ON CERTAIN PROBLEMS IN THE THEORY OF FOURIER SERIES

A THESIS SUBMITTED TO THE ALIGARH MUSLIM UNIVERSITY, ALIGARH IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS

By ABUL HASAN SIDDIQI

DEPARTMENT OF MATHEMATICS AND STATISTICS ALIGARH MUSLIM UNIVERSITY, ALIGARH 1967
P R E F A C E

The present thesis entitled "On Certain problems in the theory of Fourier series" embodies the result of my researches which I have been doing since Dec. 1966.

The thesis consists of seven chapters. In the first chapter we give a résumé of hitherto known results which have inter-connections with our investigations. Chapter II is concerned with the study of the Fourier series of functions of certain classes and some approximation problems, while Chapter III deals with the transformation of Fourier series of different classes. Chapter IV is devoted to the study of general summability involving infinite matrices, of a trigonometric sequence. In Chapter V, we have obtained a number of results concerning Walsh-Fourier coefficients of certain class of functions. Chapter VI contains some interesting results concerning Walsh-Fourier series while chapter VII is concerned with the study of summability of a sequence of Walsh functions. Towards the end we give a fairly extensive bibliography of various publications which have been referred to in the present thesis.

It may be mentioned here that the major portion of this thesis, presented in the form of papers, have been
communicated for publication in various mathematical journals. Two of them have already been accepted for publication in "American Mathematical Monthly" and the Indian Journal of Mathematics, while several others have been read at the annual conferences of Indian Mathematical Society and Indian Science Congress Association.

It is with deep sense of gratitude that I take this opportunity of acknowledging my indebtedness to Prof. J. R. Siddiqi, Head of the Department of Mathematics for his guidance, kind encouragement and help during the preparation of this thesis.

I take much pleasure in offering my thanks to Dr. C.M. Mazhar, Reader in Mathematics, as a token of great debt of gratitude which I owe to him.

November 30, 1967

Abul Hasan Siddiqi
CONTENTS

CHAPTER I: INTRODUCTION

CHAPTER II: ON THE FOURIER SERIES OF FUNCTIONS OF CERTAIN CLASSES AND SOME APPROXIMATION PROBLEMS

CHAPTER III: ON THE FOURIER COEFFICIENTS OF CERTAIN CLASSES OF FUNCTIONS

CHAPTER IV: ON THE SUITEABILITY OF A TRIGONOMETRIC SEQUENCE

CHAPTER V: ON THE WALSH-FOURIER COEFFICIENTS OF CERTAIN CLASSES OF FUNCTIONS

CHAPTER VI: ON THE WALSH-FOURIER SERIES OF CERTAIN CLASSES

CHAPTER VII: ON THE SUITEABILITY OF A SEQUENCE OF WALSH FUNCTIONS

BIBLIOGRAPHY

APPENDIX
CHAPTER I

INTRODUCTION

1.1. The present thesis entitled "On certain problems in the theory of Fourier series" is based on the study of some aspects of Fourier series and that of Walsh-Fourier series. A number of theorems of different characters have been proved. Some of the results are directly concerned with the behaviour of Fourier coefficients. Before giving a résumé of the earlier researches in the light of which various new results have been obtained by the author, it seems desirable to state here the notations and definitions which will be required in the sequel.

1.2. Cesàro Summability: Let \( \sum a_n \) be a given infinite series with \( n \)-th partial sum \( s_n \). We define \( A_n \), \( S_n \) and \( C_n \) by the following identities:

\[
\begin{align*}
\sum_0^\infty a_n x &= (1 - x)^{-\alpha - 1}, \\
\sum_0^\infty s_n x &= (1 - x)^{-\alpha - 1} \sum_0^n a_n x,
\end{align*}
\]
\[ S_n^\varsigma = \frac{q_n}{A_n^\varsigma}, \quad \varsigma > 1. \]

\( S_n^\varsigma \) and \( \sigma_n^\varsigma \) are called \( n \)-th Cesaro sum of order \( \varsigma \) and \( n \)-th Cesaro mean of order \( \varsigma \) respectively.

If \( \sigma_n^\varsigma \to s \) as \( n \to \infty \), we say that the series \( \sum \limits_{n} a_n \) with partial sum \( s_n \) is summable by Cesaro mean of order \( \varsigma \) to \( s \) or simply summable \((c, \varsigma)\) to \( s \).

**Norlund Summability.** Let \( \{p_n\} \) be a sequence of real or complex constants. We write

\[ p_n = \prod_{k=0}^{n} p_k, \]

\[ t_n = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} s_k, \quad p_n \neq 0. \]

If \( \lim_{n \to \infty} t_n = s \), we say that the series \( \sum \limits_{n} a_n \) is summable by Norlund means associated with the sequence \( \{p_n\} \) or simply summable \((N, p_n)\) to \( s \).

For \( p_n = A_n^{\varsigma-1}, \varsigma > 0 \), we get summability \((c, \varsigma)\).

---

1) The concept of Norlund summability was first introduced by Norlund (36) but is, nowadays, more closely identified with the name of Norlund (29).
Linear methods of summation: Let \( (a_{n,k}) \) be an infinite matrix of real or complex numbers and \( \{a_k\} \) be any sequence of real or complex numbers. The sequence \( \{\sigma_n\} \) defined by

\[
\sigma_n = \sum_{k=0}^{\infty} a_{n,k} s_k
\]

is called \( A \)-transform of the sequence \( \{s_k\} \) whenever the series on the right converges for \( n = 0, 1, 2, \ldots \).

If \( \lim_{n \to \infty} \sigma_n = s \), we say that the series \( \sum a_n \) with partial \( s_n \) or the sequence \( \{s_n\} \) is \( A \)-summable to the limit \( s \).

\( A \)-summability method is said to be regular if \( s_n \to s \implies \sigma_n \to s \). The necessary and sufficient conditions for the regularity of \( A \)-summability are

\[
\lim_{n \to \infty} a_{n,k} = 0, \quad k = 0, 1, 2, \ldots
\]

1) Toeplitz, O. (34 C) He considered only triangular matrix, the result for general matrix is due to Steinhaus R. (34 B).
(ii) \[ \sup_{n,k} \sum_{k=0}^{\infty} |a_{n,k}| < M, \]

(iii) \[ \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} = 1. \]

If \( a_{n,k} = 0 \) for \( k > n \), then the matrix \( A \) is called triangular and the method of summability is called triangular method of summability.

If in a triangular matrix \( A \) we take
\[ a_{n,k} = \frac{A_{n-k}}{A_n}, \quad \alpha > -1 \]
where
\[ A_{\alpha} = \left( \begin{array}{c} 1 & \alpha \\ 0 & A_{\alpha} \end{array} \right), \]
then \( A \)-summability becomes \( \sigma \)-summability \((\sigma, \alpha)\). Similarly by taking \( a_{n,k} = \frac{p_{n-k}}{p_n} \), we observed that \( (N, \sigma) \)-summability \((N, \sigma, \sigma)\) is a special case of \( A \)-summability.

It follows from the regularity conditions for \( A \)-summability that \( \sigma \)-summability \((\sigma, \sigma)\) is regular.

\[ \ast M \text{ is a constant not necessarily the same at each occurrence.} \]
A bounded sequence \( \{q_k\} \) is said to be almost convergent to \( s \) if

\[
\lim_{n \to \infty} R_n(k) = s
\]

uniformly in \( k \), where

\[
R_n(k) = \frac{1}{n} \sum_{m=0}^{n-1} q_{m+k}
\]

It is evident that every convergent sequence is almost convergent. However the converse is not necessarily true.

The sequence \( \{q_k\} \) is said to be FA-summable to the limit \( s \) if

\[
\sum_{m=0}^{\infty} a_{m+k} = \sum_{m=0}^{\infty} a_{m+n} s_n + k
\]

tends to \( s \) as \( n \to \infty \), uniformly in \( k \).

It is well known that if \( A \) is regular then

FA-summability implies almost convergence. Also for regular matrix \( A \) the condition

\[
\begin{align*}
1) & \quad \text{Lorentz, C.G. (24).} \\
2) & \quad \text{King, J.P. (22).} \quad \text{Lorentz, C.G. (24).}
\end{align*}
\]
\[ \lim_{n \to \infty} \sum_{m=0}^{\infty} (a_{m,n} - a_{m,n+1}) = 0 \]

is necessary and sufficient in order that almost convergence may imply \(A\)-summability. In particular if a sequence \( \{a_n\} \) is almost convergent, then it is also summable \((e, \infty), \rho > 0\). Lorentz has remarked that the same condition ensures that almost convergence may imply \(FA\)-summability. Thus under these conditions almost convergence and \(FA\)-summability are equivalent.

The sequence \( \{a_k\} \) is said to be almost summable to \( s \) if the \(A\)-transform \(c_n\) of \( \{a_k\} \) is almost convergent to \( s \). The matrix \( A \) is said to be almost conservative if \( s_k \to s \) implies that \( c_n \) is almost convergent to \( t \). It is said to be almost regular if \( s_k \to s \) implies that \( c_n \) is almost convergent to \( s \). The necessary and sufficient conditions for the matrix \( A \) to be almost regular \(2) \) are

\[ (1') \sup_{n=1}^{\infty} \sum_{p=1}^{n+p-1} \sum_{j=0}^{\infty} \sum_{k=0}^{n+p-1} a_{j,k} < M, \quad n = 1, 2, \ldots, p \text{ is a positive integer}, \]

\(1) \) King, J.P. (22).

\(2) \) King, J.P. (22).
\[ (ii) \quad \lim_{p \to \infty} \frac{1}{n+p-1} \sum_{j=0}^{n} a_{j,k} = 0 \]

uniformly in \( n, \ k = 0, 1, 2, \ldots \).

\[ (iii') \quad \lim_{p \to \infty} \frac{1}{n+p-1} \sum_{j=0}^{n} \sum_{k=0}^{\infty} a_{j,k} = 1, \]

uniformly in \( n \).

1) Badger has remarked that \((iii')\) can be replaced by the following more natural condition:

\[ (i) \quad \sup_{m} \sum_{k=0}^{\infty} |a_{m,k}| < M, \ n = \pm 1, \pm 2, \ldots \]

The almost \( A \)-summability was subsequently generalized by Mazhar and Siddiqi \(^2\) in the following way.

A bounded sequence \( \{s_k\} \) will be said to be \( A-B \)-summable to \( s \) if \( A \)-transform of \( \{s_k\} \) is \( FB \)-summable to \( s \), where \( B = (b_{n,k}) \) is an infinite matrix.

It is easy to see that every \( A-B \)-summable sequence is also almost \( A \)-summable provided the second matrix, namely \( B = (b_{n,k}) \) is regular.

1) \(^{1}\) King, J.P. (22) p. 1226.

1.3. Let \( f(x) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \((0, 2\pi)\).

Let

\[
\begin{align*}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \\
&= \sum_{n=0}^{\infty} a_n(x).
\end{align*}
\]

Then the series conjugate to Fourier series of \( f(x) \) is given by

\[
\begin{align*}
E \left( \sum_{n=0}^{\infty} b_n \cos nx - a_n \sin nx \right) \\
&= \sum_{n=0}^{\infty} B_n(x).
\end{align*}
\]

**Walsh–Fourier series:** The system of orthogonal functions introduced by Rademacher\(^1\) has been the subject of a great deal of study. This system is not a complete one. Its completion was effected by Walsh\(^2\), who studied some of its Fourier properties, such as convergence, summability and so on. Others, notably, Kaczmarz\(^3\), Steinhaus\(^4\), Paley\(^5\) etc., have studied various aspects of Walsh system. Walsh has pointed out the great similarity between this system and the trigonometric system.

---

1) Rademacher, H., (32 A).
2) Walsh, J.B., (39).
5) Paley, R.E., A.C. (30).
We now usually follow Paley's modification in defining functions of Walsh system. The Rademacher functions are defined by

\[
\begin{cases}
\mathcal{Q}_0(x) = 1 \quad (0 \leq x < \frac{1}{2}), \\
\mathcal{S}_0(x) = -1 \quad (\frac{1}{2} \leq x < 1),
\end{cases}
\]

\[
(1.3.1)
\]

\[
\mathcal{S}_0(x+n) = \mathcal{S}_0(x), \quad \mathcal{Q}_n(x) = \mathcal{S}_0(2^nx) \quad (n = 1, 2, \ldots)
\]

The Walsh functions are then given by

\[
(1.3.2) \quad \Psi_0(x) = 1, \quad \Psi_n(x) = \mathcal{S}_{n_1}(x) \mathcal{S}_{n_2}(x) \cdots \mathcal{S}_{n_r}(x)
\]

for \( n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r} \), where the integers \( n_1 \) are uniquely determined for \( n_{i+1} < n_1 \). Walsh proved that \{\Psi_n(x)\} forms a complete orthonormal set.

If \( f(x) \) is periodic and Lebesgue integrable in \((0,1)\), then the series

\[
f(x) \sim \sum_{n=0}^\infty c_n \Psi_n(x),
\]

with

\[
c_n = \int_0^1 f(x) \Psi_n(x) \, dx
\]

is called Walsh-Fourier series associated with the function \( f(x) \).
One of the striking differences between Walsh-Fourier series and trigonometric Fourier series is that if \( f(x) \in BV \) and \( x_0 \) is neither a dyadic rational nor a point of continuity of \( f(x) \), then the \( n \)-th partial sum of the first series diverges\(^1\) at \( x_0 \) while that of the second necessarily converges.

**Dyadic Groups** The dyadic group \( G \) \(^2\) is defined as the countable direct product of the groups with elements 0,1 in which the group operation is addition modulo 2. Thus the dyadic group \( G \) is the set of all 0,1 sequences in which the group operation, which we shall denote by \( + \), is addition modulo 2 for each element.

Just as the additive group of real numbers modulo \( 2\pi \) is fundamental in the theory of trigonometric series and classical Fourier series, so the group \( G \) is the underlying basis in the study of Walsh-Fourier series. It was observed by Fine\(^3\) that this analogy is of great importance. With this in view he investigated a number of relationship which exist between the group operation in \( G \) and the ordinary processes of analysis.

\[ \text{References:} \]
\[ 1) \text{Pontrjajin, L. (32)} \]
\[ 2) \text{Walsh, J.L. (35b)} \]
\[ 3) \text{Fine, N.J. (7): Various properties stated here are due to Fine, N.J. (7).} \]
Let \( \overline{x} \) be an element of \( G \), \( \overline{x} = \{x_1, x_2, \ldots\} \) \( x_n = 0, 1 \).

We define the function \( \lambda \) as:

\[
\lambda(\overline{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n.
\]

This function \( \lambda \) which maps \( G \) onto the closed interval \([0, 1]\)
does not have a single-valued inverse on the dyadic rationals; in such case we take only finite expansion.

Thus for all real \( x \), if we write the inverse as \( \mu \), then

\[
\lambda(\mu(x)) = x - [x]
\]

\([x]\) denoting the greatest integer in \( x \). It is not necessarily true that \( \lambda(\mu(\overline{x})) = \overline{x} \) for all \( \overline{x} \in G \). It is true, however, provided that \( \lambda(\overline{x}) \) is not a dyadic rational.

If \( \overline{x} = \{x_n\} \) and \( \overline{y} = \{y_n\} \) are elements of \( G \), we have

\[
\overline{x} + \overline{y} = \{x_n - y_n\}.
\]

We shall abbreviate \( \lambda(\mu(x) + \mu(y)) \) as \( x \ast y \) for

any real \( x \) and \( y \). Thus if \( x = \sum_{n=1}^{\infty} 2^{-n} x_n \) and

\( y = \sum_{n=1}^{\infty} 2^{-n} y_n \), \( x_n, y_n = 0, 1 \) \( n = \ast \), we have

\[
x + y = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.
\]
1.4. A function $f(x)$ is said to belong to \( \text{Lip} \prec \) class $0 < \prec \leq 1$, if
\[
f(x + h) - f(x) = O(|h|^{\prec}), \quad h \to 0,
\]
uniformly in $x$.

It is said to belong to \( \text{Lip} \prec \) class in a set $E$ (of real numbers), if
\[
f(x + h) - f(x) = O(|h|^{\prec}), \quad 0 < \prec \leq 1,
\]
as $h \to 0$, uniformly in $E$, through unrestricted real values.

If
\[
\left[ \frac{1}{p} \int_{a}^{b} |f(x + h) - f(x)|^{p} \, dx \right]^{\frac{1}{p}} = O(|h|^{\prec}),
\]
h $\to 0$, uniformly in $x$, $0 < \prec \leq 1$, $1 \leq p < \infty$, then
\[
f(x) \in \text{Lip} (<p).
\]

The function $f(x)$ is said to belong to $\wedge_{\prec}$ class if $f(x)$ is continuous and
\[
f(x + h) + f(x - h) - 2f(x) = O(h),
\]
as $h \to 0$, uniformly in $x$.

\hspace{1cm} 1) Kennedy, P.B. (21)
\hspace{1cm} 2) Zygmund, A. (43).
The concepts of Lip $\alpha$ and $\Lambda_\alpha$ classes can be generalized in the following manner.

A function $f(x)$ will be said to belong to Lip $(\alpha(x))$ class if

$$|f(x + h) - f(x)| = O(|h|)$$

$h \to 0$, uniformly in $x$, where $\alpha(x)$ is a function such that

$$\int_0^\infty \frac{|\alpha(x)|}{x} dx < \infty, \quad \alpha > 0.$$  \hfill (1.4.1)

The class Lip$<\alpha$ is the special case $\alpha(x) = x$, $0 < \alpha \leq 1$ of the above class.

Similarly a function $f(x)$ will be said to belong to $\Lambda_\alpha(\alpha(x))$ class if it is continuous and

$$f(x + h) + g(x - h) - 2f(x) = O(|h|),$$

$h \to 0$, uniformly in $x$, where $\alpha(x)$ is a function such that (1.4.1) holds.

If we take $\alpha(x) = x$, then the $\Lambda_\alpha(x)$ is the same as $\Lambda_\alpha$.

A function $f(x)$ is said to belong to the class $B$ of essentially bounded function if $|f(x)| \leq M$ almost everywhere.
If \( f(x) \) satisfies the condition

\[
|f(x + h) - f(x)| < C h^\alpha,
\]

where \( 0 \leq x < 1, \quad 0 \leq h < 1, \quad 0 < \alpha \leq 1, \) except when \( \mu(x) + \mu(h) \)
ends in a sequence of 1's, then \( f(x) \) is said to belong to Lipschitz \( \alpha \) class on \([0,1]\).

It may be observed that

\[ f \in \text{Lip}^\alpha [0,1] \implies f(x) \in \text{Lip}^\alpha (0). \]

It is said to belong to class \( \text{Lip}^\alpha (0) \) on \( G \) if

\[
\| f(x) - f(x + h) \|_p = \left( \lambda(h) \right)^\alpha,
\]

where

\[
\| g \|_p = \left( \int_0^1 |g(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty,
\]

and

\[
\| g \|_\infty = \sup_{x \in G} |g(x)|, \quad x \in G.
\]

A sequence \( \{ \lambda_n \} \) is said to be convex if

\[
\lambda_n \geq 0,
\]

1) Morgensthaler, C.W. (27)
2) Morgensthaler, C.W. (27); Watari, C. (37).
where $\triangle \lambda_n = \lambda_n - \lambda_{n-1}$ and $\triangle^2 \lambda_n = (\triangle \lambda_n)$.

It is said to be quasi-convex if

$$\sum_{n=0}^{\infty} \frac{2^n}{(n+1)!} \triangle \lambda_n < \infty.$$  

A Fourier series is said to be lacunary with Hadamard gaps if it is of the form

$$\sum_{k=1}^{\infty} (a_k \cos n x + b_k \sin n x),$$

where

$$\frac{n_k + 1}{n_k} > \lambda > 1, \quad k = 1, 2, \ldots$$

We use the following notations:

$$g(t) = f(x + t) + f(n - t) - 2f(x);$$

$$S_n(x) = \sum_{k=0}^{n} A_k(x);$$

$$t_n(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} S_k(x);$$

$$F_n(t) = \text{Im} \left\{ \sum_{m=0}^{\infty} e^{-\text{ist}} \right\};$$

$$\Phi_p = \frac{1}{p_n} \sum_{k=0}^{n} \frac{\beta - 1}{\beta} A_n - a_n, \quad \beta > 0;$$

$$A_n = \sum_{k=0}^{n} a_{n-k}.$$
we will denote by $G$ the class of series \[ \sum_{k=0}^{\infty} \epsilon_k \psi_k(x) \]
with $\epsilon_k = \int_0^1 \psi_k(x) \, dF(t)$, where $F(t)$ is continuous and of bounded variation.

We write \[ J_k(x) = \int_0^x \psi_k(t) \, dt \]
and
\[ T(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]

1.5. Concerning Fourier series of Lip$<\alpha$ class, 1)
Lorentz in 1948 obtained the following theorem.

1) Lorentz, G.G. (26).
THEOREM A. If \( f(x) = \sum_{n=0}^{\infty} a_n \cos nx \), where 
\( a_n \downarrow 0 \), then for \( f(x) \in \text{Lip} \llcorner \), \( 0 < \llcorner < 1 \), it is necessary and sufficient that

\[
(1.5.1) \quad a_n = O\left(\frac{1}{n^{\llcorner + 1}}\right).
\]

He also demonstrated that the above theorem remains true for \( g(x) = \sum_{n=0}^{\infty} a_n \sin nx \).

Very recently this theorem was generalized by Boas in the following form:

THEOREM B. Let \( a_n \geq 0 \) and \( a_n \) be the Fourier sine or cosine coefficient of \( f \). Then \( f \in \text{Lip} \llcorner \), \( 0 < \llcorner < 1 \), iff

\[
(1.5.2) \quad \sum_{k=0}^{\infty} a_k = O(n^{-\llcorner}),
\]

or, equivalently

\[
(1.5.3) \quad \sum_{k=1}^{n} k a_k = O(n^{1-\llcorner}).
\]

It is easy to see that if \( a_n \downarrow 0 \), then conditions (1.5.1), (1.5.2) and (1.5.3) are all equivalent.

1) Boas, R.P. Jr. (4)
In chapter II of the present thesis we have obtained the following general theorem which includes, as a special case for \( X(x) = x \), \( 0 < \alpha < 1 \), the above theorem of Boas for cosine series.

**Theorem 1.** Let \( a_n > 0 \) and \( a_n \) be the Fourier cosine coefficient of \( f \). Suppose \( \lambda(x) \) is a function which satisfies the following conditions:

\[
\begin{align*}
(1) & \quad \lambda(x) \text{ is positive and monotonic increasing}, \\
(2) & \quad \lim_{x \to \infty} \frac{\lambda(x)}{x^s} = 0 \quad \text{as } x \to \infty, \quad s > 0, \\
(3) & \quad \sum_{k=n}^{\infty} \frac{\lambda(k)}{k! + \delta} = O \left( \frac{\lambda(n)}{n^s} \right), \quad n \to \infty, \quad s > 0, \\
(4) & \quad \sum_{k=1}^{n} \frac{\lambda(k)}{k} = O \left( \lambda(n) \right), \quad n \to \infty.
\end{align*}
\]

Then \( f \in \text{Lip} (X(x)) \) iff

\[
\sum_{k=n}^{\infty} \frac{1}{k!} = O \left( \frac{1}{X(n)} \right),
\]

where \( X(n) \) and \( \frac{X(n)}{n^s} \) satisfy the same conditions as \( \lambda(x) \) in (1.5.4) for \( s = 1 \) and \( s = 2 \) respectively.
In the same chapter we have also examined the problem analogous to that of Lorentz for \( \Lambda \) class. In this direction we established the following theorem.

**THEOREM 2.** \[ \text{If } a_n = O\left(\frac{1}{n^2}\right), \]
then the trigonometric series \( T(x) \) will be the Fourier series of a function belonging to \( \Lambda \) class.

**THEOREM 3.** \[ \text{If } a_n \downarrow 0 \text{ and } f(x) = \sum_{n=1}^{\infty} a_n \cos nx, \]
then \( f \in \Lambda \) iff
\[ a_n = O\left(\frac{1}{n^2}\right). \]

In a recent paper Boas \(^1\) has obtained the following theorem which generalizes our theorem 3.

**THEOREM C.** \[ \text{If } a_n \downarrow 0 \text{ and } f = \sum_{n=1}^{\infty} a_n \cos nx, \text{ then } \]
\( f \in \Lambda \) iff
\[ \sum_{n=1}^{\infty} a_n = O\left(n^{-1}\right). \]

In § 2.9 of Chapter II we have generalized the above theorem of Boas in the same sense in which Theorem 1 generalizes Theorem B.

---

\(^1\) Boas, R.P. Jr. (4)
Using (C,1) mean $\sigma_n(x)$ of a trigonometric series $T(x)$ obtained necessary and sufficient condition in order that the trigonometric series may be the Fourier series of a function belonging Lip $\alpha$ class.

His result is as follows:

**THEOREM D.** A necessary and sufficient condition that a trigonometric series $T(x)$ be the Fourier series of $f(x) \in \text{Lip } \alpha$, $0 < \alpha < 1$ is that

$$\sigma_n(x) - \sigma_m(x) = O(n^{-\alpha})$$

uniformly in $[0,2\pi]$ for all $m > n$.

In Chapter VI we have extended this result to Walsh series for $f \in \text{Lip } \alpha$ (w), $p = \infty$ on the group $G$. It is natural to enquire as to whether there is any analogue of Theorem D for $f \in \wedge_*$. In this direction we have established the following theorem in Chapter II.

**THEOREM 4.** If $\sigma_n(x) - \sigma_m(x) = O(n^{-1})$ uniformly in $[0,2\pi]$ for all $m > n$, then the trigonometric series $f(x)$ is the Fourier series of a function $f \in \wedge_*$. 

---

1) DUPLESSIS, (5).

---

* Under publication in the American Mathematical Monthly.*
Theorem 5. If a trigonometric series $T(x)$ be the Fourier series of a function $f \in \mathcal{A}_{s}$, then

$$\Delta_n^{-}(x) - \Delta_n^{+}(x) = O\left(\frac{\log n}{n}\right),$$

uniformly in $[0, 2\pi]$ for all $n > n_0$.

1.6. It is a classical result of Bernstein that if $f(x) \in \text{Lip}<\alpha$, $0 < \alpha < 1$ and $\Delta_n(x)$ is the nth-$(C,1)$ mean of the Fourier series of $f(x)$ then

$$(1.6.1) \quad \Delta_n(x) - f(x) = O\left(n^{-\alpha}\right),$$

Uniformly in $x$. The fact that (1.6.1) depends only on the behaviour of $f(x)$ in the immediate neighbourhood of the particular point $x$ concerned, follows from the following well known theorem for $k = 1$.

Theorem B. Let $0 < \alpha < 1$, $0 < \beta \leq \gamma$, $k > \alpha$. If $x$ is a point such that

$$(1.6.2) \quad |\phi(t)| \leq \alpha t^{-\alpha}$$

when $0 \leq t \leq \varepsilon$, then

$$(1.6.3) \quad \Delta_n^k(x) - f(x) = O(n^{-\alpha}),$$

where $\Delta_n^k(x)$ is the $(C,k)$ mean of the Fourier series of $f(x)$.

* The most general form of Theorem B is due to Obrechkoff, M (29 A).

[1] See e.g. Zygmund A, (44) vol. I.
The condition (1.6.3) is still a local property when \( k = \kappa \), but in this case the condition (1.6.2) is no longer sufficient for the truth of (1.6.3) and therefore for \( k = \kappa \), Fleett obtained the following theorem.

**Theorem F.** Let \( 0 < \kappa < 1, \ k < \beta \), \( 0 < t \leq \varepsilon \). If \( x \) is a point such that

\[
(1.6.4) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

when \( 0 \leq t \leq \varepsilon \), then

\[
\sigma^\kappa_x(x) - f(x) = O(n^{-\kappa}).
\]

Fleett has proved that condition (1.6.4) cannot be replaced by (1.6.2), nor can we increase the order of the expression on the right of (1.6.4). For \( k < \kappa \) the relation (1.6.3) is no larger a local property. In this case he proved, among others, the following theorem.

**Theorem G.** Let \( 0 < \kappa < 1, \ k < \beta \), \( 0 < \varepsilon \leq \gamma \), \( k > \kappa - \beta \). If \( x \) is a point such that

\[
A_n(x) = O(n^{-\beta})
\]

and

\[
(1.6.5) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

\[
(1.6.6) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

\[
(1.6.7) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

\[
(1.6.8) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

\[
(1.6.9) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

\[
(1.6.10) \quad \int_0^t \left| \text{Id} g(u) \right| \leq \kappa
\]

1) Fleett, T.N. (9)

2) Fleett, T.N. (9)
when \( 0 \leq t \leq \varepsilon \), then

\[
\sigma_n^{\varepsilon}(x) - f(x) = O(\frac{1}{n^{1-k}}).
\]

In § 2.10. of Chapter II we have proved the following theorem:

**Theorem 6.** Let \( 0 < k < 1 \) and \( 0 < \varepsilon \leq 1 \). If \( x \) is a point such that

\[
\int_0^t \left| \log \varnothing(u) \right| \leq A \varphi(t),
\]

when \( 0 \leq t \leq \varepsilon \), then

\[
\sigma_n^{\varepsilon}(x) - f(x) = O(\frac{1}{n^{1-k}}) + O(\frac{1}{n^{1-k}}),
\]

where \( \varphi(t) \) is a positive increasing function such that

\[
\int_1^n \frac{\varphi(t)}{t^2} dt = O(\frac{1}{n^{1-k}}), \quad n \to \infty.
\]

**Theorem 7.** Let \( 0 < k < 1 \), \( 0 < \varepsilon \leq 1 \). If \( x \) is a point such that

\[
A_n(x) = O(\varnothing_n)
\]

and

\[
\int_0^t \left| \log \varnothing(u) \right| \leq A \varphi(t),
\]
when \( 0 \leq t \leq \varpi \), then

\[
\phi_{\sigma}^{k}(x) - f(x) = O(\psi(\frac{1}{n})) + O\left(\frac{1}{n}\right) + O\left(\frac{\lambda_{n}}{n^{k}}\right)
\]

where \( \{\lambda_{n}\} \) is a positive monotonic non-increasing sequence such that \( \{n \lambda_{n}\} \) is increasing and

\[
\sum_{n} \lambda_{k} = O(n \lambda_{n})
\]

and \((\tau)\) is the same as in Theorem 6.

It may be remarked, theorem 6 includes, Theorem 5 as a special case for \( \psi(t) = t \), \( 0 < \alpha < 1 \), \( k = \alpha \) while Theorem 7 includes theorem 4 for \( k > 0 \) and \( k > \alpha \).

We have also extended Theorem 6 to Nörlund means. The result obtained is as follows:

**Theorem 8.** Let \( \{p_{n}\} \) be a positive non-increasing sequence of real numbers such that

\[
\int_{t}^{\varpi} p_{n}(u) \, du = O\left(\frac{\psi(t)}{n}\right), \quad \frac{1}{n} \leq t \leq \varpi
\]

Also let \( 0 < \alpha < 1 \), \( 0 < \varepsilon < \varpi \). If \( x \) is a point such that

\[
\int_{0}^{t} \lambda \psi(\sigma) \, d\sigma \leq \lambda \psi(\alpha) < \lambda \psi(\alpha)
\]

when \( 0 \leq t \leq \varepsilon \), then
$$t_n(x) - f(x) = O(n^{-\alpha}) + O\left(\frac{1}{p_n}\right).$$

It is clear that if we take $p_n = n^{-\alpha - 1}$, Theorem P becomes a particular case of the above theorem.

1. In 1930 Hardy proved the following theorem concerning the Fourier coefficient of a function belonging to $L^p$ class.

**THEOREM H.** If $a_1, a_2, \ldots, a_n, \ldots$ are Fourier cosine coefficients of a function of $L^p$ class, $p \geq 1$, then $s_1, \frac{s_2}{2}, \ldots \frac{s_n}{n}, \ldots$ are also Fourier cosine coefficients of a function of $L^p$ class, where

$$s_n = a_0 + a_1 + \ldots + a_n.$$  

Later on Peterson proved the following theorem concerning $(C,K)$ mean of $\{a_n\}$.

**THEOREM L.** If $a_1, a_2, \ldots, a_n, \ldots$ are the Fourier coefficients of an even function

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$$

belonging to $L^p$ ($p > 2, \beta + \frac{1}{p} > 1$), then

1) Hardy, G.H. (17).
2) Peterson, G.M. (18).
then \( a_1^p, a_2^p, \ldots, a_n^p, \ldots \) are the Fourier coefficients of an even function \( g(x) \) belonging to 
\( L^p' \) \((p' < p)\).

1) In 1959 Goldberg generalized the above theorem of Hardy in the following form:

**Theorem J.** Let \( 1 < p < \infty \). If \( a_n \) is the Fourier cosine coefficient of a function belonging to \( L^p \) class, then \( c_n \) is also Fourier cosine coefficient of a function belonging to \( L^p \) class.

The result remains true also for sine series.

2) Young using matrix transformation obtained a number of fairly general results.

His results were subsequently extended by Goes\(^3\) who in a series of papers obtained several interesting results of different characters.

4) Bellman, replacing \( a_n \) by \( \sum_{k=1}^{\infty} \frac{a_k}{k^n} \) proved that Hardy's result remains still true. More precisely, he proved the following theorem.

---

1) Goldberg, R. R. (16)  
2) Young, F. H. (41)  
3) Goes, G. (10, 11, 12, 13, 14, 15)  
4) Bellman, R. (3) : The same theorem was also independently obtained by Sanouchi, G. (34 A).
THEOREM K. If \( a_1, a_2, \ldots, a_n, \ldots \) are the Fourier cosine coefficients of a function \( f(x) \in L^p, \ p > 1 \), then 
\[ \langle a_1, a_2, \ldots, a_n, \ldots \rangle \] 
are also Fourier cosine coefficients of a function \( F(x) \in L^p \), where 
\[ \langle a_n \rangle = \sum_{k=n}^{\infty} \frac{a_k}{k} \] 

1) Kawata proved the above result for sine series. It may also be remarked that in view of the Hardy's theorem result of Bellman becomes a particular case of a theorem of Young.

The case, \( p = 1, \infty \) in Theorems K and H have been considered by Loo. 2)

3) Konyushkov examined the corresponding problem for \( \text{Lip} \langle \alpha \rangle \) class. He also obtained certain results for \( \text{Lip} (\langle \alpha, p \rangle) \) class.

In Chapter III of the present thesis several results have been proved concerning Fourier series of various classes, like \( L^p, \text{BV}, \text{Lip} \langle \alpha \rangle, \wedge \) etc. A theorem on lacunary Fourier series for \( \text{Lip} \langle \alpha \rangle \) has also been established. Thus, for example, some of the results proved are as follows:

1) Kawata, T. (20 C)
2) Loo, C.T. (23 A)
3) Konyushkov, A.A. (23)
THEOREM 9. Let \( f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx, \) \( f(x) \in \Lambda_a. \)

If \( f(0) = 0, \) then \( \sum_{n=1}^{\infty} a_n \cos nx \) is the Fourier series of a function \( f(x) \in \Lambda_a. \)

THEOREM 10. Let \( g(x) \sim \sum_{n=1}^{\infty} a_n \cos nx, \) \( g(x) \in \Lambda_a. \)

If \( g(0) = 0, \) then \( \sum_{k=1}^{\infty} a_k \cos kx \) is the Fourier series of a function \( g(x) \in \Lambda_a, \) where \( a_k = \sum_{n=k}^{\infty} \frac{a_n}{k}. \)

In Chapter V we have examined as to how far the problem analogous to that of Hardy remains true for Walsh-Fourier series. In this connection we have established a number of theorems for classes \( L^p, B \) and \( S^p. \) Thus for example the result proved for \( B \) class states:

THEOREM 11. If \( \sum_{k=1}^{\infty} a_k \psi_k(x) \) is the Walsh-Fourier series of a function \( f(x) \in B, \) then \( \sum_{k=1}^{\infty} a_k \psi_k(x) \) is the Walsh-Fourier series of a function \( f(x) \in B, \)

where \( \psi_k = \sum_{n=k}^{\infty} \psi_n. \)

1.8. In 1913 Fejer proved the following theorem concerning Fourier series of a function of bounded variation.

1) Fejer, L. (6A).
THEOREM L. If \( f(x) \in BV [0, 2\pi] \) then the sequence
\[
\left\{ a_B^r(x) \right\} \text{ is summable (c, r), } r > 0 \text{ to } r^{-1} \left| f(x+\varepsilon) - f(x-\varepsilon) \right|
\]
\[= r^{-1} D(x) \text{ at every point of jump of the function } f(x). \]

This result was later extended by Siddiqi to certain general regular triangular methods of summability. His theorem, which sharpens Theorem L, is as follows:

THEOREM M. Let \( f(x) \in BV [0, 2\pi] \) and periodic with period \( 2\pi \). If \( (A) \) is regular and if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \triangle a_{n,k} = 0,
\]
then the sequence \( \left\{ a_B^r(x) \right\} \) is summable \( (A) \) to \( \frac{D(x)}{r} \).

In 1959, Hsiang improved the result of Siddiqi by replacing the condition (1.8.1) by
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \triangle a_{n,k} = 0.
\]

In 1961, Siddiqi further improved his theorem and also that of Hsiang on \( A \) for the validity of Theorem M and derived certain consequences for the Fourier coefficients of continuous functions of bounded variation. His main theorem states:

1) Siddiqi, J.A. (33)
2) Hsiang, C.C. (30 A)
3) Siddiqi, J.A. (34)
Theorem 2. If \((A)\) is regular, then for every 
\[ f(x) \in BV[0, 2\pi] \quad \text{and for every} \quad x \in [0, 2\pi], \]
\[ \lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} k B_k(x) = \frac{D(x)}{v} \]
iff
\[ \lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} \cos kt = 0 \]
in every \(0 < s \leq t \leq \pi\).

In Chapter IV of the thesis we have obtained necessary and sufficient conditions for the almost A summability, FA-summability and AB summability of the sequence \(\{k B_k(x)\}\) to the sum \(\frac{D(x)}{v}\). Thus, for example, in Theorem 2 we have proved the following:

Theorem 3. If \(A = (a_{n,k})\) is regular, then for every 
\[ f(x) \in BV[0, 2\pi] \quad \text{and for every} \quad x \in [0, 2\pi], \]
the sequence \(\{k B_k(x)\}\) is FA-summable to the limit \(\frac{D(x)}{v}\) iff
\[ \lim_{p \to \infty} \sum_{j=0}^{\infty} a_{p,j} \cos (j + k)t = 0 \]
uniformly in \(k\) in every interval \(0 < s \leq t \leq \pi\).

It may be observed that this theorem includes as a special case for \(k = 0, \quad a_{p,j} = 0, \quad j > p\), the above theorem of Siddiqi, L.A.
Problems analogous to that of Theorem N and other theorems proved in Chapter IV, have been studied in Chapter VII for the sequence \( \{ K \cdot C_k \cdot \gamma_k(x) \} \), where \( C_k \) is the Walsh-Fourier coefficient. Among others we have proved the following:

**Theorem 13.** If \( (A) \) is regular, then for every \( f \in BV [0,1] \) and for every \( x \in [0,1] \)

\[
\lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} \cdot K \cdot C_k \cdot \gamma_k(x) = 0.
\]

iff

\[
\lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} \cdot J_k(t) = 0
\]

in every \( 0 < s \leq t \leq 1 \), where \( (A) \) is a triangular matrix and \( s \) is small.

**Theorem 14.** If \( A = (a_{n,k}) \) is regular, then for every \( f(x) \in BV [0,1] \) and for every \( x \in [0,1] \), the sequence \( K \cdot C_k \cdot \gamma_k(x) \) is \( p \)-integrable to the limit zero iff

\[
(1.8.6) \quad \lim_{n \to \infty} \sum_{r=-k}^{n} a_{n,r-k} J_r(t) = 0
\]

uniformly in \( K \) in the interval \( 0 < s \leq t \leq 1 \), where \( s \) is small.
If we take \( x = 0, \, k = 0, \, a_n, r = \frac{1}{n} \), \( r < n \) and for \( r \geq n \), we get the following corollary of Theorem 14 which bridges the gap between Theorems N and O given below:

**Corollary:** If \( f(x) \in BV[0,1] \), then \( \{ k a_k \} \) is summable \((C,1)\) to zero iff

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} f^n(t) = 0
\]

for \( 0 < s \leq t \leq 1 \), where \( s \) is small.

1) **Theorem N.** If \( f(x) \in BV \) and \( V \) is its total variation in \((0,1)\), then

\[
\| q \| \leq \frac{V}{k}, \quad k > 0.
\]

2) **Theorem O.** Let \( f(x) \) be real valued, periodic and of mean value zero on \([0,1]\). If

\[
F(x) = \int_0^x f(t) \, dt
\]

and

\[
F(x) \sim \sum_{k=1}^{\infty} b_k k(x),
\]

then the arithmetic means of the sequence \( k! b_k \) tend to zero.

---

1) Fine, N. J. (7)

2) Morgenthaler, O. W. (27)
Chapter VI of the present thesis is devoted to
the study of Walsh-Fourier series of certain classes.
In this connection eight theorems have been proved. Some
of the theorems proved in this Chapter are as follows:

**THEOREM 15.** A function \( f(x) \in \text{Lip}^{<^p(v)} \), \( 0 < c < 1 \)
\( 1 < p < \infty \) iff
\[ \left\| \sigma_n^\beta (x) - \sigma_m^\beta (x) \right\|_p = \mathcal{O}(n^{-c}), \quad n > m, \quad \beta > 0 \]
where \( \sigma_n^\beta (x) \) is \( n \)-th \((C, \beta)\) mean of the Walsh-Fourier
series of \( f(x) \).

**THEOREM 16.** If \( c_n \downarrow 0 \) and \( \{c_n\} \) is convex or even
quasi-convex, then for the convergence of the Walsh series
\( \sum_n c_n \psi_n(x) \) in the metric space \( L \), it is necessary and
sufficient that
\[ \lim_{n \to \infty} c_n \log n = 0. \]

**THEOREM 17.** If \( c_n \downarrow 0 \) and \( \sum_n c_n \log n < \infty \),
then the Walsh series \( \sum_n c_n \psi_n(x) \) is a Walsh-Fourier series.

It may be remarked that Theorem 15 gives a new
caracterization for \( \text{Lip}^{<^p(v)} \) class while Theorem 16
is an analogue of a well known result for trigonometric
series and Theorem 17 contains a result of Yano \(^1\) proved
under different set of conditions.

\(^1\) Yano, K. (56).
CHAPTER II

ON THE FOURIER SERIES OF FUNCTIONS OF CERTAIN CLASSES
AND SOME APPROXIMATION PROBLEMS\(^1\).

2.1 Let \( f(x) \) be integrable in the sense of
Lebesgue in \((-r, r)\) and the periodic with period \(2r\),
and let

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)
\]

We denote a trigonometric series

\[
a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)
\]

its \(n\)th \((C,1)\) mean by \(\sigma_n(x)\).

A function \(g(x)\) is said to belong to class \(\Lambda\)
(or \(x\) class) if it is continuous and

\[
g(x+h) + g(x-h) - 2g(x) = \mathcal{O}(h), \quad h \to 0, \quad \text{uniformly in } x^2.
\]

---

1) A portion of this Chapter has already been accepted for publication in the American Mathematical Monthly.

2) Zygmund, A (43).
It is well known that if \( g(x) \in L^\infty \), then
\[ g(x) \in \text{Lip} < \alpha < 1 \] but the converse is not necessarily true 1).

2.2 Concerning Fourier series of Lip < class, Lorentz in 1948 obtained the following theorem.

**Theorem A.** If \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \),

where \( a_n \downarrow 0 \), then for \( f(x) \in \text{Lip} < \alpha < 1 \), it is necessary and sufficient that

\[
(2.2.1) \quad a_n = O \left( \frac{1}{n^{\alpha+1}} \right),
\]

This is also valid for \( g(x) = \sum_{n=1}^{\infty} a_n \sin nx \).

Very recently this theorem has been generalized by Boas 3) in the following form:

---

1) Zygmund, A. (45)
2) Lorentz, G.G. (29)
THEOREM 2. Let $a_n \geq 0$ and $a_n$ be the Fourier sine or cosine coefficient of $f$. Then 

$f \in \text{Lip}_\alpha$, $0 < \alpha < 1$, iff 

$$
(2.2.2) \quad \sum_{k=0}^{\infty} a_k = O(n^{-\alpha}), 
$$

or, equivalently 

$$
(2.2.3) \quad \sum_{k=1}^{n} k^\alpha a_k = O(n^{1-\alpha}). 
$$

It is easy to see that if $a_n \downarrow 0$, then conditions (2.2.1), (2.2.2) and (2.2.3) are all equivalent.

In the next section we shall study further generalizations of these theorems.

2.3. A function $\psi(x)$ is said to belong to Lip $(X(x))$ class if 

$$
|\psi(x+h) - \psi(x)| = O(\|X(h)\|), \quad h \to 0, 
$$

uniformly in $x$, where $X(x)$ is a function such that 

$$
(2.3.1) \quad \int_{0}^{\infty} \frac{X(x)}{x} \, dx < \infty, \quad \eta > 0. 
$$
Class \( \text{Lip}_< \) is a special case: \( X(x) = x^< \),
\( 0 < x < 1 \), of the class \( \text{Lip} (X(x)) \).

We shall prove the following theorem.

**Theorem 1.** Let \( a_n > 0 \) and \( a_n \) be the Fourier cosine coefficient of \( f \). Suppose \( \lambda(x) \) is a function which satisfies the following conditions:

\[
\begin{align*}
\text{i) } & \lambda(x) \text{ is positive and monotonic increasing,} \\
\text{ii) } & \frac{\lambda(x)}{x^s} \to 0, \quad x \to \infty, \quad s > 0, \\
\text{iii) } & \sum_{k=n}^{\infty} \frac{\lambda(k)}{k^1 + s} = O \left( \frac{\lambda(n)}{n^s} \right), \quad n \to \infty, \quad s > 0, \\
\text{iv) } & \sum_{k=1}^{\infty} \frac{\lambda(k)}{k} = O (\lambda(n)), \quad n \to \infty.
\end{align*}
\]

Then \( f \in \text{Lip} (X(x)) \), iff

\[
\sum_{k=n}^{\infty} \frac{a_k}{k} = O (\| X \|_1) ,
\]

where \( X(n) \) and \( n^2 \| X \|_1^2 \) satisfy the same conditions as \( \lambda(n) \) in (2.3.2) for \( s = 1 \) and 2 respectively.
If we take \( \chi(x) = x^s \), \( 0 < s < 1 \), then Theorem 1 includes, as a special case, the above theorem of Boas for cosine series.

2.4. We shall require the following lemmas for the proof of this theorem.

**Lemma 1.** Let \( \mu_k \geq 0 \) and \( \lambda(n) \) satisfy condition (2.3.2) for some \( s > 0 \).

Then

\[
\sum_{k=1}^{n} k^s \mu_k = O(\lambda(n)),
\]

iff

\[
\sum_{k=n}^{\infty} k^s \mu_k = O\left(\frac{\lambda(n)}{n^s}\right).
\]

**Lemma 2.** If \( \mu_k \geq 0 \) and \( \sum_{k=1}^{\infty} \mu_k < \infty \).

Let \( \chi(n) \) be a positive increasing sequence, and

\( a^2 (\chi(\frac{1}{n}) \) satisfy the same conditions as \( \lambda(n) \) in (2.3.2) for \( s = 2 \).

Then

\[
(2.4.1) \quad \sum_{k=1}^{\infty} \mu_k (1 - \cos kx) = O(\chi(x))
\]
iff

\[(2.4.2) \quad \sum_{k=n}^{\infty} \mu_k = O\left(\frac{1}{n} \right).\]

It may be remarked that Lemmas 1 and 2 are generalizations of Lemmas A and B respectively given below. 1)

**Lemma A.** Let \(\mu_k \geq 0, \quad t > \beta > 0\). Then

\[\sum_{k=1}^{\infty} \frac{\mu_k}{k^t} = O\left(n^{\beta - t}\right).\]

iff

\[\sum_{k=1}^{\infty} \mu_k = O\left(n^{\beta - t}\right).\]

**Lemma B.** If \(\mu_k \geq 0, \quad \sum \mu_k < \infty\) and \(0 < \beta < 2\), then

\[\sum_{k=1}^{\infty} \mu_k (1 - \cos kx) = O\left(x^\beta\right),\]

iff

\[\sum_{k=1}^{\infty} \mu_k = O\left(n^{-\beta}\right).\]

2.5. **Proof of Lemma 1.** Let

\[\sum_{k=1}^{\infty} k^t \mu_k = O\left(\lambda(k)\right).\]

---

1) Boas, R.P. Jr. (4).
Then
\[ \sum_{k=n}^{N} \frac{\mu_k}{\lambda_k} = \frac{1}{n^k} \sum_{k=n}^{N} \frac{1}{\lambda_k} \sum_{\nu=1}^{\lambda(k)} \frac{\nu}{n^\nu} \]

\[ = \frac{N-n}{n^k} \sum_{\nu=1}^{\lambda(k)} \frac{\nu}{n^\nu} \approx \sum_{\nu=1}^{\lambda(n)} \frac{\nu}{n^\nu} \approx \frac{\lambda(n)}{n^k} \]

Making \( N = \infty \), we have
\[ \sum_{k=n}^{\infty} \frac{\mu_k}{\lambda_k} = O\left( \sum_{k=n}^{\infty} \frac{1}{\lambda(k)} \right) + O\left( \frac{\lambda(n)}{n^k} \right) = O\left( \frac{\lambda(n)}{n^k} \right) \]

by virtue of condition (2.3.2).

Now suppose \( \sum_{k=n}^{\infty} \frac{\mu_k}{\lambda_k} = O\left( \frac{\lambda(n)}{n^k} \right) \). Then
\[ \frac{n}{\lambda(n)} < \infty \]. Let its sum be \( s \). Now we have
\[ \sum_{k=1}^{a-1} k^3 \mu_k = \sum_{k=1}^{a-1} \Delta k^3 \sum_{v=0}^{\infty} \mu_{v+1} + a^3 \sum_{v=0}^{\infty} \mu_{v+1} \]

\[ = \sum_{k=1}^{a-1} \Delta k^3 \left( \sum_{v=0}^{\infty} \mu_v - \sum_{v=0}^{\infty} \mu_{v+1} \right) + a^3 \left( \sum_{v=0}^{\infty} \mu_v - \sum_{v=0}^{\infty} \mu_{v+1} \right) \]

\[ = (1 - a^3) \sum_{k=1}^{a-1} \Delta (k+1) - k^3 \sum_{v=0}^{\infty} \mu_v + a^3 \sum_{v=0}^{\infty} \mu_v \]

\[ = 0 \left( \sum_{k=1}^{a-1} \frac{k^{\lambda-1} \lambda(k)}{k^3} \right) + o(\lambda(a)) \]

\[ = o(\lambda(a)), \]

by the hypothesis.

\section{Proof of Lemma 2}

Let

\[ \sum_{k=1}^{\infty} k \mu_k (1 - \cos kx) = o\left( \frac{x}{|\lambda(x)|} \right). \]

Then we have

\[ \sum_{k=1}^{[\frac{x}{k}]} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{k^2 \mu_k}{k^2 x^2} \cdot \frac{1 - \cos kx}{k^2 x^2} \]

\[ = o\left( \frac{x}{x^2} \right). \]
Since $t^{-2}(1 - \cos t)$ decreases in $(0, 1)$,
it follows that

$$\sum_{k=1}^{\infty} k^{2/\mu_k} = O\left(\frac{\chi(x)}{\varepsilon^2}\right),$$

that is to say,

$$\sum_{k=1}^{\infty} k^{2/\mu_k} = O\left(\frac{\chi(x)}{\varepsilon^2}\right), \quad \varepsilon \rightarrow 0.$$
This completes the proof of Lemma 2.

2.7. Proof of Theorem 1. Let $E a_n \cos nx$ be the Fourier series of $f$.

Necessity: The hypothesis implies that

\[ f(x) - f(0) = O(1) \]

and hence by virtue of Dini's test, the Fourier series of $f$ converges at $x = 0$ and therefore

\[ \sum_{k=1}^{\infty} a_k (1 - \cos kx) = O(1) \]

Applying Lemma 2, we have

\[ \sum_{k=n}^{\infty} \frac{\mu_k}{n} = O\left( \frac{1}{n} \right), \quad n \to \infty. \]

Sufficiency: We have

\[ |f(x+2h) - f(x)| = \frac{1}{2} \sum a_k \left[ \cos k(x+2h) - \cos kx \right] \]

\[ = 2 \sum a_k \sin k(x+h) \sin kh \]

\[ = 2 \sum a_k \sum_{k=1}^{\infty} \frac{1}{2h} + 1 \]
\[ \sum_{k=1}^{n} \frac{1}{2h} k \frac{a_k}{x} + \sum_{k=1}^{n} \frac{1}{2h} \frac{a_k}{x} + \mathcal{O}(\frac{1}{nx^2}) \]

\[ = \mathcal{O}(\frac{1}{nx^2}) + \mathcal{O}(\frac{1}{nx^2}) \]

\[ = \mathcal{O}(\frac{1}{nx^2}) , \quad h \to 0. \]

Now

\[ \int_{1}^{\infty} \frac{|f(x)|}{x} \, dx = \int_{1}^{\infty} \frac{1}{t} \, dt. \]

Since by condition (2.3.2) (iii) \( \frac{k^2}{x^3} < \infty \),

that is to say \( \frac{1}{k} \frac{|f(x)|}{x} < \infty \) and \( \frac{|f(x)|}{x} \) is decreasing, it follows that

\[ \int_{1}^{\infty} \frac{|f(x)|}{x} \, dx - \int_{1}^{\infty} \frac{|f(x)|}{x} \, dx \]

Therefore

\[ \int_{1}^{\infty} \frac{|f(x)|}{x} \, dx < \infty \]

Hence

\[ f(x) \in \text{Lip} (\frac{1}{x}). \]
2.6. In this section we shall study the problem analogous to that of Lorentz for \( \Lambda_e \) class. We first prove the following theorems:

**Theorem 2.** If

\[
\begin{align*}
a_n &= O\left( \frac{1}{n^2} \right), \\
b_n &= O\left( \frac{1}{n^2} \right),
\end{align*}
\]

then the trigonometric series \( T(x) \) will be the Fourier series of a function belonging to \( \Lambda_e \) class.

**Theorem 3.** If \( a_n \downarrow 0 \) and \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \), then \( f \in \Lambda_e \) iff

\[
a_n = O\left( \frac{1}{n^2} \right).
\]

**Proof of Theorem 2.** The hypothesis implies that \( T(x) \) is absolutely convergent. Let

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

Then \( T(x) \) is the Fourier series of \( f(x) \). We have

\[
f(x+h) + f(x-h) - 2f(x) = 2 \sum_{n=1}^{\infty} \left[ a_n \cos nx \cos nh - 1 \right] + b_n \sin nx \left( \cos nh - 1 \right)
\]
\[
46
\]

\[
\begin{align*}
46 & \sum_{n=1}^{\infty} a_n \cos nx \sin \frac{2nh}{2} - 4 \sum_{n=1}^{\infty} b_n \sin nx \sin \frac{2nh}{2} \\
& = A + B, \text{ say.}
\end{align*}
\]

Let \( h = \frac{1}{N} \), then we have writing

\[
A = -4 \left( \sum_{1}^{N} + \sum_{N+1}^{\infty} \right)
\]

\[
= \mathcal{O}(h^2) \sum_{1}^{N} + \mathcal{O}\left( \frac{1}{N+1} \right)
\]

\[
= \mathcal{O}(h) + \mathcal{O}(h) = \mathcal{O}(h), \quad h \to 0.
\]

Similarly \( B = \mathcal{O}(h) \).

Therefore \( f(x+h) + f(x-h) - 2f(x) = \mathcal{O}(h) \), uniformly in \( x \) as \( h \to 0 \).

This proves Theorem 2.

Proof of Theorem 3.

Sufficiency: It is evident from Theorem 2.

Necessity: The hypothesis implies that

\[
f(h) - f(0) = \mathcal{O}(h)\]
or
\[
\sum_{k=1}^{\infty} a_k \sin^2 \frac{kh}{2} = O(h).
\]

Writing \( h = \frac{v}{a} \), we have
\[
\sum_{k=1}^{\infty} a_k \sin^2 \frac{kv}{2n} = O\left(\frac{1}{n}\right),
\]
so that
\[
\sum_{k=1}^{n} a_k \sin^2 \frac{kv}{2n} = O\left(\frac{1}{n}\right).
\]

Since in this range \( \sin^2 \frac{kv}{2n} \geq \frac{1}{2} \), we have
\[
\sum_{k=1}^{n} a_k = O\left(\frac{1}{n}\right)
\]
and consequently
\[
a_n a_n = O\left(\frac{1}{n}\right)
\]
that is to say
\[
a_n = O\left(\frac{1}{n^2}\right).
\]

This completes the proof of Theorem 3.

2.9. In a very recent paper Boas \(^1\) obtained
the following theorem for \( \wedge \) class which generalizes
our Theorem 3.

---

1) Boas, R.P. Jr. (4)
SUMMARY

ON CERTAIN PROBLEMS IN THE THEORY OF FOURIER SERIES

A THESIS SUBMITTED TO THE ALIGARH MUSLIM UNIVERSITY, ALIGARH IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS

By

ABUL HASAN SIDDIQI

DEPARTMENT OF MATHEMATICS AND STATISTICS ALIGARH MUSLIM UNIVERSITY, ALIGARH

1967
SUMMARY

1.1 Definitions and notations: A function \( f(x) \) is said to belong \( \text{Lip} \left( X(x) \right) \) if

\[
f(x+h) - f(x) = O\left( |X(h)| \right), \quad h \to 0
\]

uniformly in \( x \), where \( X(x) \) is a function such that

\[
\int_{-\infty}^{\infty} \frac{|X(x)|}{x} \, dx < \infty, \quad M > 0.
\]

Let \( f(x) \) be periodic with periodic \( 2\pi \) and integrable in the sense of Lebesgue in \((-\pi, \pi)\). Let

\[
f(x) \sim \sum_{n} a_{n}(x)
\]

and the series conjugate to the Fourier series be \( \sum_{n} c_{n}(x) \).

We denote the Walsh-Fourier series of an integrable function in \((0,1)\) and periodic with period 1 as

\[
f(x) \sim \sum_{k} c_{k} \chi_{k}(x), \quad \text{where} \quad c_{k} = \frac{1}{2} \int_{0}^{1} f(x) \chi_{k}(x) \, dx.
\]

A bounded sequence \( \{a_{k}\} \) is said to be almost \( A \)-summable to \( s \) if its \( A \)-transform is almost convergent to \( s \), where

\[A = (a_{n,k}) \text{ is infinite matrix }[6]
\]

If \( A \)-transform of \( \{a_{k}\} \) is \( AB \)-summable to \( s \), then \( \{a_{k}\} \) is said to be \( AB \)-summable to \( s \).

We use the following notations:
\[ T(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) , \]

\[ P_n(u) = \text{Im} \left\{ \int \left( e^{i(n + \frac{1}{2})u} \sum_{m=0}^{\infty} \phi_m e^{-imu} \right) \right\} , \]

\[ \psi(u) = f(x+u) + f(x-u) - 3f(x) , \]

\[ S_k(x) = \sum_{n=0}^{N} p_k(x) , \]

\[ P_n = \sum_{k=0}^{N} p_k \quad \text{and} \quad P \left( \frac{1}{t} \right) = P \left[ \frac{1}{t} \right] \]

\[ t_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(x) , \]

\[ J_k(t) = \int_{0}^{t} \psi_k(x) \, dx , \quad J_k(t) = k J_k(t) \]

We denote by \( \mathcal{S} \) the class of series \( \sum c_k \psi_k(x) \)

with \( c_k = \int_{0}^{1} \psi_k(t) \, dF(t) \), where \( F(t) \) is continuous

and of bounded variation.

1.2. In Chapter II of the present thesis we have proved 9 theorems. The first theorem proved as follows

\textbf{Theorem 1.} Let \( a_n > 0 \) and \( a_n \) be the Fourier

cosine coefficient of \( f \). Suppose \( \lambda(x) \) is a function which

satisfies the following conditions:
(i) \( \lambda(x) \) is a positive and monotone increasing.

(ii) \( \frac{\lambda(x)}{x^s} \downarrow 0 \) as \( x \to \infty, \ s > 0 \).

(iii) \( \sum_{k=n}^{\infty} \frac{\lambda(k)}{k^{1+s}} = O\left( \frac{\lambda(x)}{x^s} \right), \ n \to \infty, \ s > 0 \).

(iv) \( \sum_{k=1}^{n} \frac{\lambda(k)}{k} = O\left( \lambda(x) \right), \ n \to \infty \).

Then \( f \in \text{Lip} X(x) \) iff

\[ \sum_{k=n}^{\infty} a_k = O\left( \left| X\left( \frac{1}{n} \right) \right| \right), \]

where \( X(n) \) and \( n^2 \left| X\left( \frac{1}{n} \right) \right| \) satisfy the same conditions as \( \lambda(x) \) in (1.2.1) for \( s = 1 \) and \( s = 2 \) respectively.

This generalizes a very recent result of Boas [1] for lip \( \ll \) class. Theorems 2 and 3 are concerned with \( \land \) class. Theorem 4 is a generalization of a theorem of Boas [1] for \( \land \) class. In Theorems 5 and 6 problems analogous to that of Durelliss [2] for Lip \( \ll \) have been considered for \( \land \) class. Thus for example in Theorem 5 we have proved:

**Theorem 2.** If \( \Omega_n(x) - \Omega_n(x) = O\left( n^{-1} \right) \)
uniformly in \( [0, \infty] \) for all \( x > n \), then the trigonometry
Theorem 7, 8 and 9 contain results concerning approximation of Fourier series by Cesaro and Nörlund mean.

The last theorem which includes as a special case a result Flett [4] is as follows:

**Theorem 3.** Let \( \{p_n\} \) be a positive non-increasing sequence of real numbers such that

\[
\int_0^1 \phi(u) \, du = O \left( \frac{p_n(1)}{n} \right), \quad \frac{1}{n} \leq t \leq \frac{k}{n},
\]

Also let \( 0 < \alpha < 1, \quad 0 < \beta \leq \gamma \). If \( x \) is a point such that

\[
\int_0^1 |\phi(u)| \, du \leq n^{-\alpha},
\]

then

\[
t_n(x) - f(x) = O(n^{-\alpha}) + O\left( \frac{1}{p_n} \right).
\]

Chapter III is devoted to the study of transformations of Fourier series of \( L^p \), \( BV \), \( Lip^{<} \) and \( \Lambda_e \) classes. In this connection a number of theorems have been proved. We mention here a few of them just to give an idea about the nature of problems discussed in this Chapter.
THEOREM 4. Let \( f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx \), \( f(x) \in \mathbb{V} \), 
\( f(0) = 0 \), then \( \sum_{n=1}^{\infty} a_n \cos nx \) is the Fourier series of 
a function \( f(x) \in \mathbb{V} \). \( a_n \) is the \( n \)th \((C,1)\) mean 
of \( \{a_n\} \).

THEOREM 5. Let \( g(x) \sim \sum_{n=1}^{\infty} a_n \cos nx \), \( g(x) \in \mathbb{V} \).
\( g(0) = 0 \), then \( \sum_{k=1}^{\infty} \langle k \rangle \cos kx \) is the Fourier series of a 
function \( g(x) \in \mathbb{V} \), where \( \langle k \rangle = \sum_{k=n}^{\infty} \frac{a_k}{k} \).

In Chapter IV we have studied almost \( A \)-summability, 
\( FA \) summability and \( A^\alpha \) summability of the trigonometric 
sequence \( \{K a_k(x)\} \). Theorem concerning \( FA \) summability 
is as follows:

THEOREM 6. If \( A = (a_n,k) \) is regular, then for 
every \( f(x) \in BV [0,2\pi] \) and for every \( x \in [0,2\pi] \), the 
sequence \( \{K a_k(x)\} \) is \( FA \)-summable to the limit 
\( D(x) = \lim_{v} \frac{f(x+v) - f(x-v)}{2} \) if and only if
\[
\lim_{p \to \infty} \sum_{j=0}^{\infty} a_p, j \cos (j+k)t = 0
\]
uniformly in \( k \) in every interval \( 0 < t \leq \pi \).
It may be remarked that this theorem includes as a special case, a theorem of J.A. Siddiqi [6] for triangular method of summability.

Chapter V is concerned with the problem of transformation of Walsh-Fourier series. In this chapter we have examined as to how for the problem analogous to that of Hardy [3] remains true for such series. In this connection we have established four theorems for classes $L^p$ essentially bounded class B and $S'$. Thus for example the result for B class states:

**Theorem 5.** If $\sum_{k=0}^{\infty} c_k \psi_k(x)$ is the Walsh-Fourier series of a function $f(x) \in B$, then $\sum_{k=0}^{\infty} \left| c_k \right| \psi_k(x)$ is the Walsh-Fourier series of a function $F(x) \in B$, where $c_k = \frac{1}{k} \left| \sum_{m=1}^{k} c_m \right|$.

Chapter VI of the thesis is devoted to the study of Walsh-Fourier series of certain classes. In this connection eight theorems have been proved. Some of the theorems prove in this chapter are follows:

**Theorem 9.** A function $f(x) \in \text{Lip } \psi^p(v)$, $0 < \psi < 1$, $1 < p < \infty$ iff:

$$\| \sigma_m^\beta (x) - \sigma_n^\beta (x) \| = O(n^{-\psi}), \quad a > n, \beta >.$$
where \( g^b_n(x) \) is \( n \)-th \((C, \beta)\) mean of the Walsh-Fourier series of \( f(x) \).

**Theorem 9.** If \( c_n \downarrow 0 \) and \( \{c_n\} \) is convergent or even quasi-convex, then for the convergence of the Walsh series \( \sum c_n y_n(x) \) in the metric space \( L \), it is necessary and sufficient that

\[
\lim_{n \to \infty} c_n \log n = 0
\]

**Theorem 10.** If \( c_n \downarrow 0 \) and \( \sum \Delta c \log n < \infty \), then the Walsh series \( \sum c_k y_k(x) \) is a Walsh-Fourier series.

It may be remarked that Theorem 7 gives a new characterization for \( \text{Lip}_p(k) \) class while Theorem 9 is an analogue of a well-known result for trigometric series and Theorem 9 contains a result of Yano [9] proved under different set of conditions.

In Chapter VII we have obtained necessary and sufficient conditions for \( A \)-summability, almost \( A \)-summability, \( FA \) summability and \( AB \)-summability of the sequence \( \{k c_k y_k(x)\} \), where \( c_k \) and \( y_k(x) \)
are the Walsh-Fourier coefficient and Walsh functions respectively. Among others, we have proved the following:

**Theorem 10.** If \((a)\) is regular, then for every \(f \in BV [0, 1]\) and for every \(x \in [0, 1]\)

\[\lim_{a \to \infty} \sum_{k=0}^{m} a_{n,k} \frac{c_k \psi_k(x)}{k} = 0,\]

iff

\[\lim_{a \to \infty} \sum_{k=0}^{m} a_{n,k} J_k(t) = 0,\]

in every \(0 < s \leq t \leq 1\), where \((A)\) is a triangular matrix and \(s\) is small.

**Theorem 11.** If \(A = (a_{n,k})\) is regular, then for every \(f(x) \in BV [0, 1]\) and for every \(x \in [0, 1]\)

the sequence \(\{k c_k \psi_k(x)\}\) is FA summable to the limit zero iff

\[\lim_{a \to \infty} \sum_{r=0}^{\infty} a_{n,r-k} J_r^* (t) = 0,\]

uniformly in \(k\) in the interval \(0 < s \leq t \leq 1\),

where \(s\) is small.
If we take \( x = 0, \ k = 0, \ a_n, r = \frac{1}{n}, \ r < a \) and \( a \) for \( r \geq a \), we get the following corollary of Theorem 11 which bridges the gap between Theorem A and B given below.

**Corollary**: If \( f(x) \in BV [0,1] \) then \( \{k c_k\} \) is summable \((C,1)\) to zero iff,
\[
\lim_{m \to \infty} \sum_{n=1}^{m-1} \frac{1}{m} \int_{[n/n]}^{[n+1/n]} f(t) \ dt = 0
\]

for \( 0 < s \leq t \leq 1 \), where \( s \) is small.

**Theorem A** [3]. If \( f(x) \in BV \) and \( V \) is its total variation in \((0,1)\), then
\[
|c_k| < \frac{V}{k}, \ k > 0.
\]

**Theorem B** [7]. Let \( f(x) \) be real valued periodic and of mean value zero on \([0,1]\). If
\[
F(x) = \int_{0}^{x} f(t) \ dt
\]
and
\[
F(x) \sim \sum_{k=0}^{\infty} b_k \chi_k(x),
\]
then the arithmetic means of the sequence \( \{k!c_k!\} \) tend to zero.
BIBLIOGRAPHY

THEOREM 3. If \( a_k \geq 0 \) and \( f \sim E a_n \cos nx \),
then \( f \in \Lambda_e \) iff
\[
\sum_{n=0}^{\infty} a_k = O(n^{-1}).
\]

In this section we shall prove a theorem which includes, as a special case, the above theorem of Boas.

A function \( f \) is said to belong to \( \Lambda_e (X(x)) \) class if it is continuous and
\[
f(x+h) + f(x-h) - 2f(x) = O(|X(h)|), \quad h \to 0
\]
uniformly in \( x \) where \( X(x) \) is a function such that
\[
\int_0^\infty \frac{|X(x)|}{x} \, dx < \infty.
\]
If we take \( X^*(x) = x \), then the class \( \Lambda_\infty(x) \) is the same as \( \Lambda_e \).

THEOREM 4. If \( a_n \geq 0 \) and \( f \sim E a_n \cos nx \),
then \( f \in \Lambda_e (X^*(x)) \) iff
\[
\sum_{k=0}^{\infty} a_k = O\left( \left| \mathcal{X}'\left( \frac{1}{n} \right) \right| \right), \quad n \to \infty,
\]

where \( \mathcal{X}(n) \) and \( n^2 \left| \mathcal{X}'\left( \frac{1}{n} \right) \right| \) satisfy the same conditions as \( \gamma(x) \) in (2.3.2) for \( \epsilon = 1 \) and \( 2 \) respectively.

**Proof of Theorem 4. Sufficiency:**

Let \( \sum_{k=0}^{\infty} a_k = O\left( \left| \mathcal{X}'\left( \frac{1}{n} \right) \right| \right) \), then

\[
f(x_{2h}) + f(x-2h) = 2f(x) \leq 4 \sum_{n} a_n \sin^2 nh \cos nx
\]

\[
= 4 \sum_{n} a_n \sin^2 nh
\]

\[
= 2 \sum_{n} a_n (1 - \cos^2 nh)
\]

\[
= O\left( \left| \mathcal{X}'(2h) \right| \right),
\]

by Lemma 2.

As shown in the proof of Theorem 1,

\[
\int_0^\infty \frac{\mathcal{X}'(x)}{x} \, dx < \infty.
\]

Hence \( f \in \wedge_\infty (\mathcal{X}^*(x)) \).
**Necessity:** \( f \in \bigwedge_{0} (X^*(x)) \) implies that

\[ f(2h) + f(-2h) - 2f(0) = O(4X^*(2h)!). \]

that is to say

\[ \sum_{m} a_{m} (1 - \cos 2m \pi h) = O(4X^*(2h)!). \]

Applying Lemma 2, we have

\[ \sum_{k=0}^{\infty} a_{k} = O(4X^*(\frac{1}{2})!). \]

This completes the proof of Theorem 4.

2-3(A). In 1954 Duplessis \(^1\) obtained a necessary and sufficient condition in order that a trigonometric series \( T(x) \) may be a Fourier series of a function belonging to \( \text{Lip} \leq \) class. His result is as follows:

**THEOREM D.** A necessary and sufficient condition that a trigonometric series \( T(x) \) be the Fourier series of \( f(x) \in \text{Lip} \leq \) on \( 0 < \alpha < 1 \), is that

\[ \delta_{n}(x) = o_{n}(x) = O(n^{-\alpha}), \]

---

\(^1\) Duplessis, N. (5).
uniformly in \([0, \pi]\) for all \(a > m\).

In this section we shall study the corresponding problem for \(\Lambda_0\) class. We prove the following theorems:

**Theorem 5.** If

\[
\tilde{c}_n(x) - \tilde{c}_m(x) = O(n^{-1}),
\]

uniformly in \([0, \pi]\) for all \(a > m\), then the trigonometric series \(T(x)\) is the Fourier series of a function \(f \in \Lambda_0\).

**Theorem 6.** If a trigonometric series \(T(x)\) be the Fourier series of a function \(f \in \Lambda_0\), then

\[
\tilde{c}_n(x) - \tilde{c}_m(x) = O\left(\frac{\log n}{n}\right),
\]

uniformly in \([0, \pi]\) for all \(a > m\).

The following lemmas will be needed in the proof of these theorems.

**Lemma 3.** If \(\tilde{c}_n(x) - f(x) = O(n^{-1})\), uniformly in \(x\), then \(f \in \Lambda_0\).
**Lemma 4.** 1) **If** \( f(x) \in \bigwedge_0 \), **then**

\[
\frac{\log n}{n} - f(x) = \mathcal{O}\left(\frac{\log n}{n}\right).
\]

**Proof of Lemma 3.** It is known 2) that

\[
f \in \bigwedge_0 \text{ iff } E_n(f) = \mathcal{O}\left(\frac{1}{n}\right),
\]

where \( E_n(f) \) is the best approximation of \( f \) of order \( n \). Let \( T_n^*(x) \) be the trigonometric polynomial which gives the least deviation, then

\[
E_n(f) = \max |T_n^*(x) - f(x)|
\]

\[
\leq \max |\pi(x) - f(x)|
\]

\[
= \mathcal{O}(n^{-1}).
\]

Hence \( f \in \bigwedge_0 \).

This completes the proof of Lemma 3.

---


2) Zygmund, A (43).
Proof of Theorem 3. From the hypothesis we have
\[ \delta_n(x) - \delta(x) \to 0, \]
uniformly in \( x \) as \( n, m \to \infty \) which implies that there exists a function \( f \) such that
\[ \delta_n(x) = f(x) \]
uniformly. Hence
\[ \delta_n(x) - f(x) = O\left(\frac{1}{n}\right). \]

Uniformly. By virtue of Lemma 3 it follows that
\[ f \in \mathcal{A}. \]

Now
\[ \delta_n(x) = \sum_{\nu=0}^{n} \left(1 - \frac{\nu}{n+1}\right) \lambda_{\nu}(x) \]
and therefore for \( k \leq n \), we have
\[ (1 - \frac{k}{n+1}) a_k = \frac{1}{\nu} \int_{-\pi}^{\pi} \delta_n(x) \cos kx dx. \]

Since \( \delta_n(x) \to f(x) \) uniformly, taking the limit as \( n \to \infty \), we have
\[ a_k = \frac{1}{\nu} \lim_{n \to \infty} \int_{-\pi}^{\pi} \delta_n(x) \cos kx dx. \]
\[
\frac{1}{v} \int_{-v}^{v} \lim_{n \to \infty} \sigma_n(x) \cos kx \, dx
\]
\[
= \frac{1}{v} \int_{-v}^{v} f(x) \cos kx \, dx.
\]

Similarly we can show that
\[
h_k = \frac{1}{v} \int_{-v}^{v} f(x) \sin kx \, dx.
\]

Thus \( T(x) \) is the Fourier series of \( f \in \Lambda \).

**Proof of Theorem 6.** By Lemma 2 we have
\[
\sigma_n(x) - f(x) = O\left(\frac{\log n}{n}\right), \quad n \to \infty,
\]
uniformly in \( x \) and therefore
\[
\sigma_n(x) - \sigma_m(x) = O\left(\sigma_n(x) - f(x) + (f(x) - \sigma_m(x))\right)
\]
\[
= O\left(\frac{\log n}{n}\right) + O\left(\frac{\log m}{m}\right)
\]
\[
= O\left(\frac{\log n}{n}\right)
\]
uniformly in \([0, 2\pi]\) for all \( n > m \).

This proves Theorem 6.
2.10. We write

\[ \mathcal{S}(t) = f(x+t) + f(x-t) - 2f(x), \]

\[ E_n = \left( \frac{n+k}{n} \right), \]

\[ D_n(t) = \frac{1}{2} + \frac{2}{\pi} \cos nt = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{t}{2}}, \]

\[ S_n(x) = \sum_{k=0}^{\infty} E_n \sum_{v=0}^{k-1} S_v(x) \]

\[ K_n(t) = \frac{1}{E_n} \sum_{v=0}^{k-1} D(t) \]

\[ (n,t) = \left\{ \left( n + \frac{k}{2} + \frac{1}{2} \right) t - \frac{kv}{2} \right\}, \]

where \( S_n(x) \) denotes the \( n \)-th partial sum of the Fourier series of \( f(x) \).

It is known that \(^1\)

\[ K_n(t) \left\{ \begin{array}{l} \leq A \cdot n \quad 0 \leq t \leq \pi \\ \leq R(t) + S(t), \quad \frac{1}{n} \leq t \leq \pi, \end{array} \right. \]

where

\[ R(t) = \frac{\sin(n,t)}{E_n(2 \sin \frac{t}{2})}, \]

\[ |S(t)| \leq A n^{-1} t^{-2}. \]

\(^{1}\) Hardy, G.H. (18).
and $A$ is a constant not necessarily the same at each occurrence.

Concerning the degree of approximation to a function by the Cesàro means of its Fourier series, Flett 1) proved a number of interesting theorems, among others he proved the following:

**Theorem E.** Let $0 < \kappa < 1$, $0 < \theta \leq r$. If $x$ is a point such that

$$\sum_{n=1}^{\infty} |a_n(x)| \leq A \kappa,$$

when $0 \leq t \leq s$, then

$$\sigma_n(x) - f(x) = O(n^{-\kappa}).$$

**Theorem F.** Let $0 < \kappa < 1$, $0 \leq \beta < 1$, $0 < \theta \leq r$, $k \geq -\beta$. If $x$ is a point such that

$$A_n(x) = O(n^{-\beta}),$$

and (2.10.1) holds when $0 \leq t \leq s$, then

1) Flett, T.M. (9)
\[ k \sigma_n(x) - f(x) = O(n^{-k}). \]

Our object in this section is to obtain certain generalizations of these theorems. In this direction the following theorems are proved.

**Theorem 7.** Let \( 0 < k < 1 \) and \( 0 < s \leq r \). If \( x \) is a point such that

\[ (2.10.8) \quad \int_0^t \lambda \sigma(u) \, du \leq \lambda \psi(t), \]

when \( 0 \leq t \leq s \), then

\[ \sigma_n^k(x) - f(x) = O(\psi(\frac{1}{n})) + O(n^{-k}), \]

where \( \psi(t) \) is a positive increasing function such that

\[ \int_0^1 \frac{\psi(t)}{t^2} \, dt = O(n \psi(\frac{1}{n})), \; n \to \infty. \]

**Theorem 8.** Let \( 0 < k < 1 \), \( 0 < s \leq r \). If \( x \) is a point such that

\[ \lambda_n(x) = O(\lambda_n), \]

and

\[ \int_0^t \lambda \sigma(u) \, du \leq \lambda \psi(t), \]
when \( 0 \leq t \leq \delta \), then

\[
\sigma_n^k(x) - f(x) = O\left( \varphi\left( \frac{1}{n} \right) \right) + O\left( \frac{1}{n} \right) + O\left( \frac{1}{n^k} \right),
\]

where \( \lambda_n \) is a positive monotonic non-increasing sequence such that \( \{ n \lambda_n \} \) is increasing and

\[
\sum_{k=1}^{n} \lambda_k = \left( \sum_{k=1}^{n} \lambda_n \right),
\]

and \( \varphi(t) \) is a positive increasing function such that

\[
\int_{1}^{t} \frac{\varphi(s)}{s^2} ds = O\left( \varphi\left( \frac{1}{n} \right) \right).
\]

It may be observed that Theorem 7 includes Theorem 8 as a special case for

\[ \varphi(t) = t^\kappa, \quad 0 < \kappa < 1, \quad k = \kappa \]

while Theorem 8 includes Theorems 9 for \( k > 0 \) and \( k \geq \kappa - \beta \).

2.11. We require the following lemmas for the proof of these theorems.
LEMMA 5. 1) If $g(t) \in L$, $0 < k < 1$, $0 < \varepsilon r$,
then
\[
\int g(t) k_n^k(t) dt = O(n^{-k}).
\]

LEMMA 6. Under the hypotheses of Theorem 7, we have
\[
\int_0^{\infty} \varphi(t) k_n^k(t) dt = o \left( \psi \left( \frac{1}{n^2} \right) \right) + o(n^{-k-1}),
\]
where $k > 0$.

LEMMA 7. Let $-1 < k \leq 1$ and
\[
(t) = \begin{cases} 
(2 \sin \frac{t}{2})^{-k-1} & 0 < t < \pi, \\
\mu t, & 0 \leq t \leq \pi,
\end{cases}
\]
where $\mu = (2 \sin \frac{\varepsilon r}{2})^{-k-1}$. If $\lambda_n(x) = O(\lambda_n)$, $\lambda_n$
satisfying the conditions of Theorem 8, then
\[
\int_0^{\infty} g(t) \varphi(t) \sin(n_0 t) dt = O(\lambda_n) + O(\frac{1}{n^2}).
\]

1) This is a slight modification of a result of Flett (9).
It may be remarked that Lemma 7 includes, as a special case, for \( \lambda_n = n^{-\beta} \), \( 0 \leq \beta < 1 \), the Lemma 3 of Flett.

**Proof of Lemma 6.** We write

\[
\int_0^t \phi(t) \xi_n(t) dt = \int_0^t \phi(t) \xi_n(t) dt + \int_0^t \phi(t) \eta_n(t) dt
\]

\[
\int_0^t \phi(t) \eta(t) dt
\]

\[
= L_1 + L_2 + L_3, \text{ say.}
\]

Since \( \phi(0) = 0 \) we get

\[
|\phi(t)| = |\phi(t) - \phi(0)| = \int_0^t |d \phi(u)|
\]

\[
\leq \int_0^t |d \phi(u)| \leq A(t),
\]

Hence by the hypotheses, \( \lambda_n \)

\[
L_1 = O \left( \int_0^t |\phi(t)| dt \right) = O \left( \int_0^t |\phi(t)| dt \right)
\]

\[
= O \left( \frac{1}{n} \right).
\]

---

1) Flett, T.M. (9), p.66.
Also under the hypothesis

\[ L_3 = O \left( \int_\frac{1}{a}^a |\Phi(t)| n^{-1} t^{-2} \, dt \right) \]

\[ = O \left( n^{-1} \int_\frac{1}{a}^a \frac{\Psi(t)}{t^2} \, dt \right) = O(\psi(\frac{1}{a})). \]

Now let

\[ \Lambda(t) = \int_t^a R(u) \, du. \]

Then

|\Lambda(t)| \leq A n^{-k-1} t^{-k-1}.\]

Hence

\[ L_2 = \int_\frac{1}{a}^a \Phi(t) \Lambda(t) \, dt = - \int_\frac{1}{a}^a \Phi \, d\Lambda \]

\[ = - \left[ \Phi \Lambda \right]_\frac{1}{a}^a + \int_\frac{1}{a}^a \Lambda \, d\Phi \]

\[ = O(n^{-k-1}) + O(\psi(\frac{1}{a})) + \int_\frac{1}{a}^a \Lambda(t) \, d\Phi(t) \]

If \( \Phi^*(t) = \int_0^t |d\Phi(u)| \), then \( \Phi^*(t) \leq A \Psi(t) \),

so that
\[ \int_0^1 \theta(x) x^k \, dx = O(n^{-k-1}) + O(\frac{1}{n}) \]

Hence

\[ \int_0^1 \theta(t) K_n(t) \, dt = O(n^{-k-1}) + O(\frac{1}{n}) \]

Proof of Lemma 7. By proceeding on the lines of Flett 1) we have on writing \( A_\nu(x) = A - \gamma(x) \)

1) Flett, T.M. (g), p. 87.
\[
\int_0^a g(t) \rho(t) \sin(x,t) \, dt = \sum_{\nu = 0}^{n} A_{\nu}(x) \int_0^a \rho(t) \sin(\nu x, t) \, dt,
\]
with \( b = a + 2f(x) \), and for

\[
n \neq 0\]

\[
\int_0^a \rho(t) \sin(n \nu x, t) \, dt = O \left( n^{-\nu - \frac{k}{2} - \frac{1}{2}} \right)^2.
\]

We write

\[
\sum_{n = -\infty}^{\infty} A_{\nu}(x) \int_0^a \rho(t) \sin(\nu x, t) \, dt
\]

\[
= \sum_{n = -\infty}^{-1} + \sum_{n = 1}^{\infty} + \sum_{n = 1}^{\infty} \sum_{m = n + 2}^{\infty} = M_1 + M_2 + M_3 + M_4, \text{ say,}
\]

where \( \Sigma' \) contains the terms corresponding to

\[
= 1, 2, 3 + 1.
\]

Now

\[
M_1 = O \left( \sum_{\nu = -1}^{\infty} A_{\nu}(x) \left( n^{-\nu} \frac{k}{2} + \frac{1}{2} \right)^2 \right)
\]

\[
= O \left( \sum_{\nu = 1}^{\infty} \lambda_{\nu} (n + \nu)^{-2} \right) = O \left( \sum_{\nu = 1}^{n} \lambda_{\nu} \right) + O \left( \sum_{\nu = n + 1}^{\infty} \nu^{-2} \right)
\]

\[
= O \left( \sum_{\nu = 1}^{n} n^{-2} \lambda_{\nu} \right) + \sum_{\nu = n + 1}^{\infty} \nu^{-2}
\]
\[ = \mathcal{O}\left( \frac{\lambda_n}{n} \right) + \mathcal{O}\left( \frac{\lambda_n}{n} \right) = \mathcal{O}\left( \frac{\lambda_n}{n} \right). \]

Similarly,

\[ M_2 = \mathcal{O}\left( \sum_{\nu=1}^{n-1} \lambda_{\nu} \left( n - \nu + \frac{k}{2} + \frac{1}{2} \right)^{-2} \right) \]
\[ = \mathcal{O}\left( \sum_{\nu=1}^{n-1} \lambda_{\nu} \left( n - \nu \right)^{-2} \right) = \mathcal{O}\left( \sum_{\nu=1}^{n-1} \lambda_{\nu} \left( n - \nu \right)^{-2} \right) \]
\[ = \mathcal{O}\left( \sum_{\nu=1}^{n-1} \lambda_{\nu} \left( n - \nu - \frac{k}{2} - \frac{1}{2} \right)^{-2} \right) \]
\[ = \mathcal{O}\left( \lambda_n \sum_{\nu=1}^{n-1} \lambda_{\nu} \left( n - \nu \right)^{-2} \right) \]
\[ = \mathcal{O}\left( \lambda_n \right) \cdot \mathcal{O}\left( \lambda_n \right) = \mathcal{O}\left( \lambda_n \right). \]

Also,

\[ M_3 = \mathcal{O}\left( \lambda_n \sum_{\nu=1}^{n+2} \left( \nu - n - \frac{k-1}{2} \right)^{-2} \right) \]
\[ = \mathcal{O}\left( \lambda_n \sum_{\nu=1}^{n+2} \left( \nu - n - 1 \right)^{-2} \right), \quad -1 < k \leq 1, \]
\[ = \mathcal{O}\left( \lambda_n \right). \]
Next

\[ n' = A_0(x) \int_0^r \varphi(t) \sin(n,t) \, dt + A_0(x) \int_0^r \varphi(t) \sin(o,t) \, dt \]

\[ + A_{n+1}(x) \int_0^r \varphi(t) \sin(-1,t) \, dt \]

\[ = O(n^{-2}) + O(\lambda_n) + O(\lambda_n) = (\_\_\_\_\_\_\_) \]

\[ = O(\lambda_n) + O\left( \frac{1}{n^2} \right) \]

This proves Lemma 7.

2.12. **Proof of Theorem 7.** We have

\[ \delta_n^k(x) - f(x) = \frac{1}{r} \int_0^r \varphi(t) K_n^k(t) \, dt \]

\[ = \int_0^s + \int_s^r = I_1 + I_2, \text{ say}. \]

Lemma 5 gives \( I_2 = O(n^{-k}) \), while from

Lemma 6 we have \( I_1 = O(\psi(\frac{1}{n})) + O(n^{-k-1}) \).

Hence \( \delta_n^k - f(x) = O(n^{-k}) + O(\psi(\frac{1}{n})) \).

**Proof of Theorem 8.** We have

\[ w(\delta_n^k(x) - f(x)) = \int_0^s \varphi(t) K_n^k(t) \, dt \]

\[ + \int_s^r \varphi(t) R(t) \, dt + \int_0^r \varphi(t) S(t) \, dt. \]
= N_1 + N_2 + N_3, \text{ say.}

Since

\[ \int_0^1 \varphi(t) \rho(t) \sin(n, t) \, dt = \int_0^1 \varphi(t) \mu(t) \sin(n, t) \, dt + \int_0^1 \varphi(t) \frac{\sin(n, t)}{(2\sin \frac{t}{2})^{k+1}} \, dt \]

It is clear that

\[ N_2 = \frac{1}{2} \int_0^1 \varphi(t) \rho(t) \sin(n, t) \, dt - \mu \int_0^1 \varphi(t) \sin(n, t) \, dt \]

\[ = O\left(\frac{\lambda}{\sqrt{n}}\right) + O\left[\left(\frac{1}{n^2} + k\right) \right] \]

Since \( \rho(t) \in BV(0, 1) \) we write \( \varphi = \varphi^1 - \varphi^\mu \), where \( \varphi^1 \) and \( \varphi^\mu \) are non-negative and non-decreasing functions.

By second mean value theorem

\[ \int_0^1 \varphi^1(t) \sin(n, t) \, dt = O\left(\frac{1}{n}\right) \]

Similarly,

\[ \int_0^1 \varphi^\mu(t) \sin(n, t) \, dt = O\left(\frac{1}{n}\right) \]
Hence

\[ \frac{1}{k} \int_0^s t \vartheta(t) \sin(n,t) dt = O\left( \frac{1}{n^k+1} \right). \]

Thus

\[ N_2 = O\left( \frac{2n}{n^k} \right) + O\left( \frac{1}{n^k+1} \right). \]

By virtue of Lemma 6 we have

\[ N_1 = O\left( \varphi\left( \frac{1}{n} \right) \right) + O\left( n^{-k-1} \right). \]

Also

\[ N_3 = O(n^{-1}). \]

Thus

\[ \sigma_n^k - f(x) = O(n^{-1}) + O\left( \varphi\left( \frac{1}{n} \right) \right) + O\left( \frac{2n}{n^k} \right). \]

This completes the proof of Theorem 8.

2.13. Let \( \{p_n\} \) be a sequence of positive real numbers. We write

\[ p_n = \sum_{\nu=0}^n p_\nu, \quad p(t) = p(t). \]
where \( a_k(x) \) is the \( k \)-th partial sum of the Fourier series of \( f(x) \).

If \( p_n = \frac{\pi}{n}, k > 0 \) the Norlund mean \( t_n(x) \)
becomes Cesaro mean \( \sigma_n(x) \).

In this section we shall examine the problem as to whether \( \sigma_n(x) \) in Theorem 8 can be replaced by Norlund mean \( t_n(x) \). Concerning this problem we prove the following theorem which includes Theorem 8 as a special case for \( p_n = \frac{\pi}{n^{<1}}, 0 < \alpha < 1 \).

1) The concept of Norlund summability was first introduced by Voronoï (39) but is, now a days, more closely identified with the name of Norlund (39).
THEOREM 9. Let \( \{p_n\} \) be a positive non-increasing sequence of real numbers such that

\[
\int_0^R p_n(u) \, du = O\left( \frac{P\left( \frac{1}{n} \right)}{n} \right), \quad \frac{1}{n} \leq t \leq R.
\]

Also let \( 0 < \alpha < 1, \quad 0 < \varepsilon \leq r \). If \( x \) is a point such that

\[
\int_0^t |f(u)| \, du < \varepsilon t,
\]

when \( 0 \leq t \leq \varepsilon \), then

\[
t_n(x) - f(x) = O(n^{-\alpha}) + O\left( \frac{1}{P_n} \right).
\]

2.14. The following lemmas are pertinent for the proof of this theorem.

LEMMA 8. We have

\[
k_n(t) = \begin{cases} 
\Theta(n), & 0 \leq t \leq \varepsilon, \\
\frac{P_n(t)}{2\pi} + \Theta\left( \frac{P_n}{P_n t^2} \right), & \frac{1}{n} \leq t \leq R.
\end{cases}
\]

LEMMA 9. If \( \varphi(t) \in L, \quad 0 < \alpha < 1 \) and \( 0 < \varepsilon \leq r \), then
\[ \int_0^t \mathcal{G}(t) K_n(t) \, dt = \mathcal{O} \left( \frac{1}{p_n} \right), \quad n \to \infty. \]

**Lemma 10.** Under the hypotheses of the Theorem 9, we have

\[ \int_0^t \mathcal{G}(t) K_n(t) \, dt = \mathcal{O}(n^{-\xi}) + \mathcal{O} \left( \frac{1}{p_n} \right). \]

**Proof of Lemma 6.**

\[ K_n(t) = \frac{1}{p_n} \sum_{k=0}^{n-1} P_n - \sum_{k=0}^{n} \mathcal{G}_k(t) = \mathcal{O}(n). \]

We write

\[ K_n(t) = \frac{1}{p_n} \sum_{\nu=0}^{n} \frac{\sin(\nu + \frac{1}{2})t}{2 \sin \frac{t}{2}} = \frac{1}{2p_n \sin \frac{t}{2}} \text{Im} \left\{ \sum_{\nu=0}^{n} p_{n-\nu} e^{i(n+\frac{1}{2})t} \right\} = \frac{1}{2p_n \sin \frac{t}{2}} \text{Im} \left\{ \sum_{\nu=0}^{n} p_{n-\nu} e^{i(n-\nu + \frac{1}{2})t} \right\}. \]
\begin{equation}
\frac{1}{2P_n \sin \frac{t}{2}} \text{Im} \left\{ e^{i(n + \frac{1}{2})t} \sum_{\nu=0}^{n} p_{\nu} e^{-i\nu t} \right\}
\end{equation}

\begin{align*}
&= \frac{1}{2P_n \sin \frac{t}{2}} \text{Im} \left\{ e^{i(n + \frac{1}{2})t} \left( \sum_{\nu=0}^{n} - \sum_{\nu=n+1}^{\infty} \right) \right\} \\
&= \frac{P_n(t)}{2P_n \sin \frac{t}{2}} - \frac{1}{2P_n \sin \frac{t}{2}} \text{Im} \left\{ e^{i(n + \frac{1}{2})t} \sum_{\nu=n+1}^{\infty} p_{\nu} e^{-i\nu t} \right\}
\end{align*}

Since \( \{p_{\nu}\} \) is non-increasing, we have

\begin{align*}
\sum_{\nu=n+1}^{\infty} p_{\nu} e^{-i\nu t} &\leq 2p_{n+1} \max_{\nu=0}^{\infty} e^{-i\nu t} \\
&= 2p_{n+1} \left( \frac{1}{1 - e^{-i\nu t}} \right) \\
&= \frac{p_{n+1}}{\sin \frac{t}{2}}.
\end{align*}

Thus, \( I_m \sigma(z) \leq 10(z) \), it follows that the second term is

\begin{equation}
O\left( \frac{1}{P_n t} \right) = O\left( \frac{P_n}{P_n t^2} \right).
\end{equation}

This proves Lemma 8.
Proof of Lemma 2. It is well known that

if \( \{ p_n \} \) is non-negative and non-increasing, then

\[
\sum_{k=0}^{\infty} p_k e^{-ikt} \leq p\left( \frac{1}{t} \right), \quad 0 \leq t \leq v
\]

and \( n^{-1} p_n \leq t \frac{1}{t} \) for \( \frac{1}{n} \leq t \leq v \).

Since \( p_n \geq (n+1)p_n \) it follows that

\[
p_n \leq \frac{p_n}{n} \leq t \frac{1}{t} \quad \text{so that} \quad \frac{p_n}{t} \leq p\left( \frac{1}{t} \right)
\]

We have therefore for \( \frac{1}{n} \leq t \leq v \),

\[
K_n(t) = O\left( \frac{1}{t} \right) + O\left( \frac{p_n}{p_n + 2} \right)
\]

\[
= O\left( \frac{p(\frac{1}{t})}{t} \right) + O\left( \frac{p(\frac{1}{t})}{t} \right) = O\left( \frac{p(\frac{1}{t})}{t} \right).
\]

Hence

\[
\int \left| \varphi(t) K(t) dt \right| \leq A \int \left| \varphi(t) \right| \frac{1}{t} \frac{p\left( \frac{1}{t} \right)}{p_n} \frac{p(\frac{1}{t})}{t} dt
\]

\[
\leq A \frac{p\left( \frac{1}{t} \right)}{t} \int \left| \varphi(t) \right| dt
\]

\[
= O\left( \frac{1}{p_n} \right).
\]

Proof of Lemma 10. We write

\[ \int_0^t \varphi(t) K_n(t) \, dt = \frac{1}{n} \int_0^t 1 \, dt + \int_0^t \frac{1}{2n} \sin \left( \frac{t}{2} \right) \, dt + \int_0^t \left( \frac{p_n}{t^2} \right) \, dt \]

\[ = C_1 + C_2 + C_3, \ 	ext{say.} \]

Since \( \varphi(0) = 0 \), \( \varphi(t) = \varphi(t) - \varphi(0) = \int_0^t \varphi(u) \, du \)

\[ \leq \int_0^t |\varphi(u)| \, du \leq a t^\lambda, \]

we have

\[ C_1 = O(n \int_0^t |\varphi(t)| \, dt) = O(n \int_0^t t \, dt) \]

\[ = O(n^{-\lambda}). \]

Also

\[ C_3 = O(\int_0^t \frac{p_n}{t^2} \, dt) \]

\[ \leq 2. \]
\[ = O\left( \frac{p_n}{n} \right)^{-\kappa + 1} = O(n^{-\kappa}), \]

since \( n p_n < p_n \).

Next let us set

\[ H(t) = \int_0^t \frac{p_n(u)du}{2 p_n \sin \frac{u}{2}} = \frac{1}{2 p_n \sin \frac{t}{2}} \int_0^t p_n(u)du \]

\[ t < \frac{t}{2} < v, \]

\[ = O\left( \frac{P\left( \frac{1}{2} \right)}{p_n t} \right), \]

by the hypothesis. Then

\[ c_2 = \int \frac{p_n(t) \phi(t)}{2 p_n \sin \frac{t}{2}} dt = -\int \frac{\phi H'(t) dt}{1} \]

\[ = -\int \phi d H(t) = - \left[ \frac{\phi H(t)}{1} \right] + \int H(t) d \phi(t) \]

\[ = O\left( \frac{1}{n p_n} \right) + O(n^{-\kappa}) + \int H(t) d \phi(t). \]

Let \( \phi^* \) denote the total variation of \( \phi(t) \) in \((0, t)\), so that \( \phi^*(t) \leq A t^\kappa \). We have
\[
\frac{1}{n} \int H(t) \, d\phi(t) \leq \frac{1}{n} \int H(t) \, d\phi^*
\]

\[
= \mathcal{O}\left( \frac{1}{n} \int \frac{1}{t} \, d\phi^* \right)
\]

\[
= \mathcal{O}\left( \frac{n}{n^2} \int \frac{1}{n} \, d\phi^* \right)
\]

\[
= \mathcal{O}\left( \frac{1}{n} \left[ \left[ \frac{\phi^*}{t} \right]_{1/n}^{1/n} - \int \frac{\phi^*}{t^2} \, dt \right] \right)
\]

\[
= \mathcal{O}(n^{-\kappa}) + \mathcal{O}\left( \frac{1}{n} \int t^{2-\kappa} \, dt \right)
\]

\[
= \mathcal{O}(n^{-\kappa}).
\]

This completes the proof of Lemma 10.

2.15. Proof of Theorem 9. We have

\[
t_n(x) - f(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} s_k(x) - f(x)
\]

\[
= \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} (s_k(x) - f(x))
\]
\[ = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} \frac{1}{\nu} \int_{0}^{\nu} \Theta(t) \Delta_k(t) \, dt \]

\[ = \frac{1}{\nu} \int_{0}^{\nu} \Theta(t) \left( \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} \Delta_k(t) \right) \, dt \]

\[ = \frac{1}{\nu} \int_{0}^{\nu} \Theta(t) \Delta_n(t) \, dt. \]

\[ = \frac{1}{\nu} \left( \int_{0}^{s} + \int_{s}^{\nu} \right) = s_1 + s_2, \text{ say}. \]

**Applying Lemma 10, we have**

\[ s_1 = \Theta(n^{-k}) + O\left( \frac{1}{P_n} \right) \]

**and by virtue of Lemma 9 we get**

\[ s_2 = O\left( \frac{1}{P_n} \right). \]

This proves the theorem.
CHAPTER III

ON THE FOURIER COEFFICIENTS OF CERTAIN

CLASSES OF FUNCTIONS

3.1 Let \( f(x) \) be a function integrable in the

sense of Lebesgue over \((-\pi, \pi)\) and periodic with

period \( 2\pi \). Let the Fourier series associated with

\( f(x) \) be

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).
\]

A function \( f(x) \) is said to belong to \( \text{Lip} \alpha \) class

if

\[
f(x+h) - f(x) = O(h^\alpha), \quad \text{as } h \to 0,
\]

uniformly in \( x, \quad 0 < \alpha \leq 1 \).

It is said to belong to \( \text{Lip} \alpha \) in \( E \) if

\[
f(x+h) - f(x) = O(h^\alpha), \quad 0 < \alpha \leq 1,
\]

uniformly in \( E \), as \( h \to 0 \) through unrestricted real

values, where \( E \) is a set of real numbers.

If

\[
\left[ \int_a^b |f(x+h) - f(x)|^p \, dx \right]^{1/p} = O(h^\alpha), \quad h \to 0
\]

1) Kennedy, P. R. (21).
uniformly in \( x, 0 < \kappa \leq 1, \ 1 \leq p < \infty \), then

\[
f(x) \in \text{Lip}(\kappa, p).
\]

It is said to belong to \( \text{Lip}(\kappa, p) \) if it is continuous and

\[
f(x+h) + f(x-h) - 2f(x) = O(h), \text{ as } h \to 0,
\]

uniformly in \( x \).

A Fourier series is said to be lacunary with Hadamard gaps if it is of the form

\[
\sum_{k=1}^{\infty} \left( a_k \cos n_k x + b_k \sin n_k x \right),
\]

where the natural numbers \( n_k \) satisfy the inequality

\[
\frac{n_k+1}{n_k} \geq \lambda > 1 \quad (k = 1, 2, \ldots)
\]

We denote the \((C, \beta)\) mean of a sequence \( \{ a_n \} \) by \( \frac{a^\beta}{n} \), that is, we write

\[
\frac{a^\beta}{n} = \frac{1}{n\beta} \sum_{v=0}^{n-1} (n-v)^{\beta-1} a_v, \quad \beta \geq 0,
\]

and use \( \frac{b^\beta}{n} \) to denote the corresponding mean for the sequence \( \{ b_n \} \).

1) Zygmund, A. (43).
We write
\[ c^* = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \gamma \left( \frac{m}{n} \right) a_m, \gamma(x) \in BV[-1,1], \]
\[ A_n = A^* = \frac{1}{2\pi} \sum_{k=1}^{n} a_k, \]
\[ B_n = B^* = \frac{1}{2\pi} \sum_{k=1}^{n} b_k, \]
\[ A'_k = \sum_{n=1}^{\infty} \frac{s_{nk}}{n}, \]
\[ B'_k = \sum_{n=1}^{\infty} \frac{b_{nk}}{n}. \]

3.2. In 1926 Hardy proved the following theorem concerning the Fourier coefficients of a function belonging to \( L^p \) class.

**Theorem A.** If \( a_1, a_2, a_3, \ldots \) are Fourier cosine coefficients of a function of class \( L^p \), \( p \geq 1 \), then
\[ \frac{1}{2} a_1, \frac{1}{3} a_2, \frac{1}{4} a_3, \ldots \] are also Fourier coefficients of a function of class \( L^p \), where
\[ a_n = a_1 + a_2 + a_3 + \ldots + a_n. \]

---

1) Hardy, G.H. (17).
Later on Peterson proved the following theorem concerning \((C,R)\) mean of \(\{a_n\}\).

**THEOREM B.** If \(a_1, a_2, \ldots, a_n, \ldots\) are the Fourier coefficients of an even function
\[
f(x) \sim \sum_{1}^{\infty} a_n \cos nx
\]
belonging to \(L^p (p > 2, \beta + \frac{1}{p} > 1)\),
then \(a_1, a_2, \ldots, a_n, \ldots\) are the Fourier coefficients of an even function \(g(x)\) belonging to \(L^{p'} (p' < p)\).

Generalizing the result of Hardy mentioned above, Goldberg proved the following theorem.

**THEOREM C.** Let \(1 \leq p < \infty\). If \(a_n\) is the Fourier cosine coefficient of a function belonging to \(L^p\) class, then \(c_n^*\) is also Fourier cosine coefficient of a function belonging to \(L^p\) class.

The result remains true also for Fourier sine coefficients.

1) Peterson, C.M. (31)
2) Goldberg, R.R. (16).
Bellman, on the other hand, replacing $A_n$ by
\[ \sum_{k=n}^{\infty} \frac{a_k}{k}, \]
proven that Hardy's result remains still true. More precisely, he proved the following theorem.

**THEOREM D.** If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are Fourier cosine coefficients of a function $f(x) \in L^p$, $p > 1$, then $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$ are also Fourier cosine coefficients of a function $F(x) \in L^p$, where $\alpha_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$.

Konyushkov examined the corresponding problem for Lip $\alpha$ class. His result is as follows:

**THEOREM E.** If $b_1, b_2, b_3, \ldots, b_n, \ldots$ are Fourier sine coefficients of a function $g(x) \in \text{Lip} \alpha$, then $\beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots$ are also Fourier sine coefficients of a function $G(x) \in \text{Lip} \alpha$, where $\beta_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$.

Concerning Lip($\alpha$, $p$) class, Konyushkov established the following result which is an analogue of Theorem A for $L^p$ class.

---

1) Bellman, R. (3)
2) Konyushkov, L. L. (23)
3) Konyushkov, L. L. (23).
THEOREM 1. Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \) and \( f(x) \in \text{Lip}(\varepsilon, p) \),

\[ 1 \leq p < \infty, \quad \varepsilon < \frac{1}{p} \]

and let \( \varphi(x) = \sum_{n=1}^{\infty} \phi_n \cos nx \), then

\( \varphi(x) \in \text{Lip}(\varepsilon, p) \).

Regarding Theorem 1 we observe that its converse is not necessarily true. For, if we take \( a_n = (-1)^n \),

then

\[ a_n = \frac{1}{n} \sum_{k=1}^{n} (1) \]

and

\[ a_n = (-1)^n - (-1)^{n-1} = (-1)^{n-1} = 2(-1)^n \]

we have

\[ \left[ \sum_{n=1}^{\infty} \left| a_n \right|^p \right]^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} \left[ \frac{1}{n} (-1)^n \right]^p \right) \]

Hence by the Hausdorff-Young Theorem \( a_n \) is the Fourier coefficient of a function \( F(x) \) belonging to class \( L^{p'} \),

where

\[ \frac{1}{p} + \frac{1}{p'} = 1 \]

\[ 1 < q \leq 2 \text{ since } p' \geq 2 \]

and consequently \( F(x) \in L^q \). Now if \( \sum_{n=1}^{\infty} a_n \cos nx \) is the
Fourier series of $f(x)$ belonging to class $L^q$, then
we have by Hausdorff-Young Theorem necessarily,

$$
\left( \sum_{n=1}^{\infty} |a_n|^{q'} \right)^{\frac{1}{q'}} < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1.
$$

But

$$
\left[ \sum_{n=1}^{\infty} |a_n|^{q'} \right]^{\frac{1}{q'}} = \left[ \sum_{n=1}^{\infty} a_n^{q'} \right]^{\frac{1}{q'}} = \infty.
$$

Therefore $\sum_{n=1}^{\infty} a_n \cos nx$ is not the Fourier series
of a function $f(x)$ belonging to $L^q$ class.

Having seen that converse of Hardy's theorem is not
true in general, the question arises as to what additional
condition should be imposed on $a_n$ so that the same may
hold true. In this direction we have obtained a result
in Theorem 1 of the present chapter, which is of
necessary and sufficient type.

In Theorem 2 a suitable condition has been obtained
under which we may get $L^p$ class instead of $L^{p'}$, $p' < p$
in theorem B and also to relax the restriction on
$p$ in the same theorem.
Extension of the result of Theorem 1 to other classes that is to say for \( f(x) \in BV, f(x) \in \text{Lip} \), and \( f(x) \in \text{Lip}^\alpha \) have been made in Theorems 3, 4, 5 respectively. In theorems 6 and 7 we extend Theorem 5 to function of bounded variation and \( \text{Lip} \) classes respectively. Theorem 8 is concerned with lacunary Fourier series.

3.3. In what follows we shall prove the following theorems:

**Theorem 1.** Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \) with \( a_n \to 0 \).

Then a necessary and sufficient condition that \( \sum_{n=1}^{\infty} a_n \cos nx \) be the Fourier series of \( f \in L^p \) is that \( \sum_{n=1}^{\infty} a_n^p \cos nx \) be the Fourier series of a function belonging to \( L^p \) class.

**Theorem 2.** Let \( \sum_{n=1}^{\infty} a_n \cos nx \) be a trigonometric series. For \( 1 < p \leq 2 \) let \( 0 < s < 1 \) such that \( sp > 1 \).

Then the series \( \sum_{n=1}^{\infty} a_n^s \cos nx \) is the Fourier series of a function belonging to \( \text{class} L^{p'} \), where

\[
\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } s > 0.
\]
THEOREM 3. Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \), where \( f(x) \in BV [0, \pi] \) and \( \cot \frac{1}{2} x f(x) \in L(0,\pi) \), then
\[ \sum_{n=1}^{\infty} a_n \cos nx \text{ is the Fourier series of a function } f(x) \in BV [0,\pi]. \]

THEOREM 4. Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \), \( f(x) \in Lip < \), \( f(0) = 0 \), \( 0 < \alpha < 1 \), then \( \sum_{n=1}^{\infty} a_n \cos nx \text{ is the Fourier series of } f(x) \in Lip < \).

THEOREM 5. Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx \), \( f(x) \in \wedge_* \), \( f(0) = 0 \), then \( \sum_{n=1}^{\infty} a_n \cos nx \text{ is the Fourier series of } f(x) \in \wedge_* \).

THEOREM 6. Let \( g(x) = \sum_{n=1}^{\infty} a_n \cos nx \), and \( g(x) \in BV [0, \pi] \), then \( \sum_{n=1}^{\infty} a_n \cos nx \text{ is Fourier series of a function } g(x) \in BV [0,\pi] \) where \( a_n = \frac{2}{\pi} \int_0^\pi g(x) \cos nx \, dx \).

THEOREM 7. Let \( g(x) = \sum_{n=1}^{\infty} a_n \cos nx \), \( g(x) \in \wedge_* \), \( g(0) = 0 \), then \( \sum_{n=1}^{\infty} a_n \cos nx \text{ is the Fourier series of } \int_0^x g(t) \, dt \in \wedge_* \), where \( a_n = \frac{2}{\pi} \int_0^\pi g(x) \cos nx \, dx \).
THEOREM 2. Let \( g(x) \sim \sum_{k=1}^{\infty} (a_k \cos \omega_k x + b_k \sin \omega_k x) \),

\[
\lim \inf_{k \to \infty} \frac{\|a_k\|}{\|b_k\|} > 1 \quad \text{and} \quad g(x) \in \operatorname{Lip}_x^\alpha \quad \text{in some set} \ E \quad \text{of positive measure}, \quad 0 < \alpha < 1,
\]

then

\[
\sum_{k=1}^{\infty} (a_k \cos \omega_k x + b_k \sin \omega_k x)
\]

is the Fourier series of \( g(x) \in \operatorname{Lip}_x^\alpha \) in \( E \).

3.4 In the proof of these theorems the following lemmas will be required.

**Lemma 1.** Let \( f(x) = \sum_{n=1}^{\infty} a_n \cos n x \) with \( a_n \downarrow 0 \). Then \( f \in L^p \) iff \( \sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty \).

**Lemma 2.** If \( a_n \downarrow 0 \), then the convergence of

\[
\sum_{n=1}^{\infty} a_n^p n^{p-2}
\]

implies the convergence of the series

\[
\sum_{n=1}^{\infty} a_n^{p-2}.
\]

**Proof.** We have

\[
a_n^* = \frac{1}{n} \sum_{k=1}^{n} a_k \geq \frac{1}{n} (n a_n) = a_n,
\]

1) Hardy, G.H. and Littlewood, J.E. (19).
so that
\[ \sum_{n=1}^{\infty} a_n n^{-p} \geq \sum_{n=1}^{\infty} a_n n^{-p-2}, \]
and hence the result follows

**Lemma 3.** Let \( f(x) = \frac{\sin x}{x} \). Then \( f(x) \in BV \).

**Lemma 4.** Let \( f(x) = \frac{a}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \).

If \( \sum_{k=1}^{\infty} (|a_k| + |b_k|) = O\left( \frac{1}{n^{\alpha}} \right) \) \( (0 \leq \alpha < 1) \)

then \( f(x) \in \text{Lip} \alpha \).

**Lemma 5.** Let \( 0 < \alpha < 1 \) and let \( \lim \inf_{k \to \infty} \frac{n_{k+1}}{n_k} > 1 \).

Then in order that the series \( \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \)

should be the Fourier series of a function belonging to \( \text{Lip} \alpha \) in some set of positive measure, it is necessary and sufficient that

---

1) Lorentz, G. G. (25).
2) Lorentz, G. G. (25).
3) Kennedy, P. B. (21).
3.8. Proof of Theorem 1. Necessity part is a particular case of Theorem A.

Sufficiency: Since $a_n \downarrow 0$ it follows that

$\sum_{n=1}^{\infty} a_n^p < \infty$. From the proof of Lemma 1), it follows that

if $\sum_{n=1}^{\infty} A_n \cos nx$ is the Fourier series of a function $F(x) \in L^p$, then $\sum_{n=1}^{\infty} a_n^{p-2} < \infty$.

Applying Lemma 2 we have

$$\sum_{n=1}^{\infty} a_n^{p-2} < \infty.$$ 

Hence by virtue of Lemma 1, $f(x) \in L^p$ and consequently $\sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier series of $f(x)$.

This completes the proof of Theorem 1.

---

3.6. PROOF OF THEOREM 2. We have

\[ a^\beta_n = \frac{1}{n^\beta} \sum_{k=0}^{n} a_{n-k}^{\beta-1} a_k \]

\[ = O\left( \frac{1}{n^\beta} \sum_{k=0}^{n} a_{n-k}^{\beta-1} (k+1)^{-\delta} \right) \]

\[ = O\left( \frac{1}{n^\beta} \sum_{k=0}^{n} a_{n-k}^{\beta-1} a_k^{-\delta} \right) \]

\[ = O\left( \frac{1}{n^\beta} a_n^{-\delta} \right) \]

\[ = O\left( \frac{1}{n^\beta} \right), \]

so that

\[ \left( \sum_{n=1}^{\infty} \frac{a^\beta_n}{n^\beta} \right) \frac{1}{P} = O\left( \sum_{n=1}^{\infty} \frac{1}{n^\beta} \right) \frac{1}{P} \]

\[ = O(1), \text{ since } \delta > 1. \]

Hence by Hausdorff-Young theorem \( a^\beta_n \) is the Fourier coefficient of a function belonging to class \( L^{p'} \).
3.7. PROOF OF THEOREM 3. Let \( \lambda_n = \frac{a_n}{a_{n-2}} \),
then as shown by Hardy we have

\[
F_1(x) \sim E \sum_{n=1}^{\infty} \lambda_n \cos nx,
\]

where

\[
F_1(x) = \int \cot \frac{1}{2} u f(u) \, du.
\]

Since \( \cot \frac{1}{2} u f(u) \notin L \), \( F_1(x) \) is indefinite integral hence it is absolutely continuous, which, in turn, implies that \( F_1(x) \in BV \).

We have

\[
\left( \sum_{k=n}^{\infty} \frac{a_k}{k^2} \right)^{\frac{1}{p}} \leq N \left[ \sum_{k=n}^{\infty} \frac{1}{k^p} \right]^{\frac{1}{p}}, \quad 1 \leq p \leq 2,
\]

\[
= N \left[ \sum_{k=n}^{\infty} \frac{1}{k^p} \right]^{\frac{1}{p}}
\]

\[
= N \left[ \frac{1}{n^{2p-1}} \right]^{\frac{1}{p}}
\]

\[
= O\left( \frac{1}{n^{2-\frac{1}{p}}} \right).
\]

1) Hardy, G.H. (17).
\[ = O\left( \frac{1}{n} \right), \]

Since \( 2 - \frac{1}{p} \geq 1. \)

Therefore by virtue of lemma 3, \( \left\{ \frac{a_n}{2n} \right\} \) is the Fourier coefficient of a function of bounded variation.

Since \( A_n \) and \( \frac{a_n}{2n} \) are Fourier coefficients of functions of bounded variation, \( A_n \) will also be the Fourier coefficients of a function of bounded variation.

This completes the proof of the Theorem 3.

3.8. Proof of Theorem 4. Let \( \overline{A}_n = A_n - \frac{1}{2n} a_n \), then we have

\[ F_1(x) = \sum_{n=1}^{\infty} \overline{A}_n \cos nx, \]

where

\[ F_1(x) = \int_{x}^{x} \cot \frac{1}{2} u f(u) du. \]

By virtue of lemma 4, \( \frac{a_n}{2n} \) is the Fourier coefficient of a function belonging to \( \text{Lip} \alpha \) class. Hence it is sufficient to show that \( F_1(x) \in \text{Lip} \alpha. \)
Let us suppose $0 \leq x < x + h \leq r$.

We have

$$P'_1(x) = \int^{x}_{x} \cot \frac{1}{2} u f(u) \, du$$

$$P'_1(x+h) = \int^{x+h}_{x} \cot \frac{1}{2} u f(u) \, du$$

so that

$$|P'_1(x+h) - P'_1(x)| = \int^{x+h}_{x} \cot \frac{1}{2} u f(u) \, du \leq \int^{x+h}_{x} \frac{|f(u)|}{u} \, du.$$ 

Since $f(u) \in \text{Lip}^{\alpha}$ and $f(0) = 0$, $|f(u)| \leq c u^{\alpha}$, and so we have

$$|P'_1(x+h) - P'_1(x)| \leq c \int^{x+h}_{x} u^{-\alpha} \, du$$

$$= c \left[ -u^{-\alpha+1} \right]^{x+h}_{x}$$

$$= c \frac{(x+h)^{-\alpha+1} - x^{-\alpha+1}}{\alpha - 1}.$$ 

uniformly in $x$. Therefore $P'_1(x) \in \text{Lip}^{\alpha}$.

Remark. It may be observed that the condition $f(0) = 0$ is essential for the validity of the theorem, since we can construct a function belonging to $\text{Lip}^{\alpha}$ such that $f(0) \neq 0$ and for which the $P'_1(x)$ is discontinuous, which in turn implies that $P'_1(x) \notin \text{Lip}^{\alpha}$. 
3.9. Proof of Theorem 2. Let $a_n = A_n - \frac{1}{2n} a_n$, then as already remarked, $A_n$ is the Fourier cosine coefficient of $F_1(x) = \frac{1}{x} \cot \frac{x}{2} f(u)du$. Since $f(x) \in \Lambda$, $a_n = O\left(\frac{1}{n}\right)$ hence by virtue of Theorem 2 of Chapter II, $\frac{a}{2n}$ is Fourier coefficient of a function belonging to $\Lambda$ class. It is therefore, sufficient to prove that $F_1(x) \in \Lambda$.

We have

$$|F_1(x+h) + F_1(x-h) - 2F_1(x)| = \left| \int \cot \frac{1}{2x} uf(u)du \right| + \left| \int \cot \frac{1}{2x-h} uf(u)du \right|$$

$$- 2 \int \cot \frac{1}{2x} uf(u)du!$$

$$\leq \int \cot \frac{1}{2x} uf(u)du + \int \cot \frac{1}{2x-h} uf(u)du!$$

Since $f(x) \in \Lambda$, $f(0) = 0$, $|f(u)| \leq cu$, we have for $0 \leq x < x + h \leq \pi$,

\[ \text{Vol.II: Zygmund, } A \text{ (44), p. 46.} \]
Therefor* $F'(x) = r'(x)$.

We have

$$\lim_{h \to 0} \frac{F_1(x+h) + F_1(x-h) - 2F_1(x)}{x} = \frac{c}{x} \int \frac{du}{x} + c_1 \int \frac{du}{x-h} = o(h).$$

Therefore $F_1(x+h) + F_1(x-h) - 2F_1(x) = o(h)$, as $h \to 0$ uniformly in $x$. Also by virtue of Theorem 4,

$F_1(x) \in \text{Lip } \alpha$ and hence it is continuous. Therefore

$F_1(x) \in \text{Lip } \alpha$.

This completes the proof of Theorem 5.

3.10. Proof of Theorem 6. Let $\omega_n^* = \omega_n - \frac{\omega_n}{2n}$.

By a result of Loo 1) $\omega_n^*$ is the Fourier cosine coefficient of the function

$$G(x) = \frac{(x - x)}{2\pi} \cot \frac{x}{2} \int_0^x g(t)dt - \frac{x}{2\pi} \cot \frac{x}{2} \int_0^x g(t)dt$$

$$= I_1(x) - I_2(x),$$

provided this function is integrable.

Since $\int g(t)dt$ being an indefinite integral is a function

of bounded variation and also $x \cot \frac{x}{2} \in \text{BV}[0, x]$, it follows that $I_2(x) \in \text{BV}[0, x]$.

1) Loo, T.C. (83) pp. 273 - 274.
We have

\[ I_1(x) = \frac{1}{2\pi} \int x \cot \frac{x}{2} \frac{1}{x} \int g(t) dt. \]

Since \( g(t) \in BV [0,\pi] \), it follows that

\[ \frac{1}{x} \int g(t) dt \in BV [0,\pi] \]. Hence \( I_1(x) \in BV [0,\pi] \).

Also, as proved in Theorem 3, \( a_n \) is the Fourier coefficient of a function of bounded variation. Therefore \( a_n \) is the Fourier coefficients of a function of bounded variation.

**Remark 1.** It may be remarked that theorems 3 and 6 remain also true for sine series.

**Remark 2.** Let

\[ \phi(x) = \frac{x-x}{2} \quad 0 < x < 2\pi \]

\[ = 0, \quad x = 0, 2\pi, \]

then

\[ \phi(x) \sim \sum_{n=1}^{\infty} \frac{\sin nx}{n} \]

and

\[ \phi(x) \in BV [0,2\pi] \].
Therefore by a result of Loo it follows that $E B_n \sin nx$ is also a Fourier series, where 

$$B_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

But it cannot be the Fourier series of a function of bounded variation, for

$$B_n \sim \log n \neq O\left(\frac{1}{n}\right).$$

Thus we observe that in general if $E B_n \sin nx$ is a Fourier series of a function of bounded variation then it is necessary that the Fourier series $E B_n \sin nx$ is a Fourier series of a function of bounded variation.

3.11. Proof of Theorem 7. Again $A_n = \frac{E}{n} \frac{\alpha}{\nu} - \frac{1}{2n}$.

As in the previous theorem, $A_n$ is the Fourier cosine coefficient of the function

$$G(x) = \frac{1}{2} \cot \frac{1}{2} x \int_{0}^{x} g(t) \, dt - \frac{1}{2} \int_{0}^{\frac{1}{2} x} \cot \frac{1}{2} x \int_{0}^{\frac{1}{2} x} g(t) \, dt$$

$$- \frac{1}{2} \int_{0}^{\frac{1}{2} x} \cot \frac{1}{2} x \int_{0}^{x} g(t) \, dt, \tag{1}$$

1) Loo, T.C. (23 A) p. 270.
\[ I_1(x) = I_1(x) + I_2(x) + I_3(x), \]

provided \( G(x) \in L. \)

We have

\[
\begin{aligned}
I_1(x+h) - I_1(x) &= \frac{1}{2!} \cot \frac{1}{2} (x+h) \int_x^{x+h} g(t) \, dt \\
&\quad + \left( \cot \frac{x+h}{2} - \cot \frac{x}{2} \right) \int_0^x g(t) \, dt.
\end{aligned}
\]

Let \( 0 \leq x < x+h \leq v. \) Since \( g(t) \in \Lambda, \) \( g(0) = 0, \)

\[ |g(t)| \leq ct, \]

we have therefore

\[ I_1(x+h) - I_1(x) = O(h), \quad \text{as } h \to 0 \]

uniformly in \( x. \) Thus \( I_1(x) \in \text{Lip 1} \) and hence \( \in \Lambda, \)

and it follows immediately that \( I_2(x) \in \text{Lip 1} \) and

so belongs to \( \Lambda. \)

We know that \( f \in \text{Lip 1} \) iff \( f \) is indefinite integral

of a bounded function. By hypothesis it follows

that \( I_3(x) \in \Lambda, \) since \( \delta_1 = \delta_1 - \frac{a_n}{2\pi} \) and we

have already seen that \( \frac{a_n}{2\pi} \) is the Fourier coefficient

\[ 1) \text{ Zygmund, A (44), Vol. I p.43.} \]
of a function of \( \wedge \) class. Hence \( \alpha_n \) is also

Fourier coefficient of a function \( \in \wedge \).

3.12. **Proof of Theorem 8**. We have

\[
A_n^* = \sum_{v=k}^{\infty} \frac{a_{nv}}{n_v} = \sum_{v=k}^{\infty} \frac{1}{(n_v)^k + 1}
\]

\[
= \sum_{v=k}^{\infty} \frac{1}{v^k + 1}
\]

\[
= \sim (n_k^{-k}).
\]

By lemma 5 \( A_n^* \) is Fourier coefficient of

\( G(x) \in \text{Lip}^* \).

Similarly \( B_n^* \) is Fourier coefficient of

\( G'(x) \in \text{Lip}^* \).

Thus proof of Theorem 8 is concluded.
CHAPTER IV

ON THE SUMMABILITY OF A TRIGONOMETRIC SEQUENCE

4.1. Let $A = (a_{n,k})$ be an infinite matrix of real or complex numbers and let $\{s_k\}$ be any sequence of complex numbers. The sequence $\{s_n\}$ defined by

$$\tag{4.1} s_n = \sum_{k=0}^{\infty} a_{n,k} s_k,$$

is called $A$-transform of $\{s_k\}$ whenever the series converges for $n = 0, 1, 2, \ldots$ The sequence $\{s_n\}$ is said to be $A$-summable to $s$ if $\{s_n\}$ converges to $s$.

The matrix $A$ is called regular if it satisfies the following conditions:

(i) $\lim_{n \to \infty} a_{n,k} = 0$, for $k = 0, 1, 2, \ldots$

(ii) $\sup_{n} \sum_{k=0}^{\infty} |a_{n,k}| \leq M$

(iii) $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} = 1$

A bounded sequence $\{s_k\}$ is said to be almost convergent to $l$ if

1) Lorentz, G. G. (24)

* A portion of this chapter has already been accepted for publication in the Indian Journal of Mathematics.
\[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=0}^{p-1} a_{n+k} = l,
\]
uniformly in \( n \).

It is easy to see that a convergent sequence is almost convergent and the limits are the same.

A sequence \( \{a_k\} \) is said to be almost \( A \)-summable \(^1\) to \( s \) if the \( A \)-transform of \( \{a_k\} \) is almost convergent to \( s \), and the matrix \( A \) is said to be almost regular if \( a_k \to s \) implies that \( \{a_k\} \) is almost convergent to \( s \). The necessary and sufficient conditions for the matrix \( A \) to be almost regular \(^2\) are:

\[
(1) \quad \sup \left\{ \frac{1}{p} \sum_{j=0}^{p-1} a_{j,k} \right\} < M, \quad n=0,1,2,\ldots,
\]
where \( M \) is a positive constant and \( p \) is a positive integer,

\[
(11) \quad \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p-1} a_{j,k} = 0
\]
uniformly in \( n \), \( k = 0,1,2,\ldots \),

\[
(111) \quad \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} a_{j,k} = 1, \quad \text{uniformly in } n.
\]

---

1) King, J.P. (22).
2) King, J.P. (22).
A bounded sequence \( \{a_n\} \) is said to be \( F_4 \)-summable to the limit \( s \) if

\[
\delta_{m,k} = \sum_{n=0}^{\infty} a_{m,n} s_{n+k}
\]

tends to \( s \) as \( m \to \infty \), uniformly in \( k \).

If

\[
a_{n,n} = \begin{cases} 
\frac{1}{m}, & n \leq m, \\
0, & n > m,
\end{cases}
\]

\( F_4 \)-summability is the same as almost convergence of sequence \( \{a_k\} \). It is also known that a \( F_4 \)-summable sequence is almost convergent if the matrix \( A \) is regular. Moreover, if we take \( k = 0 \), the above definition reduces to that of \( A \)-summability.

A bounded sequence \( \{a_k\} \) will be said to be \( AB \)-summable to the limit \( s \), if its \( A \)-transform is \( FB \)-summable to the limit \( s \) where \( B = (b_{n,k}) \) is an infinite matrix.

It is easy to see that every \( A-B \)-summable sequence is also almost \( A \)-summable provided the second matrix, namely \( (b_{n,k}) \) is regular.

---

1) Lorentz, G.G. (24).
2) Lorentz, G.G. (24).
Let \( f(x) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \((0, 2\pi)\). Let

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

be the Fourier series of \( f(x) \) and let

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x),
\]

be its conjugate series.

We write

\[
\begin{align*}
t_p &= \sum_{k=0}^{\infty} a_{p,k} \mathcal{B}_k(x), \\
t_{p,k} &= \sum_{j=0}^{\infty} a_{p,j} (j+k) \mathcal{B}_{j+k}(x), \\
t_{q,v} &= \sum_{j=0}^{\infty} b_{q,j} t_{j+v}, \\
\psi(t) &= \begin{cases} 
    f(x+t) - f(x-t) & 0 < t \leq \pi, \\
    f(x+0) - f(x-0) & t = 0,
\end{cases}
\end{align*}
\]

and

\[
\delta(x) = \begin{cases} 
    f(x+0) - f(x-0) 
\end{cases}.
\]
4.2. Generalizing a theorem of Fejer in 1954, Siddiqi proved the following theorem for summability (A), where $A = (a_{n,k})$ is a triangular matrix of real or complex numbers.

**THEOREM A.** Let $f(x) \in BV [0, 2\pi]$ and periodic with period $2\pi$. If (A) is regular and if

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k}^* = 0, \quad a_{n,k}^* = a_{n,k} - a_{n,k+1},$$

then the sequence $\{n B_n(x)\}$ is summable (A) to $D(x)$.

Later on in 1961 he obtained necessary and sufficient condition on $A$ for the validity of Theorem A and derived certain consequences for the Fourier coefficients of continuous functions of bounded variation. His main theorem is as follows:

**THEOREM B.** If (A) is regular, then for every $f(x) \in BV [0, 2\pi]$ and for every $x \in [0, 2\pi]$,

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} k B_n(x) = \frac{D(x)}{v}$$

1) Fejer, L. (64).
2) Siddiqi, J.A. (33).
3) Siddiqi, J.A. (34).
iff

\[(4.22) \quad \lim_{n \to \infty} \sum_{k=0}^{n} a_n \cos kt = 0,\]

in every \(0 < s \leq t \leq \pi\).

The object of the present chapter is to obtain necessary and sufficient conditions in order that the sequence \(\{k E_k(x)^f\}\) be almost \(A\)-summable, 
\(PA\) - summable, and \(AB\) - summable to \(\frac{D(x)}{v}\). It may be remarked that Theorem B is a particular case 
k = 0, \(a_p = 0, j > p\), of Theorem 2 of the present chapter.

4.3 In what follows we shall take the matrices \(A\) and \(B\) to be real, prove the following theorems:

**Theorem 1.** If \(A\) be almost regular, then for every 
\(f(x) \in BV[0, 2\pi]\) and for every \(x \in [0, 2\pi]\),

\[\lim_{p \to \infty} \frac{1}{p} \sum_{r=0}^{p-1} \sum_{n=0}^{r} \gamma = \frac{D(x)}{v}\]

uniformly in \(x\), iff

\[(4.3.1) \quad \lim_{p \to \infty} \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} \cos kt = 0,\]

uniformly in \(a\) for every \(0 < s \leq t \leq \pi\).
THEOREM 2. If \( A = (a_{n,k}) \) is regular, then for every \( f(x) \in BV[0,\pi] \) and for every \( x \in [0,\pi] \), the sequence \( \{ k A_k(x) \} \) is \( PA \)-summable to the limit \( \frac{D(x)}{n} \) if

\[
\lim_{p \to \infty} \sum_{j=0}^{\infty} a_{p,j} \cos(j+k)t = 0,
\]

uniformly in \( k \) in every interval \( 0 < s < t < \pi \).

THEOREM 3. If \( (a_{n,k}) \) and \( (b_{n,k}) \) are two infinite matrices satisfying the following conditions:

(1) \[
\lim_{p \to \infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} = 1,
\]

(2) \[
\sup_{p \to \infty} \max_{n=0}^{\infty} \max_{k=0}^{\infty} |a_{n,k}| < \infty,
\]

then for every \( f(x) \in BV[0,\pi] \) and for \( x \in [0,\pi] \), the sequence \( \{ k A_k(x) \} \) is \( AB \)-summable to the limit \( \frac{D(x)}{n} \) if

\[
\sum_{j=0}^{\infty} b_{p,j} \sum_{k=0}^{\infty} a_{j+n,k} \cos kt = 0
\]

uniformly in \( v \) in every interval \( 0 < s < t < \pi \).
4.4 Proof of the theorem 1. We have

\[
\frac{1}{p} \sum_{r=0}^{p-1} t_{n+r} = \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} k B_k(x)
\]

\[
= \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} \frac{1}{\pi} \int_0^\pi k \sin kt \gamma(t) dt
\]

\[
= \frac{D(x)}{v} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} + \frac{1}{\pi} \int_0^\pi d\gamma(t) \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} \cos kt
\]

\[
= I_1 + I_2, \text{ say.}
\]

By virtue of the condition (iii),

\[I_1 = \frac{D(x)}{v}, \text{ as } p \to \infty \text{ uniformly in } n. \text{ It, therefore, suffices to show that}
\]

\[(4.4.1) \quad I_2 = \frac{1}{v} \int_0^\pi d\gamma(t). K_{n,p}(t) \to 0, p \to \infty,
\]

uniformly in \( n \), where

\[(4.4.2) \quad K_{n,p}(t) = \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} \cos kt.
\]
We shall first show that condition (4.4.1) is equivalent to the following condition:

\[(4.4.3) \quad \frac{1}{v} \int_0^v k_{n,p}(t) \psi_x(t) \, dt \to 0, \quad p \to \infty, \]

uniformly in \(n, \quad \varepsilon < \delta \leq v\) for every \(f \in BV[0,2\varepsilon]\) and for every \(x \in [0,2\varepsilon]\).

If \(f \in BV[0,2\varepsilon]\) and \(x \in [0,2\varepsilon]\), given any \(\varepsilon > 0\) there exists an \(\delta > 0\) such that

\[(4.4.4) \quad \int_0^\varepsilon \lambda_x(t) \, dt < \frac{\varepsilon}{2\varepsilon} .\]

By virtue of condition (4.7)' we have

\[|k_{n,p}(t)| = \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{n+r} \cos nt \]

\[\leq \frac{1}{p} \sum_{k=0}^{n+p-1} \cos nt \leq \sum_{j=n}^{n+p-1} a_{j,k} \]

\[(4.4.5) \quad \sum_{k=0}^{n+p-1} \sum_{j=n}^{n+p-1} a_{j,k} \leq N , \]

uniformly in \(n\) and so that
\[
\int_0^1 \frac{1}{n} \int_0^n K_{n,p}(t) d\gamma_p(t) - \int_0^1 \int_0^n K_{n,p}(t) d\gamma_p(t) \leq \epsilon
\]
uniformly in \( n \).

Thus \( \lim_{p \to \infty} \int_0^1 \frac{1}{n} \int_0^n K_{n,p}(t) d\gamma_p(t) = 0 \) uniformly in \( n \)
iff \( \int_0^1 \frac{1}{n} \int_0^n K_{n,p}(t) d\gamma_p(t) \to 0 \) as \( p \to \infty \),
uniformly in \( n \). Thus (4.4.3), (4.4.7) are equivalent.

Following the lines of Banach\(^1\) it can be easily verified that a sequence of continuous functions \( \{x_n^k(t)\} \) converges weakly (uniformly in \( k \)) to a continuous function \( x(t) \), that is to say

\[
\lim_{n \to \infty} \int_0^1 x_n^k(t) dg(t) = \int_0^1 x(t) dg(t)
\]
uniformly in \( k \) for every \( g \in M^0 \), if and only if

(i) \( x_n^k(t) \) is bounded uniformly in \( k \), \( n = 1,2,\ldots \)

(ii) \( \lim_{n \to \infty} x_n^k(t) = x(t) \) uniformly in \( k = 0,1,\ldots \ldots \)

for every \( t \in [0,1] \).

\(^1\) Banach, \& (1) p. 134.
Thus it follows that (4.4.5) holds iff

(4.4.6) \( |K_{n,p}(t)| \leq K \) for all \( n,p \) and \( t \in [\varepsilon, r] \)

for every \( \varepsilon > 0 \), and uniformly in \( n \).

(4.4.7) (4.3.1) holds.

Since by virtue of (4.4.5), (4.4.6) automatically holds it follows that (4.4.5) holds iff (4.3.1) holds.

This completes the proof of the theorem.

4.6. Proof of Theorem B

\[
t'_{p,k} = \sum_{j=0}^{\infty} a_{p,j} (j+k) B_{j+k}(x)
\]

\[
= \sum_{j=0}^{\infty} a_{p,j} (j+k) \frac{v}{v} \int_{0}^{v} \Psi(t) \sin(j+k)t \ dt
\]

\[
= \sum_{j=0}^{\infty} a_{p,j} \frac{D(x)}{v} + \frac{1}{v} \int_{0}^{v} \sum_{j=0}^{\infty} a_{p,j} \cos(k+j) d\Psi(t)
\]

\[
= I_1 + I_2, \text{ say.}
\]

By virtue of the condition (iii) of regularity,

\[
I_1 = \frac{D(x)}{v}, \quad \text{as } p \to \infty \text{ uniformly in } k. \text{ It,}
\]

therefore, suffices to show that
\begin{equation}
I_k = \frac{1}{\pi} \int_0^\infty d\gamma(t) K_{k,p}(t) \rightarrow 0, \quad p \rightarrow \infty
\end{equation}

uniformly in \( k \), where

\begin{equation}
K_{k,p}(t) = \sum_{j=0}^{\infty} a_{p,j} \cos(k+j)t.
\end{equation}

Proceeding on the lines of the proof of Theorem 1 we observe that (4.5.1) is equivalent to

\begin{equation}
\int_0^\infty d\gamma(t) K_{k,p}(t) \rightarrow 0, \quad p \rightarrow \infty
\end{equation}

uniformly in \( k \), \( 0 < s \leq \tau \) for \( f \in BV[0,2\pi] \)

and for every \( x \in [0,2\pi] \).

By virtue of regularity condition (ii)

\begin{equation}
|K_{k,p}(t)| \leq \sum_{j=0}^{\infty} |a_{p,j}| \leq M
\end{equation}

for all \( p \) and \( t \in [s,\tau] \) for every \( s > 0 \) and uniformly in \( k \).

Following the lines of Banach \footnote{Banach, A. (1930) p. 134.} it can be easily verified that a sequence of continuous functions \( x_k(t) \) converges weakly (uniformly in \( k \)) to a continuous function \( x(t) \), that is to say.

\footnote{Banach, A. (1930) p. 134.}
\[ \lim_{n \to \infty} \int_0^k x_n(t) \, dg(t) = \int_0^k x(t) \, dg(t) \]

uniformly in \( k \) for every \( g \in BV \), if and only if

(i) \( x_n(t) \) is bounded, uniformly in \( k, \ n = 1, 2, \ldots \)

(ii) \( \lim_{n \to \infty} x_n(t) = x(t) \) uniformly in \( k = 0, 1, \ldots \)

for every \( t \in [0, 1] \).

Thus it follows that (4.5.5) holds iff

(4.5.5) \[ |K_{k,p}(t)| \leq K \text{ for all } k, p \text{ and } t \in [s, v] \]

for every \( s > 0 \),

(4.5.6) (4.5.5) holds.

Since by virtue of (4.5.4), (4.5.4) always holds it follows that (4.5.5) holds iff (4.5.2) holds.

This completes the proof of theorem 1.

4.6. **Proof of Theorem 3.**

\[ t''_{p,v} = \sum_{j=0}^{\infty} b_{p,j} t_{j+p} \]
\[\begin{align*}
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{p,j} \sum_{k=0}^{\infty} a_{j+\nu,k} k B_k(x) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j+\nu,k} k \frac{1}{v} \int_{0}^{v} \psi_x(t) \sin kt \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j+\nu,k} \frac{D(x)}{v} \\
&= \int_{0}^{v} \int_{0}^{\infty} d\psi_x(t) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{p,j} a_{j+\nu,k} \cos kt, \\
&= I_1 + I_2, \text{ say.}
\end{align*}\]

By condition (1) of the theorem, \( I_1 = \frac{D(x)}{v} \) as \( p \to \infty \) uniformly in \( k \). It, therefore, suffices to show that \( I_2 \to 0 \) uniformly in \( k \). The proof of which is similar to that of Theorem 2 and hence is omitted.

This completes the proof of theorem 3.
CHAPTER V

ON THE WALSH - FOURIER COEFFICIENTS OF CERTAIN
CLASSES OF FUNCTIONS

5.1. Let the Rademacher function be defined

by \( \psi_0(x) = 1 \) (0 \( \leq x < \frac{1}{2} \)), \( \psi_0(x) = -1 \) (\( \frac{1}{2} \leq x < 1 \))

\( \psi_0 \) is defined recursively for \( n = 1, 2, \ldots \).

The Walsh function \( \psi(x) \) is defined as follows:

(a) \( \psi_0(x) = 1 \),

(b) If \( n \) has the unique dyadic expansion \( n = \sum_{i=0}^{\infty} a_i x_i \)

where \( x_i = 0, 1 \) and \( x_i = 0 \), for \( i > n \), then

\( \psi_n(x) = \psi_{a_1}(x) \psi_{a_2}(x) \cdots \psi_{a_n}(x) \), where

\( a_1, a_2, a_3, \ldots, a_n \) correspond to the coefficients

\( x_{a_1} = 1 \). Every function \( f(x) \) which is of period 1

and Lebesgue integrable on \([0,1]\) may be expanded

in a Walsh-Fourier series,

\[ f(x) \sim \sum_{k=0}^{\infty} c_k \psi_k(x), \]

where \( c_k = \int_0^1 f(x) \psi_k(x) dx \), \( k = 0, 1, 2, \ldots \).

A function \( f(x) \) is said to belong to the class \( B \)

of essentially bounded functions if \( |f(x)| \leq M \) almost

everywhere.
Let $\mathcal{S}^1$ denote the class of series $\sum_{k=0}^{\infty} c_k y_k(x)$ with coefficients $c_k = \frac{1}{2} \int_{-1}^{1} y_k(t) dF(t)$, where $F(t)$ is continuous and of bounded variation.

We denote by $\mathcal{K}_k$ the $k$-th $(C,1)$ mean of the sequence of Walsh-Fourier coefficients $c_k$.

A.2. Hardy proved the following result concerning the Fourier coefficients of a function of $L^p$ class, $p \geq 1$.

**Theorem A.** If $a_1, a_2, \ldots, a_n$ be the Fourier coefficients of a function $f(x) \in L^p$, $p \geq 1$, then $A_1, A_2, A_3, \ldots, A_n$ are also Fourier coefficients of a function of $L^p$ class, where

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k.$$

In the present chapter we shall examine how for the above result of Hardy remains the true for Walsh-Fourier series of functions of $L^p$, $B$ and $S$ classes.

The first three theorems are connected with Walsh-Fourier coefficients while the last theorem is Fourier series with respect to orthonormal system of functions. In what follows we prove the following theorems:

1) Morgensthaler, O.W. (27)
2) Hardy, G.H. (17).
THEOREM 1. If $\sum_{k=0}^{\infty} y_k(x)$ is the Walsh-Fourier series of a function $f(x) \in L^p(0,1)$, $1 < p < 2$, then
$\sum_{k=0}^{\infty} <2k y_k(x)$ is the Walsh-Fourier series of a function $f(x) \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 2. If $\sum_{k=0}^{\infty} y_k(x)$ is the Walsh-Fourier series of a function $f(x) \in L^2$, then $\sum_{k=0}^{\infty} <2k y_k(x)$ is the Walsh-Fourier series of a function $f(x) \in L^2$.

THEOREM 3. If $\sum_{k=0}^{\infty} e_k y_k(x)$ is the series of class $S'$ with function $f(x)$ satisfying the condition

$$\lim_{t \to 0} \int_0^t |df(x)| = 0(t),$$

then
$\sum_{k=0}^{\infty} <2k y_k(x)$

belongs to $S'$.

THEOREM 4. Suppose $f(x) \in L^2$ and $f(x) = \sum_{k=0}^{\infty} c_k \varphi_k(x)$, where $\varphi_k(x)$ is an orthonormal system. Then
$\sum_{k=0}^{\infty} <2k c_k \varphi_k(x)$ is also Fourier coefficient of a function $F(x) \in L^2$ with respect to this orthonormal system.
8.3. We require the following lemmas for the proof of these theorems.

1) \text{LEMMMA 1.}

\[ \sum_{k=0}^{2^n-1} \chi_k(x) = \begin{cases} 2^n \text{ in } (0, 2^{-(n+1)}) \\ 2 \text{ in } (2^{-(n+1)}, 2^{-n}) \\ 0 \text{ in } (2^{-n}, 1). \end{cases} \]

2) \text{LEMMMA 2. A necessary and sufficient condition that a Walsh series should belong to the class } \mathcal{B} \text{ of essentially bounded periodic functions on } (0, 1), \text{ is the existence of a constant } M \text{ such that the } (C, 1) \text{ mean } \sigma_k \text{ of the series satisfies } |\sigma_k(x)| \leq M \text{ for all } k \text{ and all } x.

3) \text{LEMMMA 3.} \sum_{k=0}^{2^n-1} a_k \chi_k(x) \in \mathcal{S}

if and only if

\[ \int_0^1 |\sigma_n(x)| \, dx = O(1) \]

and

\[ \sum_{k=0}^{2^n-1} a_k \chi_k(x) \to 0, \]

uniformly in \([0, 1], \sigma_k \text{ is the } k\text{-th } (C, 1) \text{ mean of the Walsh series.}

1) Fine, N.J. (7) p. 386
2) Morgensthaler, C.W. (27) p. 487
LEMMA 4. If \( \sum_{k=1}^{\infty} b_k < \infty \), then \( \sum_{k=1}^{\infty} \left( \frac{1}{k} \right)^n < \infty \),

where
\[
B_k = \frac{1}{k} \sum_{v=0}^{k-1} b_v.
\]

5.4. Proof of Theorem 1. We have
\[
e_k = \int_0^1 f(x) \psi_k(x) \, dx
\]

and therefore by virtue of lemma 1
\[
s_n = \frac{1}{2^n} \sum_{k=0}^{2^n-1} c_k
\]

\[
= \frac{1}{2^n} \int_0^{2^n} \left[ \sum_{k=0}^{2^n-1} f(x) \psi_k(x) \right] \, dx
\]

\[
= \frac{1}{2^n} \left[ \int_0^{2^n} f(x) \, dx + \int_{-2^{n+1}}^{-2^n} f(x) \, dx \right]
\]

\[
= \frac{1}{2^n} \left[ \int_0^{2^n} f(x) \, dx + \int_{-2^{n+1}}^{-2^n} f(x) \, dx \right]
\]

\[
= \int_0^{2^n} f(x) \, dx.
\]

---

1) E. Elliot, E. B. (6)
Applying Hölder's inequality, we have
\[
1 \leq \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \left( \int_0^1 \, dx \right)^{1/q} \cdot \frac{1}{\frac{1}{p} + \frac{1}{q} - 1} = o\left( \frac{n}{q} \right),
\]
since \( f(x) \in L^p(0,1), \ 1 < p < 2 \), so that
\[
\sum_{n=1}^\infty \left( \frac{1}{2^n} \right)^p = \left( \sum_{n=1}^\infty \left( \frac{1}{2^n} \right)^q \right)^{1/p} \cdot \left( \sum_{n=1}^\infty \left( \frac{1}{2^n} \right)^q \right)^{1/q} < \infty.
\]
By virtue of Weiss's theorem we conclude that \( \varphi_n \) is the Walsh-Fourier coefficient of a function belonging to class \( L^q \).

This completes the proof of Theorem 1.

5.5. Proof of Theorem 2. Consider the series
\[
\sum_{n=0}^\infty \varphi_k \psi_k(x) \quad \text{with}
\]
\[
\varphi_k = \frac{1}{2} \sum_{n=0}^{2^n-1} \varphi_k \psi_k(x).
\]

Denoting the $m$-th $(C,1)$ mean and $m$-th partial sum of this series by $\phi_m(x,f)$ and $S_m(x,f)$ respectively, we have

$$|\phi_m(x,f)| = \frac{1}{m} \left[ \sum_{v=0}^{m-1} \frac{1}{v!} S_v(x,f) \right]$$

$$\leq \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \left| \psi_k(x) \right|$$

$$\leq \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \int_0^1 f(t) \left| \frac{d^r}{dt^r} \psi_r(t) \right| dt$$

$$\leq \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \int_0^1 \int_0^{\frac{1}{r+1}} f(t) \left| \frac{d^r}{dt^r} \psi_r(t) \right| dt$$

$$\leq \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \int_0^1 \int_0^{\frac{1}{r+1}} f(t) \left| \frac{d^r}{dt^r} \psi_r(t) \right| dt$$

$$\leq \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \int_0^1 \int_0^{\frac{1}{r+1}} f(t) \left| \frac{d^r}{dt^r} \psi_r(t) \right| dt$$

$$= \frac{1}{m} \sum_{v=0}^{m-1} \frac{1}{v!} \sum_{k=0}^{v-1} \frac{1}{g^k} \left\{ \frac{-(k+1)^{-k}}{2} \right\},$$

since $f(x) \in B$. 
By virtue of lemma 1 we conclude that

\[
\sigma_m(x, p) \leq \frac{M}{m} \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \frac{2^k}{\Sigma} \delta \int \frac{\delta}{\Sigma} \text{d}t
\]

\[
= \frac{M}{m} \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \frac{1}{\delta k} \frac{\delta}{\Sigma}
\]

\[
= \frac{M}{m} \sum_{v=0}^{m-1} \delta \left(\sum_{k=0}^{v-1} \frac{1}{\delta k}\right) = \frac{M}{m} \left[ O(n) \right]
\]

\[
= O(1),
\]

for all \( m \) and \( x \).

Applying lemma 2, we get the required result.

5.6. Proof of Theorem 3. Denoting by \( \sigma_m(x) \) the

\[
(\mathcal{C}, 1) \text{ mean of the series } \sum_{k=0}^{\infty} g_k x_k(x), \text{ we have}
\]

\[
\sigma_m(x) \leq \frac{1}{m} \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \frac{2^{-v}}{\Sigma} \int \frac{2^{-v}}{\Sigma} \sum_{r=0}^{2^{-v}} \gamma_r(t) \delta P(t) \text{d}x
\]

so that

\[
\int_{0}^{\infty} \sigma_m(x) \text{d}x \leq \frac{1}{m} \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \frac{2^{-v}}{\Sigma} \int \frac{2^{-v}}{\Sigma} \sum_{r=0}^{2^{-v}} \gamma_r(t) \delta P(t) \text{d}x
\]

\[
= \frac{1}{m} \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \frac{2^{-v}}{\Sigma} \int \frac{2^{-v}}{\Sigma} \sum_{r=0}^{2^{-v}} \gamma_r(t) \delta P(t) \text{d}x
\]

\[
= O(1),
\]
By virtue of lemma 1 and the hypothesis we have

\[ \int_0^{\infty} \rho_0(x) \, dx \leq \frac{1}{n} \sum_{v=0}^{n-1} \sum_{k=0}^{v-1} \left( \frac{1}{2^k} \right) \]

\[ = \frac{1}{n} \sum_{v=0}^{n-1} \sum_{k=0}^{v-1} \left( \frac{1}{2^k} \right) \]

\[ = \frac{1}{n} \sum_{v=0}^{n-1} \sum_{k=0}^{v-1} \left( \frac{1}{2^k} \right) \]

\[ = \frac{1}{2} \left( \frac{1}{2^k} \right) \]

Also,

\[ \sum_{k=0}^{\infty} 2^k \rho_k(x) ! = \sum_{k=0}^{\infty} \frac{1}{2^k} \left( \sum_{r=0}^{2^k} \psi(t) \right) \rho_k(x) ! \]

\[ \leq \sum_{k=0}^{\infty} 2^k \left( \sum_{r=0}^{\infty} \psi(t) \right) \rho_k(x) ! \]

\[ = \frac{1}{2} \left( \frac{1}{2^k} \right) \]

\[ = \frac{1}{2} \left( \frac{1}{2^k} \right) \]
\[ = \mathcal{O}(1), \quad n \to \infty \]

uniformly in \( z \).

Hence by virtue of lemma 3 \( \tilde{\zeta}^N \to \gamma_k(x) \) belongs to \( S' \).

This completes the proof of theorem 3.

3.7. Proof of Theorem 4. Since \( f(x) \in L^2 \).

By Bessel's inequality the series

\[ \sum_{k=1}^{\infty} c_k^2 < \infty. \]

Applying lemma 2 we obtain that

\[ \sum_{k=1}^{\infty} A_k^2 < \infty. \]

Now applying Riess - Fisheber theorem we conclude that \( \alpha_n \) is Fourier coefficient with respect to orthonormal system \( A_k(x) \) of a function \( f(x) \in L^2 \).
CHAPTER VI

ON THE WALSH-FOURIER SERIES OF CERTAIN CLASSES

6.1. Let Rademacher function be defined by
\[ \psi_0(x) = 1 \quad (0 \leq x < \frac{1}{2}), \quad \varphi_0(x) = -1 \quad (\frac{1}{2} \leq x < 1) \]
\[ \psi_n(x+1) = \psi_n(x), \quad \varphi_n(x) = \varphi_n(2^n x_n) \quad (n = 1, 2, 3, \ldots) \]

The Walsh function \( \psi(x) \) is defined as follows:

(a) \( \psi_0(x) = 1 \)

(b) If \( n \) has the unique dyadic expansion \( n = 2^j x_j; x_j = 0, 1 \) and \( x_j = 0 \), for \( j > m \), then \( \psi_n(x) = \psi_{m_1}(x) \psi_{m_2}(x) \ldots \)

\( \psi_{m}(x), \) where \( m_1, m_2, m_3, \ldots m_r \) correspond to the coefficients \( x_{m_j} = 1 \). Every function \( f(x) \) which is of period 1 and Lebesgue integrable on \([0, 1]\) may be expanded in a Walsh-Fourier series

\[ f(x) \sim \sum_{k=0}^{\infty} a_k \psi_k(x), \text{where } a_k = \int_0^1 f(x) \varphi_k(x) dx, \quad k = 0, 1, 2, \ldots \]

We denote by \( \psi_n^\beta(x) \) the n-th \((C, \beta)\) mean of the Walsh-Fourier series. We write

\[ s_n(x) = \sum_{k=0}^{n} a_k \psi_k(x), \quad \psi_n(x) = \psi_n^\beta(x) \]
The dyadic group \( G \) is the set of all sequences 
\[
\vec{x} = \{x_n\}, \quad x_n = 0, 1, \quad n = 1, 2, 3, \ldots
\]
the operation of \( G \) being addition modulo 2 in each coordinate and it 
is denoted by \( + \). Let \( \vec{x} \) be an element of \( G \). We 
define the function 
\[
\lambda(\vec{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n
\]
The function \( \lambda \) maps \( G \) onto the closed interval \([0, 1]\). 
Let \( \mu \) be the inverse mapping of \( \lambda \).

For each fixed \( x \) and for almost all \( t \) the equation 
\[
(6.1.1) \quad \gamma_n(x + t) = \gamma_n(x) \gamma_n(t).
\]

For each fixed \( x \) and all \( t \) we have
\[
(6.1.2) \quad \int_0^1 f(t) \gamma_n(x + t) dt = \int_0^1 f(x + t) \gamma_n(t) dt.
\]

A function \( f(x) \) is said to belong to class 
\( \text{Lip} (\kappa, q) \) if 
\[
\left[ \frac{1}{b-a} \int_a^b |f(x+h)-f(x)|^q \right]^{1/q} = O(h^\kappa), \quad h \to 0
\]
uniformly in \( x \), \( 0 < \kappa < 1 \) and \( q \geq 1 \).
If \( f(x) \) satisfies the condition
\[
|f(x + h) - f(x)| < ch, \quad 0 \leq x < 1, \quad 0 \leq h < 1, \quad c < \infty,
\]
except when \( (\mu(x) + \mu(h)) \) ends in a sequence of \( 1 \)'s,
then \( f(x) \) is said to belong to Lipschitz \( \langle \nu \rangle \) on \([0,1] \).

It may be observed that \( f \in \text{Lip} \langle [0,1] \rangle \Rightarrow f \in \text{Lip} \langle \nu \rangle \).

It is said to belong to \( \text{Lip}_p \langle \nu \rangle \) on \( \mathcal{C} \) if
\[
\|f(x) - f(x + h)\|_p = O \left( \left\langle \nu \right\rangle(h) \right),
\]
where
\[
\|g\|_p = \left( \int_0^1 |g(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty
\]
and
\[
\|g\|_\infty = \sup_{x \in \mathcal{C}} |g(x)|.
\]

A Walsh polynomial of degree \( n \) is defined as an expression of the form
\[
\sum_{\nu=0}^{n-1} c_{\nu} \mathcal{V}_{\nu}(x), \quad \text{where} \quad c_{n-1} \neq 0.
\]
The set of all polynomials with degree not greater than \( n \) is denoted by \( \mathcal{B}_n \).

We write
\[
\mathcal{B}_n(p)(f) = \inf \left[ \|f(x) - p(x)\|_p, \quad p(x) \in \mathcal{B}_n \right]
\]

2) Morgenthaler, C.W. (27); Watarai, C. (37).
We denote by $R_n(n)$ the sequence $\frac{1}{n+p-1} \sum_{k=n}^{p} x_k$.

A bounded sequence $x_n$ is said to be almost convergent to a limit $s$ if $\lim_{p \to \infty} R_n(p) = s$ uniformly in $n$.

It is known that every almost convergent sequence is summable $(c, p)$, \( p > 0 \).

A sequence $\{\lambda_n\}$ is said to be convex if

$$2 \lambda_n - \lambda_n > 0,$$

where

$$\Delta \lambda_n = \lambda_n - \lambda_{n+1} \quad \text{and} \quad 2 \lambda_n - \lambda_n = \Delta(\lambda_n)$$

The sequence $\{\lambda_n\}$ is said to be quasi-convex if

$$\sum_{n=0}^{x} (n+1)! \Delta \lambda_n! < \infty.$$

\[6.2\] This chapter is devoted to the study of Walsh-Fourier series of certain classes. In this connection a number of theorems have been proved. Theorem 1 gives a new characterization for the class $\text{Lip}_p^{(p)}(w)$. In theorem 2 a sufficient condition has been obtained on the coefficient of Walsh-Fourier series in order that it is

---

1) Lorentz, G.G. (24).
generating function \( f(x) \in \text{Lip} \ll (w) \) while Theorem 3 gives a necessary and sufficient condition in order that a Walsh-series may be the Fourier series of a function belonging to \( \text{Lip}^\infty (w) \) on \( G \).

In Theorem 4 we have obtained a necessary and sufficient condition in order that the sequence \( \left\{ s_n(x) \right\} \) be almost convergent. Theorem 5 deals with the problem of approximation of \( f(x) \) by the \( R_n(p) \) mean of the \( \left\{ s_n(x) \right\} \). In Theorem 6 we have investigated the condition on the coefficients of Walsh-series in order that this series may become a Fourier series of an integrable function. Theorem 7 gives a necessary and sufficient condition for the convergence of the series \( \sum_{n=1}^{\infty} c_n \psi_n(x) \) in the metric space \( L \), while in Theorem 8 we have investigated certain condition under which

\[
\lim_{n \to \infty} \int_0^L |f(x) - s_n(x)|^p \, dx = 0, \quad \text{for } 0 < p < 1.
\]

6.3. In what follows we shall prove the following theorem:

**Theorem 1.** A function \( f(x) \in \text{Lip} \ll (w), \, 0 < \ll < 1, \, 1 < p < \infty \)
\[
\| \sigma^\beta_n(x) - \sigma^\beta_m(x) \|_p = O(n^{-\xi}), \quad \xi > 0, \quad \beta > \alpha,
\]
where \(\sigma^\beta_n(x)\) is the \(n\)-th \((\alpha, \beta)\) mean of Walsh-Fourier series of \(f(x)\).

**Theorem 2.** If
\[
\lim_{k \to \infty} |a_k| = O\left(\frac{1}{n^\alpha}\right), \quad 0 < \alpha < 1,
\]
then \(f(x) \in \text{Lip} \alpha(\omega)\), provided \(\mathcal{M}(x) + \mathcal{M}(\frac{1}{2^{n+1}})\)
does not end in the sequence of 1's, 0 \(\leq x < 1\).

**Theorem 3.** A necessary and sufficient condition that a Walsh-series be the Fourier series of a function
\(f(x) \in \text{Lip} \alpha(\omega)\) on \(G\), \(0 < \alpha < 1\), is that
\[
\sigma^\alpha_n(x) - \sigma^\alpha_m(x) = O(n^{-\alpha})
\]
uniformly in \(G\) for all \(n > m\), \(\sigma^\alpha_n(x)\) is the \(n\)-th \((\alpha, \beta)\) mean of Walsh-series.

**Theorem 4.** The sequence \(\{s_{\sigma^\alpha_n(x)}\}\) is almost convergent to limit \(f(x)\) iff
\[
\int_0^\infty \int_{-2^r}^{2^r} (f(x + t) - f(x)) dt = o(p), \quad p \to \infty
\]
uniformly in \(n\).

**Theorem 5.** If \(f(x) \in \text{Lip} \alpha, \quad 0 < \alpha < 1\) then
where $s^{*}$ is the $R_n(p)$ mean of the sequence $\{s^{*}_n(x)\}_p$.

**Theorem 6.** If $c_n \to 0$ and $E (\triangle c_n) \log n < b$, then the series $\sum_{k=0}^{\infty} c_k \psi_k(x)$ is a Walsh-Fourier series.

**Theorem 7.** If $c_n \to 0$ and $c_k$ is convex or even quasi-convex, then for the convergence of the series $\sum_{n=0}^{\infty} c_n \psi_n(x)$ in the metric space $L$, it is necessary and sufficient that

$$\lim_{n \to \infty} c_n \log n = c.$$  

**Theorem 8.** If $c_k \to 0$ and $\sum_{k=0}^{\infty} |\triangle c_k| < b$, then we have

$$\lim_{n \to \infty} \int_{-\infty}^{1} |f(x) - s_n(x)|^p \, dx = 0,$$

where $f(x) = \sum_{k=0}^{\infty} c_k \psi_k(x)$ and $s_n(x)$ is its $n$-th partial sum.

It may be remarked that Theorem 3 is an analogue of a theorem of Duplessis for trigonometric series.

---

1) Duplessis, N. (5).
while in Theorem 6 a result of Yano has been proved under different set of conditions. Theorem 7 and 8 are analogous of known results for trigonometric series 2).

6.4 In the proof of these theorems we require the following lemmas.

3) **Lemma 1** For \( \beta > 0 \), the Walsh-Fourier series of an integrable function \( f(x) \) over \( G \) is \((C, \beta)\) summable to \( f(x) \) almost everywhere.

Fine has proved the above result for \( f \in L(\phi, 1) \).

The general case for \( G \) follows from a theorem of Morgenstaller (p. 475).

4) **Lemma 2**

\[ \text{If } f(x) \in \text{Lip } \phi(w), 1 < p < \infty, \]

\[ \alpha < \phi < 1, \text{ then for any } \beta > 0, \quad \left\| \frac{1}{m} \sum_{k=0}^{m} f(x) \right\|_p = O(m^{-\alpha}), \]

where \( \frac{1}{m} \sum_{k=0}^{m} f(x) \) denotes the \( m \)-th \((C, \beta)\) means of the Walsh-Fourier series of \( f(x) \).

---

1) Yano, S. (39).
3) Fine, N. J. (37).
LEMMA 3. 1) If $f(x)$ has the Fourier coefficients $a_n$ and if $h$ is any fixed number, then the Fourier coefficients of $f(x + h)$ are $\sum_{n=0}^{\infty} a_n \psi_n(h)$.

LEMMA 4.

$$D_n(t) = \sum_{k=0}^{2^n-1} \psi_k(t) = 2^n \left( \begin{array}{cc} 2^n & -1 \end{array} \right) \left( \begin{array}{c} t \in (0, 2^{-n}) \\
0 \in (2^{-n}, 1) \end{array} \right)$$

LEMMA 5. If $c_n \rightarrow c_{n+1} \rightarrow 0$ and the series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

converges, except for $x = 0$, to an integrable function $f(x)$, then the series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

is the Walsh-Fourier series of $f(x)$.

LEMMA 6. (a) If $D_n(x) = \sum_{k=0}^{2^n-1} \psi_k(x)$, then

$$|D_n(x)| < \frac{2}{x} \quad \text{for} \quad 0 < x < 1.$$
(b) If \( L_k(x) = \int_0^1 D_k(x) \, dx \),

then \( L_k(x) = O(\log k) \)

(c) If \( K_n(x) = \sum_{k=1}^n \frac{1}{n+1} D_k(x) \),

then \( |K_n(x)| \leq \frac{2}{x} \) for all \( n \).

**Lemma 7.** \( \int_0^1 K_n(x) \, dx \leq 2 \),

where \( K_n(x) = \sum_{k=1}^n \frac{1}{n} D_k(x) \).

**Lemma 8.** If \( \{ c_k \} \) is a bounded and convex sequence, then

\[
\sum_{k=1}^\infty (k+1) \Delta c_k < \infty \quad \text{and} \quad n \Delta c_n \to 0,
\]

and if \( c_k \to 0 \) and \( c_k \) is quasi-convex, then

\( n! \Delta c_n ! = o(1), \quad n \to \infty \).

---

1) Yano, S. (39).

LEMMA 9. The following statements are equivalent

(i) \( f(x) \in \text{Lip}^p \alpha(v) \),

(ii) \( \epsilon^{(p)}_n(x) = \inf \left\{ \| f - p_n \|_p : p_n \in B_n \right\} = O(n^{-\alpha}) \).

LEMMA 10. If \( f(x) \in \text{Lip}^p_{\alpha}(v) \) \( p = 1 \) or \( p = \infty \),

\[ 0 < \alpha < 1, \text{ then for } \beta > \alpha \]

\[ \| \phi_\theta(x) - f(x) \|_p = O(n^{-\beta}) \], where \( p = 1 \) or \( p = \infty \)

Remark. Proceeding on the lines of Theorem 2 and Theorem 3 of Yano, it is easy to show that they remain valid for \( f(x) \in \text{Lip}^{\infty}_{\alpha}(v) \) on \([0,1]\). Now Lemma 10 for group 0 follows by Theorem 1(2) of Morgenthaler and the fact that \( f(x) \in \text{Lip}^{\infty}_{\alpha}(v) \) on 0, then

\[ f(x) = \overline{f}(\mu(x)) \] belongs to \( \text{Lip}^{\infty}_{\alpha}(v) \) on \([0,1]\).

2) Yano, S. (40).
3) Yano, S. (40).
6.5. **Proof of the Theorem 1. Sufficiency:** We have

$$\|a_{n}(x) - a_{m}(x)\|_{p} = O(n^{-\alpha}), \ n \rightarrow \infty.$$ 

By virtue of Lemma 1, $a_{n}(x) \rightarrow f(x), \ m \rightarrow \infty$ almost everywhere, so that

$$\lim_{m \rightarrow \infty} \|a_{n}(x) - a_{m}(x)\|_{p} = \|f(x) - a_{n}(x)\|_{p}.$$ 

Now

$$\begin{eqnarray*}
\|f(x)\|_{p} &=& \inf_{P \in B_{n}} \left\{ \int \|f(y) - P(y)\|_{p} \right\}, \\
&\leq& \|f(x) - a_{n}(x)\|_{p} \\
&=& O(n^{-\alpha}).
\end{eqnarray*}$$

Hence by virtue of Lemma 2 we see that $f(x) \in \text{Lip}_{\alpha}(v)$. 

**Necessity:** Now let $f(x) \in \text{Lip}_{\alpha}(v), \ 1 < p < \infty, \ 0 < \alpha < 1$

By virtue of Lemma 2 we have

$$\|f(x) - a_{n}(x)\|_{p} = O(n^{-\alpha}).$$

Hence

$$\|a_{n}(x) - a_{m}(x)\|_{p} \leq \|a_{n}(x) - f(x)\|_{p} + \|f(x) - a_{n}(x)\|_{p}.$$
\[ = \mathcal{O}(a^{-k}), \quad a > n. \]

This completes the proof of Theorem 1.

6.6. Proof of Theorem 2. It is well known \(^1\) that

\[ f_k\left(\frac{1}{g+1}\right) = 1 \text{ if } k < 2^n. \]

From this it follows that

\[ (1 - f_k(h)) = 1 \text{ iff } k < 2^n, \quad h = \frac{1}{2+1} \]

so that by Lemma 3 we have

\[ \frac{n}{2} - 1 \leq \sum_{k=0}^{n-1} \lambda_k \left(1 - f_k(h)\right) + \sum_{k=2^n}^{\infty} \lambda_k \left(1 - f_k(h)\right) \]

\[ = \sum_{k=2^n}^{\infty} \lambda_k \left(1 - f_k(h)\right) \]

\[ \leq \sum_{k=2^n}^{\infty} \lambda_k = \mathcal{O}\left(\frac{1}{2^n}\right) \]

\[ = \mathcal{O}(h^n). \]

This completes the proof of Theorem 2.

---

6.7. Proof of Theorem 3.

**Sufficiency**: Since \( c_n(x) - \delta_m(x) = \bigcirc(n^{-\alpha}) \), we have \( c_n(x) - \delta_m(x) \to 0 \) uniformly in \( a \) as \( m,n \to \infty \) which implies that there exists a function \( f(x) \) such that \( \delta_m(x) \to f(x) \) uniformly as \( a \to \infty \). From this it follows that

\[
\delta_n(x) - f(x) = \bigcirc(n^{-\alpha}).
\]

Also,

\[
E_n(f) \leq \sup |f(x) - \delta_n(x)| = \bigcirc(n^{-\alpha})
\]

By Lemma 9 it follows that \( f(x) \in \text{Lip}_{-\alpha}^{\infty}(w) \).

Now we will show that \( \sum_{v=0}^{\infty} c(x) \varphi(x) \) is Walsh-Fourier series of \( f(x) \). We have

\[
\delta_n(x) = \sum_{v=0}^{n} \left(1 - \frac{v}{n+1}\right)c_v \varphi_v(x)
\]

so that

\[
\int_0^1 \delta_n(x) f(x)dx = \sum_{v=0}^{n} \left(1 - \frac{v}{n+1}\right)c_v \int_0^1 \varphi_v(x) f(x)dx.
\]
Since \( \{ \psi_n(x) \} \) is orthonormal system, hence for \( k \leq n \)

\[
\int_0^1 \psi_n(x) \psi_k(x) \, dx = c_k \left( 1 - \frac{k}{n} \right)
\]

Taking limits as \( n \to \infty \)

\[
\int_0^1 f(x) \psi_k(x) \, dx = c_k .
\]

**Necessity:** let \( f(x) \in \text{Lip}_\infty^a(w) \), then by virtue of Lemma 10, we have

\[
\| \omega_n - f \|_\infty = O(n^{-\gamma}).
\]

Hence

\[
\| \omega_n(x) - \omega_n(x) \| \leq \| \omega_n(x) - f(x) \| +
\]

\[
+ \| f(x) - \omega_n(x) \| = O(n^{-\gamma}), \quad n > n.
\]

This completes the proof of the Theorem 3.

6.6. Proof of Theorem 4. Let

\[
s_p = \frac{\sum_{k=0}^{p} s_2^{k+1} + s_2^{n+2} + \ldots + s_2^{n+p-1}}{p} .
\]

\[
= \frac{1}{p} \left[ \sum_{k=0}^{p} \left( \sum_{k=0}^{n} \int_{0}^{t} f(t) \psi_k(t) \, dt \right) \right] .
\]
\[
\begin{align*}
&= \frac{1}{p} \left[ \sum_{k=0}^{n} \left( \frac{n}{2} \right) \sum_{k=0}^{n+p-1} \mathcal{D}_2 \mathcal{R}(x + t) f(t) \right] \\
&= \frac{1}{p} \left[ \sum_{k=0}^{n} \left( \frac{n}{2} \right) \sum_{k=0}^{n+p-1} \int_0^t \mathcal{D}_2 \mathcal{R}(x + t) f(t) \, dt \right] \\
&= \frac{1}{p} \int_0^t \sum_{r=n}^{n+p-1} \mathcal{D}_2 \mathcal{R}(x + t) f(t) \, dt \\
&= \frac{1}{p} \int_0^t \int_0^t \mathcal{D}_2 \mathcal{R}(x + t) f(t) \, dt \\
&= \frac{1}{p} \int_0^t \mathcal{D}_2 \mathcal{R}(x + t) f(t) \, dt \\
&= \frac{1}{p} \int_0^t \mathcal{D}_2 \mathcal{R}(x + t) f(t) \, dt \\
\end{align*}
\]

If we take \( f(t) = 1 \), we have

\[
1 = \frac{1}{p} \sum_{r=n}^{n+p-1} \int_0^t g^r \, dt \\
\]

\[
f(x) = \frac{1}{p} \sum_{r=n}^{n+p-1} \int_0^t f(x + t) \, dt.
\]

Thus we have

\[
(6.8.1) \quad s^p - f(x) = \frac{1}{p} \sum_{r=n}^{n+p-1} \int_0^t (f(x + t) - f(x)) \, dt
\]
By definition of almost convergence the result follows:

6.3. Proof of Theorem 5. From (6.5.1) we have

\[ s^* - f(x) = \frac{1}{p} \sum_{r=n}^{n+p-1} \int_0^r (f(x + t) - f(x)) dt \]

\[ = \frac{1}{p} \sum_{r=n}^{n+p-1} \int_0^r O(t) dt \]

\[ = O \left( \frac{1}{p} \sum_{r=n}^{n+p-1} r \right) \left( \frac{1}{g(r+1)} \right) \]

\[ = O \left( \frac{1}{p} \sum_{r=n}^{n+p-1} \frac{1}{g(r+1)} \right) \]

\[ = O(p^{-1}). \]

6.10. Proof of Theorem 6. Applying Abel's transformation, we have

\[ s_n(x) = \sum_{k=0}^{n-1} \Delta^k D_{n+1}(x) + c_n D_{n+1}(x). \]

For \( x \neq 0 \), by lemma 6 and the fact that \( c_n \to 0 \),
\( c_n D_{n+1}(x) \to 0 \), as \( n \to \infty \). Taking limit we have
\[ f(x) = \sum_{k=0}^{\infty} D_{k+1}(x) \Delta c_k, \]

Since

\[ \sum_{k=0}^{\infty} \Delta c_k \int_0^1 D_k(x) \, dx = O\left( \sum_{k=0}^{\infty} (\Delta c_k) \log k \right) < \infty, \]

it follows that \( \sum_{k=0}^{\infty} c_k \psi_k(x) \) converges, except at \( x = \infty \), to an integrable function \( f(x) \). By virtue of lemma 6, the result follows.

6.11. Proof of Theorem 7. Let \( \{c_n\} \) be convex. Applying Abel's transformation twice we have

\[ s_n(x) = \sum_{k=0}^{n-1} \Delta c_k D_{k+1}(x) + c_n D_{n+1}(x) \]

\[ = \sum_{k=0}^{n-2} (k + 1) \Delta c_k K_k(x) + \]

\[ + n \Delta c_{n-1} K_{n-1}(x) + c_n D_{n+1}(x). \]

By virtue of Lemma 6 and 8, we have for \( x \neq 0 \), on taking limit as \( n \to \infty \),

\[ f(x) = \sum_{k=0}^{\infty} (k + 1) \Delta c_k K_k(x), \]
so that
\[ f(x) - s_n(x) = \sum_{k=0}^{\infty} \triangle c_k K_k(x) - \]
\[ - n K_{n-1}(x) \triangle c_{n-1} - D_{n+1}(x)c_n. \]

By lemmas 7 and 8,
\[ \sum_{k=0}^{\infty} \triangle c_k \int_0^1 K_k(x) \leq 2 \sum_{k=0}^{\infty} \triangle c_k \to 0, \]
as \( n \to \infty \), and
\[ \int_0^1 |f(x) - s_n(x)| dx = \int_0^1 |D_{n+1}(x)| dx c_n = o(1). \]

The result follows by the virtue of the fact that
\[ L_n = O(\log n) \]
and
\[ \frac{1}{n} \sum_{k=1}^{n} l_k = \frac{\log n}{4\log 2} + o(1). \]

---

\[ s_n(x) = \sum_{k=0}^{n-1} \Delta c_k D_{k+1}(x) + c_n D_{n+1}(x), \]

so that

\[ f(x) - s_n(x) = \sum_{k=n}^{\infty} \Delta c_k D_k(x) - c_n D_{n+1}(x) \]

for \( x \not\equiv a \, (\text{mod } 1) \).

By Lemma 6 for \( x \not\equiv a \, (\text{mod } 1) \), we have

\[ \int_0^1 |f(x) - s_n(x)|^p \, dx \leq \frac{2^p}{p} \left( \sum_{k=n}^{\infty} |\Delta c_k| + |c_n| \right) \]

so that

\[ \int_0^1 |f(x) - s_n(x)|^p \, dx \leq 2 \left( |c_n| + \sum_{k=n}^{\infty} |\Delta c_k| \right) \int_0^1 x^p \, dx \to 0, \]

as \( n \to \infty \).

This completes the proof of the theorem.
CHAPTER VII
ON THE SUMMABILITY OF A SEQUENCE OF WALSH FUNCTIONS

7.1. The Rademacher function are defined by

\[ g_0(x) = 1 \ (0 \leq x < \frac{1}{2}), \quad g_0(x) = -1 \ (\frac{1}{2} \leq x < 1), \]

\[ g_0(x + 1) = g_0(x), \quad g_n(x) = g_0(2^n x), \quad (n = 1, 2, 3, \ldots). \]

The Walsh functions are given by

\[ y_0(x) = 1, \quad y_n(x) = g^{n_1}_1(x) \cdot g^{n_2}_2(x) \cdot \ldots \cdot g^{n_r}_r(x) \]

for \( n = 2^{n_1} + 2^{n_2} + 2^{n_3} + \ldots + 2^{n_r} \), where the integers \( n_i \)
are uniquely determined by \( n_{i+1} < n_i \).

Let \( f(x) \) be an integrable function in the sense of Lebesgue in \([0, 1]\) and be periodic with period 1. Let

Walsh-Fourier series of \( f(x) \) be \( \sum_{n=1}^{\infty} a_n y_n(x) \), where

\[ a_n = \frac{1}{1} \int_0 f(x) \ y_n(x) \ dx. \]

We shall now enumerate important properties and
results concerning Walsh-Functions which have been
obtained by Fine and which have played significant
role in the theory of Walsh-Fourier series.

1) Fine, N.J. (7).
The dyadic group \( G \) may be defined as the countable direct product of the groups with elements 0 and 1, in which the group operation is addition modulo 2. Thus the dyadic group \( G \) is the set of all \( 0,1 \) sequences in which the group operation, which we shall denote by \( + \), is addition modulo 2 for each element.

Let \( \overline{x} \) be an element of \( G \), \( \overline{x} = \{x_1, x_2, \ldots \} \), \( x_n = 0,1 \). We define the function

\[
(7.1.1) \quad \lambda(\overline{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n.
\]

The function \( \lambda \), which maps \( G \) on to the closed interval \([0,1]\), does have a single-valued inverse on the dyadic rationals; we shall agree to take the finite expansion in that case. Thus for all real \( x \), if we write the inverse as \( \mu^- \),

\[
(7.1.2) \quad \lambda(\mu(x)) = x - [x]
\]

If \( \overline{x} = \{x_n\} \) and \( \overline{y} = \{y_n\} \) are elements of \( G \), we have

\[
(7.1.3) \quad \overline{x} \div \overline{y} = \{x_n - y_n\}.
\]
We shall abbreviate \( \lambda(\mu(x) + \nu(y)) \) as \( x + y \)
for any real \( x \) and \( y \). Then, if \( x = \sum_{n=1}^{\infty} x_n \),
y = \( \sum_{n=1}^{\infty} y_n \), \( x_n \) and \( y_n = 0,1 \), we have by (7.1.2) and (7.1.3)

\[
(7.1.4) \quad x + y = \sum_{n=1}^{\infty} |x_n - y_n|.
\]

for any real number \( x \) and \( h \), we have

\[
(7.1.5) \quad |(x + h) - (x - [x])| \leq h - [h]
\]

In particular if \( 0 \leq x < 1, 0 \leq h < 1 \), then we have

\[
(7.1.6) \quad |(x + h) - x| \leq h.
\]

For each fixed \( x \) and for almost all \( t \), the equation

\[
(7.1.7) \quad \psi_n(x + t) = \psi_n(x) \psi_n(t)
\]

Also for each fixed \( x \) and all \( t \)

\[
(7.1.8) \quad \int_{0}^{1} f(x + t) dt = \int_{0}^{1} f(t) dt
\]
\[(7.1.9) \quad \int_0^1 f(t) \psi_n(x + t) \, dt = \frac{1}{2} \int_0^1 f(x + t) \psi_n(t) \, dt.\]

Let

\[J_k(y) = \int_0^1 \psi_k(t) \, dt, \quad k = 0, 1, 2, \ldots,\]

\[J_k^*(y) = k J_k(y).\]

For \(k \geq 1\), we write \(k = 2 + k'\), where \(0 \leq k' < 2\), \(n = 0, 1, 2, \ldots\) we have also

\[(7.1.10) \quad J_k(y) = 2^{-(n+2)} \left\{ \psi_k'(y) - \sum_{r=1}^{\infty} 2^{-(r+k)} \psi_{2^r+k}(y) \right\}\]

It is easy to see that

\[(7.1.11) \quad 2^{n+2} J_k(y) = 0, \text{ for } y = 0, 1, \ldots\]

and

\[(7.1.12) \quad |J_k^*(y)| \leq M \text{ for all } y \text{ and } k.\]

Let \(R_k(x)\) denote the sequence \(\{k a_k k(x)\}\),

where \(a_k\) is Walsh-Fourier coefficient of a functions of bounded variation.

Let \(A = (a_{m,k})\) be an infinite matrix of real or complex numbers and \(\{a_k\}\) be any sequence of real numbers. With every sequence \(\{a_k\}\) we associate a sequence \(\{a^*_k\}\) given by
\[(7.1.13) \quad a = \sum_{k=0}^{\infty} a_n k^k, \]

provided the series on the right converges for all \(a\). The sequence \(\{a_n\}\) is called \(a\)-transform of \(\{a_k\}\).

If \(a \to s\) as \(a \to \infty\),

we say that the sequence \(\{s_k\}\) is summable \(A\) to \(s\).

The matrix \(A\) is called regular if it satisfies the following conditions:

(i) \(\lim_{m \to \infty} a_m, k = 0\) for \(k = 0, 1, 2, 3, \ldots\)

(ii) \(\sup_{m,k=0}^{\infty} a_{m,k} \leq M\).

(iii) \(\lim_{m \to \infty} \sum_{k=0}^{\infty} a_{m,k} = 1\).

The matrix \(A\) is called triangular, if \(a_{m,k} = 0\) for \(k > m\).

We say that bounded sequence \(\{a_k\}\) is almost convergent to the sum if

\[(7.1.14) \quad \lim_{p \to \infty} \sum_{k=0}^{p-1} \sum_{n+k} = \ell \]

uniformly in \(a\).

\[1)\) Lorentz, G. G. (24).\]
Every almost convergent sequence is summable \((C, <)\),
\(< > o 1)\) and the limits are equal.

A sequence \(\{a_n\}\) is said to be almost \(A\)-summable to \(s\) if the \(A\)-transform of \(\{a_n\}\) is almost convergent to \(s\) and the matrix \(A\) is said to be almost regular if \(a_k = s\) implies that \(\{a_n\}\) is almost convergent to \(s\).

The necessary and sufficient conditions for the matrix \(A\) to be almost regular\(^3)\) are:

\[
\begin{align*}
(a) & \sup_{k=0}^{\infty} \left| a_{n,k} \right| < M, \quad m = \pm 1, \pm 2, \pm 3 \ldots \ldots \\
(b) & \lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{p+n} a_{j,k} = 0 \text{ uniformly in } n, k = 0, 1, 2, \ldots \\
(c) & \lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{p+n} a_{j,k} = 1, \text{ uniformly in } n.
\end{align*}
\]

where \(M\) is a positive constant.

1) Lorentz, G.C. (24).
2) King, J.P. (22).
3) King, J.P. (22); In the theorem of the author condition (a) is the following:

\[
\sup_{k=0}^{\infty} \frac{1}{p} \sum_{j=n}^{p+n} |a_{j,k}| < M, \quad n = 1, 2, \ldots
\]

and \(p\) is a positive integer. However, he has mentioned that this condition can be replaced by the condition (a) mentioned above.
A sequence \( \{a_k\} \) is said to be FA-summable \(^1\) to the limit \( s \) if
\[
\sum_{q=0}^{\infty} a_{n,q} s_{q+1}^k.
\]
tends to \( s \) as \( n \to \infty \), uniformly in \( k \).

2) It is known \(^2\) that every FA-summable sequence is almost convergent if \( A \) is a regular matrix.

A sequence \( \{a_k\} \) will be said to be \( AB \)-summable to the limit \( s \) if its \( A \)-transform is \( FB \)-summable to the limit \( s \) where \( B = (b_{n,k}) \) is a finite matrix.

It is easy to see that every \( AB \)-summable sequence is almost \( s \)-summable provided the second matrix \( B \) is regular.

3) In 1947, Fine \(^3\) proved the following theorems concerning Walsh-Fourier coefficients of functions of bounded variation and absolutely continuous function.

**THEOREM A.** If \( f(x) \) is of bounded variation, and \( V \) is its total variation over \((0,1)\), then
\[
\left| a_k \right| \leq \frac{V}{k} \quad \text{for} \quad k > 0.
\]

---

1) Lorentz, G. G. (24).
2) Lorentz, G. G. (24).
3) Fine, N. J. (7).
THEOREM B. The only absolutely continuous functions whose Fourier coefficients satisfy \( a_k = o\left( \frac{1}{k} \right) \) are the constants.

This result shows a marked difference in the behaviour of Walsh-Fourier series and ordinary Fourier series of absolutely continuous functions. Morgenthaler\(^1\), later on proved a theorem which shows that "on the average" the coefficients behave as they do in the classical system.

His result is as follows:

THEOREM C. Let \( f(x) \) be real valued, periodic, and of mean value zero on \([0,1]\). If \( F(x) = \int_0^\infty f(t) dt \) and

\[
F(x) \sim \sum_{k=0}^\infty b_k \gamma_k(x),
\]

then the arithmetic means of the sequence \( \frac{k}{b_k} \) tend to zero.

In the present chapter we shall obtain necessary and sufficient conditions in order that the sequence \( \{B_k(x)\} \) be \((A)\), almost \(A\), \(FA\) and \(AB\)-summable.

We shall also deduce certain interesting corollaries concerning Walsh-Fourier coefficients of functions of bounded variation.

\(^{1}\) Morgenthaler, G.W. (27).
7.3. In what follows we shall prove the following theorems:

**Theorem 1.** If \( A \) is regular, then for every \( f \in BV[0,1] \) and for every \( x \in [0,1] \)

\[
\lim_{m \to \infty} \sum_{k=0}^{m} a_{m,k} R_k(x) = 0
\]

iff

\[
(7.3.1) \quad \lim_{m \to \infty} \sum_{k=0}^{m} a_{m,k} J_0(t) = 0
\]

in every \( 0 < s < t \leq 1 \), where \( A \) is a triangular matrix and \( s \) is small.

**Theorem 2.** If \( A \) is almost regular, then for every \( f(x) \in BV[0,1] \) and for every \( x \in [0,1] \)

\[
\lim_{p \to \infty} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{m,r,k} B_k(x) = 0
\]

uniformly in \( m \), iff

\[
(7.3.2) \quad \lim_{p \to \infty} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{m,r,k} J_0(t) = 0,
\]

uniformly in \( m \) for every \( 0 < s \leq t \leq 1 \), \( s \) is small.

**Theorem 3.** If \( A = (a_{m,k}) \) is regular, then for every \( f(x) \in BV[0,1] \) and for every \( x \in [0,1] \), the
sequence $\{b_k(x)\}$ is FA-summable to the limit zero if and only if

$$\lim_{n \to \infty} \sum_{v=k}^{\infty} a_{n,v-k} f_v^*(t) = 0,$$

uniformly in $k$ in the interval $0 < s \leq t \leq 1$ where $s$ is small.

**Theorem 4.** If $(a_{m,k})$ and $(b_{n,k})$ be two infinite matrices satisfying the condition:

$$(7.3.4) \quad \sup_{m} \sum_{v=k}^{\infty} |b_{m,v-k}| \sum_{j=0}^{\infty} |a_{v,j}| < \infty$$

uniformly in $k$, then for $f(x) \in BV [0,1]$ and for every $x \in [0,1]$ the sequence $\{b_k(x)\}$ is AB-summable to the limit zero iff

$$\lim_{n \to \infty} \sum_{j=k}^{\infty} b_{n,j-k}^* \sum_{k=0}^{\infty} a_{j,k} f_k^*(t) = 0$$

uniformly in $k$ in every interval $0 < s \leq t \leq 1$, $s$ is small.

7.4. **Proof of Theorem 1.** We have by virtue of (7.3.7) and (7.1.9)
\[ \sum_{k=0}^{n} a_{n,k} B_k(x) = \sum_{k=0}^{n} a_{n,k} k \gamma_k(x) \]

\[ = \sum_{k=0}^{n} a_{n,k} k \int_{0}^{1} f(t) \gamma_k(t) \gamma_k(x) \, dt \]

\[ = \sum_{k=0}^{n} a_{n,k} k \int_{0}^{1} f(t) \gamma_k(x+t) \, dt \]

\[ = \sum_{k=0}^{n} a_{n,k} k \int_{0}^{1} f(x+t) \gamma_k(t) \, dt \]

\[ = \sum_{k=0}^{n} a_{n,k} k \left[ f(x+t) \gamma_k(t) \right]_0^1 - \sum_{k=0}^{n} a_{n,k} k \int_{0}^{1} \gamma_k(t) \, df(x+t) \]

\[ = 0 - \frac{1}{n} \sum_{k=0}^{n} a_{n,k} \int_{0}^{1} \gamma_k(t) \, df(x+t) \]

Let

\[ K_n(t) = \sum_{k=0}^{n} a_{n,k} \gamma_k(t) \]

We have to show that if (7.3.1) holds, then for every \( f \in BV[0,1] \) and for every \( x \in [0,1] \),

\[ \lim_{n \to \infty} \int_{0}^{1} K_n(t) \, df(x+t) = 0, \]

where \( \frac{d}{dx}(t) = f(x+t) \),

and conversely.
Condition (7.4.1) is equivalent to following condition

\[
(7.4.2) \quad \lim_{m \to \infty} \int_0^m k(t) d P_x(t) = 0
\]

for every \( f \in BV[0,1] \) and for every \( x \in [0,1] \) and for \( 0 < \varepsilon < 1 \).

For if \( f \in BV[0,1] \) and \( x \in [0,1] \), given any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
(7.4.3) \quad \int_0^\delta |d P_x(t)| < \frac{\varepsilon}{2M}
\]

By virtue of the regularity condition we have

\[
(7.4.4) \quad |K_m(t)| \leq \sum_{k=0}^m |a_{2,k}||x_k(t)| \leq M \sum_{k=0}^m |a_{2,k}| \leq M
\]

so that

\[
\int_0^\delta K_m(t) d P_x(t) = \int_0^\delta K_m(t) d P_x(t) ! = \int_0^\delta K_m(t) d P_x(t) ! < \frac{\varepsilon}{2}
\]

Thus conditions (7.4.1) and (7.4.2) are equivalent.
By a theorem on weak convergence of sequences in the Banach space of all continuous functions defined in a finite closed interval it follows that (7.4.2) holds iff

\[(i)^* \quad \|x_m(t)\| \leq M \text{ for all } m \text{ and } t \text{ in } a, b \text{ and} \]

\[(ii)^* \quad (7.3.1) \text{ holds.} \]

Since \((i)^*\) always holds by virtue of (7.4.4), it follows that (7.4.2) holds iff (7.3.1) holds.

This completes the proof of the theorem 4.

7.5. Proof of Theorem 2. We have as in the proof of the previous theorem:

\[
\frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{m-r} a_{m+r,k} R_k(x)
\]

\[= \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{m-r} a_{m+r,k} \left\{ \int f(x+t) \psi_k(t) dt \right\}
\]

\[= \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{m-r} a_{m+r,k} \left[ \int f(x+t) J_k(t) \right]_0
\]

\[= \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{m-r} a_{m+r,k} \int dx \int f(x+t) J_k(t)
\]

\[= I_1 - I_2, \text{ say.} \]

---

Since $I_1 = 0$, it is sufficient to show that

\[(7.5.1) \quad I_2 = \frac{1}{p} \int_0^c \int_a f(x + t)K_{a,p}(t) - c, \text{ as } p \to \infty \]

uniformly in $n$, where

\[(7.5.2) \quad K_{a,p}(t) = \frac{1}{p} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} a_{m+r,k} I_k(t).\]

By virtue of the conditions (4) and (7.1.12) we have

\[|K_{a,p}(t)| \leq M_2\]

uniformly in $n$ and therefore we can show, as in the proof of Theorem 1, that condition (7.5.1) is equivalent to the following condition:

\[(7.5.3) \quad \int_k K_{a,p}(t) d f(x + t) - c, \text{ as } p \to \infty,\]

uniformly in $a$.

1) Following the lines of Banach it can be easily verified that a sequence of continuous functions $\{x^k(t)\}$ converges weakly (uniformly in $k$) to a continuous function $x(t)$, that is to say

\[\lim_{n \to \infty} \int_0^c x^k(t) dg(t) = \int_0^c x(t) dg(t)\]

uniformly in $k$, for every $g \in BV$, if and only if

1) Banach, S (1) p. 134.
(i) $x_n^k(t)$ is bounded uniformly in $k$, $n = 1, 2, \ldots$

(ii) $\lim_{n \to \infty} x_n^k(t) = x(t)$ uniformly in $k = 0, 1, \ldots$

for every $t \in [0, 1]$.

Applying this theorem and by virtue of the fact

$|K_{a, p}(t)| \leq M_2$ for all $a, p$ and $t \in [0, 1]$

the theorem follows.

7.6. Proof of Theorem 3. We have by virtue of

(7.1.7) and (7.1.9)

$$\delta_{a, k} = \sum_{v=0}^a a_{v+k} a_{v+k}$$

$$= \sum_{j=k}^a a_{j-k} a_{j}$$

$$= \sum_{j=k}^a a_{j-k} a_{j} \psi_j(x)$$

$$= \sum_{j=k}^a a_{j-k} (\int_0^1 f(t) \psi_j(t) dt) \psi_j(x)$$

$$= \sum_{j=k}^a a_{j-k} \int_0^1 f(x + t) \psi_j(t) dt$$
\[
\begin{align*}
&\sum_{j=k}^n a_n, j-k \left[ f(x + t) \right] \frac{1}{J_j(t)} - \\
&= \sum_{j=k}^n \left( \frac{1}{J_j(t)} \right) d(f(x + t)) J_j(t) \\
&= \sum_{j=k}^n a_n, j-k \left[ J_j^*(t) f(x + t) \right] - \\
&= \sum_{j=k}^n a_n, j-k \left[ J_j^*(t) f(x + t) \right] - \\
&= L_1 - L_2, \text{ say}
\end{align*}
\]

\( L_1 = 0 \) uniformly in \( k \).

In order to prove the theorem we have to show that if (7.3.3) holds, then for every \( f \in BV[0,1] \) and for every \( x \in [0,1] \)

\[(7.6.1) \quad \lim_{n \to \infty} \int_0^1 K_{n,k}^* (t) df(x + t) = 0 \]

uniformly in \( k \), where

\[
K_{n,k}^* (t) = \sum_{j=k}^n a_n, j-k \left[ J_j^*(t) \right],
\]

and conversely.

Proceeding on the lines of the proof of theorem 2 we can show that condition (7.6.1) is equivalent to
uniformly in \( k \), for every \( f \in BV[0,1] \), for every \( x \in [0,1] \) and for \( 0 < \varepsilon < 1 \). Thus it follows as shown in Theorem 2, that (7.6.2) holds, iff

(a') \[ |K^{*}_{m,k}(t)| \leq M, \quad m = 1,2, \ldots \text{ and } t \in [r,t] \]

for \( \delta > 0 \) and uniformly in \( k \).

(b') (7.3.3) holds.

Since (a') always holds, it follows that (7.6.2) holds iff (7.3.3) holds.

This completes the proof of Theorem 3.

7.7. Proof of Theorem 4. We have

\[
\begin{align*}
\sum_{k=0}^{\infty} a_{m,k} k a_{k} \frac{\gamma(x)}{k} &= \sum_{m,k}^{\infty} a_{m,k} v \sum_{v+k}^{\infty} b_{n,v} v+k \\
&= \sum_{v=0}^{\infty} b_{n,v} \sum_{k=0}^{\infty} a_{v+k} k a_{k} \frac{\gamma(x)}{k}
\end{align*}
\]
\[\begin{align*}
&= \sum_{j=k}^{\infty} b_{a,j-k} \sum_{k=0}^{\infty} a_{j,k} \int_0^1 f(x + t) J_k(t) dt \\
&= \sum_{j=k}^{\infty} b_{a,j-k} \sum_{k=0}^{\infty} a_{j,k} \int_0^1 f(x + t) J_k(t) dt \\
&= \sum_{j=k}^{\infty} b_{a,j-k} \sum_{k=0}^{\infty} a_{j,k} \int_0^1 f(x + t) J_k(t) dt \\
&= N_1 - N_2, \text{ say.}
\end{align*}\]

But \(N_1 = 0\).

Proceeding on the lines of the proof of Theorem 3 we can show that \(N_2 = 0\), as \(m \to \infty\), uniformly in \(k'\) i.f.f

\[\lim_{m \to \infty} \sum_{j=k}^{\infty} b_{a,j-k} \sum_{k=0}^{\infty} a_{j,k} J_k(t) = 0\]

uniformly in \(k'\) in the interval \(0 < s \leq t \leq 1\).

This completes the proof of the theorem 4.

7.7. If we take \(x = 0\), \(k = 0\), \(a_{n,k} = \frac{1}{n}\), \(k < n\)

we get the following corollary of Theorem 3 which
bridges the gap between Theorems A and C.

**Corollary:** If \( f(x) \in BV[0,1] \), then

\[ \text{\( k \), \( m \), is summable \((C,1)\) to zero iff} \]

\[ \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} \int_{\psi(t)}^{0} = 0 \]

for \( 0 < s \leq t \leq 1 \).
BIBLIOGRAPHY


[6a] Fejér, L. Über die Bestimmung des Fourier's der Funktion aus ihrer Fourierreihe, Journal für die rein und angewandte Mathematik, 142 (1912), 165-188.


[14] _______ **Complementary space of Fourier coefficients**


[34] Nestruev, I.P. Construction theory of Functions (Translation series U.S. atomic energy commission 1961)


[34] ——— The Fourier coefficients of continuous functions of bounded variation, Mathematische Annalen, 149(1963) 103 - 108.


[34B] Steinhaus, E., Remarks on the generalizations of the idea of limit Space Matematyczno-Fizyko, 22 (1911) 131 - 134.
[34] Toemitz, O., Über allgemeine lineare Mittelbildungen, _Papre Mathematiska-Fysiska_, 22(1911)112-119


[38] Varonoi, O.F., Verallgemeinerung des Begriffs der
Summe einer unendlicher Reihen, _Proceedings (Prevail) of the XI congress(1901) of
Russian Naturalists and Physicians, St.
Peterburg, 1902, 60-61, Annals of Mathematics 13(1), 13(1912-1913)

[39] Yano, S., On Walsh-Fourier series, _Tohoku Mathematical


[42] Zait, Y.V., On a modification of the concept of the
modulus of smoothness and its use in esti-
-mating Fourier coefficients, _Doklady
Academiia Nauk SSR_ 160(1965)759-761
(_Soviet Mathematics 6(1965)129-137)._

September 13, 1967

Dear Mr. Slauiqi:

I'm happy to report that your paper,

"A Note on Fourier Series of Functions of a Class,"

is accepted for publication in the Monthly.

Sincerely yours,

[Signature]

(M. A. Rosenboum)

Acting Head
Dept. Stat
A. M. U. ALGERIA