.
\end{cases}
\]  

(A)

In case when \( m > 2 \), the exponential operator \( A_m = a^\lambda (\frac{d}{dx})^m \) provides formal solution for the generalized heat equation. It does not seem possible to associate it to any transformation of the Gauss type. We must, however, emphasize that the Hermite-Kampé de Féret polynomials [3] of the type

\[
H^{(m)}_n(x, y \ln(a)) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^{n-2r}(y \ln(a))^r}{(n-2r)!r!}
\]

or equivalently the Gould-Hopper polynomials [[199], p. 76, eq. (1.9) (6)]:

\[
g_n^m(x, y \ln(a)) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{r!(n-2r)!} x^{n-2r}(y \ln(a))^r
\]

are a solution of

\[
\begin{cases}
\frac{\partial}{\partial \lambda} F(x, \lambda \ln(a)) = \ln(a) \left( \frac{\partial}{\partial x} \right)^m F(x, \lambda \ln(a)), \\
F(x, 0) = x^n.
\end{cases}
\]  

(A')

or in other words[136] and [157]

\[
a^\lambda (\frac{d}{dx})^m x^n = H^{(m)}_n(x, y \ln(a))
\]

This last result is particularly important, since it allows the conclusion that if \( f(x) \) is an analytic function defined by the series expansion
\[ f(x) = \sum_n c_n x^n \]
then, by Taylor Theorem, we write
\[ a^\lambda \frac{\partial}{\partial \ln a} \sum_n c_n H_n^{(m)}(x, y \ln(a)) \]
The polynomials \( H_n^{(m)}(x, y \ln(a)) \) will be said to be the polynomials of index \( n \) and order \( m \).

**Particular case:** The replacement of \( a \) with \( e \) into the equations of this section give raise to the eqs. given in the first section of Dattoli et al. [39].

### 6.2 Exponential Operators on Base \( a (a > 0, \neq 1) \)

In 2003, extensive uses of exponential operators on the natural base \( e \) were used by Dattoli et al. [39]. Let us see the applications of the exponential shift operators on the base \( a (a > 0, \neq 1) \) used in [136] and [157], which play an important role in problems concerning pure and applied Mathematics [23].

\[ \hat{A} = a^{\lambda \varphi(x)} \frac{d}{dx} \]

For \( a = e \), the properties of the generalized shift operator are similar to that of discussed in ref. ([27, 192, 205]) and their importance for the solution of generalized difference equations are similar to that of stressed in ref [34]. The action of \( \hat{A} \) on a given function \( f(x) \) has been shown to be provided by [136] and [157]

\[ \hat{A} f(x) = f[F^{-1}(\lambda \ln(a) + F(x))], \]
where
Chapter 6: Shift Operators On The Base $a$ ($a > 0, \neq 1$), ...  

$F(x) = \int x \frac{d\xi}{q(\xi)}$ or $\frac{d}{dx}F(x) = \frac{1}{q(x)}$ or $q(x)\frac{d}{dx}F(x) = 1$ \hspace{1cm} (6.2.3)

defines the associated characteristic function of the generalized shift operator and $F^{-1}(\cdot)$ is its inverse. The proof of the above identity can be easily given, by Taylor Theorem noting that

$$\hat{A}F(x) = F(x) + \lambda \ln(a)$$

only if [136] and [157]

$$\left[q(x)\frac{d}{dx}, F(x)\right] = 1,$$

where $[\cdot, \cdot]$ denotes commutation brackets, that is:

$$q(x)\frac{d}{dx}F(x) = 1.$$ 

More generally, we can always write [136] and [157]

$$\hat{A}f(x) = f[F^{-1}(\lambda \ln(a) + F(x))],$$

It is evident that for $q(x) = 1$, $\hat{A}$ reduces to the ordinary shift operator, when we put $q(x) = x$ we find $F(x) = \ln(x)$ and $F^{-1}(x) = e^x$, we have

$$a^{\lambda x}f(x) = f(a^\lambda x)$$ \hspace{1cm} (6.2.4)

It is evident that the operators [136] and [157]

$$\hat{T}_x = q(x)\frac{d}{dx}$$ \hspace{1cm} (6.2.5)

can be viewed as an ordinary derivative, although $F(x)$ is a function, $[F(x)]^n$ behaves, under the action of $\hat{T}_x$, as an ordinary monomial, we obtain indeed

$$\hat{T}_x[F(x)]^n = n[F(x)]^{n-1},$$ \hspace{1cm} (6.2.6)
we can take advantage from this trivial property to discuss the rule associated with the use of operators like

$$\hat{A}_m = a^{\lambda(\hat{T})^m} = a^{\lambda(\frac{d}{dx})^m}, \quad (6.2.7)$$

for \( m \) integer or real.

According to the conclusion of the introductory section and to these last relations, we can introduce the polynomials

$$h_n^{(m)}(x, y \ln(a)) = H_n^{(m)}(F(x), y \ln(a)), \quad (6.2.8)$$

which satisfy the recurrences

$$\left[ F(x) + my \ln(a) \left( \frac{\partial}{\partial x} \right)^{m-1} \right] H_n^{(m)}(F(x), y \ln(a))$$

$$= n! \sum_{r=0}^{[n+1]} \frac{(n+1 - mr)[F(x)]^{n+1-mr} (y \ln(a))^r}{(n+1 - mr)!} + m \left[ n! \sum_{r=1}^{[n+1]} \frac{[F(x)]^{n+1-m(r+1)} (y \ln(a))^{r+1}}{(n+1 - m(r+1))!(r+1)!} \right]$$

$$= n! \sum_{r=0}^{[n+1]} \frac{(n+1 - mr)[F(x)]^{n+1-mr} (y \ln(a))^r}{(n+1 - mr)!} \frac{r!}{r!} + m \left[ n! \sum_{r=0}^{[n+1]} \frac{[F(x)]^{n+1-mr} r(y \ln(a))^r}{(n+1 - mr)! \cdot r!} \right]$$

$$= (n+1)n! \sum_{r=0}^{[n+1]} \frac{(y \ln(a))^r [F(x)]^{n+1-mr}}{r!(n+1 - mr)!}$$

or

$$\left[ F(x) + my \ln(a) \left( \frac{\partial}{\partial x} \right)^{m-1} \right] H_n^{(m)}(F(x), y \ln(a)) = H_{n+1}^{(m)}(F(x), y \ln(a)),$$
by using the eq. (6.2.8), we have

\[
\begin{align*}
F(x) + m y \ln(a) \left( \hat{T}_x \right)^{m-1} h_n^{(m)}(x, y \ln(a)) & = h_{n+1}^{(m)}(x, y \ln(a)), \\
\hat{T}_x [h_n^{(m)}(x, y \ln(a))] & = n h_{n-1}^{(m)}(x, y \ln(a)).
\end{align*}
\] (6.2.9)

Clearly \( h_n^{(m)}(x, y \ln(a)) \) are functions satisfying polynomial type identities and will therefore be called pseudo H.K.d.F.

It becomes also evident that identities of the following type

\[
\left[ F(x) + 2y \ln(a) \left( \hat{T}_x \right) \right]^n = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(F(x), y \ln(a)) \left( 2y \ln(a) \hat{T}_x \right)^s.
\] (6.2.10)

We show that eq. (6.2.10) follows from the Weyl identity. Note that since \( F(x) \) and \( 2y \ln(a) \hat{T}_x \) do not commute, therefore the use of the Newton binomial formula is not allowed. Multiplying the left-hand side of eq. (6.2.10) by \( \frac{t^n}{n!} \) and summing over \( n \), we find

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ F(x) + 2y \ln(a) \left( \hat{T}_x \right) \right]^n = e^{t(F(x) + 2y \ln(a) \hat{T}_x)},
\]

by using the Weyl identity, where \( \hat{P} = t F(x) \) and \( \hat{Q} = 2y t \ln(a) \hat{T}_x \)

since

\[
[[\hat{P}, \hat{Q}], \hat{P}] = [[\hat{P}, \hat{Q}], \hat{Q}] = 0
\]

further noting that

\[
[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P} = -2y \ln(a) t^2,
\]

and

\[
e^{\hat{P} + \hat{Q}} = e^{\hat{P} e^{-\frac{1}{2} [\hat{P}, \hat{Q}]}} e^{\frac{1}{2} [\hat{P}, \hat{Q}]}
\]
therefore, we can write

\[ e^{t(F(x)+2y \ln(a)\hat{T}_x)} = e^{tF(x)+y \ln(a)t^2} e^{2yt \ln(a)\hat{T}_x} \]

By expanding the exponential function, we obtain

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(F(x), y \ln(a)) \sum_{s=0}^{\infty} \frac{t^s}{s!} (2y \ln(a)\hat{T}_x)^s \]

\[ = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{t^{n+s}}{n!s!} H_n(F(x), y \ln(a))(2y \ln(a)\hat{T}_x)^s. \]

Setting \( k = n + s \) and inverting summations, we find

\[ \sum_{k=0}^{\infty} \frac{t^k}{k!} (F(x)+2y \ln(a)\hat{T}_x)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{s=0}^{k} \binom{k}{s} H_{k-s}(F(x), y \ln(a))(2y \ln(a)\hat{T}_x)^s. \]

Therefore, (6.2.10) follows from the comparison of the coefficients of \( \frac{t^k}{k!} \) in the last equation. By using the eq. (6.2.8) we find the following result

\[ \left[ F(x) + 2y \ln(a) \left( \hat{T}_x \right)^n \right] = \sum_{s=0}^{n} (2y \ln(a))^s \binom{n}{s} h_{n-s}(x, y \ln(a)) \left( \hat{T}_x \right)^s \]

and

\[ a^{y(\hat{T}_x)^m} f(F(x)) = e^{y \ln(a)\hat{T}_x} f(F(x)) \]

\[ = f(F(x) + my \ln(a) \left( \hat{T}_x \right)^{m-1}) a^{y(\hat{T}_x)^m} \]

(6.2.12)

which realize an extension of the ordinary Burchnall and Crofton [27, 192] identities valid for \( q(x) = 1 \).
Chapter 6: Shift Operators On The Base $a$ ($a > 0, \neq 1$), . . . 151

It is evident that all the wealth of properties of H.K.d.F. can be extended fairly straightforwardly to the functions $h_n^{(m)}(x, y \ln(a))$. The use of the previously discussed rules may greatly simplify the application of different types of exponential polynomials.

To give some examples, we note e.g. that

$$a_y(x \frac{\partial}{\partial x})^m (x^n) = e^{y \ln(a)}(x \frac{\partial}{\partial x})^m (x^n) = e^{y \ln(a)}(x \frac{\partial}{\partial x})^m e^n \ln(x)$$

$$= \sum_{r=0}^{\infty} \frac{n^r}{r!} H_r^{(m)}(\ln(x), y \ln(a)), \quad (6.2.13)$$

and since

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} H_r^{(m)}(x, y \ln(a)) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{k=0}^{[\frac{x}{a}]} t^r \sum_{k=0}^{[\frac{x}{a}]} \frac{x^{r-mk} (y \ln(a))^k}{(r-mk)!k!}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{[\frac{x}{a}]} \frac{x^{r-mk} (y \ln(a))^k}{(r-mk)!k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(xt)^n}{n!} \frac{(y \ln(a)t^m)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(y \ln(a)t^m)^k}{k!} = e^{xt+y \ln(a)t^m}$$

or

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} H_r^{(m)}(x, y \ln(a)) = e^{xt+y \ln(a)t^m} \quad (6.2.14)$$

or equivalently,

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} g_r^{(m)}(x, y \ln(a)) = e^{xt+y \ln(a)t^m}, \quad (6.2.15)$$

by replacing $t$ and $x$ with $n$ and $\ln(x)$ respectively, for the aforementioned Gould-Hopper polynomials [[199], p. 86, eq. (1.11) (27)], we find that

$$a_y(x \frac{\partial}{\partial x})^m (x^n) = e^{y \ln(a)}(x \frac{\partial}{\partial x})^m (x^n)$$
Chapter 6: Shift Operators On The Base \( a \) \((a > 0, \neq 1)\), …

\[ = x^n e^{j \ln(a) n m} = x^n a^{j m m}. \quad (6.2.16) \]

It is now evident that if \( f(x) \) is specified by any analytic function \((f(x) = \sum_n c_n x^n)\), then

\[ a^{j (x \frac{\partial}{\partial x})^m} f(x) = \sum_n c_n x^n a^{j m m}, \quad (6.2.17) \]

provided that the last series is convergent. Further comments on this last result will be presented in the concluding section (9.4).

A further example of exponential operator is provided by the case

\[ q(x) = (x - b)^2, \]

\[ F(x) = \int_x^x \frac{d\xi}{(x - b)^2} = -\frac{1}{(x - b)}, \]

for which we find

\[ a^{y [(x-b)^2 \frac{\partial}{\partial x}]^2} \left[ \frac{x-b}{x} \right] = e^{y \ln(a) [(x-b)^2 \frac{\partial}{\partial x}]^2} \left[ \frac{x-b}{x} \right] \]

\[ = e^{y \ln(a) [(x-b)^2 \frac{\partial}{\partial x}]^2} \left[ \frac{1}{1 + \frac{b}{x-b}} \right] \]

\[ = e^{y \ln(a) [(x-b)^2 \frac{\partial}{\partial x}]^2} \sum_{s=0}^\infty \left( -\frac{b}{x-b} \right)^s \]

\[ = \sum_{s=0}^\infty (b)^s e^{y \ln(a) [(x-b)^2 \frac{\partial}{\partial x}]^2} \left( -\frac{1}{x-b} \right)^s \]

let

\[-\frac{1}{x-b} = t, \]

therefore,
\[ \frac{\partial t}{\partial x} = \frac{1}{(x-b)^2} \]

and

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial t} \frac{\partial t}{\partial x} = \frac{1}{(x-b)^2} \frac{\partial}{\partial t} \]

or

\[ \frac{\partial}{\partial t} = (x-b)^2 \frac{\partial}{\partial x} \]

Now

\[ \sum_{s=0}^{\infty} (b)^s e^{y \ln(a)} \left( \frac{\partial}{\partial x} \right)^2 t^s = \sum_{s=0}^{\infty} (b)^s H_s(t, y \ln(a)) \]

or

\[ a^y \left[ (x-b)^2 \frac{\partial}{\partial x} \right]^2 \left[ \frac{x-b}{x} \right] = \sum_{s=0}^{\infty} (b)^s H_s \left( -\frac{1}{x-b}, y \ln(a) \right). \tag{6.2.18} \]

6.2 **Particular case:** The results determined by Dattoli et al. ([39]; in 2 section) can be obtained with the substitution \( a = e \), into the equations of this section.

6.3 **Fraction Order Exponential Operators** \( a (a > 0, \neq 1) \)

In this section we will discuss fractional shift operators of the type

\[ \%_{\mu} = a^{\left( q(x) \frac{d}{dx} \right)^\mu} \]  

(6.3.1)

with \( q(x) = 1 \) and \( \mu \) any real number such that \( 0 < \mu < 1 \).

Before entering into the main body of the discussion, we recall that the
Riemann-Liouville derivative of fractional order $m$ is defined by (see [186]; see also [[199], p. 286, eq. (5.1) (8)])

$$
\left( \frac{d}{dx} \right)^\nu f(x) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\nu-1} f(t) dt,
$$
(6.3.2)

where $m$ is a positive integer such that $m - 1 < \nu < m$.

Accordingly, we get

$$
a^\nu \left( \frac{d}{dx} \right)^\nu x^n = H_n^{(\nu)}(x, y \ln(a))
$$
(6.3.3)

where $H_n^{(\nu)}(x, y \ln(a))$ are H.K.d.F. pseudo-polynomials of fractional order, defined by

$$
H_n^{(\nu)}(x, y \ln(a)) = n! \sum_{r=0}^{\infty} \frac{x^{n-\mu r}(y \ln(a))^r}{\Gamma(n-\mu r + 1)! r!},
$$
(6.3.4)

whose validity can be easily checked by direct expansion of the operator and by the fact that [186]

$$
\left[ \left( \frac{d}{dx} \right)^\mu \right]^k x^n = \left( \frac{d}{dx} \right)^{k\mu} x^n
= \frac{\Gamma(n + 1)x^{n-\mu r}}{\Gamma(n-\mu r + 1)!}, \quad (\mu < n + 1).
$$
(6.3.5)

According to the previous discussions $H_n^{(\nu)}(x, y \ln(a))$ is the natural solution of the fractional Cauchy problem

$$
\begin{cases}
\frac{\partial}{\partial y} u(x, y \ln(a)) = \ln(a) \left[ \frac{\partial}{\partial x} \right]^\mu u(x, y \ln(a)), \\
u(x, 0) = x^n.
\end{cases}
$$
(B)

We must underline that the function $H_n^{(\nu)}(x, y \ln(a))$ is an extension of the ordinary H.K.d.F. or Gould-Hopper polynomials.

More generally we can solve the problem (B) with the general condition.
Chapter 6: Shift Operators On The Base $a$ ($a > 0, \neq 1$),...

\[ u(x, 0) = f(x) = \sum_n c_n x^n \]  \hspace{1cm} (6.3.6)

according to the following relation

\[ u(x, y \ln(a)) = f(x) = \sum_n c_n H_n^{(\mu)}(x, y \ln(a)). \]  \hspace{1cm} (6.3.7)

It is obvious that we can combine relevant to generalized shift operators exponential operators of the type

\[ \hat{A}_\mu = a^{(q(x) \frac{d}{dx})^\mu}. \]

With this purpose in view, we consider the problem

\[
\begin{align*}
\frac{\partial}{\partial y} u(x, y \ln(a)) &= \ln(a) \left[ q(x) \frac{\partial}{\partial x} \right]^\mu u(x, y \ln(a)), \\
u(x, 0) &= (F(x))^n; 
\end{align*}
\]

where

\[ F(x) = \int^x \frac{d\xi}{q(\xi)}. \]

It is fairly natural to write the solution of (C) as follows:

\[ u(x, y \ln(a)) = H_n^{(\mu)}(F(x), y \ln(a)) \]  \hspace{1cm} (6.3.8)

This last result completes the purposes of the present chapter aimed at providing a general framework for the families of exponential operators.

6.3 **Particular case:** The substitution $a = e$, reduces the eqs. (6.3.1) to (6.3.8) to the results due to Dattoli et al. [[39]; p. 220-221, section 3].
6.4 Concluding Remarks

In the section 6.3 we have seen that the theory of exponential operators can be conveniently complemented by the use of functions satisfying recurrences of quasi monomial nature.

In these concluding remarks we will discuss the introduction of a family of functions which can be viewed as a fairly natural consequence of the so far developed formalism.

We consider indeed the case of logarithmic Bessel functions, whose generating function can be cast in the form

\[ G(x, \vartheta) = x^{i \sin(\vartheta)} = \sum_{n=-\infty}^{\infty} e^{i n \vartheta} J_n(\ln(x)), \quad (6.4.1) \]

where \( J_n(x) \) denote the first kind cylinder Bessel functions, that is

\[ J_n(z) = \left( \frac{1}{2} \right)^n \sum_{k=0}^{\infty} \frac{(-1/4)^k z^k}{k!(n+k)!}. \]

It is evident that we can take advantage from the discussion of the previous sections, to consider the following problem

\[ a^y(x \frac{\partial}{\partial x})^2 x^{i \sin(\vartheta)} = e^{y \ln(a)}(x \frac{\partial}{\partial x})^2 x^{i \sin(\vartheta)} \]

\[ = e^{i \sin(\vartheta)(\ln(x) + 2y \ln(a) x \frac{\partial}{\partial x})}. \quad (6.4.2) \]

The exponential can be decoupled by means of the Weyl rule

\[ e^{\hat{P}+\hat{Q}} = e^{\hat{P}} e^{\hat{Q}} e^{-\frac{\lambda}{2} [\hat{P}, \hat{Q}]}, \]

where

\[ [\hat{P}, \hat{Q}] = \hat{P} \hat{Q} - \hat{Q} \hat{P}, \]
Chapter 6: Shift Operators On The Base $a$ ($a > 0, \neq 1$), ...

by setting indeed

$$\hat{P} = i \sin(\vartheta) \ln(x),$$
$$\hat{Q} = 2iy \ln(a)(\sin(\vartheta))x \frac{\partial}{\partial x},$$

we find

$$[\hat{P}, \hat{Q}] = 2y \ln(a)[\sin(\vartheta)]^2,$$

thus getting

$$a^y(x \frac{\partial}{\partial x})^2 x^i \sin(\vartheta) = x^i \sin(\vartheta)e^{-y \ln(a)[\sin(\vartheta)]^2}.$$ (6.4.5)

Which is the generating function of a two-variable Bessel function, namely

$$x^i \sin(\vartheta)e^{-y \ln(a)[\sin(\vartheta)]^2} = \sum_{n=-\infty}^{\infty} e^{im\theta}(\hat{h}J_n(x, y \ln(a))).$$ (6.4.6)

$$\hat{h}J_n(x, y \ln(a)) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(\ln(x), y \ln(a))}{2^{n+2r}r!(n+r)!}.$$ (6.4.7)

It is evident that we ended up with a Bessel type function generalizing those of Hermite nature discussed in ref. [33]. It is worth emphasizing that the above equations satisfy a partial differential equation of the type

$$\left\{ \frac{\partial}{\partial y}(\hat{h}J_n(x, y \ln(a))) = \ln(a) \left(x \frac{\partial}{\partial x}\right)^2 (\hat{h}J_n(x, y \ln(a))), \right. \left. \hat{h}J_n(x, 0) = J_n(\ln(x)). \right\}$$ (6.4.8)

It is evident that the above considerations can be extended to any generating function of the type [39]

$$a^{iF(x) \sin(\vartheta)}.$$ (6.4.9)
Before concluding we will show how the combined use of integral transform and the previous formalism allows the derivation of further important relations. According to the identity [206]

\[ a^{\lambda \hat{p}^2} = e^{\lambda \ln(a) \hat{p}^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2\sqrt{\lambda \ln(a)} \xi} d\xi. \]  

\[ (6.4.10) \]

we can easily conclude that the polynomials \( h_n^{(2)}(x, y) \) can also be realized in terms of the integral representation

\[ a^y \left[ q(\xi) \right] ^2 [F(x)]^n = e^{y \ln(a)} \left[ q(\xi) \right] ^2 [F(x)]^n \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} [F(x) + 2\sqrt{y \ln(a)} \xi]^n d\xi. \]  

\[ (6.4.11) \]

Going back to eq. (6.2.17) and specializing for \( m = 2 \), the use of the above relations allows to state the following identity

\[ \sum_{n=0}^{\infty} \frac{(-1)^n a^{-ym^2}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-y \ln(a)n^2}}{n!} \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{-\cos(2\sqrt{y} \xi)} \cos(\sin(2\sqrt{y} \xi)) d\xi. \]

Finally since

\[ e^{-yd} = \frac{y}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{y^2}{4t}} e^{-d^2t} dt \]  

\[ (6.4.12) \]

by replacing \( d \) with \( (\ln(a) T_x)^{\frac{1}{2}} \) and by using the previously discussed rules we find

\[ e^{-y(\ln(a) T_x)^{\frac{1}{2}}} = \frac{y}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{y^2}{4t}} e^{-(\ln(a) T_x)^{t}} dt \]

\[ = \frac{y}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{y^2}{4t} a^{-T_x^{t}}} dt. \]
now

\[ e^{-y(\ln(a)T_x)^{\frac{1}{2}}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t} e^{-\frac{y^2}{4t}} a^{-\frac{1}{2}(T_x)t} f(x) \, dt \]

or

\[ e^{-y(\ln(a)T_x)^{\frac{1}{2}}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t} e^{-\frac{y^2}{4t}} f(F^{-1}(F(x) - t \ln(a))) \, dt. \]  

(6.4.13)

Which realizes the transform providing the action of a fractional generalized shift operator on a given function, the validity of both (6.4.11) and (6.4.12) is limited to the case in which the integral converges.

6.4 Particular case: The substitution of \( a = e \) into the eqs. (6.4.1) to (6.4.13) give raise to the eqs. (15) to (26) due to Dattoli et al. [39].

The sections 6.1, 6.2 and 6.4 of this chapter have been published in *International Transactions in Mathematical Sciences and Computer*, 2, No. 2, (2009), 453-462 and the section 6.3 has been accepted for publication in International Journal of Mathematical Analysis.
Chapter 7

Generalized Operational Methods, Fractional Operators And Special Polynomials

ABSTRACT: The aim of the present chapter is to introduce and use the generalized exponential operators, operators on the base $a$ ($a > 0, \neq 1$), to deal with the families of partial differential equations of evolution type, to treat the problems involving fractional operators. Further, the properties of the families of special polynomials or special functions (like the Riemann $\zeta$ function), naturally associated with the proposed formalism.

7.1 Introduction

In 2003, Dattoli [37] used operators on the natural base $e$ for determining fractional operators, integral transforms and new family of special polynomials. For determining the new family of polynomials, we introduce and use operators on the base $a$ ($a > 0, \neq 1$).

It is well known that Hermite and Laguerre polynomials are defined through operational identities (see[33]) i.e. through the exponential operators defined on natural base $e$ [37].
We define Hermite polynomial through the operational identities, the exponential operators defined on the base $a$ ($a > 0, \neq 1$), as follows:

$$a^y \frac{\partial^2}{\partial x^2} x^n = e^{y \ln(a)} \frac{\partial^2}{\partial x^2} x^n$$

$$= \sum_{r=0}^{\infty} \frac{(\ln(a))^{\frac{\partial^2}{\partial x^2}})^r}{r!} x^n = \sum_{r=0}^{\infty} \frac{\left(y \ln(a)\right)^r n! x^{n-2r}}{(n-2r)! r!}$$

or

$$a^y \frac{\partial^2}{\partial x^2} x^n = H_n(x, \ln(a)y). \quad (7.1.1)$$

Similarly, we define Laguerre polynomial through the operational identities, the exponential operators defined on the base $a$ ($a > 0, \neq 1$), given below:

$$a^{-y} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left[ \frac{(-1)^n}{n!} x^n \right] = e^{-y \ln(a)} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left[ \frac{(-1)^n}{n!} x^n \right]$$

$$= \sum_{r=0}^{\infty} \frac{(-y \ln(a))^r \frac{\partial}{\partial x} \frac{\partial}{\partial x}}{r!} \left[ \frac{(-1)^n}{n!} x^n \right] = \sum_{r=0}^{\infty} \frac{(-1)^{n+r} (y \ln(a))^r \left( \frac{\partial}{\partial x} \right)^r x^n}{r!(n-r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{n+r} (y \ln(a))^r n! x^{n-r}}{r![(n-r)!]^2} = n! \sum_{r=0}^{\infty} \frac{(-1)^r (y \ln(a))^{n-r} x^r}{(n-r)![(r!)^2}$$

or

$$a^{-y} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left[ \frac{(-1)^n}{n!} x^n \right] = L_n(x, y \ln(a)). \quad (7.1.2)$$

which will play crucial role in applications [208, 209].

The polynomials (7.1.1) and (7.1.2) are generalized forms of Hermite and Laguerre polynomials and are linked to the ordinary case by
\[ (-i)^n (y \ln(a))^{\frac{3}{2}} H_n \left( \frac{ix}{2 \sqrt{y \ln(a)}} \right) = (-i)^n (y \ln(a))^{\frac{3}{2}} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^r n! \left( \frac{2x}{2 \sqrt{y \ln(a)}} \right)^{n-2r}}{r!(n-2r)!} \]

\[\begin{align*}
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^2 n! (y \ln(a))}{r!(n-2r)!} x^{n-2r} &= n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))}{r!(n-2r)!} x^{n-2r} \\
&\quad\text{or} \\
(-i)^n (y \ln(a))^{\frac{3}{2}} H_n \left( \frac{ix}{2 \sqrt{y \ln(a)}} \right) = H_n(x, y \ln(a)). \quad (7.1.3) \end{align*}\]

and

\[\begin{align*}
(y \ln(a))^n L_n \left( \frac{x}{y \ln(a)} \right) &= (y \ln(a))^n \sum_{r=0}^{n} \frac{(-1)^r n! \left( \frac{x}{y \ln(a)} \right)^r}{(r!)^2 (n-r)!} \\
&\quad\text{or} \\
&\quad\text{or} \\
(y \ln(a))^n L_n \left( \frac{x}{y \ln(a)} \right) = L_n(x, y \ln(a)). \quad (7.1.4) \end{align*}\]

The use of the following identity [209]

\[ a^{B^2} = e^{B^2 \ln(a)} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2 + 2B \sqrt{\ln(a)} \xi) d\xi. \quad (7.1.5) \]

enables us to concentrate on the eq. (7.1.1), in particular on the generalized exponential operator, which, according to the standard procedure [27], can be written in the following form

\[ a^{y \frac{\partial^2}{\partial x^2}} = e^{y \ln(a) \frac{\partial^2}{\partial x^2}} \]

\[\begin{align*}
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\xi^2 + 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) d\xi. \quad (7.1.6) \end{align*}\]
The use of the above identity and of the following fact

\[ a^{\frac{\partial}{\partial x}} f(x) = f(x + \lambda \ln(a)) \]  

(7.1.7)

allows to conclude that the polynomials \( H_n(x, y \ln(a)) \) satisfy the integral representation

\[ H_n(x, y \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\xi^2 + 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) x^n d\xi \]

or

\[ H_n(x, y \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2) \exp \left( 2\xi \sqrt{y \ln(a)} \frac{\partial}{\partial x} \right) x^n d\xi \]

or

\[ H_n(x, y \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2)[x + 2\xi \sqrt{y \ln(a)}]^n d\xi. \]  

(7.1.8)

The above example shows

(a) it is possible to define polynomials by means of an operational identity,
(b) such operational identity can in turn be used to derive integral representation.

1(a). Particular case: The substitution \( a = e \), into the eqs. (7.1.1), (7.1.2), (7.1.3), (7.1.4), (7.1.6), (7.1.7) and (7.1.8) reduce to the results (1), (2), (3), (4), (5) and (6) due to Dattoli [37].

Methods employing the combined use of generalized exponential operators and integral transforms provide a powerful tool for the solution of P.D.E. of evolution type. An appropriate example follows from the equation associated with the Black-Scholes financial model [210]

\[
\left\{ \begin{array}{l}
\frac{1}{(\ln(a))^2} \frac{\partial}{\partial \tau} A = S^2 \frac{\partial^2}{\partial S^2} A + \lambda S \frac{\partial}{\partial S} A - \lambda A \\
A(S, 0) = f(S)
\end{array} \right. 
\]  

(7.1.9)
which can be rewritten as

\[
\frac{1}{(\ln(a))^2} \frac{\partial}{\partial \tau} A = \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 A - \left( \frac{\lambda + 1}{2} \right)^2 A
\]

or

\[
\frac{\partial}{\partial \tau} A - \left[ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right] (\ln(a))^2 A = 0 \quad (7.1.10)
\]

which is a linear differential equation in \(A\).

whose Integrating Factor is

\[
\exp \left[ - \frac{1}{(\ln(a))^2} \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right] (\ln(a))^2 \tau
\]

and which admits the formal solution

\[
A(S, \tau(\ln(a))^2) = \exp \left[ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 - \left( \frac{\lambda + 1}{2} \right)^2 \right] (\ln(a))^2 \tau \int f(S) dS
\]

\[
= \exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 (\ln(a))^2 \tau \right] \times \exp \left[ \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right)^2 \right] (\ln(a))^2 \tau \int f(S) dS
\]

\[
= \exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 \tau(\ln(a))^2 \right] \frac{\sqrt{\pi}}{\sqrt{\tau^2 \ln(a)^2}} \int_{-\infty}^{\infty} \exp \left[ - \xi^2 + 2 \left( S \frac{\partial}{\partial S} + \frac{\lambda - 1}{2} \right) \sqrt{\tau \ln(a)} \right] f(S) d\xi,
\]

or

\[
A(S, \tau(\ln(a))^2) = \frac{\exp \left[ - \left( \frac{\lambda + 1}{2} \right)^2 \tau(\ln(a))^2 \right]}{\sqrt{\pi}}
\]

\[
\times \int_{-\infty}^{\infty} \exp \left[ - \xi^2 + (\lambda - 1) \xi \sqrt{\tau \ln(a)} \right] \exp \left( 2\sqrt{\tau \ln(a)} S \frac{\partial}{\partial S} \right) f(S) d\xi, \quad (7.1.12)
\]
by the dilatation operator, we have
\[ a^{\lambda \frac{\partial}{\partial x}} f(x) = \exp \left( \lambda \ln(a) x \frac{\partial}{\partial x} \right) = f(x a^\lambda) \] \quad (7.1.13)

from the eqs. (7.1.12) and (7.1.13), we have
\[ A(S, \tau(\ln(a))^2) = \frac{\exp \left[ - \left( \frac{\lambda+1}{2} \right)^2 \tau(\ln(a))^2 \right]}{\sqrt{\pi}} \]

\[ \times \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + (\lambda - 1)\xi \sqrt{\tau} \ln(a) \right] f(\exp(2\xi \sqrt{\tau} \ln(a))S) d\xi \]

or
\[ A(S, \tau(\ln(a))^2) = \frac{\exp \left[ - \left( \frac{\lambda+1}{2} \right)^2 \tau(\ln(a))^2 \right]}{\sqrt{\pi}} \]

\[ \times \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + (\lambda - 1)\xi \sqrt{\tau} \ln(a) \right] f(a^{2\xi \sqrt{\tau}}S) d\xi. \] \quad (7.1.15)

This last result shows that methods employing operational techniques can be used in fairly wide context and allow noticeable flexibility.

In this chapter we introduce new families of special polynomials starting from a suitable definition. We shall show that the concept we develop is useful in different variety including the theory of fractional derivatives.

1(b). Particular case: When we put \( a = e \), into the eqs. (7.1.9), (7.1.10), (7.1.11), (7.1.13) and (7.1.15) reduce to the results (7), (8), (9), (10) and (11) due to Dattoli [37].
Chapter 7: Generalized Operational Methods, Fractional Operators...

7.2 Fractional Operators and a New Class of Special Polynomials

As we know from the theory of fractional operators, i.e. operators raised to a fractional power, is the identity [[199], p.218]

\[ \lambda^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\lambda t) t^{\nu-1} dt \]  

(7.2.1)

It is therefore evident that

\[ \left[ \alpha - \ln(a) \frac{\partial^2}{\partial x^2} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp \left[ \left\{ - \left( \alpha - \ln(a) \frac{\partial^2}{\partial x^2} \right) t \right\} \right] t^{\nu-1} f(x) dt \]

or

\[ \left[ \alpha - \ln(a) \frac{\partial^2}{\partial x^2} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} a^\frac{2}{\partial x^2} f(x) dt. \]  

(7.2.2)

In the case in which \( f(x) = e^{-x^2} \) and using the eq. (7.1.5), produces

\[ \left[ \alpha - \ln(a) \frac{\partial^2}{\partial x^2} \right] e^{-x^2} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} a^\frac{2}{\partial x^2} e^{-x^2} dt, \]  

(7.2.3)

let us consider the under-brace function of eq. (7.2.3)

\[ a^\frac{2}{\partial x^2} e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\xi^2 + 2\sqrt{\ln(a)} \xi \frac{\partial}{\partial x} \right] e^{-x^2} d\xi \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\xi^2 \right] \exp \left[ 2\sqrt{\ln(a)} \xi \frac{\partial}{\partial x} \right] e^{-x^2} d\xi \]  

(7.2.4)
since under the brace is the shift operator, therefore, from eqs. (7.1.7) and (7.2.4), we have

\[
a^t \frac{\partial^2}{\partial x^2} e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\xi^2 \right] \exp \left[ -(2\sqrt{t \ln(a)}\xi + x^2) \right] d\xi
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\xi^2 \right] \exp \left[ -(4t \ln(a)\xi^2 + x^2 + 2\sqrt{t \ln(a)}\xi x) \right] d\xi
\]

\[
= \frac{e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -(1 + 4t \ln(a))\xi^2 + 2(2\sqrt{t \ln(a)}\xi x) \right] d\xi
\]

\[
= \frac{e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -(1 + 4t \ln(a)) \left( \xi + \frac{2\sqrt{t \ln(a)}x}{1 + 4t \ln(a)} \right)^2 \right] d\xi
\]

\[
= \exp \left[ -\frac{x^2}{1 + 4t \ln(a)} \right] \int_{-\infty}^{\infty} \exp \left[ - \left( \frac{1 + 4t \ln(a)}{\sqrt{1 + 4t \ln(a)}} \xi + \frac{2\sqrt{t \ln(a)}x}{\sqrt{1 + 4t \ln(a)}} \right)^2 \right] d\xi,
\]

let us suppose that

\[
(\sqrt{1 + 4t \ln(a)})\xi + \frac{2\sqrt{t \ln(a)}x}{\sqrt{1 + 4t \ln(a)}} = z,
\]

therefore

\[
a^t \frac{\partial^2}{\partial x^2} e^{-x^2} = \frac{\exp \left[ -\frac{x^2}{1 + 4t \ln(a)} \right]}{\sqrt{1 + 4t \ln(a)}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-z^2] dz
\]

\[
= \frac{\exp \left[ -\frac{x^2}{1 + 4t \ln(a)} \right]}{\sqrt{1 + 4t \ln(a)}} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \exp[-z^2] dz = \frac{\exp \left[ -\frac{x^2}{1 + 4t \ln(a)} \right]}{\sqrt{1 + 4t \ln(a)}}
\]
finally, substituting this value into the eq. (7.2.3), we have

\[
\left[ \alpha - \ln(a) \frac{\partial^2}{\partial x^2} \right]^{-\nu} e^{-x^2} = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{e^{-\alpha t} t^{\nu-1}}{\sqrt{1 + 4t \ln(a)}} \exp \left[ -\frac{x^2}{1 + 4t \ln(a)} \right] dt.
\]

(7.2.5)

Let us now consider the simpler case

\[ f(x) = x^n. \]

According to eq. (7.2.2) and eq. (7.1.1), we have

\[
\left[ 1 - y \ln(a) \frac{\partial^2}{\partial x^2} \right]^{-\nu} x^n = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t^{\nu-1}} \frac{a^{\nu/2} x^n}{\sqrt{1 + 4t \ln(a)}} dt,
\]

(7.2.6)

substituting the value of the under-brace from the eq. (7.1.1) into the eq. (7.2.6), we get the desired result

\[
\left[ 1 - y \ln(a) \frac{\partial^2}{\partial x^2} \right]^{-\nu} x^n = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t^{\nu-1}} H_n(x, y \ln(a)) dt,
\]

(7.2.7)

where

\[
\left( y \ln(a) \right)^{\nu} \sum_{r=0}^{\lfloor n/\nu \rfloor} \frac{\nu^r (\nu + r - 1)}{r! (n - r)!} t^r x^{n-r} dt
\]

(7.2.8)

where \( (\nu)_r \) is the pochhammer symbol.

2(a). Particular case: If we put \( a = e \), into the eqs. (7.2.2), (7.2.5), (7.2.7) and (7.2.8) lead to the eqs. (13), (14), (15) and (16) due to Dattoli [37].
By introducing the notation (see [37], p. 154, eqs. (17), (18), (19))

\[(\nu y)^r = (\nu)_r y^r \quad (7.2.9)\]

and the operators \(\hat{Y}_\nu\) and \(\hat{D}_{y, \nu}\)

such that

\[\hat{Y}_\nu[(\nu y)^r] = (\nu y)^{r+1}, \quad (7.2.10)\]
\[\hat{D}_{y, \nu}[(\nu y)^r] = r(\nu y)^{r-1}. \quad (7.2.11)\]

It is not difficult to realize that the polynomials (7.2.8) are an umbral image of the ordinary Hermite polynomials and that satisfy the recurrences

\[
\frac{\partial}{\partial x}(\nu H_n(x, y \ln(a))) = \frac{\partial}{\partial x} \left[ n! \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \frac{(\nu)_r (y \ln(a))^r x^{n-2r}}{r!(n-2r)!} \right] 
= n \nu H_{n-1}(x, y \ln(a))
\]

or

\[
\frac{\partial}{\partial x}(\nu H_n(x, y \ln(a))) = n \nu H_{n-1}(x, y \ln(a)). \quad (7.2.12)
\]

and

\[
[n + 2 \ln(a) \hat{Y}_\nu \frac{\partial}{\partial x}] \nu H_n(x, y \ln(a)) 
= n \left[ \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \frac{(\nu y \ln(a))^r x^{n-1-2r}}{r!(n-2r)!} \right] + 2 \ln(a) n \left[ \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \frac{\hat{Y}_\nu(\nu y)^r (\ln(a))^r x^{n-1-2r}}{(r-1)!(n-1-2r)!} \right]
\]
\[ n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{(n + 1 - 2r)}{(n + 1 - 2r)!} x^{n+1-2r} \right] + n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{2r(\nu y)^{r+1}(\ln(a))^{r+1}}{r!(n - 1 - 2r)!} x^{n-1-2r} \right] \]

\[ = n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{(n + 1 - 2r)}{(n + 1 - 2r)!} x^{n+1-2r} \right] + n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{2r(\nu y)^r(\ln(a))^r}{r!(n + 1 - 2r)!} x^{n+1-2r} \right] - n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(\nu y)^{r+1}(\ln(a))^{r+1}}{r!(n - 1 - 2r)!} x^{n-1-2r} \right] \]

\[ = (n+1)! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!(n + 1 - 2r)!} x^{n+1-2r} \right] - n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{2r(\nu y \ln(a))^r}{r!(n + 1 - 2r)!} x^{n+1-2r} \right] \]

\[ + n! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{2r(\nu y \ln(a))^r}{r!(n + 1 - 2r)!} x^{n+1-2r} \right] = (n+1)! \left[ \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!(n + 1 - 2r)!} x^{n+1-2r} \right] \]

or

\[ \left[ x + 2 \ln(a) \nu y \frac{\partial}{\partial x} \right] \nu H_n(x, y \ln(a)) = \nu H_{n+1}(x, y \ln(a)) \quad (7.2.13) \]

and the following differential equations as well

\[ \left[ 2 \ln(a) \nu y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n \right] \nu H_n(x, y \ln(a)) \]

\[ = 2 \ln(a) \left[ n! \sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{(\nu y \ln(a))^r (\ln(a))^r}{r! (n - 2 - 2r)!} x^{n-2-2r} \right] + \left[ n! \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(\nu y)^{r+1}(\ln(a))^{r+1}}{r!(n - 1 - 2r)!} x^{n-1-2r} \right] \]

\[ - n \left[ n! \sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r! (n - 2r)!} x^{n-2r} \right] \]

\[ = 2n! \left[ \sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{(r + 1)(\nu y)^{r+1}(\ln(a))^{r+1}}{(r+1)! (n - 2 - 2r)!} x^{n-2-2r} \right] + n! \left[ \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(\nu y \ln(a))^r (n - 2r)}{r! (n - 2r)!} x^{n-2r} \right] \]
\[-n \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] \]

\[= n! \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{2r(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] + \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{(n-2r)x^{n-2r}}{(n-2r)!} \right] + n! \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] \]

\[-n \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] \]

\[= n! \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{2r(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] + \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] \]

\[-n! \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{2r(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] - n! \left[ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^r}{r!} \frac{x^{n-2r}}{(n-2r)!} \right] = 0 \]

or

\[\left[ 2 \ln(a) \tilde{Y}_{\nu} \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n \right] \nu H_n(x, y \ln(a)) = 0. \]

(7.2.14)

and

\[\tilde{D}_{y,\nu} \left[ \nu H_n(x, y \ln(a)) \right] = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\hat{D}_{y,\nu}(\nu y)^r}{r!(n-2r)!} \frac{(\ln(a))^r}{x^{n-2r}} \]

\[= n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y)^{r-1}(\ln(a))^r}{r!(n-2r)!} \frac{x^{n-2r}}{(n-2r)!} = (\ln(a)) \sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{(-1)^{r-1}(\ln(a))^{r-1}}{(r-1)!(n-2r)!} \frac{x^{n-2r}}{r!(n-2-2r)!} \]

\[= \ln(a) \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\nu y \ln(a))^{r-1}}{(r-1)!(n-2r)!} \frac{x^{n-2r}}{r!(n-2-2r)!} \]

\[= \ln(a) n! \sum_{r=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{(\nu y \ln(a))^{r}}{r!(n-2-2r)!} \frac{x^{n-2-2r}}{r!(n-2-2r)!} \]
\[
\ln(a) \frac{\partial^2}{\partial x^2} \left[ n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\nu)_r (y \ln(a))^r}{r! (n - 2r)!} x^{n-2r} \right] = \ln(a) \frac{\partial^2}{\partial x^2} \left[ H_n(x, y \ln(a)) \right]
\]
or
\[
\mathcal{D}_{\nu, \nu} \left[ H_n(x, y \ln(a)) \right] = \ln(a) \frac{\partial^2}{\partial x^2} \left[ H_n(x, y \ln(a)) \right]. \tag{7.2.15}
\]
It is worth noting that the \(H_n(x, y \ln(a))\) can be derived from the operational rule

\[
a^\nu \frac{\partial^2}{\partial x^2} x^n = \exp \left[ \ln(a) \frac{\partial}{\partial x^2} \right] x^n = \sum_{r=0}^{\infty} \frac{(\ln(a))_r (\nu)_r}{r!} \frac{\partial^r}{\partial x^r} x^n
\]

\[
= \sum_{r=0}^{\infty} \frac{(\ln(a))_r (\nu)_r}{r!} \frac{n!}{(n - 2r)!} x^{n-2r}
\]

\[
= n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\nu)_r (y \ln(a))^r}{r! (n - 2r)!} x^{n-2r}
\]
or
\[
a^\nu \frac{\partial^2}{\partial x^2} x^n = H_n(x, y \ln(a)). \tag{7.2.16}
\]
Analogous results can be obtained for a new family of polynomials, which can be viewed as an umbral image of the Laguerre family.

We consider indeed the operational definition in the following manner

\[
\left( 1 + y \ln(a) \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right)^{-\nu} \left[ \frac{(-1)^n x^n}{n!} \right]
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t} t^{\nu-1} \exp \left[ -ty \ln(a) \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right] \left[ \frac{(-1)^n x^n}{n!} \right] dt
\]
\[
\frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t\nu-1} x^r \left\{ \frac{(-1)^n x^n}{n!} \right\} dt
\]

the under-brace has been determined in eq. (1.2), therefore

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t\nu-1} L_n(x, ty \ln(a)) dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t\nu-1} \left[ n! \sum_{r=0}^n \frac{(-1)^r x^r (ty \ln(a))^{n-r}}{(n-r!)(r!)^2} \right] dt
\]

\[
= \frac{1}{\Gamma(\nu)} \left[ n! \sum_{r=0}^n \frac{(-1)^r x^r (ty \ln(a))^{n-r}}{(n-r!)(r!)^2} \right] \int_0^\infty e^{-t\nu+n-r-1} dt
\]

\[
= \frac{1}{\Gamma(\nu)} \left[ n! \sum_{r=0}^n \frac{(-1)^r x^r (y \ln(a))^{n-r} x^r}{(n-r!)(r!)^2} \right] \Gamma(\nu + n - r)
\]

\[
= \left[ n! \sum_{r=0}^n \frac{(-1)^r (\nu)_{n-r} (y \ln(a))^{n-r} x^r}{(n-r!)(r!)^2} \right] = \nu L_n(x, y \ln(a)).
\]

or

\[
\left( 1 + y \ln(a) \frac{\partial}{\partial x} x^r \frac{\partial}{\partial x} \right)^{-\nu} \left[ \frac{(-1)^n x^n}{n!} \right] = \nu L_n(x, y \ln(a)).
\]

(7.2.17)

The use of the same operators as before allows the derivation of the following

\[
\hat{D}_{y,\nu} \left[ \nu L_n(x, y \ln(a)) \right] = \left[ n! \sum_{r=0}^n \frac{(-1)^r \hat{D}_{y,\nu} (\nu y)^{n-r} (\ln(a))^{n-r} x^r}{(r!)^2 (n-r)!} \right]
\]

\[
= \ln(a) \left[ n! \sum_{r=0}^{n-1} \frac{(-1)^r (n-r)(\nu y)^{n-r-1} (\ln(a))^{n-r-1} x^r}{(r!)^2 (n-r)!} \right]
\]

\[
= \ln(a) \left[ n! \sum_{r=0}^{n-1} \frac{(-1)^r (\nu y \ln(a))^{n-r-1} x^r}{(r!)^2 (n-r-1)!} \right]
\]
\begin{align*}
&= -\ln(a) \left[ n! \sum_{r=0}^{n-1} \frac{(-1)^{r+1} (\nu y \ln(a))^{n-r-1}(r+1)^2}{((r+1)!)^2(n-r)!} x^r \right] \\
&= -\ln(a) \left[ n! \sum_{r=1}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}r^2}{(r!)^2(n-r)!} x^{r-1} \right]
\end{align*}

\begin{align*}
&= -\ln(a) \left[ n! \sum_{r=0}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}r}{(r!)^2(n-r)!} x^{r} \right] \\
&= -\ln(a) \frac{\partial}{\partial x} x \left[ n! \sum_{r=0}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}}{(r!)^2(n-r)!} x^{r} \right]
\end{align*}

or

\begin{align*}
\hat{D}_{y,\nu} \left[ \nu L_n(x, y \ln(a)) \right] &= -\ln(a) \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \left[ \nu L_n(x, y \ln(a)) \right].
\end{align*}

(7.2.18)

and the use of the following operators we find that

\begin{align*}
&\left[ \ln(a) \tilde{Y}_\nu x \frac{\partial^2}{\partial x^2} + (\ln(a) \tilde{Y}_\nu - x) \frac{\partial}{\partial x} + n \right] \nu L_n(x, y \ln(a)) \\
&= \ln(a) \tilde{Y}_\nu x \left[ n! \sum_{r=2}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}r(r-1)}{(r!)^2(n-r)!} x^{r-2} \right] \\
&\quad + (\ln(a) \tilde{Y}_\nu - x) \left[ n! \sum_{r=1}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}r}{(r!)^2(n-r)!} x^{r-1} \right] + n \left[ n! \sum_{r=0}^{n} \frac{(-1)^{r}(\nu y \ln(a))^{n-r}}{(r!)^2(n-r)!} x^{r} \right]
\end{align*}
\[ n! \ln(a) \left[ \sum_{r=1}^{n} \frac{(-1)^r Y_n(\nu y)^{n-r}(\ln(a))^{n-r} r(r-1)}{(r!)(n-r)!} x^{r-1} \right] \]

\[ + \left[ n! \ln(a) \sum_{r=1}^{n} \frac{(-1)^r Y_n(\nu y)^{n-r}(\ln(a))^{n-r} x^{r-1}}{(r!)(n-r)!} \right] - \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} r}{(r!)(n-r)!} x^{r} \right] \]

\[ + n \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} x^{r}}{(r!)(n-r)!} \right] \]

\[ = \left[ n! \sum_{r=1}^{n} \frac{(-1)^r (\nu y)^{n-r+1}(\ln(a))^{n-r+1} r(r-1)}{(r!)(n-r)!} x^{r-1} \right] \]

\[ + \left[ n! \sum_{r=1}^{n} \frac{(-1)^r (\nu y)^{n-r+1}(\ln(a))^{n-r+1} r(r-1)}{(r!)(n-r)!} x^{r-1} \right] - \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} r}{(r!)(n-r)!} x^{r} \right] \]

\[ + n \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} r}{(r!)(n-r)!} x^{r} \right] \]

\[ = \left[ n! \sum_{r=0}^{n} \frac{(-1)^{r+1} (\nu y)^{n-r}(\ln(a))^{n-r} (r+1) r}{((r+1)!)^2(n-r-1)!} x^{r} \right] \]

\[ + \left[ n! \sum_{r=0}^{n} \frac{(-1)^{r+1} (\nu y)^{n-r}(\ln(a))^{n-r} (r+1) r}{((r+1)!)^2(n-r-1)!} x^{r} \right] - \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} r}{(r!)(n-r)!} x^{r} \right] \]

\[ + n \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu y)^{n-r}(\ln(a))^{n-r} r}{(r!)(n-r)!} x^{r} \right] \]
Chapter 7: Generalized Operational Methods, Fractional Operators...

\[ \begin{align*}
&= - \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] \\
&- \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] - \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] \\
&+ n \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] \\
&- \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] \\
&+ n \left[ n! \sum_{r=0}^{n} \frac{(-1)^r (\nu)(y \ln(a))^{n-r} (r+1)^r x^r}{((r+1)!)^2 (n-r)!} \right] = 0
\end{align*} \]

or

\[ \ln(a) \frac{\partial^2}{\partial x^2} + (\ln(a) \ln x) \frac{\partial}{\partial x} + n \cdot L_n(x, y \ln(a)) = 0. \]

(7.2.19)

A particularly interesting case arises when the highest order of the derivative appearing in the fractional operator is 1, namely, when
\[
\left[1 - y \ln(a) \frac{\partial}{\partial x}\right]^{\nu} x^n = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-t) t^{\nu-1} e^{yt \ln(a)} x^n \, dt
\]

for the value of the under-brace we use the eq. (7.1.7), we have

\[
\left[1 - y \ln(a) \frac{\partial}{\partial x}\right]^{\nu} x^n = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-t) t^{\nu-1} (x + yt \ln(a))^n \, dt, \quad (7.2.20)
\]

it is evident that the corresponding polynomial

\[
\nu H_n^{(1)}(x, y \ln(a)) = n! \sum_{r=0}^{n} \frac{(\nu y \ln(a))^r x^{n-r}}{r!(n-r)!}
\]

\[
= n! \sum_{r=0}^{n} \frac{(\nu y \ln(a))^r x^{n-r}}{r!(n-r)!}. \quad (7.2.21)
\]

is an umbral image of the ordinary binomials.

The above example provides an indication of the implications offered by the method we have proposed. In the next concluding section, we will discuss further examples aimed at proving that the combined use of operational rules and integral representations may provide unsuspected links between apparently disconnected fields.

2(b). Particular case: Substituting \(a = e\), into the eqs. (7.2.15), (7.2.16) leads to (20) and (7.2.14), (7.2.15), (7.2.16), (7.2.17), (7.2.18), (7.2.19), (7.2.20), (7.2.21) produce the results (21), (22), (23), (24), (25), (26), (27), (31), (32) determined by Dattoli [37].

7.3 Concluding Remarks

One of the advantages offered by the use of the integral representation in dealing with differential operators is the possibility of giving a meaning to
apparently meaningless operations. This is indeed the case of the Riemann Liouville definition of fractional derivative (see [188]; see also [[199], Chapter 5]).

By taking the advantage from the definition of the Euler Γ function (see [[37], p. 153, eq. (12)]) with \( a = 1 \), Dattoli determined the results (see [[37], p. 156, eq. (34),(35)]). The use of the identity (see [[37], p. 153, eq. (12)]) allows to conclude that

\[
\left[ x \ln(a) \frac{d}{dx} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp \left[ -tx \ln(a) \frac{d}{dx} \right] t^{\nu-1} f(x) dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} a^{-xt \frac{dx}{dx}} f(x) dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} f(\exp(-t \ln(a) + \ln(x))) dt
\]

or

\[
\left[ x \ln(a) \frac{d}{dx} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} f(xa^{-t}) dt. \tag{7.3.1}
\]

If we set \( f(x) = \left( \frac{x}{1-x} \right) \) in eq. (7.3.1) we find that

\[
\left[ x \ln(a) \frac{d}{dx} \right]^{-\nu} \left( \frac{x}{1-x} \right) = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} a^{-xt \frac{dx}{dx}} \left( \frac{x}{1-x} \right) dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} a^{-xt \frac{dx}{dx}} \sum_{n=0}^\infty x^{n+1} dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} \sum_{n=1}^\infty a^{-xt \frac{dx}{dx}} x^n dt
\]

use of the eq. (7.3.1), yields
\[
\begin{align*}
\frac{1}{\Gamma(\nu)} & \int_0^\infty t^{\nu-1} \sum_{n=1}^\infty (xa^{-t})^n dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=1}^\infty x^n \int_0^\infty t^{\nu-1} a^{-nt} dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=1}^\infty x^n \int_0^\infty t^{\nu-1} e^{-n \ln(a)t} dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=1}^\infty x^n \frac{\Gamma(\nu)}{[n \ln(a)]^\nu} = \sum_{n=1}^\infty \frac{x^n}{[n \ln(a)]^\nu}
\end{align*}
\]

or

\[
\left[ x \ln(a) \frac{d}{dx} \right]^{-\nu} \left( \frac{x}{1-x} \right) = \sum_{n=1}^\infty \frac{x^n}{[n \ln(a)]^\nu} = \zeta(x, \nu), \quad |x| < 1. \quad (7.3.2)
\]

According to the previous result, it becomes clear that fractional forms of the operator \( x \frac{d}{dx} \) can be used as an operational definition of the Riemann \( \zeta \) function \([1]\). A further example in this direction is provided by

\[
\left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} \frac{a^{-tx \partial}}{\partial x} f(x) dt
\]

or

\[
\left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} f(xa^{-t}) dt. \quad (7.3.3)
\]

and if \( f(x) = \exp(x) \), we end up with

\[
\left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} \exp(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} \frac{a^{-tx \partial}}{\partial x} \exp(x) dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} \exp(xa^{-t}) dt
\]
\[ f(t) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} \sum_{n=0}^\infty \frac{x^n a^{-nt}}{n!} dt \]

\[ = \frac{1}{\Gamma(\nu)} \sum_{n=0}^\infty \frac{x^n}{n!} \int_0^\infty t^{\nu-1} \exp(-\alpha t) a^{-nt} dt \]

\[ = \frac{1}{\Gamma(\nu)} \sum_{n=0}^\infty \frac{x^n n!}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} \exp[-(\alpha + n \ln(a))t] dt \]

\[ = \sum_{n=0}^\infty \frac{x^n [\alpha + n \ln(a)]^{\nu n!}}{\Gamma(\nu)} \]

or

\[ \left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} \exp(x) = \sum_{n=0}^\infty \frac{x^n}{[\alpha + n \ln(a)]^{\nu n!}}. \quad (7.3.4) \]

Eq. (7.3.4) suggests further consequences too. By replacing \( f(x) \) with \( L_n(x, y \ln(a)) \)

in eq. (7.3.3), we find that

\[ \left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} L_n(x, y \ln(a)) = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} n! \sum_{r=0}^n (-1)^r (y \ln(a))^{n-r} \frac{a^{-tx} \partial}{\partial x^r} L_n(x, y \ln(a)) dt \]

\[ = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} n! \sum_{r=0}^n (-1)^r (xa^{-t})^r (y \ln(a))^{n-r} \frac{a^{-tx} \partial}{\partial x^r} dt \]

\[ = \frac{1}{\Gamma(\nu)} \frac{n!}{(r!)^2 (n-r)!} \sum_{r=0}^n (-1)^r x^r (y \ln(a))^{n-r} \int_0^\infty \exp[-(\alpha + r \ln(a))t] t^{\nu-1} dt \]

\[ = \frac{1}{\Gamma(\nu)} \frac{n!}{(r!)^2 (n-r)!} \frac{\Gamma(\nu)}{[\alpha + r \ln(a)]^{\nu r}} \]
\[ n! \sum_{r=0}^{n-1} \frac{(-1)^r x^r (y \ln(a))^{n-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-r)!} \]

or

\[ \left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} L_n(x, y \ln(a)) = n! \sum_{r=0}^{n-1} \frac{(-1)^r x^r (y \ln(a))^{n-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-r)!} \]

(7.3.5)

Denoting the polynomial \( L_n(x, y \ln(a); \alpha, \nu) \) on the right hand side of the above eq. (7.3.5) we find the following recurrences

\[
\frac{\partial}{\partial y} L_n(x, y \ln(a); \alpha, \nu) = n! \sum_{r=0}^{n-1} \frac{(-1)^r x^r y^{n-r} (\ln(a))^{r}}{[\alpha + r \ln(a)]^\nu(r!)(n-1-r)!}
\]

\[
= n \ln(a)(n-1)! \sum_{r=0}^{n-1} \frac{(-1)^r x^r (y \ln(a))^{n-1-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-1-r)!} = n L_{n-1}(x, y \ln(a); \alpha, \nu)
\]

or

\[
\frac{\partial}{\partial y} L_n(x, y \ln(a); \alpha, \nu) = n \ln(a) L_{n-1}(x, y \ln(a); \alpha, \nu)
\]

(7.3.6)

and

\[
- \frac{\partial}{\partial x} \frac{\partial}{\partial x} L_n(x, y \ln(a); \alpha, \nu) = - \frac{\partial}{\partial x} \frac{\partial}{\partial x} n! \sum_{r=0}^{n-1} \frac{(-1)^r x^r (y \ln(a))^{n-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-r)!}
\]

\[
= - \frac{\partial}{\partial x} n! \sum_{r=0}^{n} \frac{(-1)^r (r) x^{r-1} (y \ln(a))^{n-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-r)!}
\]

\[
= - n! \sum_{r=1}^{n} \frac{(-1)^r (r) (r-1) x^{r-1} (y \ln(a))^{n-r}}{[\alpha + r \ln(a)]^\nu(r!)(n-r)!}
\]
\[\begin{align*}
\frac{\partial}{\partial x} \frac{\partial}{\partial x} L_n(x, y \ln(a); \alpha, \nu) &= nL_{n-1}(x, y \ln(a); \alpha + \ln(a), \nu). \\
(7.3.7)
\end{align*}\]

Again by replacing \( f(x) \) in the similar fashion as above with \( H_n(x, y \ln(a)) \) in eq. (7.3.3), we find that

\[\left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} H_n(x, y \ln(a))\]

\[= \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} a^{-\nu} H_n(x, y \ln(a)) dt\]

\[= \frac{1}{\Gamma(\nu)} \int_0^\infty \exp(-\alpha t) t^{\nu-1} a^{-\nu} \sum_{r=0}^{[\frac{n}{2}] \frac{r!}{r!(n-2r)!}} \frac{(y \ln(a))^r}{(n-2r)!} \int_0^\infty \exp[-(\alpha + (n-2r) \ln(a))] t^{\nu-1} a^{-\nu} dt\]

\[= \frac{1}{\Gamma(\nu)} \frac{n!}{r!(n-2r)!} \left( y \ln(a) \right)^r x^{n-2r} \int_0^\infty \exp[-(\alpha + (n-2r) \ln(a))] t^{\nu-1} dt\]

\[= \frac{1}{\Gamma(\nu)} n! \sum_{r=0}^{[\frac{n}{2}] \frac{r!}{r!(n-2r)!}} \frac{(y \ln(a))^r x^{n-2r}}{(n-2r)!} \left[ \alpha + (n-2r) \ln(a) \right] t^{\nu-1} dt\]

or

\[= \frac{1}{\Gamma(\nu)} n! \sum_{r=0}^{[\frac{n}{2}] \frac{r!}{r!(n-2r)!}} \frac{(y \ln(a))^r x^{n-2r}}{(n-2r)!} \left[ \frac{\Gamma(\nu)}{[\alpha + (n-2r) \ln(a)]^\nu} \right] \]
\[
\left[ \alpha + x \ln(a) \frac{\partial}{\partial x} \right]^{-\nu} H_n(x, y \ln(a)) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))^r x^{n-2r}}{[\alpha + (n-2r) \ln(a)]^\nu r!(n-2r)!}.
\]

(7.3.8)

Denoting by \(H_n(x, y \ln(a); \alpha, \nu)\) the polynomial defined on the right hand side of eq. (7.3.8), we find the recurrences

\[
\frac{\partial}{\partial x} H_n(x, y \ln(a); \alpha, \nu) = \frac{\partial}{\partial x} \left[ n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))^r x^{n-2r}}{[\alpha + (n-2r) \ln(a)]^\nu r!(n-2r)!} \right]
\]

\[
= n \left( n - 1 \right) \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))^r x^{n-1-2r}}{[\alpha + \ln(a) + (n-1-2r) \ln(a)]^\nu (n-1-2r)!}
\]

or

\[
\frac{\partial}{\partial y} H_n(x, y \ln(a); \alpha, \nu) = n H_{n-1}(x, y \ln(a); \alpha + \ln(a), \nu)
\]

(7.3.9)

and

\[
\frac{\partial}{\partial y} H_n(x, y \ln(a); \alpha, \nu) = \frac{\partial}{\partial y} \left[ n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))^r x^{n-2r}}{[\alpha + (n-2r) \ln(a)]^\nu r!(n-2r)!} \right]
\]

\[
= \left[ n! \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{y^{r-1} \ln(a))^r x^{n-2r}}{[\alpha + (n-2r) \ln(a)]^\nu (r-1)! (n-2r)!} \right]
\]

\[
= \ln(a) n(n-1) \left( n - 2 \right) \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(y \ln(a))^r x^{n-2-2r}}{[\alpha + (n-2-2r) \ln(a)]^\nu r!(n-2-2r)!}
\]

or

\[
\frac{\partial}{\partial y} H_n(x, y \ln(a); \alpha, \nu) = \ln(a) n(n-1) H_{n-2}(x, y \ln(a); \alpha, \nu).
\]

(7.3.10)
We consider, therefore, the differential equations of the type

$$\sqrt{\left(\alpha \ln(a)\right)^2 \frac{d^2}{dx^2} + 1} f(x) = S(x)$$  \hspace{1cm} (7.3.11)$$

where $S(x)$ denotes a known function.

The formal solution of eq. (7.3.11) can be cast in the following form

$$f(x) = \frac{1}{\sqrt{\left(\alpha \ln(a)\right)^2 \frac{d^2}{dx^2} + 1}} S(x).$$  \hspace{1cm} (7.3.12)$$

By recalling from the theory of Laplace transforms (see [46])

$$\frac{1}{\sqrt{A^2 + 1}} = \int_0^\infty J_0(t)e^{-At}dt,$$  \hspace{1cm} (7.3.13)$$

or

$$\frac{1}{\sqrt{\left(\ln(a)A\right)^2 + 1}} = \int_0^\infty J_0(t)a^{-At}dt,$$  \hspace{1cm} (7.3.14)$$

and on replacing $A$ with $\alpha \frac{d}{dx}$, we find that

$$\frac{1}{\sqrt{\left(\ln(a)\alpha \frac{d}{dx}\right)^2 + 1}} = \int_0^\infty J_0(t)a^{-\alpha \frac{d}{dx}}dt,$$

or

$$\frac{1}{\sqrt{\left(\alpha \ln(a) \frac{d}{dx}\right)^2 + 1}} S(x) = \int_0^\infty J_0(t)a^{-\alpha \frac{d}{dx}}S(x)dt$$

$$= \int_0^\infty J_0(t)S(x - \alpha t \ln(a))dt,$$

i.e.

$$f(x) = \int_0^\infty J_0(t)a^{-\alpha \frac{d}{dx}}S(x)dt$$

$$= \int_0^\infty J_0(t)S(x - \alpha t \ln(a))dt. \hspace{1cm} (7.3.15)$$
The solution of eq. (7.3.11) in the form (7.3.15) holds only if the integral is convergent and can be viewed as a kind of convolution of $S(x)$ on the $0^\text{th}$-order cylindrical Bessel function. 

As a final example, we will consider the solution of the fractional diffusive equation

$$ \begin{cases} \frac{\partial}{\partial y}f(x, y \ln(a)) = -\ln(a)\frac{\partial^{1/2}}{\partial x^{1/2}}f(x, y \ln(a)), \\ f(x, 0) = g(x). \end{cases} \quad (7.3.16) $$

Which can be treated using an identity valid within the framework of the Laplace transform theory [46], that can be written in the following form

$$ e^{-y\sqrt{\ln(a)d}} = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\left(\frac{y^2}{4t}\right)-t\ln(a)d}}{t^{3/2}} dt $$

By replacing $d$ with $\frac{\partial}{\partial x}$ and by proceeding as before, we find the solution of eq. (7.3.17) as

$$ e^{-y(\ln(a)\frac{\partial}{\partial x})^{1/2}}g(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\left(\frac{y^2}{4t}\right)}t^{3/2}a^{-t\frac{\partial}{\partial x}}g(x)dt}{t^{3/2}a^{-t}\frac{\partial}{\partial x}g(x)dt} $$

This result can be viewed as the analogue of the Gauss transform for the solution of the heat diffusion equation. In this paper it has been shown that methods of operational nature can provide a fairly useful tool to solve a large body of problems including fractional propagation equations.

3. Particular case: The substitution $a = e$, transforms the eqs. (7.3.1), (7.3.2), (7.3.3), (7.3.4), (7.3.5), (7.3.6), (7.3.7), (7.3.11), (7.3.12), (7.3.15),
(7.3.16), (7.3.17) and (7.3.18) to (36), (37), (38), (39), (40), (41), (42), (43), (45), (46), (47) and (48) due to Dattoli [37].
Bibliography


for Special Functions and Their Applications (SSFA), Chennai (2003), 13-26.


[57] Khan, M. A.; *Certain fractional q-Integrals and q-derivative*, Ganita, **24** no.1, (1973), 83-94.


[70] Khan, M. A.; *On a calculus for the $T_{k,q,x}$-operator*, Mathematica Balkanica, New Series, 6, Fasc.1, (1992), 83-93.


[99] Khan, M. A.; Ahmad, K.; *On a generalization of Bessel polynomials suggested by the polynomials L_n^{a,b}(x) of Prabhakar and Rekha*, J. Analysis, 6, (1998), 151-161.


[102] Khan, M. A.; Shukla. A. K.; *On some generalized sister celine’s polyno-

[103] Khan, M. A.; Ahmad, K.; *On a two variables analogue of konhauser’s
225-240.

Ciencia Indica, xxv, M.no.2, (1999), 221-224.

[105] Khan, M. A.; *On exact D.E.of three variables*, Acta Ciencia Indica,
xxv M. No.3, (1999), 281-284.

[106] Khan, M. A.; Ahmad, K.; *A study of Bessel polynomials suggested by
the polynomials of Prabhakar and Rekha*, Souchow Journal of Mathematics,

[107] Khan, M. A.; Najmi, M.; *Discrete analytic continuation of a (p,q)-

[108] Khan, M. A.; Ahmad, K.; *On a two variables analogue of besssels

[109] Khan, M. A.; Najmi, M.; *On a convolution operator for (p, q) analytic

[110] Khan, M. A.; *A note on a polynomial set generated by G(2xt – t^2) for
the choice G(u) = 0F1(−,−;α;u)*, Pro Mathematica XV/no.29-30,

by the polynomials K_n(x) of M. A. Khan, G. S. Abuhammash*, Acta


[136] Khan, M. A.; Asif, M.; Shift Operators on the Base $a (a > 0, \neq 1)$ and Monomial Type Functions, International Transactions in Mathematical Sciences and Computer, 2, No. 2, (2009), 453-462.


[144] Khan, M. A.; Alidad, B.; *Polynomials set generated by e\(c^i\phi(\phi(x))\psi(y))*, Proyecciones Journal of Mathematics, 29 No.3, (2010), 201-207.


[158] Khan, M. A.; Asif, M.; Shift operators on the base a (a > 0, ≠ 1) and pseudo-polynomials of fractional order, Accepted for publication in International Journal of Mathematical Analysis.

[159] Khan, M. A.; Khan, A. H.; Ahmad, N.; A note on new two variable analogue of modified Hermite polynomials, Accepted for publication in Mathematical Sciences Research Journal.

[160] Khan, M. A.; Khan, A. H.; Abbas, S. M.; A note on pseudo two variables Jacobi polynomials, Accepted for publication in Int. Trans. in App. Sci.

[161] Khan, M. A.; Asif, M.; Generalized operational methods, fractional operators and special polynomials, Communicated for publication.

[162] Khan, M. A.; Asif, M.; q-Analogue of exponential operators and difference equations, Communicated for publication.
[163] Khan, M. A.; Asif, M.; *q-Analogue shift operators and pseudopolynomials of fractional order*. Communicated for publication.

[164] Khan, M. A.; Asif, M.; *q-Analogue of the operational methods, fractional operators and special polynomials*. Communicated for publication.

[165] Khan, M. A.; Asif, M.; *Generalized q-shift operators and monomial type functions*. Communicated for publication.


[167] Khan, M. A.; Asif, M.; *Jacobi type and Gegenbauer type generalization of certain polynomials*. Communicated for publication.


[170] Khan, M. A.; Nisar, K. S.; *Bilinear and bilateral summation formulae of certain polynomials in the form of operator representations*. Communicated for publication.


[172] Khan, M. A.; Nisar, K. S.; *On generalization of Appell’s type functions of two variables and their expansion*. Communicated for publication.


[175] Khan, M. A.; Nisar, K. S.; *On a generalization of Lauricilla’s functions of several variables*, Communicated for publication.

[176] Khan, M. A.; Nisar, K. S.; *Some Rodrigues formulae for $F_1$ and $\phi_1$ type polynomials*, Communicated for publication.

[177] Khan, M. A.; Nisar, K. S.; *The Lagrange polynomials of several variables and the Umbral Calculus*, Communicated for publication.


Shift Operators on the Base $a \ (a > 0, \neq 1)$ and Monomial Type Functions

Mumtaz Ahmad Khan* and Mohammad Asif
Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh - 202002, India

ABSTRACT
The aim of the present paper is to introduce and use the generalized exponential shift operators, operators on the base $a \ (a > 0, \neq 1)$, to deal with the formalism of quasi monomiality for introducing new families of functions with peculiar properties, recalling those of quasi monomials, given by Dattoli et al. [6]. We show that these functions can be exploited in the solution of wide classes of differential equations, including cases of fractional type.

Keywords : Hermite-Kamp’e de Feriet polynomials; Bessel functions; Monomiality principle; Shift operators; Differential equations; Fractional calculus.

1. INTRODUCTION
In 2003, extensive uses of exponential operators, operators on the natural base $e$ were used by Dattoli et al. (2003). We introduce and use the exponential shift operators on the base $a \ (a > 0, \neq 1)$, which would play an important role in problems concerning pure and applied Mathematics. The properties of the generalized shift operator

$$\hat{A} = a^2 q(x) \frac{d}{dx}$$

are similar to that of discussed in Dattoli et al. (1997) and their importance for the solution of generalized difference equations are similar to that of stressed in Dattoli and Levi (2000).

* Corresponding author; E-mail: mumtaz_ahmad_khan_2008@yahoo.com
The action of \( A(x) \) on a given function \( f(x) \) has been shown to be provided by
\[
A(x) = f\left[F^{-1}(\lambda \ln(a) + F(x))\right], \tag{1.2}
\]
where
\[
F(x) = \int q(\xi) \frac{dx}{q(\xi)} \tag{1.3}
\]
defines the associated characteristic function of the generalized shift operator and
\( F^{-1}(\cdot) \) its inverse.

It is evident that for \( q(x) = 1 \), \( A \) reduces to the ordinary shift operator, while for \( q(x) = x \), we find \( F(x) = \ln(x) \) and \( F^{-1}(x) = e^x \), thus finding
\[
A = a^{\lambda q(x)} \frac{d}{dx} f(x) = f(a^x) \tag{1.4}
\]
within such context, the operator
\[
\hat{T}_x = q(x) \frac{d}{dx} \tag{1.5}
\]
can be viewed as an ordinary derivative, so that the following commutation relation holds true
\[
[\hat{T}_x, F(x)] = 1. \tag{1.6}
\]
Although \( F(x) \) is a function, \( [F(x)]^n \) behaves, under the action of \( \hat{T}_x \), as an ordinary monomial (Dattoli et al., 2003), we obtain indeed
\[
\hat{T}_x [F(x)]^n = n[F(x)]^{n-1}, \tag{1.7}
\]
we can take advantage from this fairly naive property to discuss the rule associated with the use of operators like
\[
A_m = a^{\lambda \hat{T}_x m}, \tag{1.8}
\]
and make a brief comment for the extension to non-integer \( m \).

Particular case: The substitution \( a = e \) into the eqs. (1.1), (1.2), (1.4) and (1.8) reduce to the eqs. (1), (2), (4), and the last equation on the page no. 122 due to Dattoli et al. (2003).

2. SHIFT OPERATORS ON THE BASE \( a (a > 0, \neq 1) \) AND PSEUDO POLYNOMIALS

Arbitrary order Hermite-Kamp'e de F'eriet (H.K.d.F.) (or Gould-Hopper) polynomials (see Appell and Kamp'e de F'eriet, 1926; Gould and Hopper, 1962; Srivastava and Manocha,
Appendix

Shift Operators on the Base \( a (a > 0, \neq 1) \) and Monomial Type Functions

\[ H_n^{(m)}(x, y \ln(a)) = n! \sum_{r=0}^{n} \frac{(y \ln(a))^r x^{n-r}}{r!(n-mr)!} \] \((2.1)\)

are known to satisfy the identities

\[ \left[ x + my \ln(a) \left( \frac{\partial y}{\partial x} \right)^{m-1} \right] H_n^{(m)}(x, y \ln(a)) = n! \sum_{r=0}^{n} \frac{(n+1-mr)x^{n+1-mr}(y \ln(a))^r}{(n+1-mr)! r!} + m \left[ \sum_{r=0}^{n} \frac{x^{n+1-mr}(r+1)(y \ln(a))^r}{(n+1-1)(r+1)! (r+1)!} \right] \]

\[ = n! \left( \sum_{r=0}^{n} \frac{(n+1-mr)x^{n+1-mr}(y \ln(a))^r}{(n+1-mr)! r!} \right) + \left[ \sum_{r=0}^{n} \frac{x^{n+1-mr}(y \ln(a))^r}{(n+1-mr)! r!} \right] \]

\[ = (n+1)! \sum_{r=0}^{n} \frac{(y \ln(a))^r x^{n+1-mr}}{r!(n+1-mr)!} \]

or

\[ \left[ x + my \ln(a) \left( \frac{\partial y}{\partial x} \right)^{m-1} \right] H_n^{(m)}(x, y \ln(a)) = H_n^{(m+1)}(x, y \ln(a)) \] \((2.2)\)

and

\[ \frac{\partial}{\partial x} H_n^{(m)}(x, y \ln(a)) = n(n-1)! \sum_{r=0}^{n} \frac{(y \ln(a))^r x^{n+1-mr}}{r!(n-1-mr)!} \]

\[ \frac{\partial}{\partial x} H_n^{(m)}(x, y \ln(a)) = nH_n^{(m+1)}(x, y \ln(a)) \]

We define Hermite polynomial through the operational identities, the exponential operators defined on the base \( a (a > 0, \neq 1) \), as follows

\[ \frac{y e^{\frac{\partial}{\partial x}}}{a e^{\frac{\partial}{\partial x}}} x^n = e^{\frac{y \ln(a)}{a}} = \sum_{r=0}^{\infty} \frac{(y \ln(a))^r e^{\frac{\partial^2}{\partial x^2}}}{r!} x^n = n! \sum_{r=0}^{n/2} \frac{(y \ln(a))^r x^{n-2r}}{r!(n-2r)!} \]

or

\[ \frac{y e^{\frac{\partial}{\partial x}}}{a e^{\frac{\partial}{\partial x}}} x^n = H_n(x, y \ln(a)). \] \((2.3)\)
Appendix

The first two relations (2.1) and (2.2) ensure the quasi monomial nature of H.K.d.F. polynomials, while the third, consequence of the identity
\[
\frac{\partial}{\partial y} H_n^{(m)}(x, y \ln(a)) = \frac{\partial}{\partial y} \left( \frac{e}{\partial x} \right)^m x^n = \frac{\partial}{\partial y} e^{y \ln(a)} \left( \frac{e}{\partial x} \right)^m x^n
\]
\[
= \ln(a) \frac{e^m}{\partial x^m} e^{y \ln(a)} \left( \frac{e}{\partial x} \right)^m x^n
\]
or
\[
\frac{\partial}{\partial y} H_n^{(m)}(x, y \ln(a)) = \ln(a) \frac{e^m}{\partial x^m} H_n^{(m)}(x, y \ln(a))
\]
....(2.4)

and
\[
H_n^{(m)}(x, 0) = x^n,
\]

provides us with their operational definition, while its inverse can be written as
\[
a^y \left( \frac{e}{\partial x} \right)^m H_n^{(m)}(x, y \ln(a)) = x^n.
\]
....(2.5)

According to the conclusion of the introductory section and to these last relations, we can introduce the “polynomials”
\[
h_n^{(m)}(x, y \ln(a)) = H_n^{(m)}(F(x), y \ln(a))
\]
....(2.6)

which satisfy the recurrences
\[
[F(x) + my \ln(a)(F_x)^{m-1}] h_n^{(m)}(x, y \ln(a)) = h_{n+1}^{(m)}(x, y \ln(a)),
\]
....(2.7)

\[
\hat{T}_x[h_n^{(m)}(x, y \ln(a))] = nh_n^{(m)}(x, y \ln(a))
\]

It is therefore evident that the \(h_n^{(m)}(x, y \ln(a))\) are functions satisfying polynomial type identities and will be therefore called pseudo H.K.d.F.

It becomes also evident that identities of the following type
\[
[F(x) + 2y \ln(a)(\hat{T}_x)]^n = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(F(x), y \ln(a)(\hat{T}_x))^s.
\]
....(2.8)

We show that eq. (2.8) follows from the Weyl identity. Note that since \(F(x)\) and \(2y \ln(a)(\hat{T}_x)\) do not commute, therefore the use of the Newton binomial formula is not allowed.

Multiplying the left-hand side of eq. (2.8) by \(\frac{\ln^n}{n!}\) and summing over \(n\), we find
\[
\sum_{n=0}^{\infty} \frac{\ln^n}{n!} [F(x) + 2y \ln(a)(\hat{T}_x)]^n = e^{(F(x) + 2y \ln(a)(\hat{T}_x))}.
\]
Appendix

Shift Operators on the Base a (a > 0, ≠ 1) and Monomial Type Functions

by using the Weyl identity, where \( \hat{P} = tF(x) \) and \( \hat{Q} = 2y t \ln(a) \hat{T}_x \) since

\[
[\hat{P}, \hat{Q}] = [\hat{P}, \hat{Q}] = 0
\]

and noting that

\[
[\hat{P}, \hat{Q}] = \hat{P} \hat{Q} - \hat{Q} \hat{P} = -2y \ln(a) t^2 \quad \text{and} \quad e^{\hat{P} + \hat{Q}} = e^{\hat{P}} e^{\hat{Q} - \frac{1}{2}[\hat{P}, \hat{Q}]}
\]

Therefore we can write

\[
e^{tF(x) + 2y t \ln(a) \hat{T}_x} = e^{tF(x) + y \ln(a) t^2} e^{2y t \ln(a) \hat{T}_x}
\]

By expanding the exponential function, we obtain

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(F(x), y \ln(a)) \sum_{s=0}^{\infty} \frac{t^s}{s!} (2y \ln(a) \hat{T}_x)^s
\]

\[= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{t^{n+s}}{n! s!} H_n(F(x), y \ln(a)) (2y \ln(a) \hat{T}_x)^s
\]

Setting \( k = n + s \) and inverting summations, we find

\[
\sum_{k=0}^{\infty} \frac{k^k}{k!} \left[ F(x) + 2y \ln(a) \hat{T}_x \right]^k = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \frac{k^k}{k! s! (s-k)!} H_{k-s}(F(x), y \ln(a)) (2y \ln(a) \hat{T}_x)^s
\]

Therefore, (2.8) follows from the comparison of the coefficients of \( k^k \) in the last equation.

By using the eq. (2.6) we find the following result

\[
\left[ F(x) + 2y \ln(a) \hat{T}_x \right]^n = \sum_{s=0}^{n} (2y \ln(a) \hat{T}_x)^s \binom{n}{s} \frac{H_s(F(x), y \ln(a))(\hat{T}_x)^s}{s!}
\]

and

\[
a^{y \ln(a) \hat{T}_x^m} f(F(x)) = e^{y \ln(a) \hat{T}_x^m} f(F(x)) = f(F(x) + my \ln(a) \hat{T}_x^{m-1}) q^y \hat{T}_x^m
\]

which realize an extension of the ordinary Burchanl and Crofton identities valid for \( q(x) = 1 \).

It is evident that all the wealth of properties of H.K.d.F. can be extended fairly straightforwardly to the functions \( h_n^{(m)}(x, y \ln(a)) \). The use of the previously discussed rules may greatly simplify the application of different types of exponential polynomials.

To give some examples, we note e.g. that

\[
y^{\frac{d}{da}} \left( x^{\frac{d}{da}} \right)^m (x^n) = y^{\ln(a)} \left( x^{\frac{d}{da}} \right)^m (x^n) = e^{y \ln(a) \left( x^{\frac{d}{da}} \right)^m} e^{n \ln(x)}
\]

\[= \sum_{r=0}^{\infty} \frac{n^r}{r!} H_r^{(m)}(\ln(x), y \ln(a))
\]

\[\text{\copyright HACS} \]
and since
\[ \sum_{r=0}^{\infty} \frac{f^r}{r!} H_r(x, y \ln(a)) = e^{x + y \ln(a) m} \quad \text{...(2.12)} \]
we find
\[ a \left( x \frac{\partial}{\partial x}\right)^m (x^n) = e^{y \ln(a)} (x \frac{\partial}{\partial x})^m (x^n) = x^n e^{y \ln(a) m} = x^n a^m \quad \text{...(2.13)} \]
It is now evident that if \( f(x) \) is specified by any analytic function \( f(x) = \sum c_n x^n \), then
\[ a \left( x \frac{\partial}{\partial x}\right)^m f(x) = \sum c_n x^n a^m \quad \text{...(2.14)} \]
provided that the last series is convergent. Further comments on this last result will be presented in the concluding section.

A further example of exponential operator is provided by the case
\[ q(x) = (x - b)^2, \quad F(x) = \int x \frac{d\xi}{(\xi - b)} = -\frac{1}{x - b} \]
for which we find
\[ a \left( x - b \frac{\partial}{\partial x}\right)^2 \left( x \frac{\partial}{\partial x}\right) = e^{y \ln(a)} (x - b)^2 \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x}\right) = e^{y \ln(a)} (x - b)^2 \frac{\partial}{\partial x} \left( \frac{1}{1 + \frac{b}{x - b}} \right) = e^{y \ln(a)} (x - b)^2 \frac{\partial}{\partial x} \left( \sum_{s=0}^{\infty} \left( -\frac{b}{x - b} \right)^s \right) = \sum_{s=0}^{\infty} (b)^s e^{y \ln(a)} (x - b)^2 \frac{\partial}{\partial x} \left( \left( -\frac{1}{x - b} \right)^s \right) \]
Let \( \frac{1}{x - b} = t \), therefore \( \frac{\partial t}{\partial x} = \frac{1}{(x - b)^2} \) and \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial x} = \frac{1}{(x - b)^2} \frac{\partial}{\partial x} \) or \( \frac{\partial}{\partial t} = (x - b)^2 \frac{\partial}{\partial x} \).

Now \[ \sum_{s=0}^{\infty} (b)^s e^{y \ln(a)} \left( \frac{\partial}{\partial t} \right)^2 t^s = \sum_{s=0}^{\infty} (b)^s H_s(t, y \ln(a)) \]
Appendix

Shift Operators on the Base \( a \) \((a > 0, \neq 1)\) and Monomial Type Functions

or

\[
\begin{align*}
\left( x-b \right)^2 \frac{\partial}{\partial x} \left[ \frac{x-b}{x} \right] &= \sum_{s=0}^{\infty} (b)^s H_s \left( -\frac{b}{x-b}, y \ln(a) \right) \\
\text{....(2.15)}
\end{align*}
\]

**Particular case:** The replacement of \( a \) with \( e \) into the eqs. (2.1) to (2.15) give raise to the eqs. (8) to (21) due to Dattoli et al. [6] respectively.

3. CONCLUDING REMARKS

In the previous section we have seen that the theory of exponential operators can be conveniently complemented by the use of functions satisfying recurrences of quasi monomial nature.

In these concluding remarks we will discuss the introduction of a family of functions which can be viewed as a fairly natural consequence of the so far developed formalism.

We consider indeed the case of logarithmic Bessel functions, whose Generating function can be cast in the form

\[
G(x, \theta) = x^{i \sin(\theta)} = \sum_{n=-\infty}^{\infty} e^{i n \theta} J_n(\ln(x)),
\]

where \( J_n(x) \) denote the first kind cylinder Bessel functions.

It is evident that we can take advantage from the discussion of the previous sections, to consider the following problem

\[
\begin{align*}
\left( \frac{x}{i \sin(\theta)} \right)^2 x^{i \sin(\theta)} &= e^{\frac{i \sin(\theta)}{\ln(a)}} x^{i \sin(\theta)} = e^{i \sin(\theta) \left( \ln(x) + 2y \ln(a) \frac{\partial}{\partial x} \right)} \\
\text{....(3.2)}
\end{align*}
\]

The exponential can be decoupled by means of the Weyl rule

\[
e^{\hat{P}} e^{\hat{Q}} = e^{\hat{P}} e^{\frac{1}{2}[\hat{P}, \hat{Q}]},
\]

where, \([\hat{P}, \hat{Q}] = \hat{P} \hat{Q} - \hat{Q} \hat{P}\),

by setting indeed

\[
\begin{align*}
\hat{P} &= i \sin(\theta) \ln(x) \\
\hat{Q} &= 2iy \ln(a) (\sin(\theta)) x \frac{\partial}{\partial x}
\end{align*}
\]

we find

\[
[\hat{P}, \hat{Q}] = 2y \ln(a) (\sin(\theta))^2
\]

thas getting
Appendix

\[ a \left(x, \frac{\partial}{\partial x}\right)^2 x^i \sin(\theta) = x^i \sin(\theta) e^{-y \ln(a)(\sin(\theta))^2} \]

which is the generating function of a two-variable Bessel function, namely

\[ x^i \sin(\theta) e^{-y \ln(a)(\sin(\theta))^2} = \sum_{n=-\infty}^{\infty} e^{\ln(a)} h_n(x, y \ln(a)) \]

\[ h_n(x, y \ln(a)) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(\ln(x), y \ln(a))}{2^{n+2r} r!(n+r)!}. \]

It is evident that we ended up with a Bessel type function generalizing those of Hermite nature discussed in Dattoli et al. (2001). It is worth emphasizing that the above equations satisfy a partial differential equation of the type

\[ \frac{\partial}{\partial y} h_n(x, y \ln(a)) = \ln(a) \left(x, \frac{\partial}{\partial x}\right)^2 h_n(x, y \ln(a)), \]

\[ h_n(x, 0) = I_n(x \ln(a)). \]

It is evident that the above considerations can be extended to any generating function of the type

\[ a^p(x) \sin(\theta) \]

Before concluding we will show how the combined use of integral transform and the previous formalism allows the derivation of further important relations. According to the identity (Witschel, 1985)

\[ a^\gamma = e^{\gamma \ln(a)} b^\gamma = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 + 2x \gamma} \ln(a) \, \, d\xi, \]

we can easily conclude that the polynomials \( h_n^{(2)}(x, y) \) can also be realized in terms of the integral representation

\[ a \left[ q(x) \frac{\partial}{\partial x}\right]^2 [F(x)]^n = e^{\ln(a)} \left[ q(x) \frac{\partial}{\partial x}\right]^2 [F(x)]^n \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} [F(x) + 2 \sqrt{\gamma} \ln(a) \xi]^n d\xi. \]

Going back to eq. (2.14) and specializing for \( m=2 \), the use of the above relations allows to state the following identity

\[ \sum_{n=0}^{\infty} \frac{(-1)^n a^{-yn^2}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-y \ln(a) n^2}}{n!} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{-\cos(2 \sqrt{2} \xi \ln(a)) \xi} d\xi. \]
Finally since Dattoli et al. (2004)

\[ e^{-yd} = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t^{\frac{1}{2}}} e^{-\frac{(y)^2}{4t}} e^{-d^2} \, dt \]

by replacing \( d \) with \((\ln(a)\Gamma_x)^{1/2}\) and by using the previously discussed rules we find

\[ e^{-y(\ln(a)\Gamma_x)^{1/2}} = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t^{\frac{1}{2}}} e^\left(-y(\ln(a)\Gamma_x)^{1/2}\right) \, dt = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t^{\frac{1}{2}}} e^{-\frac{(y)^2}{4t} a^{-\Gamma_x} f(x)} \, dt \]

now

\[ e^{-y(\ln(a)\Gamma_x)^{1/2}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t^{\frac{1}{2}}} e^{-\frac{(y)^2}{4t} a^{-\Gamma_x} f(x)} \, dt \]

or

\[ e^{-y(\ln(a)\Gamma_x)^{1/2}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t^{\frac{1}{2}}} e^{-\frac{(y)^2}{4t} f\left(\int_{-1}^{1} (F(x) - t \ln(a)) \right)} \, dt . \]

which realizes the transform providing the action of a fractional generalized shift operator on a given function. The validity of both (3.11) and (3.13) is limited to the case in which the integral converges.

**Particular case:** The substitution of \( a = e \) into the eqs. (3.1) to (3.13) give raise to the eqs. (22) to (34) due to Dattoli et al. (2003).

**ACKNOWLEDGEMENT**

The second author wishes to express his heartfelt thanks to the Human Resource Development Group Council of Scientific & Industrial Research of India for awarding Senior Research Fellowship (NET) (F.No. 10-2(3)/ 2005(i)- E.U.II).

**REFERENCES**


Appendix


Khan M. A. and Asif M. Generalized Exponential Operators and Difference Equations, communicated for publication.

Khan M. A. and Asif M. Generalized Operational Methods, Fractional Operators and Special Polynomials, communicated for publication.

Khan M. A. and Asif M. Shift Operators on the Base $a (a > 0; \neq 1)$ and Pseudo-Polynomials of Fractional Order, communicated for publication.


Dear Sirs,

I have the pleasure to let you know that your paper: "Generalized exponential operators and difference equations" after the positive report of the Editorial Board, has been accepted for publication in n. 30 of our journal.

Best regards,

Piergiulio Corsini

Publisher:
FORUM
Editrice Universitaria Udinese
Via Palladio 8, 33100 Udine, Italy
Appendix

From: Emil Minchev <minchev@m-hikari.com>
To: mumtaz ahmad khan <mumtaz.ahmad.khan.2008@yahoo.com>
Sent: Wed, August 18, 2010 1:18:27 AM
Subject: payment of page charge

Dr. Emil Minchev
Managing Editor,
President of Hikari Ltd

Dear Professor Khan,

Thank you very much for your kind e-mail and prompt cooperation. The files of your paper:

"SHIFT OPERATORS ON THE BASE a (a > 0, \neq 1) AND PSEUDO-POLYNOMIALS OF FRACTIONAL ORDER" - 13 pages

have been received and prepared according to the respective journal's style file.

To process further your paper toward publication, you are most kindly asked to pay the respective page charge which is: 195 Euro.

The method of payment is by bank transfer or by bank cheque as it is described below: (Kindly notice that other methods of payment are not acceptable.)

Method of payment: (I) by BANK TRANSFER to the following bank:

Beneficiary bank: Unicredit Bulbank
Address of the bank: Sveta Nedelya Square 7, Sofia 1000, Bulgaria

SWIFT: UNCRBGSF
IBAN: BG11UNCR70001505689259

Beneficiary's name: HIKARI Ltd
Beneficiary's address: ul. Rui planina 4, vh.7, et.5, Ruse 7005, Bulgaria

Correspondent bank of the beneficiary’s bank:
EUR SSI:
BKAUATWW
HYVEDEMM
Appendix

Please note that the page charge doesn’t include the bank charges for the transfer. The bank charges must be paid by the author and should not be deducted from the page charge.

Method of payment: (II) by BANK CHEQUE: Please send a bank cheque for 215 Euro (195 Euro page charge + 20 Euro tax for cashing the cheque). The cheque must be payable to HIKARI LTD and should be sent by registered airmail letter to the FOLLOWING ADDRESS: Hikari Ltd, P.O. Box 15, Ruse 7005, Bulgaria.

Your paper will be published within about three months after the payment of the page charge. By that time reprints will be sent to your address by air mail as well as you will be provided with the electronic file of the final form of your paper by e-mail.

You are most kindly asked to pay the page charge as soon as possible. After the payment of the page charge, please let me know by e-mail the date when the payment has been done. Your cooperation is greatly appreciated.

Yours sincerely,

Emil Minchev

Managing Editor and
President of Hikari Ltd