SOME MOMENT PROPERTIES OF GENERALIZED ORDER STATISTICS

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Dedicated
To
My Parents
And
Grand Parents
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## CONTENTS

Acknowledgement

Preface

I Preliminaries and Basic Concepts 1-27

1.1 Order Statistics 1
1.2 Distribution of Order Statistics 1-5
1.3 Record Values and Record Times 6-7
1.4 Distribution of Record Values 7-9
1.5 k-Records 9-10
1.6 Sequential Order Statistics 10-12
1.7 Generalized order statistics 12-16
1.8 Dual (lower) Generalized Order Statistics 16-17
1.9 Some Important Results 18-21
1.10 Some continuous distributions 21-27

II Moments of Single Generalized Order Statistics 28-56

1.1 Introduction 28-29
1.2 Moments of Some Specific Distributions 29-44
1.3 Moments of General Class of Distributions 45-56

III Moments of Joint Generalized Order Statistics 57-88

1.1 Introduction 57
1.2 Moments of Some Specific Distributions 58-75
1.3 Moments of General Class of Distributions 75-88

IV Moments of Dual (lower) Generalized Order Statistics 89-123

1.1 Introduction 89-90
1.2 Single Moments of Dual Generalized Order Statistics 90-107
1.3 Product Moments of Dual Generalized Order Statistics 107-123
V Characterization of Distribution through Generalized Order Statistics and Dual (Lower) Generalized Order Statistics

1.1 Introduction 124

1.2 The Characterization of Weibull distribution by Conditional Variance of Generalized Order Statistics 124-128

1.3 Characterization of General Class of Distributions by Conditional Variance of Generalized Order Statistics 128-132


References
The present dissertation entitled "Some Moment Properties of Generalized Order Statistics" is a brief collection of the work done so far on the subject. I have tried my best to include sufficient and relevant materials in the systematic way, which are contained in five chapters.

Chapter I is introductory in nature and deals with the basic concepts and results about order statistics, records, sequential order statistics, generalized order statistics, lower generalized order statistics, which may be helpful to grasp the ideas contained in the subsequent chapters.

Chapter II consists the results based on moments and recurrence relations of single generalized order statistics for some specific continuous distribution and general class of distributions. Further, various deductions, examples and particular cases are also discussed.

Chapter III also deals with the results of moments and recurrence relations of joint generalized order statistics for some specific and general class of distributions as given in Chapter II.

Chapter IV is based on results on single and product moments of dual (lower) generalized order statistics, whereas in Chapter V, some characterization results are given.

In the end, a comprehensive list of references referred into this dissertation is given.
In this chapter we have introduced those concepts/results which are needed to grasp the idea in subsequent chapters.

1. **Order Statistics**

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a continuous population having probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \). Let these be arranged in ascending order of magnitude as

\[
X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{r:n} \leq \ldots \leq X_{n:n}
\]

then \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) are collectively called the order statistics and \( X_{r:n} (r = 1, 2, \ldots, n) \) is called the \( r^{th} \) order statistic. \( X_{1:n} = \min(X_1, X_2, \ldots, X_n) \) and \( X_{n:n} = \max(X_1, X_2, \ldots, X_n) \) are called extreme order statistics or the smallest and the largest order statistics.

David and Nagaraja (2003) is the basic book on order statistics dealing in detail with its different aspects. Asymptotic theory of extremes and related developments of order statistics are well described in an applausible work of Galambos (1987). Also, references may be made to Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991), Arnold et al. (1992) and the references therein.

2. **Distribution of Order Statistics**

Here in this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous.
The pdf of $X_{r,n}$, the $r$-th order statistics is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x), \quad -\infty < x < \infty$$

(1.2.1)

The pdf of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1 - F(x)]^{n-1} f(x), \quad -\infty < x < \infty$$

(1.2.2)

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x), \quad -\infty < x < \infty$$

(1.2.3)

The df of $X_{r,n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \leq x)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i}, \quad -\infty < x < \infty$$

(1.2.4)

$$= \frac{n!}{(r-1)!(n-r)!} \int_{0}^{F(x)} u^{r-1} (1-u)^{n-r} du$$

(1.2.5)

$$= B(r, n-r+1)$$

(1.2.6)

RHS is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991)
Preliminaries and basic concepts

\[ F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{n-1-i}{r-1} [F(x)]^i [1-F(x)]^{n-r-i}; \quad -\infty < x < \infty \quad (1.2.7) \]

For continuous case the pdf of \( X_{r:n} \) may also be obtained by differentiating (1.2.5) w.r.t. \( x \).

From the density function given in (1.2.1), we may obtain the \( k \)-th moment of \( X_{r:n} \) as below

\[ \alpha_{r:n}^{(k)} = E[X_{r:n}^k] = \int_{-\infty}^{\infty} x^k f_{r:n}(x) \, dx \quad (1.2.8) \]

The joint pdf of \( X_{r:n}, X_{s:n}, 1 \leq r < s \leq n \) is given by

\[ f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{r-1} \]

\[ \times [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x) f(y), \quad -\infty < x < y < \infty \quad (1.2.9) \]

The joint df of \( X_{r:n} \) and \( X_{s:n}, (1 \leq r < s \leq n) \) can be obtained as follows:

\[ F_{r,s:n}(x,y) = P(X_{r:n} \leq x, X_{s:n} \leq y) \]

\[ = P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x) \]

\[ \quad \text{and at least } s \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \]

\[ = \sum_{j=s}^{n} \sum_{i=r}^{j} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x) \]

\[ \quad \text{and exactly } j \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \]
Preliminaries and basic concepts

\[ n! \sum_{i=r}^{n} \frac{1}{i!(n-i)!} [F(x)]^i [F(y) - F(x)]^j - [1 - F(y)]^{n-j} \]

(1.2.10)

We can write the joint pdf of \( X_{r:n} \) and \( X_{s:n} \) in (2.10) equivalently as:

\[
F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^x \int_0^y u^{r-1} (v-u)^{s-r-1} \times (1-v)^{n-s} \, du \, dv
\]

\[
= \int F(x), F(y) (r,s-r,n-s+1) ; -\infty < x < y < \infty
\]

(1.2.11)

which is incomplete bivariate beta function.

It may be noted that for \( x > y \)

\[
F_{r,s:n}(x,y) = F_{s:n}(y)
\]

(1.2.12)

The product moments of the \( j-th \) and \( k-th \) order of \( X_{r:n} \) and \( X_{s:n} \) respectively, \((1 \leq r < s \leq n)\) is given by:

\[
\alpha_{r,s:n}^{(j,k)} = E[X_r^j X_s^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{r,s:n}(x,y) \, dx \, dy
\]

(1.2.13)

In general, the joint pdf of \( X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n} \) for \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \) is given by:

\[
f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n}, x_{i_2:n}, \ldots, x_{i_k:n})
\]

\[
= n! \left\{ \prod_{j=1}^{k} f(x_{i_j}) \right\} \prod_{j=0}^{\frac{k(k+1)}{2}} \left[ \frac{[F(x_{i,j+1}) - F(x_{i,j})]^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!} \right]
\]
\[-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty\]

where \( x_0 = -\infty, x_{k+1} = +\infty, i_0 = 0, i_{k+1} = n + 1 \)

**(Remarks):**

a. The ranking of random variables \( X_1, X_2, \ldots, X_n \) is preserved under any monotonic increasing transformation of the random variables.

b. Regarding the probability integral transformation, if \( X_{r:n}, 1 \leq r \leq n \), are the order statistics from a continuous distribution \( F(x) \), then the transformation \( U_{r:n} = F(X_{r:n}) \) produces a random variable which is the \( r^{th} \) order statistics from a uniform distribution on \( U(0,1) \).

c. Even if \( X_1, X_2, \ldots, X_n \) are independent random variables, order statistics are not independent random variables.

d. Let \( X_1, X_2, \ldots, X_n \) be iid random variables from a continuous distribution, then the set of order statistics \( \{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\} \) is both sufficient and complete (Lehmann, 1986).

e. Let \( X \) be a continuous random variable with \( E[X_{r:n}] = \alpha_{r:n} \),

(i) If \( \alpha = E(X) \) exists then \( \alpha_{r:n} \) exists, but converse is not necessarily true. That is, \( \alpha_{r:n} \) may exist for certain (but not all) values of \( r \), even though \( \alpha \) does not exist.

(ii) \( \alpha_{r:n} \) for all \( n \) determine the distribution completely.
3. Record Values and Record Times

Suppose that \( X_1, X_2, \ldots, X_n \) is a sequence of independent and identically distributed random variables with \( df \ F(x) \). Let \( Y_n = \max (\min) \{ X_1, X_2, \ldots, X_n \} \) for \( n \geq 1 \). We say \( X_j \) is an upper (lower) record values of \( \{ X_n, n \geq 1 \} \), if \( Y_j > (\leq) Y_{j-1}, j > 1 \). By definition \( X_1 \) is an upper as well as lower record values. One can transform the upper record by replacing the original sequence of \( \{ X_j \} \) by \( \{-X_j, j \geq 1\} \) or if \( P(X_i > 0) = 1 \) for all \( i \) by \( \left\{ \frac{1}{X_i}, i \geq 1 \right\} \), the lower record value of this sequence will correspond to the upper record values of the original sequence (Ahsanullah, 1995)

The indices at which upper record values occur are given by the record times \( \{ U(n) \}, n > 0 \). That is \( X_{U(n)} \) is the \( n-th \) upper record, where \( U(n) = \min \{ j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1 \} \) and \( U(n) = 1 \). The distribution of \( U(n), n \geq 1 \) does not depend on \( F \). Further, we will denote \( L(n) \) as the indices where the lower record values occur. By assumption \( U(1) = L(1) = 1 \). The distribution of \( L(n) \) also does not depend on \( F \).

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recoding them: e.g. Olympic records or world records in sports.

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables.
Preliminaries and basic concepts

It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.

To be precise, record values are defined by means of record times. That is, those times have to be described at which successively largest values appear.

Chandler (1952) shows several properties of record values and notes their Markovian structure. Two recent books on records by Ahsanullah (1995) and Arnold et al. (1998) are worth mentioning.

4. Distribution of Record Values

Let \( R(x) \) be a continuous function of \( x \) with \( R(x) = -\ln \bar{F}(x) \) and \( 0 < \bar{F}(x) = 1 - F(x) \), where 'ln' is the natural logarithm.

If we define \( F_n(x) \) as the df of \( X_{U(n)} \) for \( n \geq 1 \), then we have (Ahsanullah, 1995)

\[
F_n(x) = P(X_{U(n)} \leq x) = \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty
\]

(1.4.1)

and the pdf \( f_n(x) \) of \( X_{U(n)} \) is

\[
f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty
\]

(1.4.2)

The joint pdf of \( X_{U(i)} \) and \( X_{U(j)} \) is

\[
f_{i,j}(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j)
\]
Preliminaries and basic concepts

The joint pdf of the $n$ record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \cdots r(x_{n-1}) f(x_n),$$

$$-\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty$$

where

$$r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{1 - F(x)}, \quad 0 < F(x) < 1$$

is known as hazard rate.

In particular at $i = 1$, $j = n$, we have

$$f_{1, n}(x_1, x_n) = r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} f(x_n), \quad -\infty < x_1 < x_2 < \infty.$$
Preliminaries and basic concepts

\[ f(X_{U(i)} \mid X_{U(j)} = x_j) = \frac{(j-1)!}{(i-1)!(j-i-1)!} \left( \frac{R(x_i)}{R(x_j)} \right)^{j-i-1} \left( 1 - \frac{R(x_i)}{R(x_j)} \right)^{j-i-1} \frac{r(x_i)}{R(x_j)} \]

\(-\infty < x_i < x_{i+1} < \infty\)

(1.4.6)

5. k-Records

In some situations record values themselves are viewed as ‘outlier’ and hence second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example.

Let \(X_1, X_2, \ldots, X_n\) be an identically and independent sequence of random variables with a continuous distribution function \(F(x)\) and let \(k\) be a positive integer.

Then the random variables \(L^{(k)}(n)\) is given by (Kamps, 1995b)

\[ L^{(k)}(n) = 1 \]

\[ L^{(k)}(n+1) = \min\{j \in N; X_{j+k-1} > X_{L^{(k)}(n), L^{(k)}(n)+k-1}, n \in N, \} \]

are called \(k\)-th record times and the quantities \(X_{L^{(k)}(n)}, n \in N\) are called \(k\)-th record values or \(k\)-records.

We can obtain ordinary record values at \(k = 1\).

The joint density of the \(k\)-records \(X_{L^{(k)}(1)}, \ldots, X_{L^{(k)}(r)}\) is given as

\[ f_{X_{L^{(k)}(1)}, \ldots, X_{L^{(k)}(r)}}(x_1, \ldots, x_r) \]
Preliminaries and basic concepts

\[ f_r = k^r \left( \prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)} \right) \left[ 1 - F(x_r) \right]^{k-1} f(x_r) \]  

(1.5.1)

and the marginal densities and marginal distribution functions are given by:

\[ f_{X_{(k)(r)}}(x) = \frac{k^r}{(r-1)!} \left[ R(x) \right]^{r-1} \left[ 1 - F(x) \right]^{k-1} f(x) \]  

(1.5.2)

\[ F_{X_{(k)(r)}}(x) = 1 - \left[ 1 - F(x) \right]^k \sum_{j=0}^{r-1} \frac{1}{j!} \left[ k R(x) \right]^j \]  

(1.5.3)

6. Sequential Order Statistics

A \( k \)-out-of-\( n \) systems are important technical structures which are often considered in the literature. Such systems consist of \( n \) components of the same kind with independent and identically distributed (\( iid \)) life lengths. All components start working simultaneously, and the system will work as long as \( k \) components function. Parallel and series systems are particular cases of \( k \)-out-of-\( n \) systems corresponding to \( k = 1 \) and \( k = n \), respectively. In the conventional modeling of these structures it is supposed that the failure of any component does not affect the remaining ones. Hence, the (\( n-k+1 \))-th order statistic from an \( iid \) sample describes the lifetime of some \( k \)-out-of-\( n \) system.

In \( k \)-out-of-\( n \)-system, it is generally assumed that we have components of the same kind without any interactions with respect to life-length distributions. Hence, the system failure is modelled by an order statistics based on \( iid \) r.v.'s.

However, the failure of some components can more or less strongly influence the remaining components. This can be thought of as damage caused by the \( i \)-th failure in the system. Thus, a more flexible model,
that is more general and therefore more applicable to practical situations, must take some dependence structure into account.

In this model, the life length distribution of the remaining components in the system may change after each failure of the components. If we observe the \(i\)-th failure at time \(x\), the remaining components are now supposed to have a possibly different life-length distribution. This distribution is truncated on the left at \(x\) to ensure realizations arranged in ascending order of magnitude (Kamps, 1995b).

Let \( (Y_j^{(i)})_{1 \leq i \leq n, 1 \leq j \leq n-i+1} \) be independent random variables with

\[
(Y_j^{(i)})_{1 \leq j \leq n-i+1} - F_i, 1 \leq i \leq n
\]  

(1.6.1)

where \(F_1, F_2, \ldots, F_n\) are strictly increasing and continuous distribution functions

with \(F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)\)

Moreover, let \( X_j^{(1)} = Y_j^{(1)}, 1 \leq j \leq n, \)

\(X_\star^{(1)} = \min\{X_1^{(1)}, \ldots, X_n^{(1)}\}\)

and for \(2 \leq i \leq n:\)

\(X_j^{(i)} = F_i^{-1}\left[F_i(Y_j^{(i)})(1 - F_i(X_\star^{(i-1)})\right], \)

\(X_\star^{(i)} = \min\{X_j^{(i)}, 1 \leq j \leq n-i+1\}\)

Then random variables \(X_\star^{(1)}, \ldots, X_\star^{(n)}\) are called sequential order statistics.
If we have absolutely continuous distribution functions $F_1, \ldots, F_n$ with densities $f_1, \ldots, f_n$ respectively, the joint density of the first $r$ sequential order statistics $X_{(1)}^*, \ldots, X_{(r)}^*$ is given by

$$f_{X_{(1)}^*, \ldots, X_{(r)}^*}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \frac{(1-F_i(x_i))}{(1-F_i(x_{i-1}))} \frac{f_i(x_i)}{1-F_i(x_{i-1})},$$

$$r \leq n, x_0 = -\infty \quad (1.6.2)$$

Sequential order statistics form a Markov chain with transition probabilities

$$P(X_{(r)}^* > r \mid X_{(r-1)}^* = s) = \left(\frac{1-F_r(t)}{1-F_r(s)}\right)^{n-r+1}, 2 \leq r \leq n \quad (1.6.3)$$

**Remark 1.6.1:** Choosing $F_1 = \ldots = F_n = F$, we can obtain the joint density function of the order statistics $X_{1:n}, \ldots, X_{n:n}$ based on $n$ iid r.v.'s with distribution function $F$.

**Remark 1.6.2:** Distribution function of $r$-th sequential order statistics is given by

$$F_r(t) = 1 - [1 - F(t)]^{\alpha_r}, 1 \leq r \leq n \quad (1.6.4)$$

where $F()$ is a distribution function and $\alpha_1, \ldots, \alpha_n$ are positive real numbers.

**7. Generalized order statistics**

Let $F(x)$ be an absolutely continuous df with pdf $f(x)$ of r.v. $X$. 

Let \( n \in \mathbb{N}, n \geq 2, k > 0, \tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \mathbb{R}^{n-1}, M_r = \sum_{j=r}^{n-1} m_j \), such that \( \gamma_r = k + n - r + M_r > 0 \) for all \( r \in \{1, 2, ..., n-1\} \). Then \( X(r, n, \tilde{m}, k), r = 1, 2, ..., n \) are called gos if their joint pdf is given by

\[
\begin{align*}
X &\left( r, n, \tilde{m}, k \right) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \prod_{i=1}^{n-1} \left[ \left( 1 - F(x_i) \right)^{m_i} f(x_i) \left( 1 - F(x_n) \right)^{n-r+M_r-1} f(x_n) \right]^{k-r} \\
&= C_{r-1} \left( \prod_{i=1}^{r-1} \left[ 1 - F(x_i) \right]^{m_i} f(x_i) \right) \left[ 1 - F(x_r) \right]^{k+n-r+M_r-1} f(x_r)
\end{align*}
\]

on the cone \( F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1) \) of \( \mathbb{R}^n \).

Choosing the parameters appropriately, models such as ordinary order statistics \( (\gamma_i = n - i + 1; \ i = 1, 2, ..., n \ i.e. \ m_1 = m_2 = ... = m_{n-1} = 0, k = 1) \), \( k^{th} \) record values \( (\gamma_i = k \ i.e. \ m_1 = m_2 = ... = m_{n-1} = -1, k \in \mathbb{N}) \), sequential order statistics \( (\gamma_i = (n - i + 1)\alpha_i; \alpha_1, \alpha_2, ..., \alpha_n > 0) \), order statistics with non-integral sample size \( (\gamma_i = \alpha - i + 1; \alpha > 0) \), Pfeifer’s record values \( (\gamma_i = \beta_1; \beta_1, \beta_2, ..., \beta_n > 0) \) and progressive type II censored order statistics \( (m_i \in \mathbb{N}_0, k \in \mathbb{N}) \) are obtained [Kamps (1995a), Kamps and Cramer (2001)].

The joint density of the first \( r \) generalized order statistics (gos) is given by:

\[
f_X(1, n, \tilde{m}, k), ..., X(r, n, \tilde{m}, k)(x_1, x_2, ..., x_r)
\]

\[
= C_{r-1} \left( \prod_{i=1}^{r-1} \left[ 1 - F(x_i) \right]^{m_i} f(x_i) \right) \left[ 1 - F(x_r) \right]^{k+n-r+M_r-1} f(x_r)
\]

on the cone \( F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1) \).

Here we may consider two cases:
Case I: \( m_1 = m_2 = \ldots = m_{n-1} = m \)

Case II: \( \gamma_i \neq \gamma_j \; ; \; i, j = 1, 2, \ldots, n-1 \)

For Case I, the marginal density of the \( r \)-th generalized order statistics (gos) is given by [Kamps, 1995b]

\[
f_X(r,n,m,k)(x) = \frac{C_{r-1}}{(r-1)!} \left[ 1 - F(x) \right]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))
\]

(1.7.3)

and the joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k) \), \( 1 \leq r < s \leq n \) is

\[
f_X(r,n,m,k), X(s,n,m,k)(x,y)
= \frac{C_{s-1}}{(r-1)! (s-r-1)!} \left[ 1 - F(x) \right]^m g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{\gamma_s - 1} f(x) f(y)
\]

(1.7.4)

where \( C_{r-1} = \prod_{i=1}^{r} \gamma_i \), \( \gamma_i = k + (n-i)(m+1) \)

\[
h_m(x) = \begin{cases} 
- \frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\
- \log(1-x), & m = -1 
\end{cases}
\]

\[
g_m(x) = \int_0^x (1-t)^m \, dt = h_m(x) - h_m(0), \; x \in [0,1)
\]

The conditional pdf of \( X(s,n,m,k) \) given \( X(r,n,m,k) = x \), \( 1 \leq r < s \leq n \) is given by

\[
f_X(s,n,m,k) \mid X(r,n,m,k)(y \mid x)
\]
\[
\frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[1 - F(y)]^{r-1} f(y)}{[1 - F(x)]^{r-1}},
\]
\[x < y \quad (1.7.5)\]

and the conditional pdf of \( X(r,n,m,k) \) given \( X(s,n,m,k) = y \), \( 1 \leq r < s \leq n \) is

\[
f_{X(r,n,m,k)\mid X(s,n,m,k)}(x \mid y) = \frac{(s-1)! (m+1)}{(s-r-1)! (r-1)!} \times \frac{[\bar{F}(x)]^m [1 - (\bar{F}(x))^{m+1}]^{r-1} \left[ (\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1} \right]^{s-r-1} f(x)}{[1 - (\bar{F}(y))^{m+1}]^{s-1}}
\]
\[x < y \quad (1.7.6)\]

For Case II, the pdf of \( X(r,n,\tilde{m},k) \) is [Kamps and Cramer, 2001]

\[
f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i - 1}
\]
\[ (1.7.7)\]

and the joint pdf of \( X(r,n,\tilde{m},k) \) and \( X(s,n,\tilde{m},k) \), \( 1 \leq r < s \leq n \) is

\[
f_{X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{1 - F(y)}{1 - F(x)} \right)^{\gamma_i}
\times \left[ \sum_{i=1}^{r} a_i(r) (1 - F(x))^{\gamma_i} \right] \frac{f(x)f(y)}{(1 - F(x))(1 - F(y))}
\]
\[ (1.7.8)\]

where

\[
C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + M_i
\]
Preliminaries and basic concepts

\begin{align*}
a_i(r) &= \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_i)} , 
\quad 1 \leq i \leq r \leq n
\end{align*}

and

\begin{align*}
a_i^{(r)}(s) &= \prod_{j=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)} , 
\quad r+1 \leq i \leq s \leq n
\end{align*}

Thus, the conditional pdf of \( X(s, n, m, k) \) given \( X(r, n, m, k) = x \), \( 1 \leq r < s \leq n \) is given by

\begin{align*}
f_{X(s, n, m, k) | X(r, n, m, k)}(y | x) &= \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{1 - F(y)}{1 - F(x)} \right]^{\gamma_i} \frac{f(y)}{[1 - F(y)]} , 
\quad x \leq y \tag{1.7.9}
\end{align*}

and the conditional pdf of \( X(r, n, m, k) \) given \( X(s, n, m, k) = y \), \( 1 \leq r < s \leq n \) is given by

\begin{align*}
f_{X(r, n, m, k) | X(s, n, m, k)}(x | y) &= \frac{s}{s} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ \sum_{i=1}^{r} a_i(r) \left[ \frac{F(y)}{F(x)} \right]^{\gamma_i} \right\} \frac{f(x)}{F(x)} 
= \frac{\sum_{i=r+1}^{s} a_i(s)(F(y))^{\gamma_i}}{\sum_{i=1}^{s} a_i(s)(F(y))^{\gamma_i}} \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(x)}{F(x)} 
\tag{1.7.10}
\end{align*}

8. Dual (lower) Generalized Order Statistics

The pdf of the dual generalized order statistics (dgos) \( X'(r, n, m, k) \) is obtained by replacing \( 1 - F(x) \) with \( F(x) \).
For \( m_1 = m_2 = \ldots = m_{n-1} = m \),
its joint pdf is (Burkschat et al., 2003)
\[
\frac{f_x'(1, n, m, k), \ldots, f_x'(n, n, m, k)}{[F(x)]^m f(x)}
= k \left( \prod_{r=1}^{n-1} \gamma_r \right) \left( \prod_{j=1}^{n-1} [F(x)]^m f(x_j) \right) [F(x_n)]^{k-1} f(x_n) \tag{1.8.1}
\]
for \( F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0) \)
The joint density function of \( r \)-th and \( s \)-th dgos is
\[
f_x'(r, n, m, k), x'(s, n, m, k) (x, y)
= \frac{C_{s-1}}{(r-1)! (s-r-1)!} [F(x)]^m f(x) g_m^{r-1} (F(x))
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1} f(y), \quad x > y, \tag{1.8.2}
\]
where
\[
h_m(x) = \begin{cases} 
- \frac{1}{m+1} x^{m+1}, & m \neq -1 \\
- \log x, & m = 1
\end{cases} \tag{1.8.3}
\]
\[
g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1). \tag{1.8.4}
\]
and the density function of \( r \)-th dgos is given by
\[
f_x'(r, n, m, k)(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{r-1} g_m^{r-1}(F(x)) f(x). \tag{1.8.5}
\]
9. Some Important Results

**Result 1 (David and Nagaraja, 2003):** Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with the df $F(x)$ and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the $r$-th order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at $y$.

**Result 2 (David and Nagaraja, 2003):** Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with the df $F(x)$ and pdf $f(x)$, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)$-th order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at $x$.

**Result 3 (David and Nagaraja, 2003):** Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with df $F(x)$ and pdf $f(x)$, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{k:n} = z$ for $1 \leq r < s < k \leq n$, is the same as the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at $x$ and on the right at $z$.

**Result 4:** Order statistics in a sample from a continuous distribution form a Markov chain, that is
So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

**Result 5 (Ali and Khan, 1997):** Let \( g(x) \) be a Borel measurable function of \( x \) in the interval \([\alpha, \beta]\) then, for \( 1 \leq r \leq n, \quad n = 1, 2, \ldots \)

(i) \[ E[g(X_{r:n})] - E[g(X_{r-1:n-1})] \]

\[= \left( \frac{n-1}{r-1} \right) \int_{Q_1} g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \quad (1.9.1) \]

(ii) \[ E[g(X_{r:n})] - E[g(X_{r-1:n})] \]

\[= \left( \frac{n}{r-1} \right) \int_{Q_1} g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \quad (1.9.2) \]

(iii) \[ E[g(X_{r-1:n-1})] - E[g(X_{r-1:n})] \]

\[= \left( \frac{n-1}{r-2} \right) \int_{Q_1} g'(x)[F(x)]^{r-1} [1 - F(x)]^{n-r+1} \, dx. \quad (1.9.3) \]

In view of (1.9.1), (1.9.2) and (1.9.3), we have

\[(n - r + 1)E[g(X_{r-1:n})] + (r - 1)E[g(X_{r:n})] = nE[g(X_{r-1:n-1})]. \quad (1.9.4)\]

At \( g(x) = x \) in (1.9.4), we get the well known relation established by (David and Nagaraja, 2003).

**Result 6 (Ali and Khan, 1998):** If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \quad n = 1, 2, \ldots \)

\[E[g(X_{rn}, X_{sn})] - E[g(X_{rn}, X_{s-1n})] \]

\[= \frac{C_{r,s,n}}{(n-s+1)} \int_{Q_1} \int_{y < x \leq \rho} \frac{\partial}{\partial y} g(x, y)[F(x)]^{r-1} \, dxdy. \]
\[ x[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1} f(x) dy dx. \quad (1.9.5) \]

**Result 7 (Khan et al., 2001):** If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)

\[
E[g(X_{r,n}, X_{s:n})] - E[g(X_{r-1,n}, X_{s:n})]
= \frac{C_{r,s:n}}{s-r} \int_{Q_1} \int_{Q_1} \frac{\partial}{\partial x} g(x, y)[F(x)]^{r-1}
\times [F(y) - F(x)]^{s-r}[1 - F(y)]^{n-s} f(y) dy dx.
\quad (1.9.6)
\]

**Result 8 (Khan et al., 2007):**

**Case I:** \( m_i = m_j = m, \ i, j = 1, 2, \ldots, n - 1. \)

For \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1, 2, \ldots \).

i) \( E[X^j (r, n, m, k)] - E[X^j (r-1, n, m, k)] \)
\[
= \frac{C_{r-2}}{(r-1)!} \int_{Q_1} x^{j-1} [\overline{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx. \quad (1.9.7)
\]

ii) \( E[X^j (r-1, n, m, k)] - E[X^j (r-1, n-1, m, k)] \)
\[
= \frac{(m + 1)C_{r-2}}{\gamma_1(r-2)!} \int_{Q_1} x^{j-1} [\overline{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx. \quad (1.9.8)
\]

iii) \( E[X^j (r, n, m, k)] - E[X^j (r-1, n-1, m, k)] \)
\[
= \frac{C_{r-1}}{\gamma_1(r-1)!} \int_{Q_1} x^{j-1} [\overline{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx. \quad (1.9.9)
\]

**Result 9 (Athar and Islam, 2004):**

For \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1, 2, \ldots \)
Case I: \( m_1 = m_2 = \ldots = m_{n-1} = m \)

\[
E[\xi \{ X(r, n, m, k) \}] - E[\xi \{ X(r-1, n, m, k) \}]
= \frac{C_{r-2}}{(r-1)!} \int \xi'(x) [1 - F(x)]^{\gamma-1} g^{r-1}_m(F(x)) dx
\tag{1.9.10}
\]

Result 10 (Athar and Islam, 2004):

For \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1, 2, \ldots \)

\textbf{i} \) \( E[\xi \{ X(r-1, n, m, k) \}] - E[\xi \{ X(r-1, n-1, m^*, k) \}]
= -\frac{(m+1)}{\gamma_1} \int \xi'(x) [1 - F(x)]^{\gamma'} g^{r-1}_m(F(x)) dx
\tag{1.9.11}
\]

\textbf{ii} \) \( E[\xi \{ X(r, n, m, k) \}] - E[\xi \{ X(r-1, n-1, m^*, k) \}]
= \frac{C_{r-2}}{(r-1)!} \int \xi'(x) [1 - F(x)]^{\gamma'} g^{r-1}_m(F(x)) dx
\tag{1.9.12}
\]

where

\( m^* = (m_2 = m_3 = \ldots = m_{n-1}) \in \mathbb{R} \) and \( C_{r-2}^{(n)} = \prod_{i=1}^{r-1} [k + (n-i)(m+1)] \)

10. Some continuous distributions

1. Pareto distribution

A random variable \( X \) is said to have the Pareto distribution if its probability density function (pdf) \( f(x) \) and distribution function (df) \( F(x) \) are of the form given below:

\[
f(x) = p \lambda^p x^{-(p+1)}; \quad \lambda \leq x < \infty; \quad \lambda, p > 0
\]

\[
F(x) = 1 - \lambda^p x^{-p}; \quad \lambda \leq x < \infty; \quad \lambda, p > 0
\]
Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

II. Power function distribution

A random variable $X$ is said to have a power function distribution if its pdf and df are of the form given below:

$$f(x) = p \lambda^{-p} x^{p-1}; \quad 0 \leq x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p} x^p; \quad 0 \leq x < \lambda; \quad \lambda, p > 0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if $X$ has a power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

III. Weibull distribution

A random variable $X$ is said to have a Weibull distribution if its pdf is given by:

$$f(x) = \theta p x^{p-1} e^{-\theta x^p}; \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

and the df is given by

$$F(x) = 1 - e^{-\theta x^p}; \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0$$

Remark 1.10.1: If we put $p = 1$ in Weibull distribution, we get the pdf of exponential distribution.

Remark 1.10.2: If we put $p = 2$, it gives pdf of Rayleigh distribution.

Remark 1.10.3: If $X$ has a Weibull distribution, then the pdf of
Y = -p \log \left( \frac{X}{\alpha} \right) is

f(y) = e^{-\gamma} e^{-e^{-\gamma}}

which is a form of an Extreme Value distribution.

Remark 1.10.4: The pdf and the cdf of inverse Weibull distribution is given by

\[ f(x) = \theta px^{-(p+1)} e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \ \theta > 0, \ p > 0 \]

\[ F(x) = e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \ \theta > 0, \ p > 0 \]

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

IV. Exponential distribution

A random variable \( X \) is said to have an exponential distribution if its pdf is given by

\[ f(x) = \theta e^{-\theta x}; \quad 0 \leq x < \infty; \ \theta > 0 \]

and the df is given by

\[ F(x) = 1 - e^{-\theta x}; \quad 0 \leq x < \infty; \ \theta > 0 \]

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact,
whenever a continuous random variable $X$ assuming non-negative values satisfies the assumption,

$$P(X > s + t | X > s) = P(X > t)$$

for all $s$ and $t$,

then $X$ will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

V. Rectangular distribution

A random variable $X$ is said to have a rectangular distribution if its pdf is given by

$$f(x) = \frac{1}{\lambda - \beta}; \beta \leq x \leq \lambda$$

and the df is given by

$$F(x) = \frac{x - \beta}{\lambda - \beta}; \beta \leq x \leq \lambda$$

The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(x)$ follows rectangular distribution $R(0,1)$. This distribution is used in “rounding off” errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

VI. Burr distribution

Let $X$ be a continuous random variable, then different forms of cumulative distribution function of $X$ are listed below (Johnson and Kotz, 1970):

1. $F(x) = x$, $0 < x < 1$
2 \( F(x) = (1 + e^{-x})^{-k}, \quad -\infty < x < \infty \)

3 \( F(x) = (1 + x^{-c})^{-k}, \quad 0 \leq x < \infty \)

4 \( F(x) = \left[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right]^{-k}, \quad 0 \leq k \leq c \)

5 \( F(x) = [1 + ce^{-\tan x}]^{-k}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \)

6 \( F(x) = [1 + ce^{-k \sinh x}]^{-k}, \quad -\infty < x < \infty \)

7 \( F(x) = 2^{-k} (1 + \tanh x)^k, \quad -\infty < x < \infty \)

8 \( F(x) = \left( 2 \tan^{-1} e^x \right), \quad -\infty < x < \infty \)

9 \( F(x) = 1 - \frac{2}{c[(1+e^x)^k - 1] + 2}, \quad -\infty < x < \infty \)

10 \( F(x) = (1 + e^{-x^2})^k, \quad 0 \leq x < \infty \)

11 \( F(x) = \left( x - \frac{1}{2\pi} \sin 2\pi x \right)^k, \quad 0 \leq x \leq 1 \)

12 \( F(x) = 1 - (1 + x^c)^{-k}, \quad 0 \leq x < \infty \)

where \( k \) and \( c \) are positive parameters.

Special attention is given to type XII, whose pdf is given as:

\( f(x) = kcx^{c-1}(1+x^c)^{-(k+1)}, \quad 0 \leq x < \infty; \quad k, c > 0 \)

This distribution is frequently used for the purpose of graduation and in reliability theory. At \( c = 1 \), it is called Lomax distribution whereas at \( k = 1 \), it is known as Log-logistic distribution.
VII. Cauchy distribution

The special form of the Pearson type VII distribution, with pdf

\[ f(x) = \frac{1}{\pi \lambda} \frac{1}{1 + \left\{ \frac{(x - \theta)}{\lambda} \right\}^2} \quad -\infty < x < \infty; \lambda > 0; -\infty < \theta < \infty \]

is called the Cauchy distribution.

The cdf is given by

\[ F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left( \frac{x - \theta}{\lambda} \right) \quad -\infty < x < \infty; \lambda > 0; -\infty < \theta < \infty \]

The distribution is symmetrical about \( x = \theta \). The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, \( \theta \) and \( \lambda \) are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting \( \theta = 0, \lambda = 1 \). The standard probability density function is given by

\[ f(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \quad -\infty < x < \infty \]

and the standard cumulative distribution function is

\[ F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x \quad -\infty < x < \infty. \]

VIII. Beta distribution

a) Beta distribution of the first kind
A random variable $X$ is said to have the beta distribution of first kind if its pdf is of the form

$$f(x) = \frac{1}{B(p,q)} x^{p-1} (1-x)^{q-1}; \ 0 \leq x \leq 1, \ p, q > 0$$

Beta distribution arises as the distribution of an ordered variable from a uniform distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $Beta(r,n-r+1)$. The standard uniform distribution $U(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents $p$ and $q$ equal to 1. If $q = 1$, the distribution reduces to power function distribution.

**b) Beta distribution of the second kind**

The continuous random variable $X$ which is distributed according to probability law:

$$f(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}} \quad p, q > 0, \ 0 \leq x < \infty$$

is known as a beta variate of the second kind with parameters $p$ and $q$.

Beta distribution of second kind reduces to beta distribution of first kind if we replace $1 + x$ by $\frac{1}{y}$.

The beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.
1. Introduction

In this chapter a review of moments and recurrence relations of some specific and general class of distribution of generalized order statistics is given.

Moments and recurrence relations of generalized order statistics are investigated by many authors in literatures. The concept of some recurrence relations for moments of generalized order statistics based on non-identically distributed random variables was first introduced by Kamps (1995). Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics with in a class of distributions including variety of identities for single moments of ordinary order statistics and record values as particular cases.

Pawals and Szynal (2001 a) established recurrence relations for single moments of generalized order statistics from Burr distribution.


Athar and Islam (2004) established some recurrence relations between expectations of function of single generalized order statistics from general class of distribution

Athar et al. (2007) obtained the ratio and inverse moments of single generalized order statistics from Weibull distribution whereas Khan et al. (2007) obtained recurrence relations for single moments of generalized order statistics from doubly truncated Weibull distribution.
Ahmad (2008) established recurrence relations for single moments of
generalized order statistics from linear exponential distribution.

Khan et al. (2007) and Faizan and Athar (2009) obtained exact moments
of single generalized order statistics from general class of distribution. For
some additional results on generalized order statistics, one may refer to
Kesseling (1999), Cramer and Kamps (2003), Cramer (2003) and
references therein.

2. Moments of Some Specific Distributions

Burr Distribution

The pdf of distribution is given as

\[ f(x) = \lambda \beta^\lambda x^r \frac{1}{(\beta + x^r)^{\lambda+1}}, \quad x > 0, \ \beta > 0, \ \lambda > 0, \ r > 0 \]  

(2.2.1)

For Burr distribution, we have

\[(\beta + x^r)f(x) = \lambda \tau (1 - F(x))x^{(r-1)} \]

(2.2.2)

Theorem 2.2.1: (Pawals and Szynal, 2001a)

Fix a positive integer \( k \). For \( n \in \mathbb{N} \), \( m \in \mathbb{Z} \), \( 1 \leq r \leq n \) and \( j = 0, 1, 2, \ldots \) such
that \( \lambda \gamma_r, \tau > (j + \tau) \)

\[ E[X^{(r,n,m,k)}] = \frac{\beta(j + \tau)}{\lambda \gamma_r, \tau - (j + \tau)} E[X^{(r,n,m,k)}] \]

\[ + \frac{\lambda \gamma_r, \tau}{\lambda \gamma_r, \tau - (j + \tau)} E[X^{(r-1,n,m,k)}] \]

(2.2.3)

and consequently for \( 0 \leq r \leq n - 1 \)

\[ E[X^{(r,n,m,k)}] = \prod_{i=s+1}^{r} \frac{\lambda \gamma_i, \tau}{\lambda \gamma_i, \tau - (j + \tau)} E[X^{(j+\tau,s,n,m,k)}] \]

\[ + \sum_{p=s+1}^{r} \left[ \frac{\beta(j + \tau)}{\lambda \gamma_p, \tau - (j + \tau)} \right] \prod_{i=p+1}^{r} \left[ \frac{\lambda \gamma_i, \tau}{\lambda \gamma_i, \tau - (j + \tau)} \right] \]
Moments of single generalized order statistics

Proof: From equation (2.2.2), we note that for $1 \leq r \leq n$ and $j = 0, 1, 2, \ldots,$

$$E[X^{j+\tau}(r, n, m, k)] + \beta E[X^{j}(r, n, m, k)]$$

$$= \frac{\lambda \tau}{(r-1)!} \int x^{j+\tau-1}[1-F(x)]^{\gamma} g(x)^{-1}(F(x))dx$$

Integrating by parts, treating $X^{j+\tau-1}$ as part for integration, and taking into account that $\lambda \gamma x^{\tau} > (j+\tau),$ we get

$$E[X^{j+\tau}(r, n, m, k)] + \beta E[X^{j}(r, n, m, k)]$$

$$= \frac{\lambda \gamma}{(j+\tau)} E[X^{j+\tau}(r, n, m, k)] - E[X^{j+\tau}(r-1, n, m, k)]$$

which gives (2.2.3).

Remark 2.2.1: The recurrence relations for single moments of order statistics from the Burr distribution have the form

$$E[X^{j+\tau}_{r:n}] = \frac{\beta(j+\tau)}{\lambda(n-r+1)\tau - (j+\tau)} E[X^{j}_{r:n}]$$

$$+ \frac{\lambda(n-r+1)\tau}{\lambda(n-r+1)\tau - (j+\tau)} E[X^{j+\tau}_{r-1:n}]$$

Remark 2.2.2: The recurrence relations for single moments of $k$-th record values from the Burr distribution have the form

$$E(Y^{(k)}_{n})^{j+\tau} = \frac{\beta(j+\tau)}{\lambda k\tau - (j+\tau)} E(Y^{(k)}_{n})^{j} + \frac{\lambda k\tau}{\lambda k\tau - (j+\tau)} E(Y^{(k)}_{n-1})^{j}.$$  

Power Function Distribution

The pdf of the distribution is given as

$$f(x) = \delta (1-x)^{-\delta-1}, \quad 0 < x < 1, \quad \delta > 0$$  (2.2.4)
and the cdf of the distribution is given as

\[ F(x) = 1 - (1-x)^\delta, \quad 0 < x < 1, \quad \delta > 0 \]  \hspace{1cm} (2.2.5)

In view of (2.2.4) and (2.2.5), we have

\[ \delta(1 - F(x)) = (1 - x)f(x) \]  \hspace{1cm} (2.2.6)

**Theorem 2.2.2: (Saran and Pandey, 2003)**

Fix a positive integer \( k \). For \( n \in \mathbb{N}, m \in \mathbb{Z}, 1 \leq r \leq n \) and \( j = 1, 2, 3, \ldots \)

\[ (\delta \gamma_{r+1} + j - t) M^{(j)}_{X(r+1, n, m, k)}(t) \]

\[ = \delta \gamma_{r+1} M^{(j)}_{X(r, n, m, k)}(t) + j M^{(j-1)}_{X(r+1, n, m, k)}(t) \]

\[- t M^{(j+1)}_{X(r+1, n, m, k)}(t) \]  \hspace{1cm} (2.2.7)

**Proof:** For \( n \in \mathbb{N}, m \in \mathbb{Z}, 1 \leq r \leq n \), we have

\[ M_{X(r+1, n, m, k)}(t) = \frac{C_r}{r!} \int_{-\infty}^{\infty} e^{tx} (1-F(x))^{r+1} f(x) g_{m}^{r} (F(x)) dx. \]  \hspace{1cm} (2.2.8)

Integrating (2.2.8) by parts taking \( ((1-F(x))^{r+1} f(x)) \) as the part to be integrated, we get

\[ M_{X(r+1, n, m, k)}(t) = M_{X(r, n, m, k)}(t) + \frac{C_r}{r!} \int_{-\infty}^{\infty} e^{tx} (1-F(x))^{r+1} \]

\[ \times f(x) g_{m}^{r} (F(x)) dx \]

the constant of integration vanishes since the integral considered in (2.2.8) is a definite integral. On using (2.2.6), we obtain

\[ M_{X(r+1, n, m, k)}(t) = M_{X(r, n, m, k)}(t) + \frac{C_r}{r!} \int_{0}^{t} \frac{1}{\delta \gamma_{r+1}} e^{\delta t} (1-F(x))^{r+1} (1-x) \]

\[ \times f(x) g_{m}^{r} (F(x)) dx \]
\[ M_{X(r,n,m,k)}(t) + \frac{t}{\delta \gamma_{r+1}} M_{X(r+1,n,m,k)}(t) \]

Differentiating both sides of (2.2.9) \( j \) times with respect to \( t \), we obtain for \( n \in N, m \in Z, 1 \leq r \leq n \) that

\[ M^{(j)}_{X(r+1,n,m,k)}(t) = M^{(j)}_{X(r,n,m,k)}(t) + \frac{t}{\delta \gamma_{r+1}} M^{(j)}_{X(r+1,n,m,k)}(t) \]

\[ + \frac{j}{\delta \gamma_{r+1}} M^{(j-1)}_{X(r+1,n,m,k)}(t) - \frac{t}{\delta \gamma_{r+1}} M^{(j+1)}_{X(r+1,n,m,k)}(t) \]

\[ - \frac{j}{\delta \gamma_{r+1}} M^{(j)}_{X(r+1,n,m,k)}(t) \]

which gives the recurrence relation as given in (2.2.7).

**Remark 2.2.3:** Setting \( m = -1 \) in theorem 2.2.2, we get the recurrence relation for marginal moment generating function of the \( k \)-th upper record values from power function distribution, thus verifying the results of Raqab and Ahsanullah (2000).

**Weibull Distribution**

The pdf of distribution is given as

\[ f(x) = \frac{p}{\theta} x^{p-1} e^{-x^{p}/\theta} \quad x > 0, \quad p, \theta > 0 \]  

(2.2.10)

and the corresponding distribution function (d.f.) is

\[ F(x) = 1 - e^{-x^{p}/\theta} \quad x > 0, \quad p, \theta > 0 \]  

(2.2.11)
In view of (2.2.10) and (2.2.11), we have

\[ 1 - F(x) = \frac{\theta}{p} x^{1-p} f(x) \]  

(2.2.12)

**Lemma 2.2.1: (Athar et al., 2007)**

For Weibull distribution as given in (2.2.10) and and non-negative finite integers \( a \) and \( b \).

\[ J_j(a,b) = \frac{1}{(m+1)} \sum_{l=0}^{b} (-1)^l \binom{b}{l} J_j[a + l(m + 1),0] \]  

(2.2.13)

\[ = \frac{\theta^{(j/p)+1} \Gamma((j/p)+1)}{p(m+1)^b} \sum_{l=0}^{b} (-1)^l \binom{b}{l} \frac{1}{[a + l(m + 1)]^{(j/p)+1}}, \quad m \neq -1 \]  

(2.2.14)

\[ = \frac{1}{p} \theta^{(j/p)+1} \Gamma((j/p)+b+1) \frac{1}{a^{(j/p)+b+1}}, \quad m = -1 \]  

(2.2.15)

where

\[ J_j(a,b) = \int_0^\infty x^{j/p-1} [1 - F(x)]^a g_m^b(F(x)) \, dx \]  

(2.2.16)

and

\[ J_j(a,0) = \left( \frac{1}{p} \right)^{(j/p)+1} \left( \frac{\theta}{a} \right)^{(j/p)+1} \Gamma((j/p)+1) \]  

(2.2.17)

**Proof: Case when \( m \neq -1 \)**

This can be proved by expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \{1 - [1 - F(x)]^{m+1} \} \right]^b \) binomially in (2.2.16) and then using the result (2.2.17).

**Case when \( m = -1 \):**
Moments of single generalized order statistics

At \( m = -1 \) in (2.2.14)

\[
J_j(a,b) = \frac{0}{0} \quad \text{as} \quad \sum_{i=0}^{b} (-1)^i \binom{b}{i} = (1-1)^b = 0
\]

Since (2.2.14) is of the form \( 0/0 \) at \( m = -1 \), so after applying L-Hospital's rule, we have

\[
\lim_{m \to -1} J_j(a,b) = \frac{1}{p} \theta^{(j/p)+1} \Gamma((j/p) + b) \frac{1}{a^{(j/p) + b+1}} \sum_{i=0}^{b} (-1)^{b+i} \binom{b}{i} \frac{i^b}{b!}, b > 0
\]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!
\]

therefore,

\[
\left\{ \sum_{i=0}^{b} (-1)^{b+i} \binom{b}{i} \frac{i^b}{b!} \right\} = 1.
\]

hence

\[
\lim_{m \to -1} J_j(a,b) = \frac{1}{p} \theta^{(j/p)+1} \Gamma((j/p) + b+1) \frac{1}{a^{(j/p) + b+1}}.
\]

**Theorem 2.2.3:** (Athar et al., 2007)

For Weibull distribution as given in (2.2.10) and \( \gamma_r \geq 1, k \geq 1, 1 \leq r \leq n \),

\( m \neq -1 \).

\[
\alpha_{r,n,m,k}^{(j-p)} = \frac{p}{\theta (r-1)!} J_{j-p}(\gamma_r, r-1)
\]  

(2.2.18)
\[ \alpha_r^{(j-p)} = \frac{C_{r-1}}{(r-1)!} \left( \frac{\theta^{(j/p)}}{(m+1)^{r-1}} \sum_{l=0}^{r-1} (-1)^l \binom{r-1}{l} \frac{1}{(\gamma_{r-l})^{j/p}} \right) \quad (2.2.19) \]

**Proof:** We have

\[ \alpha_r^{(j-p)} = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^{j-p}[1-F(x)]^{r-1} f(x) g_{m}^{r-1}(F(x))dx \quad (2.2.20) \]

Now on application of (2.2.12), we get

\[ \alpha_r^{(j-p)} = \frac{p}{\theta} \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^{j-1}[1-F(x)]^{r} g_{m}^{r-1}(F(x))dx \quad (2.2.21) \]

and hence the theorem, in view of (2.2.17).

**Remark 2.2.4:** If we put \( m = 0, k = 1 \) in (2.2.19), we get the result for order statistics

\[ \alpha_r^{(j-p)} = \alpha_{r:n}^{(j-p)} \]

\[ = C_{r:n} \theta^{(j/p)-1} \frac{\Gamma(j/p)}{\sum_{l=0}^{r-1} (-1)^l \binom{r-1}{l} \frac{1}{[n+l-r+1]^{j/p}}} \quad (2.2.22) \]

where \( C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \)

by noting that \( \gamma_i = n-i+1 \) and \( C_{r-1} = \frac{n!}{(n-r)!} \)

as obtained by Khan et al. (1984) and Ali and Khan (1996).

**Remark 2.2.5:** Moment of \( k-th \) record values from the Weibull distribution may be obtained in view of (2.2.15) and (2.2.18) at \( m = -1 \).

\[ \alpha_r^{(j-p)} = \frac{1}{(r-1)!} \left( \frac{\theta}{k} \right)^{(j/p)-1} \Gamma[(j/p) + r - 1] \quad (2.2.23) \]
by noting $\gamma_j = k$ and $C_{r-1} = k^r$.

**Remark 2.2.6:** For $m = 0$ and $k = \alpha - n + 1$, $\alpha \in \mathbb{R}_+$, we get the moment of order statistics with non-integral sample size

$$
\alpha_{r;\alpha}^{(j/p)} = C_{r;\alpha} \theta^{(j/p)-1} \Gamma(j/p) \sum_{l=0}^{r-1} (-1)^l \binom{r-1}{l} \frac{1}{[\alpha + l - r + 1]^{j/p}}
$$

(2.2.24)

**Remark 2.2.7:** At $j = p$ in (2.2.19), we have

$$
\sum_{l=0}^{r-1} (-1)^l \frac{1}{l!(r-l-1)! \gamma_{r-l}} = \frac{(m+1)^{r-1}}{\prod_{j=1}^{r} \gamma_j}, \ m \neq -1
$$

(2.2.25)

a useful combinatorial identity.

**Doubly Truncated Weibull Distribution**

The pdf of distribution is given as

$$
f_1(x) = px^{p-1} e^{-x^{p}}, \ x > 0, \ p > 0
$$

(2.2.26)

and the corresponding df is

$$
F_1(x) = 1 - e^{-x^{p}}, \ x > 0, \ p > 0
$$

(2.2.27)

Now if for given $P_1$ and $Q_1$

$$
\int_{0}^{Q_1} f_1(x)dx = Q \quad \text{and} \quad \int_{0}^{P_1} f_1(x)dx = P
$$

then the truncated pdf is given by

$$
f(x) = \frac{px^{p-1} e^{-x^{p}}}{P - Q}, \ -\log(1 - Q) \leq x^{p} \leq -\log(1 - P), \ p > 0
$$
and the corresponding truncated df $F(x)$ is

$$F(x) = -P_2 + \frac{1}{p}x^{1-p} f(x),$$  \hspace{1cm} \text{(2.2.29)}$$

where

$$Q_1^P = -\log(1-Q), \quad P_1^P = -\log(1-P), \quad Q_2 = \frac{1-Q}{P-Q} \quad \text{and} \quad P_2 = \frac{1-P}{P-Q}$$

**Theorem 2.2.4:** (Khan et al., 2007)

For the given Weibull distribution and $n \in N, m \in \mathbb{R}, 2 \leq r \leq n$.

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)]$$

$$= -P_2 K \{E[X^j(r,n-1,m,k + m)] - E[X^j(r-1,n-1,m,k + m)]}\]$$

$$+ \frac{j}{p \gamma_1} E[X^{j-p}(r,n,m,k)]$$  \hspace{1cm} \text{(2.2.30)}$$

where

$$K = \frac{C_{r-2}^{(n-1)}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m} \right), \quad \gamma_i^{(n-1)} = k + (n-1-i)(m+1).$$

**Proof:** From equations (2.2.29) and (1.11.3), we have

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)]$$

$$= \frac{C_{r-1}}{\gamma_1(r-1)!} \int_{Q_1} \cdots \int_{Q_1} x^{j-1} [F(x)]^{r-1} \left\{ -P_2 + \frac{1}{p}x^{1-p} f(x) \right\}$$

$$\times g_m^{-1}(F(x)) \, dx$$
Moments of single generalized order statistics

\[ \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_{\mathcal{Q}_1} x^{j-1} [\mathcal{F}(x)]^{r-1} g_m^{r-1}(F(x)) \, dx \]

\[ + \frac{j}{p \gamma_1 (r-1)!} \int_{\mathcal{Q}_1} x^{j-p} [\mathcal{F}(x)]^{r-1} f(x) g_m^{r-1}(F(x)) \, dx \]

\[ = -P_2 \frac{C_{r-2}}{(r-1)!} j \int_{\mathcal{Q}_1} x^{j-1} [\mathcal{F}(x)]^{r-1} g_m^{r-1}(F(x)) \, dx \]

as \( \gamma_r - 1 = \gamma_r^{(n-1,k+m)} = (k + m) + (n - 1 - r)(m + 1) \), \( C_{r-1} = \gamma_1 C_{r-2}^{(n-1)} \) and hence the required result.

If we put \( p = 1 \) in the above expression, we get corresponding result for the exponential distribution. For the non-truncated case one has to put \( P = 1, \mathcal{Q} = 0 \).

**Remark 2.2.8:** Recurrence relation for single moments of order statistics \((m = 0, k = 1)\) is

\[ E(X^j_{r:n}) - E(X^j_{r-1:n-1}) = -P_2 \{ E(X^j_{r:n-1}) - E(X^j_{r-1:n-1}) \} \]

\[ + \frac{j}{np} E(X^j_{r:n} - p) \] \hspace{1cm} (2.2.31)

or

\[ E(X^j_{r:n}) = Q_2 E(X^j_{r-1:n-1}) - P_2 E(X^j_{r-1:n-1}) + \frac{j}{np} E(X^j_{r:n} - p) \] \hspace{1cm} (2.2.32)

For \( r = 1 \)

\[ E(X^j_{1:n}) = Q_2 Q_1^j - P_2 E(X^j_{1:n-1}) + \frac{j}{np} E(X^j_{1:n} - p). \] \hspace{1cm} (2.2.33)

For \( n = n \)
where by convention we use $X_{n:n-1} = P_1$ and $X_{0:n} = Q_1$ as obtained by Khan et al. (1983a).

**Remark 2.2.9:** For $k$-th record statistics ($m = -1$) recurrence relation for single moments reduces as

$$E(X^j_r) - E(X^j_{r-1}) = -P_2 \left( \frac{k}{k-1} \right)^{r-1} \left\{ E(X^j_r) - E(X^j_{r-1}) \right\}$$

$$+ \frac{j}{p^k} E(X^{j-p}_r)^k$$

as $K = C_{r-2}^{(n-1)} / C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1} \left( \frac{k}{k-1} \right), \gamma_1 = k$, for $m = -1$.

Similarly, the recurrence relations for single moments of order statistics with non-integral sample size for $m = 0$, $k = \alpha - n + 1$, $\alpha \in \mathbb{R}_+$ and for sequential order statistics for $m = \alpha - 1$, $k = \alpha$ may be obtained.

**Theorem 2.2.5:** (Khan et al., 2007)

For the given Weibull distribution and $n \in N, m \in \mathbb{R}, 2 \leq r \leq n$.

$$E[X^j (r,n,m,k)] - E[X^j (r-1,n-1,m,k)]$$

$$= \frac{(P-Q)}{p^{\gamma_1}} K^* \; j \; E[\phi(X(r,n,m,k+1))]$$

$$= \frac{j}{p^{\gamma_1}} \left\{ -(1-P) E[\phi(X(r,n,m,k))] \right\}$$

$$+ E[X^{j-p} (r,n,m,k)]$$

$$= \frac{(P-1)}{p^{\gamma_1}} \; j \; E[\phi(X(r,n,m,k))]$$

where
Moments of single generalized order statistics

\[
\phi(x) = x^{j-p}e^{x^p}, \quad K^* = \frac{C_{r-1}}{C_{r-1}^{(k+1)}} = \prod_{i=1}^{r} \left( \frac{\gamma_i}{\gamma_i+1} \right).
\]

**Proof:** In view of equation (2.2.28), (1.9.9) becomes

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)]
\]

\[
= \frac{C_{r-1}}{\gamma_1(r-1)!} \int_0^1 x^{j-1} [\overline{F}(x)]^{r-1} \cdot
\]

\[
\left\{ \left( \frac{P-Q}{p} \right) f(x) \right\} \cdot g_m^{r-1}(F(x))dx
\]

\[
= \frac{(P-Q)C_{r-1}}{p\gamma_1 C_{r-1}^{(k+1)}} \cdot \int_0^1 \left[ \frac{C_{r-1}}{(r-1)!} \cdot \int_0^1 \phi(x) [\overline{F}(x)]^{r-1} \cdot
\]

\[
f(x) g_m^{r-1}(F(x))dx \right\}
\]

where \( \gamma_r^{(k+1)} = (k+1) + (n-r)(m+1) \) and hence the Theorem.

To prove (2.2.36), note that

\[
\frac{\overline{F}(x)}{f(x)} = -\frac{1}{p} \{ (1-P)x^{j-p}e^{x^p} - x^{1-p} \}
\]

and the result follows from (1.9.7).

**Theorem 2.2.6:** (Khan et al., 2007)

For the given Weibull distribution and \( n \in \mathbb{N}, m \in \mathbb{R}, 2 \leq r \leq n \).

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]
\]

\[
= -P_2 K^{**} \{ E[X^j(r,n-1,m,k+m)]
\]

\[
- E[X^j(r-1,n-1,m,k+m)] \}
\]

\[
+ \frac{j}{pg_r} E[X^{j-p}(r,n,m,k)]
\]

(2.2.37)
where \( K^{**} = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i^{(n-1)} + m} \right) = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i - 1} \right) \).

**Proof:** Proof follows on the lines of Theorem 2.2.5 using (2.2.29) and (1.9.7).

**Remark 2.2.10:** The recurrence relation for the non-truncated exponential distribution given by Pawlas and Syznal (2001 a) are obtained by setting \( p = 1, \ Q = 0 \) and \( j = j + 1. \)

**Theorem 2.2.7:** (Khan et al., 2007)

For the given Weibull distribution and \( n \in N, \ m \in \mathfrak{R}, \ 2 \leq r \leq n. \)

\[
E[X^j(r-1,n,m,k)] - E[X^j(r-1,n-1,m,k)]
= P_2 \frac{(m+1)(r-1)K^{**}}{\gamma_1} \left\{ E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)] \right\}
= \frac{(m+1)(r-1)}{p \gamma_1} j E[X^{j-p}(r,n,m,k)]
\]

**Proof:** Proof follows from (2.2.29) and (1.9.8).

**Case II:** \( \gamma_i \neq \gamma_j, \ i, j = 1, 2, \cdots, n-1. \)

**Theorem 2.2.8:** (Khan et al., 2007)

For distribution given (2.2.27) and \( n \in N, \ m \in \mathfrak{R}, \ 2 \leq r \leq n, \ k \geq 1. \)

\[
E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n-1,\tilde{m},k)]
= \frac{j}{p \gamma_1} \{- (1 - P) E[\phi(X(r,n,\tilde{m},k))] + E[X^{j-p}(r,n,\tilde{m},k)]\}
\]

\[\text{(2.2.40)}\]
\[
\frac{(P - Q)}{\rho_{\gamma_1}} K^* j E[X^r_{n,m,k+1}]_{\gamma_1} \]  (2.2.41)

**Proof:** In view of Athar and Islam (2004), we have

\[
E[X^r_{n,m,k}] - E[X^r_{n-1,m-1,k}] = \frac{\gamma_1}{\gamma_1} C_{r-2} j \int_0^1 \sum_{i=1}^r a_i(r) [F(x)]^{\gamma_i} \, dx \]  (2.2.42)

On using equation (2.2.29), RHS of (2.2.42) becomes

\[
= \frac{\gamma_1}{\gamma_1} C_{r-2} j \int_0^1 \sum_{i=1}^r a_i(r) [F(x)]^{\gamma_i-1} \left\{ -P_2 + \frac{1}{p} x^{1-p} f(x) \right\} \, dx
\]

and hence the required result.

Result (2.2.41) can be proved by using (2.2.28) in (2.2.42).

**Remark 2.2.11:** Theorem 2.2.5 can be deduced from Theorem 2.2.8 by replacing \(m\) with \(\tilde{m}, m \neq -1\). Remaining results for case II, \((\gamma_i \neq \gamma_j)\) can also be obtained by replacing \(m\) with \(\tilde{m}\) in Theorem 2.2.6 and 2.2.7.

**Linear Exponential Distribution**

The pdf of distribution is given as

\[
F(x) = 1 - \exp[-(\lambda x + \nu x^2/2)], \quad 0 \leq x < \infty \]  (2.2.43)

and the corresponding df is

\[
F(x) = 1 - \exp[-(\lambda x + \nu x^2/2)], \quad 0 \leq x < \infty \]  (2.2.44)

In view of (2.2.43) and (2.2.44), we have

\[
f(x) = (\lambda + \nu x) F(x) \]  (2.2.45)

**Theorem 2.2.9:** (Ahmad, 2008)
Moments of single generalized order statistics

If $X$ is a rv distributed as the df (2.2.44), then for real $m, k$ with $m \geq -1, k \geq 1$, integers $r \geq 1, j \geq 1$, the following recurrence relations:

$$
\mu_{r,n,m,k}^{(j+2)} - \mu_{r-1,n,m,k}^{(j+2)} = \frac{\lambda (j+2)}{\nu (j+1)} \mu_{r-1,n,m,k}^{(j+1)}
$$

$$
- \frac{(j+2)}{\nu} \left[ \frac{\lambda}{(j+1)} \mu_{r,n,m,k}^{(j+1)} - \frac{1}{\nu} \mu_{r,n,m,k}^{(j)} \right]
$$

(2.2.46)

is satisfied.

**Proof:** If $X$ has the df (2.2.44), then we have

$$
\mu_{r,n,m,k}^{(j)} = \frac{C_{r-1}}{\Gamma (r)} \int_0^\infty x^j g_m^{r-1} (F(x)) \bar{F}^{r-1} f(x) dx
$$

or, equivalently, from (2.2.45),

$$
\mu_{r,n,m,k}^{(j)} = \frac{C_{r-1}}{\Gamma (r)} \int_0^\infty x^j g_m^{r-1} (F(x)) \bar{F}^{r-1} (\lambda + \nu x) f(x) dx
$$

which can be rewritten as

$$
\mu_{r,n,m,k}^{(j)} = \frac{\lambda C_{r-1}}{\Gamma (r)} \int_0^\infty x^j g_m^{r-1} (F(x)) \bar{F}^{r} f(x) dx
$$

$$
+ \frac{\nu C_{r-1}}{\Gamma (r)} \int_0^\infty x^{j+1} g_m^{r-1} (F(x)) \bar{F}^{r} f(x) dx
$$

$$
= \lambda Q_{r,n,m,k}^j + \nu Q_{r,n,m,k}^{j+1}
$$

(2.2.47)

where

$$
Q_{r,n,m,k}^j = \frac{C_{r-1}}{\Gamma (r)} \int_0^\infty x^j g_m^{r-1} (F(x)) \bar{F}^{r} f(x) dx
$$
Moments of single generalized order statistics

\[Q_{r,n,m,k}^{j} = \frac{\gamma}{j+1} \left[ \mu_{r,n,m,k}^{(j+1)} - \mu_{r-1,n,m,k}^{(j+1)} \right] \]

(2.2.48)

Substituting from (2.2.48) in (2.2.47), we can obtain the result (2.2.46).

**Remark 2.2.12:** Putting \( r = 1 \) in (2.2.46) and using \( \mu_{0,n,m,k}^{(j)} = 0 \), we obtain

\[\mu_{1,n,m,k}^{(j+2)} = \frac{j+2}{\nu} \left[ \frac{1}{\gamma} \mu_{1,n,m,k}^{(j)} - \frac{\lambda}{j+1} \mu_{1,n,m,k}^{(j)} \right] \]

(2.2.49)

**Remark 2.2.13:** Putting \( m = 0, k = 1 \) in (2.2.46), we obtain a recurrence relation for single moments of order statistics from the linear exponential distribution of the form

\[\mu_{r,n}^{(j+2)} - \mu_{r-1,n}^{(j+2)} = \frac{\lambda(j+2)}{\nu(j+1)} \mu_{r-1,n}^{(j+1)} - \frac{(j+2)}{\nu} \left[ \frac{\lambda}{j+1} \mu_{r,n}^{(j+1)} - \frac{1}{(n-r+1)} \mu_{r,n}^{(j)} \right] \]

(2.2.50)

The recurrence relations (2.2.50) coincide with the results of Balakrishnan and Malik (1986).

**Remark 2.2.14:** Letting \( \nu \to 0 \) and \( \lambda = 0 \) in (2.2.46) and (2.2.49), we obtain recurrence relations for the exponential and Rayleigh distributions, respectively.
3. Moments of general class of distributions

a) Let the general form of distribution be

\[ 1 - F(x) = [ah(x) + b]c, \quad \alpha \leq x \leq \beta \]  

(2.3.1)

where \( a, b \) and \( c \) are s.t \( F(\alpha) = 0, F(\beta) = 1 \) and \( h(x) \) is a monotonic and differentiable function of \( x \) in the interval \([\alpha, \beta]\). then,

\[ 1 - F(x) = -\frac{ah(x) + b}{cah'(x)} f(x) \]  

(2.3.2)

**Theorem 2.3.1: (Athar and Islam 2004)**

For general class of distribution given in (2.3.1) and \( n \in \mathbb{N}, m \in \mathbb{R}, 2 \leq r \leq n \).

\[ E[\xi\{X(r,n,m,k)\}] = E[\xi\{X(r-1,n,m,k)\}] - \frac{1}{\gamma_r ca} E[\psi\{X(r,n,m,k)\}] \]  

(2.3.3)

where \( \psi(x) = [ah(x) + b], w(x) = \frac{\xi'(x)}{h'(x)} \).

**Proof:** In view of equation (2.3.2) and (1.9.11), we have

\[
E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}] = -\frac{C_{r-2}}{(r-1)!} \beta \xi'(x) [1 - F(x)]^{r-1} \left\{-\frac{ah(x) + b}{cah'(x)} f(x)\right\} \\
\times g_m r^{-1}(F(x)) dx
\]

\[
= -\frac{1}{\gamma_r ca (r-1)!} \beta \psi(x) [1 - F(x)]^{r-1} g_m r^{-1}(F(x)) \\
\times f(x) dx
\]

rearranging the terms, we find the result.
Remark 2.3.1: Recurrence relation for single moments of order statistics 
(at \( m = 0, k = 1 \)) is

\[
E[\xi(X_{r,n})] = E[\xi(X_{r-1,n})] - \frac{1}{(n - r + 1)c_{\alpha}} E[\nu(X_{r,n})]
\]

as obtained by Ali and Khan (1997).

Remark 2.3.2: Recurrence relation for single moments of \( k^{th} \) record values will be

\[
E[\xi\{X(r,n-1,k)\}] = E[\xi\{X(r-1,n-1,k)\}] - \frac{1}{k_{\alpha}} E[\nu\{X(r,n-1,k)\}]
\]

or

\[
E[\xi(X_{r^{(k)}})] = E[\xi(X_{r-1}^{(k)})] - \frac{1}{k_{\alpha}} E[\nu(X_{r}^{(k)})]
\]

where \( X_{r^{(k)}}, r = 1,2,... \) is \( r-th \) \( k \) records.

Remark 2.3.3: For \( m = 0 \) and \( k = \alpha - n + 1, \alpha \in \mathbb{R}_+ \), we obtain the recurrence relation for single moment of order statistics with non-integral sample size as

\[
E[\xi(X_{r,\alpha})] = E[\xi(X_{r-1,\alpha})] - \frac{1}{(\alpha - r + 1)c_{\alpha}} E[\nu(X_{r,\alpha})]
\]

Remark 2.3.4: For \( m = \alpha - 1, k = \alpha \), the recurrence relation of sequential order statistics is

\[
E[\xi\{X(r,n,\alpha - 1,\alpha)\}] = E[\xi\{X(r-1,n,\alpha - 1,\alpha)\}]
\]

\[
- \frac{1}{\alpha(n - r + 1)c_{\alpha}} E[\nu\{X(r-1,n,\alpha - 1,\alpha)\}]
\]

Theorem 2.3.2: (Athar and Islam 2004)
Moments of single generalized order statistics

For given distribution and \( n \in \mathbb{N} \), \( m^* = (m_2 = m_3 = \ldots = m_{n-1}) \in \mathcal{R} \), \( 2 \leq r \leq n \).

(i) \( \gamma_1 \gamma_r E[\xi\{X(r-1,n,m,k)\} - E[\xi\{X(r-1,n-1,m^* ,k)\}]
\]
\( \quad = \frac{((m+1)(r-1))}{ca} E[\psi\{X(r,n,m,k)\}] \)

(ii) \( \gamma_i E[\xi\{X(r,n,m,k)\} - E[\xi\{X(r-1,n-1,m^* ,k)\}]
\]
\( \quad = -\frac{1}{ca} E[\psi\{X(r,n,m,k)\}] \)

**Proof:** Result can be estabilized in view Result 10 and equation (2.3.2).

**Case I:** \( \gamma_i \neq \gamma_j \).

**Theorem 2.3.3:** (Athar et al., 2004)

For distribution given in (2.3.1) and \( n \in \mathbb{N} \), \( \bar{m} \in \mathcal{R} \), \( 2 \leq r \leq n \).

\( E[\xi\{X(r,n,\bar{m},k)\}] = E[\xi\{X(r-1,n,\bar{m},k)\}] - \frac{1}{\gamma_r ca} E[\psi\{X(r,n,\bar{m},k)\}] \)

where

\( \psi(x) = [ah(x) + b]\omega(x) \quad \omega(x) = \frac{\xi'(x)}{h'(x)} \)

**Proof:** see reference.

**Theorem 2.3.4:** (Athar and Islam 2004)

For distribution given in (2.3.1) and \( \bar{m}^* = (m_2 = m_3 = \ldots = m_{n-1}) \in \mathcal{R} \),

\( 2 \leq r \leq n \).

(i) \( \gamma_1 \gamma_r [E(\xi\{X(r-1,n,\bar{m},k)\})] - [E(\xi\{X(r-1,n-1,\bar{m}^* ,k)\})]
\]
\( \quad = \frac{1}{ca} \left\{(r-1) + \sum_{j=1}^{r-1} m_j\right\} E[\psi\{X(r,n,m,k)\}] \)
Moments of single generalized order statistics

(ii) \( \gamma_1 [E(\xi \{ X(r, n, \tilde{m}, k) \})] - [E(\xi \{ X(r - 1, n - 1, \tilde{m}^*, k) \})] \)

\[ = - \frac{1}{ca} E[\psi \{ X(r, n, \tilde{m}, k) \}]. \]

**Proof:** see reference.

**Remark 2.3.3:** Recurrence relations for product moments of generalized order statistics from Pareto, Power function, Weibull, Burr type XII, beta of first kind and Cauchy distributions may be obtained with proper choice of \( a, b, c \) and \( h(x) \) as given by Khan and Abouammoh (2000).

b) Let the general form of the distribution

\[ \bar{F}(x) = (ax + b)^c, \quad \alpha < x < \beta \quad (2.3.5) \]

and

\[ X(r) = \frac{1}{b} \left[ \prod_{j=1}^{r} B_j^{\frac{1}{c}} - b \right] \quad (2.3.6) \]

**Theorem 2.3.5: (Khan et al., 2007)**

For the distribution given in (2.3.5),

\[ E[X^p(r, n, m, k)] = \frac{1}{a^p} \sum_{i=0}^{p} (-b)^{p-i} \binom{p}{i} \frac{C_r^{(k)}}{C_{r-1}^{(k+i)}} \quad (2.3.7) \]

where \( p \) is a positive integer and

\[ C_r^{(k+i)} = \prod_{j=1}^{r} \gamma_j^{(k+i)}, \quad \gamma_j^c = k + \frac{j}{c} + (n - j)(m + 1) \quad (2.3.8) \]

**Proof:** From (2.3.6), we have

\[ E[X^p(r, n, m, k)] = E\left[ \frac{1}{a^p} \left( \prod_{j=1}^{r} B_j^{\frac{1}{c}} - b \right)^p \right]. \]
Moments of single generalized order statistics

\[ = (-1)^p \left( \frac{b}{a} \right)^p E \left( 1 - \frac{1}{b} \prod_{j=1}^{r} B_j^{i/c} \right)^p \]

\[ = (-1)^p \left( \frac{b}{a} \right)^p \sum_{i=0}^{p} (-1)^i \left( \frac{p}{i} \right) \frac{1}{b^i} \prod_{j=1}^{r} E(B_j^{i/c}) \]

\[ = (-1)^p \left( \frac{b}{a} \right)^p \sum_{i=0}^{p} (-1)^i \left( \frac{p}{i} \right) \frac{1}{b^i} \prod_{j=1}^{r} \frac{c y_j}{c y_j + i} \]

\[ = (-1)^p \left( \frac{b}{a} \right)^p \sum_{i=0}^{p} (-1)^i \left( \frac{p}{i} \right) \frac{1}{b^i} \prod_{j=1}^{r} \frac{y_j}{(k + i c) \prod_{j=1}^{n-1} y_j} \]

and hence the result.

**Remark 2.3.4:** For order statistics \((m = 0, k = 1)\), we have

\[ E(X_r^{a,n}) = \frac{1}{a} \sum_{i=0}^{p} (-b)^{p-i} \left( \frac{p}{i} \right) \frac{\Gamma(n+1) \Gamma(n+1+i-r)}{\Gamma(n-r+1) \Gamma(n+1+i)} \]  

(2.3.9)

**Remark 2.3.5:** For record values \((m = -1)\), we have

\[ E(X^p(r,n,-1,k)) = \frac{1}{a} \sum_{i=0}^{p} (-b)^{p-i} \left( \frac{p}{i} \right) \left( \frac{k}{k + \frac{i}{c}} \right) \]  

(2.3.10)

Similarly, single moments of order statistics with non-integral sample size \((m = 0, y_r = \alpha - r + 1, \alpha \in \mathbb{R}_+\)) may also be obtained.

**Examples**

(I) Power function distribution

\[ \bar{F}(x) = \left( \frac{\beta - x}{\beta - \alpha} \right)^\theta = \left[ - \frac{1}{\beta - \alpha} x + \frac{\beta}{\beta - \alpha} \right]^\theta, \ \alpha \leq x \leq \beta \]
Moments of single generalized order statistics

\[
\begin{align*}
\alpha &= -\frac{1}{\beta - \alpha}, \quad b = \frac{\beta}{\beta - \alpha}, \quad c = \theta
\end{align*}
\]

Thus, from (2.3.7), we have

\[
E[X^{\beta}(r,n,m,k)] = (\beta - \alpha)^{\beta} \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \left( \frac{\beta}{\beta - \alpha} \right)^{p-i} \binom{p}{i} \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k+i)}}
\]

(2.3.11)

If we take \( \alpha = 0 \) and \( \theta = 1 \), we have the results for uniform \( U(0,\beta) \) distribution as

\[
E[X^{\beta}(r,n,m,k)] = \beta^p \sum_{i=0}^{\beta} (-1)^i \binom{p}{i} \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k+i)}}
\]

(2.3.12)

as obtained by Kamps (1995).

The \( p - \text{th} \) moments of order statistics (\( m = 0, k = 1 \)) is

\[
E[X^{\beta}(r,n,m,k)] = \beta^p \sum_{i=0}^{\beta} (-1)^i \binom{p}{i} \frac{n!}{(n+i)!} \frac{(n+i-r)!}{(n-r)!}
\]

(2.3.13)

\[
E[X_{r:n}] = \frac{\beta r}{(n+1)}, \text{ as obtained by Malik (1967).}
\]

The \( p - \text{th} \) moments of record values (at \( m = -1 \)) is

\[
E[X^{\beta}(r,n,-1,k)] = \beta^p \sum_{i=0}^{\beta} (-1)^i \binom{p}{i} \left( \frac{k}{k+i} \right)^r
\]

as obtained by Kamps (1995) and

\[
E[X^{\beta}(r,n,-1,k)] = \beta^p \sum_{i=0}^{\beta} (-1)^i \binom{p}{i} \left( \frac{k}{k+i} \right)^r
\]

[See Grudzień and Szynal (1983) and Nagaraja (1978)].
Moments of single generalized order statistics

(II) Pareto distribution

\[ F(x) = \left( \frac{\mu + \delta}{x + \delta} \right)^\theta = \left[ \frac{1}{\mu + \delta} \left( x + \frac{\delta}{\mu + \delta} \right) \right]^\theta , \quad \mu \leq x \leq \infty \]

Here

\[ a = \frac{1}{\mu + \delta}, \quad b = \frac{\delta}{\mu + \delta}, \quad c = \theta \]

Then from (2.3.7), we have

\[ E[X^p(r, n, m, k)] = (\mu + \delta)^p \sum_{i=0}^{p} \left( \frac{-\delta}{\mu + \delta} \right)^{p-i} \binom{p}{i} \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k-l)}} \]  

(2.3.14)

at \( \delta = 0 \), we have

\[ E[X^p(r, n, m, k)] = \mu^p \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k-l)}} \]

as obtained by Kamps (1995).

The \( p \)-th moments of order statistics \((m = 0, k = 1)\) is

\[ E[X_{r:n}^p] = \mu^p \frac{n!}{(n-r)!} \frac{\Gamma(n-r+1-P)}{\Gamma(n+P)} \]

as obtained by Malik (1966).

The \( p \)-th moments of record values (at \( m = -1 \)) is

\[ E[X^p(r, n, -1, k)] = \mu^p \left( \frac{k}{k - \frac{P}{\theta}} \right)^r \quad \text{and} \quad E[X \ (r, n, -1, k)] = \mu \left( \frac{k}{k - \frac{1}{\theta}} \right)^r \]
Moments of single generalized order statistics

[See Grudzień and Szynal (1983), Nagaraja (1978) and Ahsanullah and Houchens (1989)].

(III) Exponential distribution

\[ F(x) = (ax + b)^c \]

Let \( a = -\frac{\lambda}{c} \), \( b = 1 \), then we have

\[ \lim_{{c \to \infty}} F(x) = e^{-\lambda x} \]

We have, (Athar et al., 2007)

\[ \sum_{{u=0}}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k)}} = \frac{(m+1)^{r-1}(r-1)!}{\prod_{j=1}^{r} \gamma_{j}^{(k)}} \]  \hspace{1cm} (2.3.15)

Thus using (2.3.15) in (2.3.7), we have

\[ E[X^p(r,n,m,k)] = \frac{(-1)^p (\lambda)^{-p}}{c^p(m+1)^{r-1}(r-1)!} \sum_{{i=0}}^{p} (-1)^{p-i} \binom{p}{i} C_{r-1}^{(k)} \]  \hspace{1cm} (2.3.16)

At \( c' = \frac{1}{c} = 0 \), (2.3.16) is of the form \( \frac{0}{0} \) as \( \sum_{{i=0}}^{p} (-1)^i \binom{p}{i} = 0 \).

Therefore applying L-Hospital’s rule, we have

\[ E[X^p(r,n,m,k)] = \frac{(\lambda)^{-p}}{p!(m+1)^{r-1}(r-1)!} \sum_{{i=0}}^{p} (-1)^i \binom{p}{i} C_{r-1}^{(k)} \]

\[ \times \sum_{{u=0}}^{r-1} (-1)^u \binom{r-1}{u} \frac{(-i)^p p!}{\gamma_{r-u}^{(k)} p+1} \]  \hspace{1cm} (2.3.17)

Now using the result (Ruiz, 1996)

\[ \sum_{{u=0}}^{n} (-1)^u \binom{n}{u} (x-u)^n = n! \]  \hspace{1cm} (2.3.18)
We have,

$$E[X^p(r,n,m,k)] = \frac{(\lambda)^{-p} p! C^{(k)}_{r-1}}{(m + 1)^{r-1}(r - 1)!} \times \sum_{u=0}^{r-1} (-1)^u \binom{r - 1}{u} \frac{1}{(n - r + u + 1)^{p+1}}, \ m \neq -1$$  \hspace{1cm} (2.3.19)

This result was given by Kamps (1995).

For $m = 0$, $k = 1$, we have

$$E[X^p(r,n,m,k)] = r \sum_{u=0}^{n-r} \frac{(\lambda)^{-p} p!}{(r - 1)!(r - 1)!} \times \sum_{u=0}^{r-1} (-1)^u \binom{r - 1}{u} \frac{1}{(n - r + u + 1)^{p+1}}$$  \hspace{1cm} (2.3.20)

as obtained by Lieblein (1955).

When $m = -1$, applying L-Hospital's rule in (2.3.19), we have

$$E[X^p(r,n,m,k)] = \frac{(\lambda)^{-p} p! k^r}{(r - 1)!(r - 1)!} \times \sum_{u=0}^{r-1} (-1)^u \binom{r - 1}{u} (-1)^{r-1}(p + 1)(p + 2)\cdots(p + r - 1)$$

$$\times(n - r + u)^{r-1}(k)^{-p-r}$$  \hspace{1cm} (2.3.21)

Applying (2.3.18) in (2.3.21), we get

$$E[X^p(r,n,m,k)] = \frac{(\lambda)^{-p} p! k^r}{(r - 1)!(r - 1)!} \frac{(p + 1)(p + 2)\cdots(p + r - 1)(r - 1)!}{k^{p+r}}$$

$$= \frac{(\lambda)^{-p} \Gamma(p + r)}{(r - 1)! k^r}, \ m = -1$$  \hspace{1cm} (2.3.22)

as obtained by Kamps (1995).
Moments of single generalized order statistics

and

\[ E[X (r,n,-1,k)] = \frac{\lambda^{-1}}{(r-1)!} \frac{\Gamma(1+r)}{k} = \frac{r}{\lambda k} \]

[See Grudzień and Szynal (1983), Nagaraja (1978)].

c) Let the general class of the distribution be

\[ \overline{F}(x) = [ax^p + b]^c, \quad p > 0, \eta < x < \omega, \]

(2.3.23)

where \(a, b\) and \(c\) are so chosen that \(F(x)\) is a df over \((\eta, \omega)\).

For \(m_1 = m_2 = \cdots = m_n = m\),

\[ X(r,n,m,k) \sim \left[ \frac{1}{a} \left( \prod_{j=1}^{r} B_j^{1/c} - b \right) \right]^{1/p}. \]

(2.3.24)

**Theorem 2.3.6: (Faizan and Athar, 2009)**

For the distribution given in (2.2.23) and \(c^* = 1, 2, \ldots\)

\[ E[X^\alpha (r,n,m,k)] = \left( -\frac{b}{a} \right)^{[\alpha / p]} \sum_{i=0}^{[\alpha / p]} (-1)^i \frac{1}{b^i} \left( \frac{\alpha / p}{i} \right) \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k+i)}} \]

(2.3.25)

where \([\alpha / p]\) represent the integer part of \(\alpha / p\) and

\[ C_{r-1}^{(k+i)} = \prod_{j=1}^{r} \gamma_j^{(k+i)}, \quad \gamma_j^{(k+i)} = k + \frac{i}{c} + (n-j)(m+1) \]

**Proof:** From (2.3.24), we have

\[ E[X^\alpha (r,n,m,k)] = E \left[ \frac{1}{a} \left( \prod_{j=1}^{r} B_j^{1/c} - b \right)^{[\alpha / p]} \right] \]
Moments of single generalized order statistics

\[
\left(-\frac{b}{a}\right)^{[\alpha/p]} \sum_{i=0}^{[\alpha/p]} (-1)^i \frac{1}{b^i} \left(\begin{array}{c} [\alpha/p] \\ i \end{array}\right) \prod_{j=1}^{r} \frac{c \gamma_j}{\gamma_j + i}
\]

and hence the result.

**Remark 2.2.1:** At \( p = 1 \) in (2.3.25), we get

\[
E[X^\alpha(r,n,m,k)] = (-1)^\alpha \left(\frac{b}{a}\right)^\alpha \sum_{i=0}^{\alpha} (-1)^i \frac{1}{b^i} \left(\begin{array}{c} \alpha \\ i \end{array}\right) \frac{C_r^{(k)}}{C_{r-1}^{(k-\frac{1}{r})}}
\]

as obtained by Khan et al. (2008).

**Examples**

(I) **Burr distribution**

\[
F(x) = [\theta x^p + 1]^{-\mu}, \quad 0 < x < \infty
\]

where \( p = 1/\xi > 0 \) and \( \xi \) in an integer.

At \( a = \theta, b = 1 \) and \( c = -\mu \) in (2.3.23), we get

\[
E[X^\alpha(r,n,m,k)] = \left(-\frac{1}{\theta}\right)^{[\alpha/p]} \sum_{i=0}^{[\alpha/p]} (-1)^i \frac{1}{b^i} \left(\begin{array}{c} \alpha \\ i \end{array}\right) \frac{C_r^{(k)}}{C_{r-1}^{(k-\frac{1}{r})}}
\]

(II) **Weibull distribution**

\[
F(x) = [ax^p + b]^c, \quad p > 0, \quad 0 < x < \infty.
\]

Let \( a = -\frac{\lambda}{c}, b = 1 \), then we have

\[
\lim_{c \to \infty} F(x) = e^{-\lambda x^p}
\]

by an application of result (Athar et al., 2009),
\[
\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k)}} = \frac{(m+1)^{r-1}(r-1)!}{\prod_{j=1}^{r} \gamma_{j}^{(k)}}
\]

we have

\[
E[X^\alpha (r, n, m, k)] = \frac{(-1)^{[\alpha / p]}(\lambda)^{-[\alpha / p]}}{e^{[\alpha / p]}(m+1)^{r-1}(r-1)!} \times \sum_{i=0}^{[\alpha / p]} (-1)^i \binom{[\alpha / p]}{i} C_{r-1}^{(k)} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k+ic)}}
\]

at \( c' = \frac{1}{c} = 0 \), is of the form \[ \sum_{i=0}^{[\alpha / p]} (-1)^i \binom{[\alpha / p]}{i} = 0 \).

Therefore applying L'Hospital rule and using the result (Ruiz, 1996),

\[
\sum_{u=0}^{n} (-1)^u \binom{n}{u} (x-u)^n = n! 
\tag{2.3.26}
\]

we have,

\[
E[X^\alpha (r, n, m, k)] = \frac{(\lambda)^{-[\alpha / p]}[\alpha / p]! C_{r-1}^{(k)}}{(m+1)^{r-1}(r-1)!} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k+1/c)}} \quad m \neq -1
\]

as obtained by Kamps (1995).
1. Introduction

In this chapter a review of moments and recurrence relations of some specific and general class of distribution of joint generalized order statistics is given.

Pawals and Szynal (2001 a) established recurrence relations for joint moments of generalized order statistics from Burr and Pareto distributions.

Saran and Pandey (2003) established some recurrence relation satisfied by marginal and joint moment generating function of joint generalized order statistics from power function distribution.

Athar and Islam (2004) established some recurrence relations between expectations of function of joint generalized order statistics from general class of distribution

Athar et al. (2007) obtained the ratio and inverse moments of joint generalized order statistics of different order from Weibull distribution whereas Khan et al. (2007) obtained recurrence relations for joint moments of generalized order statistics from doubly truncated Weibull distribution.

Ahmad (2008) established recurrence relations for product moments of generalized order statistics from linear exponential distribution.

Khan et al. (2007) and Faizan and Athar (2009) obtained exact moments of joint generalized order statistics from general class of distribution.
2. Moments of Some Specific Distributions

Burr Distribution

Theorem 2.1: (Pawlas and Szynal, 2001a)

For the distribution as given in (2.2.1), fix a positive integer $k \geq 1$.

For $n \in \mathbb{N}$, $m \in \mathbb{Z}$, $1 \leq r \leq n$, and $i, j = 0, 1, 2, \ldots$, such that

\[ \lambda y_r \tau > (j + \tau) \]

\[
E[X^i(r, n, m, k)X^{j+\tau}(r+1, n, m, k)]
= \frac{\beta(j + \tau)}{\lambda y_{r+1} \tau - (j + \tau)} E[X^i(r, n, m, k)X^j(r+1, n, m, k)]
+ \frac{\lambda y_{r+1} \tau}{\lambda y_{r+1} \tau - (j + \tau)} E[X^{i+j+\tau}(r, n, m, k)]
\]

(3.2.1)

and for $1 \leq r \leq s - 2 \leq n$

\[
E[X^i(r, n, m, k)X^{j+\tau}(s, n, m, k)]
= \frac{\beta(j + \tau)}{\lambda y_{s} \tau - (j + \tau)} E[X^i(r, n, m, k)X^j(s, n, m, k)]
+ \frac{\lambda y_{s} \tau}{\lambda y_{s} \tau - (j + \tau)} E[X^i(r, n, m, k)X^{j+\tau}(s-1, n, m, k)]
\]

(3.2.2)

Proof: Note that for $1 \leq r \leq s - 2 \leq n$ and $i, j = 0, 1, 2, \ldots,$

\[
E[X^i(r, n, m, k)X^{j+\tau}(s, n, m, k)]
+ \beta E[X^i(r, n, m, k)X^j(s, n, m, k)]
= \frac{c_{s-1} \lambda \tau}{(r - 1)!(s - r - 1)!}
\]
\begin{equation}
\times [x^i [1 - F(x)]^m f(x) g_{r-1}^m (F(x)) I(x) dx \tag{3.2.3}
\end{equation}

where

\[ I(x) = \left \{ y^{j+\tau-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} (1 - F(y))^{\gamma_s} dy \right \}
\]

Integrating \( I(x) \) by parts we obtain

\[ I(x) = \frac{\gamma_s}{j + \tau} \left \{ y^{j+\tau} [h_m(F(y)) - h_m(F(x))]^{s-r-1} \right \}
\]

\[ \times [1 - F(y)]^{\gamma_s-1} f(y) dy \]

\[ - \frac{s-r-1}{j + \tau} \left \{ y^{j+\tau} [h_m(F(y)) - h_m(F(x))]^{s-r-2} \right \}
\]

\[ \times [1 - F(y)]^{\gamma_s-1} f(y) dy \]

Substituting this expression into (3.2.3) and simplifying we obtain (3.2.2). When \( s = r + 1 \), we have (3.2.1).

**Remark 3.2.1:** Under the assumptions of Theorem 3.2.1 with \( k = 1, m = 0 \) we get the relations for product moments of order statistics, and when \( m = -1 \) we have the relations for product moments of \( k - \text{th} \) record values from a Burr distribution.

**Remark 3.2.2:** Under the assumptions of Theorem 3.2.1 with \( \tau = 1, \beta = \lambda = \alpha^{-1} \), we get the corresponding recurrence relations for the generalized Pareto distribution.

\[ E[X^i(r, n, m, k) X^{j+1}(r + 1, n, m, k)] \]

\[ = \frac{(j + 1)}{\gamma_{r+1} - \alpha (j + 1)} E[X^i(r, n, m, k) X^j(r + 1, n, m, k)] \]
\begin{align*}
&+ \frac{\gamma_{r+1}}{\gamma_{r+1} - \alpha(j + 1)} E[X^{i+j+1}(r, n, m, k)] \\
\text{and for } 1 \leq r \leq s - 2 \\
E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)] \\
&= \frac{(j + 1)}{\gamma_s - \alpha(j + 1)} E[X^i(r, n, m, k)X^j(s, n, m, k)] \\
&+ \frac{\gamma_s}{\gamma_s - \alpha(j + 1)} E[X^i(r, n, m, k)X^{j+1}(s - 1, n, m, k)] \\
\text{Remark 3.2.3:} \text{ Under the assumptions of Remark 3.2.2 we get the recurrence relations for the exponential distribution with pdf } e^{-x}. \\
E[X^i(r, n, m, k)X^{j+1}(r + 1, n, m, k)] \\
&= \frac{(j + 1)}{\gamma_{r+1}} E[X^i(r, n, m, k)X^j(r + 1, n, m, k)] \\
&+ E[X^{i+j+1}(r, n, m, k)] \\
\text{and for } 1 \leq r \leq s - 2 \\
E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)] \\
&= \frac{(j + 1)}{\gamma_s} E[X^i(r, n, m, k)X^j(s, n, m, k)] \\
&+ E[X^i(r, n, m, k)X^{j+1}(s - 1, n, m, k)]
\end{align*}
Pareto distribution

The pdf of distribution is given as

\[ f(x) = \frac{\alpha \sigma^\alpha}{x^{\alpha+1}}, \quad x > \sigma, \quad \sigma > 0, \quad \alpha > 0 \]  

(3.2.4)

for a simple Pareto distribution we have

\[ xf(x) = \alpha [1 - F(x)] \]  

(3.2.5)

**Theorem 3.2.2: (Pawlas and Szynal, 2001a)**

Fix a positive integer \( k \geq 1 \). For \( n \in \mathbb{N}, \ m \in \mathbb{Z}, \ 1 \leq r \leq n, \) and \( i, j = 0,1,2,\ldots, \) such that \( \alpha \gamma_s > s \)

\[
E[X^i(r,n,m,k)X^j(r+1,n,m,k)]
\]

\[ = \frac{\alpha \gamma_{r+1}}{\alpha \gamma_{r+1} \gamma_s - (r + 1)} E[X^{i+j}(r,n,m,k)] \]

and for \( 1 \leq r \leq s - 2 \leq n \)

\[
E[X^i(r,n,m,k)X^j(s,n,m,k)] = \prod_{j=r}^{s} \left( \frac{\alpha \gamma_j}{\alpha \gamma_j - s} \right) E[X^{i+j}(r,n,m,k)]
\]

**Proof:** see reference.

**Remark 3.2.4:** Under the assumptions of Theorem 3.2.2 with \( k = 1, m = 0 \) we get the relations for product moments of order statistics, and when \( m = -1 \) we have the relations for product moments of \( k - \text{th} \) record values.

**Power function distribution**

**Theorem 3.2.3: (Saran and Pandey, 2003)**
For the distribution as given in (2.2.4) and $1 \leq r \leq s - 1 \leq n$, $n \in N$, $m \in Z$, $i, j = 0, 1, 2, \ldots$, and a fixed positive integer $k \geq 1$,

$$\{\delta \gamma_{s+1} - t_2 + (j + 1)\} M^{(i, j+1)}_{X(r, n, m, k), X(s+1, n, m, k)}(t_1, t_2)$$

$$= \delta \gamma_{s+1} M^{(i, j+1)}_{X(r, n, m, k), X(s, n, m, k)}(t_1, t_2)$$

$$- t_2 M^{(i, j+2)}_{X(r, n, m, k), X(s+1, n, m, k)}(t_1, t_2)$$

$$+(j+1) M^{(i, j)}_{X(r, n, m, k), X(s+1, n, m, k)}(t_1, t_2),$$

(3.2.6)

and, for $1 \leq r \leq n$

$$\{\delta \gamma_{r+1} - t_2 + (j + 1)\} M^{(i, j+1)}_{X(r, n, m, k), X(r+1, n, m, k)}(t_1, t_2)$$

$$= \delta \gamma_{r+1} M^{(i, j+1)}_{X(r, n, m, k), X(r+1, n, m, k)}(t_1, t_2)$$

$$- t_2 M^{(i, j+2)}_{X(r, n, m, k), X(r+1, n, m, k)}(t_1, t_2)$$

$$+(j+1) M^{(i, j)}_{X(r, n, m, k), X(r+1, n, m, k)}(t_1, t_2),$$

(3.2.7)

**Proof:** The joint moment generating function of $X(r, n, m, k)$ and $X(s+1, n, m, k)$ is given by

$$M_{X(r, n, m, k), X(s+1, n, m, k)}(t_1, t_2)$$

$$= \frac{C_s}{(r-1)!(s-r)!} \int_{x<y} e^{t_1 x + t_2 y} \left[1 - F(x)\right]^m f(x) g_{s-r}^{-1}(F(x))$$

$$\times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r} \left[1 - F(y)\right]^{s-r-1} f(y) dy dx$$

(3.2.8)
\[ I(x) = \int_{\infty}^{\infty} e^{t_1 x + t_2 y} [h_m(F(y)) - h_m(F(x))]^{s-r} [1-F(y)]^{\gamma_{s+1}-1} f(y) dy \]

solving the integral in \( I(x) \) by parts and substituting the resulting expression in (3.2.8), we get

\[
M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2) = M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) + \frac{t_2 C_s}{(r-1)!(s-r)!} \int_{-\infty}^{\infty} \left[ 1 - F(x) \right]^m f(x) g_m^{r-1} (F(x)) I(x) dx,
\]

where,

\[
I(x) = \int_{\infty}^{\infty} e^{t_1 x + t_2 y} [h_m(F(y)) - h_m(F(x))]^{s-r} [1-F(y)]^{\gamma_{s+1}-1} f(y) dy
\]

the constant of integration vanishes since the integral in \( I(x) \) is a definite integral. On using the relation (2.2.6), we obtain

\[
M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2) = M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) + \frac{t_2 C_s}{(r-1)!(s-r)!} \int_{\infty}^{\infty} \left[ 1 - F(x) \right]^m f(x) g_m^{r-1} (F(x)) \times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r} [1-F(y)]^{\gamma_{s+1}-1} dy dx, \]

Differentiating both sides of (3.2.9) \( i \) times with respect to \( t_1 \) and then \( j+1 \) times with respect to \( t_2 \), we get

\[
M_{X(r,n,m,k),X(s+1,n,m,k)}^{(i,j+1)}(t_1,t_2)
\]
\[ \begin{align*}
&= M^{(i,j+1)}_{X(r,n,m,k),X(s,n,m,k)}(t_1, t_2) \\
&+ \frac{t_2}{\delta y_{s+1}} M^{(i,j+1)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1, t_2) \\
&+ \frac{(j+1)}{\delta y_{s+1}} M^{(i,j)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1, t_2) \\
&- \frac{t_2}{\delta y_{s+1}} M^{(i,j+2)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1, t_2) \\
&- \frac{(j+1)}{\delta y_{s+1}} M^{(i,j+1)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1, t_2)
\end{align*} \]

which, when rewritten gives the recurrence relation in (3.2.6). Proceeding in a similar manner for the case \( s = r \), the recurrence relation given in (3.2.7) can easily be established. One can also note that Theorem (2.2.2) can be deduced from Theorem 3.2.3 by letting \( t_1 \) tend to zero.

**Remark 3.2.5:** Putting \( m = -1 \) in Theorem 3.2.3, we get the recurrence relations for joint moment generating function of the \( k \)th upper record values from power function distribution, which verify the result of Raqab and Ahsanullah (2000).

**Weibull Distribution**

**Lemma 3.2.1:** (Athar et al., 2007)

For the Weibull distribution as given in (2.2.10) and non-negative finite integers \( a, b, c \) with \( m \neq -1 \)

\[ J_{i,j}(a,0,c) = \frac{\theta^{t_i+t_j} \Gamma(t_i) \Gamma(t_j)}{\gamma^2 a^{t_i} c^{t_j}} I_\phi(t_i, t_j) \quad (3.2.10) \]

where,
\[ J_{i,j}(a,b,c) = \int_{0 \leq x < y \leq \infty} x^{i+p-1} y^{j+p-1} [1 - F(x)]^a \]  

(3.2.11)

\[ I_{\phi}(\mu, \nu) = \frac{1}{B(\mu, \nu)} \int_0^\phi x^{\mu-1} (1-x)^{\nu-1} \, dx \]

and

\[ t_i = 1 + \frac{i}{p}, \phi = \frac{a}{a+c} \]

**Proof:** From (2.2.11) and (3.2.11), we have

\[ J_{i,j}(a,0,c) = \int_0^\infty y^{i+p-1} e^{-cy^{p}/\theta} \left( \int_0^y x^{i+p-1} e^{-ax^{p}/\theta} \, dx \right) dy \]

Now

\[ \int_0^y x^{i+p-1} e^{-ax^{p}/\theta} \, dx = \frac{y^{i+p}}{pa} \left( \frac{w}{a} \right)^{i/p} e^{-w^{p}/\theta} \, dw \]

if we put \( w = \frac{ax^p}{yp} \)

Changing the order of integration, we get

\[ J_{i,j}(a,0,c) = \frac{1}{p^2 a^{1+(i/p)}} \int_0^a \left[ \int_0^{(2+i+j)/p} t(2+i+j)/p e^{-t(c+w)/\theta} \, dt \right] \, dw \]

where

\[ t = \frac{(2+i+j)}{p} \]

\[ = \frac{\theta}{p^2 a^{1+(i/p)}} \Gamma(2 + \frac{i+j}{p}) \int_0^a w^{i/p} (c + w)^{- (2+i+j)/p} \, dw \]

(3.2.12)

Now put \( z = \frac{w}{c+w} \) in (3.2.12), to get
Moments of joint generalized order statistics

\[ J_{i,j}(a,0,c) = \frac{\theta^{i+j}}{p^2 a^i c^j} \Gamma \left( 2 + \frac{i+j}{p} \right) \left[ \frac{a}{c} \right]^{i+j} \int_0^1 z^{i/p} (1-z)^{j/p} dz \]

\[ = \frac{\theta^{i+j}}{p^2 a^i c^j} \Gamma t_i \Gamma t_j I_\phi(t_i,t_j) \]

and hence the lemma.

**Lemma 3.2.2: (Athar et al., 2007)**

For the condition as stated in lemma 3.2.1 and \( m \neq -1 \).

\[ J_{i,j}(a,b,c) \]

\[ = \frac{1}{(m+1)^b} \sum_{l=0}^{b} (-1)^l \binom{b}{l} J_{i,j}[a + (m+1)(b-l),0,c + (m+1)l] \]

\[ = \frac{\theta^{i+j}}{p^2 (m+1)^b} \Gamma t_i \Gamma t_j \]

\[ \times \sum_{l=0}^{b} (-1)^l \binom{b}{l} \frac{I_\phi(l)(t_i,t_j)}{[a + (m+1)(b-l)]^{l+1} [c + (m+1)l]^{l+1}} \]

where

\[ \frac{1}{\phi(l)} = 1 + \frac{c + (m+1)l}{a + (m+1)(b-l)} \]

**Proof:** It can be proved by expanding \( [h_m(F(y)) - h_m(F(x))]^b \) in (3.2.11) binomially, after noting that

\[ h_m(F(y)) - h_m(F(x)) = g_m(F(y)) - g_m(F(x)) \]

\[ = \frac{1}{m+1} \left[ (1 - F(x))^{m+1} - (1 - F(y))^{m+1} \right] \]
and then using the lemma 3.2.1.

**Theorem 3.2.4**: (Athar et al., 2007)

For Weibull distribution as given in (2.2.10) and 
\( \gamma_r, \gamma_s \geq 1, \ k \geq 1, 1 \leq r < s \leq n, \ m \neq -1. \)

\[
\alpha^{(i,j-p)}_{r,s,n,m,k} = \left( \frac{p}{\theta} \right)^2 \frac{1}{(m+1)^{r-1}} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \mathcal{J}_{i,j-p}[(m+1)(t+1), s-r-1, \gamma_s] 
\]

\[
= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \theta^{l_j^x+t_j^x-3} \Gamma_i \Gamma_{j-p} \frac{(m+1)^{t-2}}{(m+1)^{t-2}} \times \sum_{i=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{t+l} \binom{r-1}{t} \binom{s-r-1}{l} \times \frac{1}{(\gamma_r-t - \gamma_s-t)^{l_i} (\gamma_s-t)^{l_j-p}} (3.2.13)
\]

where

\[
\phi_2(l) = 1 - \frac{\gamma_{s-l}}{\gamma_{r-t}}
\]

**Proof**: We have

\[
\alpha^{(i,j-p)}_{r,s,n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times \int_{0 \leq x < y \leq \infty} x^i y^{j-p} [1 - F(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{\gamma_s-1} f(y) \, dx \, dy
\]

Since
\[ g_r^{m-1}(F(x)) = \left\{ \frac{1}{m+1} [1 - (1 - F(x))^{m+1}] \right\}^{r-1} \]

\[ = \frac{1}{(m+1)^{r-1}} \sum_{t=0}^{r-1} (-1)^t \binom{r-1}{t} [1 - F(x)]^{(m+1)t} \]

Therefore in view of (2.2.12), we have

\[ \alpha_r^{(i,j-p)} = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \frac{p^2}{\theta^2 (m+1)^{r-1}} \]

\[ \times \sum_{t=0}^{r-1} (-1)^t \binom{r-1}{t} \int_0^\infty \int_0^\infty x^{i+p-1} y^{j-1} [1 - F(x)]^{(m+1)(t+1)} \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^s \, dx \, dy \]

Thus the theorem is proved by an application of lemma 3.2.1 and lemma 3.2.2.

**Remark 3.2.6:** At \( m = 0 \) and \( k = 1 \), the product moment of order statistics is:

\[ \alpha_r^{(i,j-p)} = C_{r,s:n} \frac{(s-r+t-3)^{t_l} t_{i} t_{j-p}}{(r-1)!(s-r-1)!(n-s)!} \]

\[ \times \sum_{t=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{t+l} \binom{r-1}{t} \binom{s-r-1}{l} \]

\[ \times \frac{I_{\phi_n}(t_{i}, t_{j-p})}{(s-r+t-l)^{t_l} (n-s+1+l)^{t_{j-p}}} \]

where,

\[ C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \]

and
\[ \phi_3(l) = 1 - \frac{n-s+l+1}{n-r+t+1} \]

as obtained by Ali and Khan (1996).

**Remark 3.2.7:** As \( m \to -1 \) in (3.2.13), the moment of \( k \)-th record value is given by

\[
\alpha_{r,s,n,m,k}^{(i,j-p)} = \frac{1}{(r-1)!(s-r-1)!} \left( \frac{\theta}{k} \right)^{t_i+t_j-3} \Gamma(t_i + t_j + s - 3) \\
\times \sum_{l=0}^{s-r-1} (-1)^l \binom{s-r-1}{l} \frac{1}{t_i+p(r+l-1)}
\]

**Remark 3.2.8:** At \( i = 0 \) and \( j = p \) in (3.2.13), we get an identity

\[
\sum_{t=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{t+l} \binom{r-1}{t} \binom{s-r-1}{l} \frac{1}{(y_{s-t})(y_{r-l})} = \frac{(m+1)^{s-2}(r-1)!(s-r-1)!}{C_{s-1}}
\]

**Doubly Truncated Weibull Distribution**

**Theorem 3.2.5:** (Khan et al., 2007)

For Weibull distribution as in (2.2.26) \( 1 \leq r < s \leq n-1, \ m \in \mathbb{R}, \ n \geq 2 \) and \( k = 1,2,\cdots \)

\[
E[X^i(r,n,m,k), X^j(s,n,m,k)] - E[X^i(r,n,m,k), X^j(s-1,n,m,k)]
= -P_2 K_1 \{ E[X^i(r,n-1,m,k+m), X^j(s,n-1,m,k+m)] \\
- E[X^i(r,n-1,m,k+m), X^j(s-1,n-1,m,k+m)] \}
+ \frac{j}{p \gamma_s} E[X^i(r,n,m,k), X^{j-p}(s,n,m,k)] \tag{3.2.14}
\]

where,
\[ K_1 = \frac{C_{s-2}}{C_{s-2}} = \prod_{i=1}^{s-1} \left( \frac{\gamma_i}{\gamma_i^{(n-1)} + m} \right) = \frac{\gamma_1}{\gamma_s} \prod_{i=1}^{s-1} \left( \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m} \right). \]

**Proof:** In view of Athar and Islam (2004), we have

\[
E[X^j(r,n,m,k), X^j(s,n,m,k)] - E[X^j(r,n,m,k), X^j(s-1,n,m,k)] = \frac{C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} j \\
\times \int_Q^{\gamma_i} \int_{Q_{r-1}}^{\gamma_i} x^j y^{r-1} [F(x)]^m f(x) g_1^{r-1}(F(x)) \\
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} dy \, dx \quad (3.2.15)
\]

Now using (2.2.29) in (3.2.15), we get

\[
E[X^j(r,n,m,k), X^j(s,n,m,k)] - E[X^j(r,n,m,k), X^j(s-1,n,m,k)] = \frac{C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} j \\
\times \left\{ -P_2 + \frac{1}{p} y^{1-p} f(y) \right\} dy \, dx \\
= -P_2 \frac{C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} j \\
\times \int_Q^{\gamma_i} \int_{Q_{r-1}}^{\gamma_i} x^j y^{r-1} [F(x)]^m f(x) g_1^{r-1}(F(x)) \\
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} dy \, dx \\
+ \frac{C_{s-1}}{p \gamma_s (r-1)! (s-r-1)!} j \int_Q^{\gamma_i} \int_{Q_{r-1}}^{\gamma_i} x^j y^{1-p} [F(x)]^m g_1^{r-1}(F(x)) \\
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} f(x) f(y) dy \, dx
\]
\[ \frac{-P_2}{(r-1)!(s-r-1)!} \]

\[ \times \int_0^1 \int_0^1 x^i y^{j-1} \left[ \bar{F}(x) \right]^m f(x) g_m^{r-1}(F(x)) \]

\[ \times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ \bar{F}(y) \right]^{\gamma_s^{(n-1,k+m)}} dy dx \]

\[ + \frac{j}{P \gamma_s} E[X^i(r,n,m,k), X^{j-1}(r,n,m,k)], \]

where,

\[ \gamma_s - 1 = \gamma_s^{(n-1,k+m)}, \quad C_{s-1} = \gamma_s C_{s-2} \]

and hence the result.

**Remark 3.2.9:** At \( p = 1, Q = 0, P = 1, \) \( s = r + 1 \) and \( j = j + 1, \)

Theorem 3.2.5 reduces to

\[ E[X^i(r,n,m,k), X^{j+1}(r+1,n,m,k)] \]

\[ - E[X^i(r,n,m,k), X^{j+1}(r,n,m,k)] \]

\[ = \frac{j + 1}{\gamma_{r+1}} E[X^i(r,n,m,k), X^j(r+1,n,m,k)] \]

Or

\[ E[X^i(r,n,m,k), X^{j+1}(r+1,n,m,k)] \]

\[ = \frac{j + 1}{\gamma_{r+1}} E[X^i(r,n,m,k), X^j(r+1,n,m,k)] \]

\[ + E[X^{i+j+1}(r,n,m,k)] \]

which is the result given by Pawals and Szynal (2001 a) for the non-truncated exponential distribution.
**Remark 3.2.10:** Recurrence relations between product moments of order statistics \((m = 0, k = 1)\) is

\[
E(X_{r,s;n}^{(i,j)}) - E(X_{r,s-1;n}^{(i,j)})
\]

\[
= -P_2 \frac{n}{n-s+1} \left\{ E(X_{r,s;n-1}^{(i,j)}) - E(X_{r,s-1;n-1}^{(i,j)}) \right\}
\]

\[
+ \frac{j}{p(n-s+1)} E(X_{r,s;n}^{(i,j-p)}),
\]

as \( K_1 = \frac{n}{n-s+1} \)

and \( \gamma_s = n-s+1 \) for \( m = 0, k = 1 \).

which is the relation obtained by Khan et al. (1983 b).

**Linear Exponential Distribution**

**Theorem 3.2.6:** (Ahmad, 2008)

For the distribution as as given in (2.2.43) and \( j, \ell = 0, 1, \ldots \), the following recurrence relations hold

(i) for integers \( r \leq n - 1 \)

\[
\mu_{r,r+1;n,m,k}^{(j,\ell+2)} = \mu_{r,n,m,k}^{(j+\ell+2)}
\]

\[
+ \frac{\ell + 2}{\nu \gamma_{r+1}^{\ell+1}} \left[ \mu_{r,r+1;n,m,k}^{(j,\ell)} - \frac{\lambda \gamma_{r+1}}{\ell + 1} [\mu_{r,r+1;n,m,k}^{(j,\ell+1)} - \mu_{r,n,m,k}^{(j+\ell+1)}] \right]
\]

(ii) for integers \( s > r + 1 \),

\[
\mu_{r,s;n,m,k}^{(j,\ell+2)} = \mu_{r,s-1;n,m,k}^{(j,\ell+2)}
\]
Moments of joint generalized order statistics

\[ + \frac{\ell + 2}{1 + 2} \sum_{r,s,n,m,k}^{\lambda} \left( \mu_{r,s;n,m,k} \frac{\gamma_s}{\ell + 1} \left[ \mu_{r,s;n,m,k} - \mu_{r,s-1;n,m,k} \right] \right) \]

(3.2.17)

**Proof:** From equation (1.8.4), we obtain for \(1 \leq r < s\)

\[
\mu_{r,s;n,m,k} = \frac{C_s}{\Gamma(r)\Gamma(s-r)} \int_0^\infty \int_0^y x^j y^\ell f(x)[1-F(x)]^m \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} g_m^{-1}(F(x))[1-F(y)]^{r+1} f(y) dy dx
\]

\[
= \frac{C_s}{\Gamma(r)\Gamma(s-r)} \int_0^\infty x^j [1-F(x)]^m f(x) g_m^{-1}(F(x)) B(x) dx,
\]

(3.2.18)

where,

\[
B(x) = \int_x^\infty y^\ell [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{r+1} f(y) dy
\]

making use of (2.2.45) we can write

\[
B(x) = \lambda J_\ell(x) + \nu J_{\ell+1}(x),
\]

(3.2.19)

where

\[
J_\ell(x) = \int_x^\infty y^\ell [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{r+1} dy
\]

\[
= \frac{1}{\ell + 1} \int_x^\infty \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [1-F(y)]^{r+1} d(y^{\ell+1})
\]

Integrating by parts, we have for \(s = r+1\)
moments of joint generalized order statistics

\[ j_{\ell}(x) = \frac{\gamma_{\ell+1}}{\ell + 1} \int_x^{y_{\ell+1}} \left[ 1 - F(y) \right]^{r+1} f(y) dy - \frac{X^{\ell+1}}{\ell + 1} \left[ 1 - F(y) \right]^{r+1}, \quad (3.2.20) \]

and for \( s > r + 1, \)

\[ j_{\ell}(x) = \frac{\gamma_{s}}{\ell + 1} \int_x^{y_{s+1}} \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ 1 - F(y) \right]^{r+1} f(y) dy \]

\[ - \frac{s - r - 1}{\ell + 1} \int_x^{y_{s+1}} \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-2} \left[ 1 - F(y) \right]^{r+1} f(y) dy. \]

\[(3.2.21)\]

substituting from (3.2.20), (3.2.21) in (3.2.19), and then in (3.2.18), we have, after simplifications, the recurrence relations (3.2.16) and (3.2.17).

**Remark 3.2.11:** Putting \( r = 1 \) in (3.2.16), we obtain

\[ \mu_{1,2,n,m,k}^{(j+1,2)} = \mu_{1,n,m,k}^{(j+1,2)} + \frac{\lambda}{\nu} \left( \mu_{1,n,m,k}^{(j+1,2)} - \frac{\lambda}{\nu} \left[ \mu_{1,2,n,m,k}^{(j+1,2)} - \mu_{1,n,m,k}^{(j+1,2)} \right] \right) \]

**Remark 3.2.12:** Putting \( m = 0, k = 1 \) in (3.2.16) and (3.2.17), we obtain the recurrence relations for product moments of ordinary order statistics of the linear exponential distribution in the form

\[ \mu_{r,r+1,n}^{(j+1,2)} = \mu_{r,n}^{(j+1,2)} + \frac{\lambda}{\nu} \left( \mu_{r,n}^{(j+1,2)} - \frac{\lambda}{\nu} \left[ \mu_{r,r+1,n}^{(j+1,2)} - \mu_{r,n}^{(j+1,2)} \right] \right) \]

\[(3.2.22)\]

and

\[ \mu_{r,s,n}^{(j+1,2)} = \mu_{r,s-1,n}^{(j+1,2)} + \frac{\lambda}{\nu} \left( \mu_{r,s-1,n}^{(j+1,2)} - \frac{\lambda}{\nu} \left[ \mu_{r,s,n}^{(j+1,2)} - \mu_{r,s-1,n}^{(j+1,2)} \right] \right) \]

\[(3.2.23)\]

Letting \( \nu \rightarrow 0 \) and \( \lambda = 0 \) in (3.2.22) and (3.2.23) we obtain results for exponential and Rayleigh distributions.
**Remark 3.2.13:** Putting \( m = -1, k \geq 1 \) in (3.2.16) and (3.2.17), we obtain the recurrence relations for product moments of upper \( k \) – records of the linear exponential distribution in the form

\[
\mu_{(r, r+1; k)}^{(j, \ell+2)} = \mu_{(r, k)}^{(j, \ell+2)} + \frac{\ell + 2}{\nu k} \left( \mu_{(r, r+1; k)}^{(j, \ell)} - \frac{\lambda k}{\ell + 1} \left[ \mu_{(r, r+1; k)}^{(j, \ell+1)} - \mu_{(r, k)}^{(j, \ell+1)} \right] \right) \tag{3.2.24}
\]

and

\[
\mu_{(r, s; k)}^{(j, \ell+2)} = \mu_{(r, s-1; k)}^{(j, \ell+2)} + \frac{\ell + 2}{\nu k} \left( \mu_{(r, s; k)}^{(j, \ell)} - \frac{\lambda k}{\ell + 1} \left[ \mu_{(r, s; k)}^{(j, \ell+1)} - \mu_{(r, s-1; k)}^{(j, \ell+1)} \right] \right) \tag{3.2.25}
\]

When \( k = 1 \) in (3.2.24) and (3.2.25), these results reduce to Raqab (2001) results for the uniform random variables.

### 3. Moments of general class of distribution

#### Case I: \( m_1 = m_2 = \ldots = m_{n-1} = m \).

**Theorem 3.3.1:** (Athar and Islam, 2004)

For distribution given in (2.3.1) and \( n \in N, m \in R \),

\[ 1 \leq r < s \leq n - 1. \]

\[
E\left[ \xi \{X(r, n, m, k), X(s, n, m, k)\} \right] - E\left[ \xi \{X(r, n, m, k), X(s-1, n, m, k)\} \right] = -\frac{1}{\gamma_s c \alpha} E\left[ \psi \{X(r, n, m, k), X(s, n, m, k)\} \right] \tag{3.3.1}
\]

where

\[
\psi(x, y) = \left[ a h(y) + b \right] \frac{\partial}{\partial y} \frac{\xi(x, y)}{h'(y)}. \]

**Proof:** see reference.

**Remark 3.3.1:** Under the assumption given in Theorem 3.3.1 with \( k = 1, m = 0 \), we get the recurrence relations for product moments of
order statistics (Ali and Khan, 1998) and at \( m = -1 \); we have the recurrence relations for product moments of \( k \textsuperscript{th} \) record values.

**Case II:** \( \gamma_1 \neq \gamma_j \)

**Theorem 3.3.2:** (Athar and Islam, 2004)

For distribution given in (2.3.1) and for \( k \),
\[ n \in \mathbb{N}, \tilde{m}^* = (m_1, m_2, \ldots, m_{n-1}) \in \mathcal{R}, \quad 1 \leq r < s \leq n - 1. \]

\[
E[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}] - E[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}] = -\frac{1}{\gamma_s} E[y\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}] 
\]

**Proof:** see reference.

**Remark 3.3.2:** For \( \xi(x,y) = \xi_2(y) \), Theorem 3.3.1 and 3.3.2 reduce to Theorem (2.3.1) and (2.3.3) respectively.

**Remark 3.3.3:** Recurrence relations for product moments of generalized order statistics from Pareto, Power function, Weibull, Burr type XIIe, beta of first kind and Cauchy distributions may be obtained with proper choice of \( a, b, c \) and \( h(x) \) as given by Khan and Abouammoh (2000).

**Theorem 3.3.3:** (Khan et al., 2007)

For distribution given in (2.3.5)

\[
E[X^p(r,n,m,k).X^q(s,n,m,k)] = (-1)^{p+q} \frac{b}{a} \left( \frac{b}{a} \right)^{p+q}
\]
where \( p \) and \( q \) are positive integers.

**Proof:** In view of (2.3.6), we have

\[
E[X^p(r,n,m,k)X^q(s,n,m,k)]
\]

\[
= E\left[\left\{\frac{1}{a}\left(\prod_{j=1}^{r}B_j^{\frac{1}{c}} - b\right)\right\}^p \left\{\frac{1}{a}\left(\prod_{j=1}^{s}B_j^{\frac{1}{c}} - b\right)\right\}^q\right]
\]

\[
= (-1)^{p+q} \left(\frac{b}{a}\right)^{p+q} E\left(1 - \frac{1}{b}\prod_{j=1}^{r}B_j^{\frac{1}{c}}\right)^p E\left(1 - \frac{1}{b}\prod_{j=1}^{s}B_j^{\frac{1}{c}}\right)^q
\]

\[
= (-1)^{p+q} \left(\frac{b}{a}\right)^{p+q}
\]

\[
\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v} \frac{1}{b^{u+v}} \prod_{j=1}^{r} E\left(B_j^{\frac{u}{c}}\right) \prod_{j=r+1}^{s} E\left(B_j^{\frac{v}{c}}\right)
\]

\[
= (-1)^{p+q} \left(\frac{b}{a}\right)^{p+q}
\]

\[
\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v} \frac{1}{b^{u+v}} \prod_{j=1}^{r} c\gamma_j \prod_{j=r+1}^{s} c\gamma_j + (u + v)
\]

\[
= (-1)^{p+q} \left(\frac{b}{a}\right)^{p+q}
\]
\[
\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v} \frac{1}{b^{u+v}} \frac{\prod_{j=1}^{r} \gamma_j^{(k)}}{\prod_{j=r+1}^{s} \gamma_j^{(k)}} \frac{1}{u^{u+v}} \frac{\prod_{j=1}^{r} (k+v)}{\prod_{j=r+1}^{s} (k+v)}
\]

and hence the result.

**Remark 3.3.4:** For order statistics \( (m = 0, k = 1) \), we have

\[
E[X_{r,n-1} X_{s,n}^q] = (-1)^{p+q} \left( \frac{b}{a} \right)^{p+q} \frac{n!}{(n-s)!} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \frac{1}{b^{u+v}} \binom{p}{u} \binom{q}{v}
\]

\[
\frac{\Gamma(n+1+u-v-s)}{c} \frac{\Gamma(n+1+u+v-r)}{c} \frac{\Gamma(n+1+v-r)}{c} \frac{\Gamma(n+1+u+v)}{c}
\]

**Remark 3.3.5:** For record values \( (m = -1) \), we have

\[
E[X_{r,n-1} X_{s,n}^q] = (-1)^{p+q} \left( \frac{b}{a} \right)^{p+q}
\]

\[
\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \frac{1}{b^{u+v}} \binom{p}{u} \binom{q}{v} \left( \frac{k}{c} \right)^{k+v} \left( \frac{k}{c} \right)^{u+v}
\]

Similarly, product moments of order statistics with non-integral sample size \( (m = 0, k = \alpha - r + 1, \alpha \in \mathbb{R}_+) \) may also be obtained.

**Examples**

(i) Power function distribution
Moments of joint generalized order statistics

\[
\bar{F}(x) = \left(\frac{\beta - x}{\beta - \alpha}\right)^\theta = \left[-\frac{1}{\beta - \alpha}x + \frac{\beta}{\beta - \alpha}\right]^\theta, \quad \alpha \leq x \leq \beta
\]

Here

\[
a = -\frac{1}{\beta - \alpha}, \quad b = \frac{\beta}{\beta - \alpha} \quad \text{and} \quad c = \theta.
\]

From (3.3.2), we have

\[
E[X^p(r,n,m,k)X^q(s,n,m,k)] = \beta^{p+q}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\left(\frac{\beta - \alpha}{\beta}\right)^{u+v}\binom{p}{u}\binom{q}{v}\frac{C^{(k)}_{s-1}C^{(k+v)}_{r-1}}{C^{(k+\frac{v}{\theta})}_{s-1}C^{(k+\frac{v+\theta}{\theta})}_{r-1}}
\]

Product moment of two order statistics \((m = 0, k = 1)\) is

\[
E[X^p_{r:n}X^q_{s:n}] = (-1)^{p+q}\beta^{p+q}\frac{n!}{(n-s)!}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\left(\frac{\beta - \alpha}{\beta}\right)^{u+v}\binom{p}{u}\binom{q}{v}
\]

\[
\times \frac{\Gamma(n+1+\frac{v}{\theta}-s)\Gamma(n+1+\frac{u+v}{\theta}-r)}{\Gamma(n+1+\frac{v}{\theta}-r)\Gamma(n+1+\frac{u+v}{\theta})} \tag{3.3.3}
\]

Taking \(\alpha = 0\) and \(\theta = 1\) in (3.3.3), we have the results for uniform \(U(0, \beta)\) distribution as

\[
E[X^p_{r:n}X^q_{s:n}] = (-1)^{p+q}\beta^{p+q}\frac{n!}{(n-s)!}\sum_{u=0}^{p}\sum_{v=0}^{q}(-1)^{u+v}\binom{p}{u}\binom{q}{v}
\]

\[
\times \frac{\Gamma(n+1+v-s)\Gamma(n+1+(u+v)-r)}{\Gamma(n+1+v-r)\Gamma(n+1+u+v)} \tag{3.3.4}
\]
For $\beta = 1$ (3.3.4) becomes
\[
E[X_{r:n}X_{s:n}] = \frac{r(s + 1)}{(n + 1)(n + 2)} \quad \text{as given by Malik (1967)}.
\]

Product moments of two record values ($m = -1$) is
\[
E[X^p (r,n,-1,k)X^q (s,n,-1,k)]
\]
\[
= \beta^{p+q} \frac{p}{\sum_{u=0}^{q} \sum_{v=0}^{q} (-1)^{u+v} \left(\frac{\beta - \alpha}{\alpha}\right)^{u+v} \binom{p}{u} \binom{q}{v}} \left(\frac{k}{k + \frac{v}{\theta}}\right)^s \left(\frac{k + \frac{v}{\theta}}{k + \frac{u+v}{\theta}}\right)^r
\]

(II) Pareto distribution
\[
\bar{F}(x) = \left(\frac{\mu + \delta}{x + \delta}\right)^{\theta} = \left[\frac{1}{\mu + \delta} x + \frac{\delta}{\mu + \delta}\right]^{-\theta}, \quad \mu \leq x \leq \infty
\]

Here
\[
a = \frac{1}{\mu + \delta}, \quad b = \frac{\delta}{\mu + \delta} \quad \text{and} \quad c = -\theta
\]

\[
E[X^p (r,n,m,k)X^q (s,n,m,k)]
\]
\[
= (-1)^{p+q} \delta^{p+q} \frac{p}{\sum_{u=0}^{q} \sum_{v=0}^{q} (-1)^{u+v} \left(\frac{\mu + \delta}{\delta}\right)^{u+v} \binom{p}{u} \binom{q}{v}} \frac{C^{(k)}_{s-1} C^{(k-v)}_{r-1}}{C^{(k-v)}_{s-1} C^{(k-u+v)}_{r-1}}
\]

(3.3.5)
Taking $\delta = 0$ in (3.3.5), we have

$$E[X^p(r, n, m, k) \cdot X^q(s, n, m, k)] = \mu^{p+q} \frac{C^{(k)}_{s-1} C^{(k-q)}_{r-1} \theta^{(k-p+q)}_{s-1}}{C^{(k-q)}_{s-1} C^{(k-p+q)}_{r-1} \theta^{(k-p+q)}}$$

Product moment of two order statistics (i.e. $m = 0$, $k = 1$) is

$$E[X^p_{r;n} \cdot X^q_{s;n}] = \mu^{p+q} \frac{\Gamma(n+1)}{\Gamma(n+1-s)} \frac{\Gamma(n+1-\frac{q}{\theta} - s) \Gamma(n+1-\frac{p+q}{\theta} - r)}{\Gamma(n+1-\frac{q}{\theta} - r) \Gamma(n+1-\frac{p+q}{\theta})}$$

as obtained by Huang (1975) and

$$E[X^p_{r;n} \cdot X^q_{s;n}] = \mu^2 \frac{\Gamma(n+1)}{\Gamma(n+1-s)} \frac{\Gamma(n+1-\frac{1}{\theta} - s) \Gamma(n+1-\frac{2}{\theta} - r)}{\Gamma(n+1-\frac{1}{\theta} - r) \Gamma(n+1-\frac{2}{\theta})}$$

as obtained by Malik (1966).

Product moments of two record values ($m = -1$) is

$$E[X^p(r, n, -1, k) \cdot X^q(s, n, -1, k)] = (-1)^{p+q} \delta^{p+q}$$

$$\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \left(\frac{\mu + \delta}{\theta} \right)^u \left(\frac{p}{u} \right) \left(\frac{q}{v} \right) \left(\frac{k}{k - \frac{v}{\theta}} \right)^s \left(\frac{k - \frac{v}{\theta}}{k - \frac{u + v}{\theta}} \right)^r$$

(III) Exponential distribution

$$F(x) = [ax + b]^c$$
Let $a = \frac{-\lambda}{c}$, $b = 1$, then we have

$$\lim_{c \to \infty} F(x) = e^{-\lambda x}$$

$$E[X^p(r,n,m,k) X^q(s,n,m,k)]$$

$$= \left( \frac{1}{\lambda c'} \right)^{p+q} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v} \frac{C_{s-1}^{(k)} C_{r-1}^{(k+vc')}}{C_{s-1}^{(k+vc')}} \frac{1}{C_{s-1}^{(k+vc')}}$$

$$= \left( \frac{1}{\lambda c'} \right)^{p+q} C_{s-1}^{(k)}$$

$$\times \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v} \frac{1}{\prod_{j=r+1}^{s} \gamma_{j}^{(k+vc')}} \prod_{j=1}^{r} \gamma_{j}^{(k+(u+v)c')}}$$

We have (Athar et al., 2007).

$$\sum_{i=0}^{s-r-1} (-1)^{i} \binom{s-r-1}{i} \frac{1}{\gamma_{s-i}^{(k)}} \prod_{j=r+1}^{s} \gamma_{j}^{(k)} = \frac{(m+1)^{s-r-1}(s-r-1)!}{\prod_{j=r+1}^{s} \gamma_{j}^{(k)}}$$

(3.3.6)

Thus in view of (2.3.15) and (3.3.6), we have

$$E[X^p(r,n,m,k) X^q(s,n,m,k)]$$

$$= \left( \frac{1}{\lambda c'} \right)^{p+q} C_{s-1}^{(k)} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \binom{p}{u} \binom{q}{v}$$

$$\times \frac{1}{(m+1)^{s-r-1}(s-r-1)!} \sum_{i=0}^{s-r-1} (-1)^{i} \binom{s-r-1}{i} \frac{1}{\gamma_{s-i}^{(k+vc')}}$$
Moments of joint generalized order statistics

\[
\frac{1}{(m+1)^{r-1}(r-1)!} \sum_{i=0}^{r-1} (-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \frac{1}{\gamma_{r-i}^{(k+(u+v)c')}}
\]

\[
= \left( \frac{1}{\lambda} \right)^{p+q} C_s^{(k)} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \left( \begin{array}{c} p \\ u \end{array} \right) \left( \begin{array}{c} q \\ v \end{array} \right)
\]

\[
\times \frac{1}{(m+1)^{s-r-1}(s-r-1)!} \sum_{i=0}^{s-r-1} (-1)^i \left( \begin{array}{c} s-r-1 \\ i \end{array} \right) \left[ \gamma_{s-i}^{(k)} + (u+v)c' \right]^{-1}
\]

\[
\times \frac{1}{(m+1)^{r-1}(r-1)!} \sum_{i=0}^{r-1} (-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \left[ \gamma_{r-i}^{(k)} + (u+v)c' \right]^{-1} c'^p
\]

(3.3.7)

Applying L-Hospital’s rule and taking \( c' \to 0 \) in (3.3.7), we have

\[
E[X^p(r,n,m,k).X^q(s,n,m,k)]
\]

\[
= \left( \frac{1}{\lambda} \right)^{p+q} \frac{1}{(m+1)^{s-2}(s-r-1)(r-1)!} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \left( \begin{array}{c} p \\ u \end{array} \right) \left( \begin{array}{c} q \\ v \end{array} \right)
\]

\[
\times \left[ \sum_{i=0}^{s-r-1} (-1)^i \left( \begin{array}{c} s-r-1 \\ i \end{array} \right) (-1)^q q! [\gamma_{s-i}^{(k)}]^{-q-1} \gamma_q^q \right]
\]

\[
\times \left[ \sum_{i=0}^{r-1} (-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) (-1)^p p! [\gamma_{r-i}^{(k)}]^{-p-1} (u+v)^p \right]
\]

(3.3.8)

Using (2.3.18) in (3.3.8), we have

\[
E[X^p(r,n,m,k).X^q(s,n,m,k)]
\]
Moments of joint generalized order statistics

\[ \left( \frac{1}{\lambda} \right)^{p+q} \frac{C^{(k)}_{s-1} p! q!}{(m+1)^{s-2} (s-r-1)! (r-1)!} \]

\[ \times \left[ \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{[\gamma^{(k)}_{s-i}]^{q+1}} \right] \]

\[ \times \left[ \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{[\gamma^{(k)}_{r-i}]^{p+1}} \right], \quad m \neq -1 \]

(3.3.9)

For \( m = -1 \), (3.3.9) can be written as

\[ E[X^p(r,n,m,k)X^q(s,n,m,k)] = \left( \frac{1}{\lambda} \right)^{p+q} \frac{C^{(k)}_{s-1} p! q!}{(s-r-1)! (r-1)!} \]

\[ \times \left[ \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \left[ k + (n-s+i)(m+1) \right]^{-(q+1)} \frac{1}{(m+1)^{s-r-1}} \right] \]

\[ \times \left[ \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left[ k + (n-r+i)(m+1) \right]^{-(p+1)} \frac{1}{(m+1)^{r-1}} \right] \]

(3.3.10)

Applying L-Hospital's rule in (3.3.10), we have

\[ E[X^p(r,n,m,k).X^q(s,n,m,k)] \]

\[ = \left( \frac{1}{\lambda} \right)^{p+q} \frac{C^{(k)}_{s-1} p! q!}{(s-r-1)! (r-1)!} \left[ \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \right] \]

\[ \times \frac{(-1)^{s-r-1} k^{-(q+1)-(s-r-1)} (n-s+i)^{s-r-1} (q+1)(q+2)\ldots(q+s-r-1)}{(s-r-1)!} \]

\[ \times \left[ \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \right] \]
Moments of joint generalized order statistics

\[
(-1)^{r-1}k^{-(p+1)-(r-1)}(n-r+i)^{-1}(p+1)(p+2)\ldots(p+r-1)
\]
\[
(r-1)!
\]

(3.3.11)

Using (2.3.18) in (3.3.11), we have

\[
E[X^p(r,n,m,k)X^q(s,n,m,k)] = \left(\frac{1}{\lambda}\right)^{p+q} \frac{C_s^{(k)}}{s-1} \frac{\Gamma(p+r)\Gamma(q+s-r)}{(s-r-1)!(r-1)!} \frac{k^{-(p+q+s)}}{}
\]

At \( q = 0 \), we get the result for single moments as obtained in (2.3.19).

**Theorem 3.3.4:** (Faizan and Athar, 2009)

For the distribution given in (2.3.23),

\[
E[X^\alpha(r,n,m,k)X^\beta(s,n,m,k)] = \left(-\frac{b}{a}\right)^{\frac{\alpha+\beta}{p}}
\]

\[
\sum_{u=0}^{[\alpha/p]} \sum_{v=0}^{[\beta/p]} (-1)^{u+v} \frac{1}{b^{u+v} \binom{[\alpha/p]}{u} \binom{[\beta/p]}{v}} \frac{C_s^{(k)}}{s-1} \frac{C_r^{(k+v)}}{r-1} \frac{k^{-(k+u+v)}}{}
\]

(3.3.12)

where \([\alpha/p]\) and \([\beta/p]\) are the integer parts of \(\alpha/p\) and \(\beta/p\) respectively.

**Proof:** We have from (2.3.24),

\[
E[X^\alpha(r,n,m,k)X^\beta(s,n,m,k)] = \left(-\frac{b}{a}\right)^{\frac{\alpha+\beta}{p}} \sum_{u=0}^{[\alpha/p]} \sum_{v=0}^{[\beta/p]} (-1)^{u+v} \binom{[\alpha/p]}{u} \binom{[\beta/p]}{v} \frac{1}{b^{u+v}}
\]
\[
\times \prod_{j=1}^{r} \frac{c\gamma_j}{c\gamma_j + (u + v)} \prod_{j=r+1}^{s} \frac{c\gamma_j}{c\gamma_j + v}
\]

\[
= \left(\frac{-b}{a}\right)^{\alpha+\beta \over p} \sum_{u=0}^{\alpha/p} \left[\frac{\beta/p}{u}\right] (-1)^{u+v} \binom{\alpha/p}{u} \binom{\beta/p}{v}
\]

\[
\times \frac{1}{b^{u+v}} \frac{\prod_{j=1}^{r} \gamma_j^{(k)}}{\prod_{j=r+1}^{s} \gamma_j} \frac{\prod_{j=1}^{r} \gamma_j^{(k)}}{\prod_{j=r+1}^{s} \gamma_j^{(k)}}
\]

and hence the result.

**Remark 3.3.6:** At \( p = 1 \) in (3.3.12), we get

\[
E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)] = (-1)^{\alpha+\beta} \left(\frac{b}{a}\right)^{\alpha+\beta}
\]

\[
\times \sum_{u=0}^{\alpha} \sum_{v=0}^{\beta} (-1)^{u+v} \frac{1}{b^{u+v}} \binom{\alpha}{u} \binom{\beta}{v} \frac{C_{s-1}^{(k)}}{C_{s-1}^{(k + u + v)}} \frac{C_{r-1}^{(k + v)}}{C_{r-1}^{(k + u + v)}}
\]

as obtained by Khan et al. (2008).

**Examples**

(I) **Burr distribution**

\[
\bar{F}(x) = [\theta x^p + 1]^{-\mu}, \quad 0 < x < \infty
\]

where \( p = 1/\xi > 0 \) and \( \xi \) in an integer.

Here \( a = \theta, b = 1 \) and \( c = -\mu \),

From (2.3.23), we have
\[ E[X^\alpha (r, n, m, k)X^\beta (s, n, m, k)] = \left( \frac{-1}{\theta} \right)^{\frac{\alpha + \beta}{p}} \sum_{u=0}^{\alpha/p} \sum_{v=0}^{\beta/p} (-1)^{u+v} \]

\[ \times \frac{1}{b^{u+v}} \left( \binom{\alpha/p}{u} \binom{\beta/p}{v} \right) \prod_{j=1}^{r} \gamma_j^{(k)} \prod_{j=r+1}^{s} \gamma_j^{(k)} \prod_{j=1}^{r} \frac{1}{\mu^{(k-u-v)}} \prod_{j=r+1}^{s} \frac{1}{\mu^{(k-v)}} \]

(II) Weibull distribution

\[ \bar{F}(x) = [ax^p + b]^c \]

here \( a = -\frac{\lambda}{c} \), \( b = 1 \)

and \( c' = \frac{1}{c} \to 0 \),

then \( E[X^\alpha (r, n, m, k)X^\beta (s, n, m, k)] = \left( \frac{1}{\lambda c'} \right)^{\frac{\alpha + \beta}{p}} \]

\[ \times C_{s-1}^{(k)} \sum_{u=0}^{\alpha/p} \sum_{v=0}^{\beta/p} (-1)^{u+v} \binom{\alpha/p}{u} \binom{\beta/p}{v} \]

\[ \times \frac{1}{\prod_{j=r+1}^{s} \gamma_j^{(k+vc')}} \prod_{j=1}^{r} \gamma_j^{(k)} \prod_{j=r+1}^{s} \gamma_j^{(k+vc')} \]

In view of the relation (Athar et al., 2009),

\[ \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{\gamma_{s-i}^{(k)}} = \frac{(m+1)^{s-r-1}(s-r-1)!}{\prod_{j=r+1}^{s} \gamma_j^{(k)}} \]

\[ E[X^\alpha (r, n, m, k)X^\beta (s, n, m, k)] \]
\[\begin{align*}
&= \left(\frac{1}{\lambda}\right)^{[\alpha + \beta]/p} C_{s-1}^{(k)} \sum_{u=0}^{[\alpha / p]} \sum_{v=0}^{[\beta / p]} (-1)^{u+v} \binom{[\alpha / p]}{u} \binom{[\beta / p]}{v} \\
&\times \frac{1}{(m+1)^{s-r-1}(s-r-1)!} \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \left[\frac{\gamma_{r-i}^{(k+v+1)}}{c^{v(\alpha/p)}}\right]^{-1} \\
&\times \frac{1}{(m+1)^{r-1}(r-1)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left[\frac{\gamma_{r-i}^{(k+u+v)}}{c^{(\alpha/p)+1}}\right]^{-1}
\end{align*}\]

Taking the limit and using the relation \((2.3.26)\), we get

\[E[X^\alpha (r,n,m,k)X^\beta (s,n,m,k)]\]

\[= \left(\frac{1}{\lambda}\right)^{[\alpha + \beta]/p} \frac{C_{s-1}^{(k)} [\alpha / p]! [\beta / p]!}{(m+1)^{s-2}(s-r-1)! (r-1)!} \]

\[\times \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{\gamma_{r-i}^{(k+[\alpha / p]+1)}} \]

\[\times \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}^{(k+(\alpha/p)+1)}}, \quad m \neq -1\]
1. Introduction

In this chapter, a review of some moments and recurrence relations of single and joint lower generalized order statistics (Lgos) for some specific and general class of distribution is given.

Pawlas and Szynal (2001 b) introduced the concept of dual (lower) generalized order statistics (Lgos) and gave the recurrence relations for single and product moments of lower generalized order statistics (Lgos) from inverse Weibull distribution. The work of Burkchat et al. (2003) may also be referred for dual (lower) generalized order statistics.

Athar et al. (2008 b) established the some recurrence relations between expectation of function of single and joint lower generalized order statistics from a general class of distribution $F(x) = [ah(x) + b]^c$. whereas Khan et al. (2009) obtained single and joint moments of lower generalized order statistics (Lgos) from a general class of distribution $F(x) = [ax + b]^c$.

Athar et al. (2009) established the simple expression for single and product moments of lower generalized order statistics from power function distribution.

Athar and Faizan (2011) obtained explicit expressions for single and product moments of lower generalized order statistics from power function distribution [generalized uniform distribution (Proctor, 1987)].

2. Single Moments of Dual Generalized Order Statistics

a) Let the general form of the distribution

\[ F(x) = (ah(x) + b)^c, \quad \alpha \leq x \leq \beta \]  

(4.2.1)

where \( a, b \) and \( c \) are such that \( F(\alpha) = 0, F(\beta) = 1 \) and \( h(x) \) is monotonic and differentiable function of \( x \) in the interval \([\alpha, \beta]\).

then

\[ f(x) = ach'(x) (ah(x) + b)^{c-1} \]

and

\[ F(x) = \frac{ah(x) + b}{ach'(x)} f(x) \]  

(4.2.2)

Lemma 4.2.1:

For \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1,2,\ldots \)

Case I: \( m_1 = m_2 = \ldots = m_{n-1} = m \)

\[ E[\xi^{r}X'(r, n, m, k)] - E[\xi^{r}X'(r-1, n, m, k)] \]

\[ = -\frac{C_{r-2}}{(r-1)!} \beta \xi(x)[1 - F(x)]^{r-1} g^{r-1}_{m}(F(x)) f(x) \]  

(4.2.3)

Proof: We have

\[ E[\xi^{r}X'(r, n, m, k)] - E[\xi^{r}X'(r-1, n, m, k)] \]

\[ = \frac{C_{r-2}}{(r-1)!} \beta \xi(x)[1 - F(x)]^{r-1} g^{r-1}_{m}(F(x)) f(x) \]
Moments of dual generalized order statistics

\[ \times [\gamma_r g_m(F(x)) - (r - 1)(F(x))^{m+1}] \, dx \quad (4.2.4) \]

Let

\[ v(x) = [F(x)]^{\gamma_r} g_{r-1}^{m}(F(x)) \quad (4.2.5) \]

Then

\[ v'(x) = [F(x)]^{\gamma_r-1} g_{r-2}^{m}(F(x)) f(x) [\gamma_r g_m(F(x)) - (r - 1)(F(x))^{m+1}] \]

Thus from (4.2.4)

\[ E[\xi \{X'(r, n, m, k)\}] - E[\xi \{X'(r - 1, n, m, k)\}] = \frac{C_{r-2}}{(r - 1)!} \int_{\alpha}^{\beta} \xi'(x) v'(x) \, dx \quad (4.2.6) \]

Now integrating in (4.2.6) by part and using the value of \( v(x) \) from (4.2.5), we get have result.

**Lemma 4.2.2:**

For \( 2 \leq r \leq n, \ n \geq 2 \) and \( k = 1,2,... \)

i) \( E[\xi \{X'(r - 1, n, m, k)\}] - E[\xi \{X'(r - 1, n - 1, m, k)\}] \)

\[ = \frac{(m + 1)}{\gamma_1} \frac{C_{r-2}^{(n)}}{(r - 2)!} \int_{\alpha}^{\beta} \xi'(x) [F(x)]^{\gamma_r} g_{r-1}^{m}(F(x)) \, dx \]

ii) \( E[\xi \{X'(r, n, m, k)\}] - E[\xi \{X'(r - 1, n - 1, m, k)\}] \)

\[ = - \frac{C_{r-2}^{(n-1)}}{(r - 1)!} \int_{\alpha}^{\beta} \xi'(x) [F(x)]^{\gamma_r} g_{r-1}^{m}(F(x)) \, dx \]

where \( \frac{C_{r-2}^{(n)}}{r-1} = \prod_{i=1}^{r-1} [k + (n - i)(m + 1)] \)

**Proof:** Proof is easy.
Theorem 4.2.1:

For the given distribution in (4.2.1) and $n \in \mathbb{N}, m \in \mathbb{R}, 2 \leq r \leq n$.

$$E[\xi \{X^{*}(r, n, m, k)\}] = E[\xi \{X^{*}(r-1, n, m, k)\}] - \frac{1}{\gamma_r^c a} E[\psi \{X^{*}(r, n, m, k)\}]$$

(4.2.7)

where $\psi(x) = [ah(x) + b] w(x)$,

and $w(x) = \frac{\xi'(x)}{h'(x)}$

Proof: In view of (4.2.2) and (4.2.3), we have

$$E[\xi \{X^{*}(r, n, m, k)\}] - E[\xi \{X^{*}(r-1, n, m, k)\}]$$

$$= -\frac{C_{r-2}}{(r-1)!} \int \xi'(x)[1 - F(x)]^{\gamma_r - 1} \left\{ \frac{ah(x) + b}{ca h'(x)} f(x) \right\}$$

$$\times g_m^{r-1}(F(x)) dx$$

$$= -\frac{1}{\gamma_r^c a} \frac{C_{r-1}}{(r-1)!} \psi(x)[1 - F(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x) dx$$

$$= -\frac{1}{\gamma_r^c a} E[\psi \{X^{*}(r, n, m, k)\}]$$

and hence the result.

Remark 4.2.1: For $m = 0$ and $k = 1$, the recurrence relation for dual (lower) generalized order statistics reduces to the recurrence relation of ordinary order statistics as
Moments of dual generalized order statistics

\[ E[\xi(X_{n-r+1:n})] = E[\xi(X_{n-r+2:n})] - \frac{1}{ca(n-r+1)} E[\psi(X_{n-r+1:n})] \]

as obtained by Ali and Khan (1997).

**Remark 4.2.2:** The recurrence relation for single moments of \( k \)-th lower records will be

\[ E[\xi\{X(r,n-1,k)\}] = E[\xi\{X(r-1,n-1,k)\}] - \frac{1}{ka} E[\psi\{X(r,n-1,k)\}] \]

**Remark 4.2.3:** For \( m = 0 \) and \( k = \alpha - n + 1, \alpha \in \mathbb{N}_+ \), we obtain the recurrence relation for single moments of order statistics with non-integer sample size as

\[ E[\xi(X_{\alpha-r+\frac{1}{\alpha}})] = E[\xi(X_{\alpha-r+\frac{2}{\alpha}})] - \frac{1}{ca(\alpha-r+1)} E[\psi(X_{\alpha-r+\frac{1}{\alpha}})]. \]

**Remark 4.2.4:** For \( m = \alpha - 1 \) and \( k = \alpha \), we obtain the recurrence relation for sequential order statistics as

\[ E[\xi\{X(r,n,\alpha-1,\alpha)\}] = E[\xi\{X(r-1,n,\alpha-1,\alpha)\}] - \frac{1}{ca(\alpha-r+1)} E[\psi\{X(r,n,\alpha-1,\alpha)\}] . \]

**Theorem 4.2.2:**

For the given distribution and \( n \in \mathbb{N}, m \in \mathbb{N}, 2 \leq r \leq n \),

\( i \) \[ E[\xi\{X'(r-1,n,m,k)\}] - E[\xi\{X'(r-1,n-1,m,k)\}] = \frac{(m+1)(r-1)}{\gamma_1\gamma_r ca} E[\psi\{X'(r,n,m,k)\}] \]

\( ii \) \[ E[\xi\{X'(r,n,m,k)\}] - E[\xi\{X'(r-1,n-1,m,k)\}] \]
\[ E\left\{ X'(r,n,m,k) \right\} = -\frac{1}{\gamma_1ca} E\left\{ \psi\{ X'(r,n,m,k) \} \right\} \]

**Proof:** Result can be established in view of lemma 4.2.2 and (4.2.2).

**Theorem 4.2.3:**

For the given distribution and \( 2 \leq r \leq n, \ n \geq 2, \)
\( \tilde{m}^* = (m_2, m_3, ..., m_{n-1}) \in \mathbb{R}, \ k = 1, 2, ... \)

i) \( E[\xi\{ X'(r,n,\tilde{m},k) \}] - E[\xi\{ X'(r-1,n,\tilde{m},k) \}] \)

\[ = -\frac{1}{\gamma_1ca} E[\psi\{ X'(r,n,\tilde{m},k) \}] \]

ii) \( E[\xi\{ X'(r-1,n,\tilde{m},k) \}] - E[\xi\{ X'(r-1,n-1,\tilde{m}^*,k) \}] \)

\[ = \frac{(r-1) + \sum_{j=1}^{r-1} m_j}{\gamma_1 \gamma_r ca} E[\xi\{ X'(r,n,\tilde{m},k) \}]. \]

iii) \( E[\xi\{ X'(r,n,\tilde{m},k) \}] - E[\xi\{ X'(r-1,n-1,\tilde{m}^*,k) \}] \)

\[ = -\frac{1}{\gamma_1ca} E[\psi\{ X'(r,n,\tilde{m},k) \}] \]

**Proof:** Result can be established on the line of Theorem 4.2.1 and Theorem 4.2.2.

**Examples**

i) **Inverse Weibull distribution**

\[ F(x) = e^{-(\theta / x)^\rho}, x > 0, \ \rho, \theta > 0. \]

We have \( a = 1, b = 0, c = 1 \)
Moments of dual generalized order statistics

and \( h(x) = e^{-(\theta / x)^p} \)

Let \( \xi(x) = x^{j+1} \), then

\[
\psi(x) = [a_h(x) + b] \quad w(x) = \frac{j+1}{p\theta^p} x^{j+p+1}.
\]

Thus from relation (4.2.7), we have

\[
E[X^{j+1}(r,n,m,k)] = E[\xi X^{j+1}(r-1,n,m,k)] \
- \frac{j+1}{\gamma_r p\theta^p} E[X^{j+p+1}(r,n,m,k)]
\]

as obtained by Pawlas and Szyal (2001).

**ii) Power function distribution**

\[
F(x) = \lambda^{-p} x^p, \quad 0 \leq x \leq \lambda
\]

here we have \( a = \lambda^{-p}, b = 0, c = 1 \)

and \( h(x) = x^p \)

Let \( \xi(x) = x^{j+1} \), then \( \psi(x) = \frac{\lambda^{-p} (j+1)}{p} x^{j+1} \)

Therefore, from relation (4.2.7), we get

\[
E[X^{j+1}(r,n,m,k)] = E[X^{j+1}(r-1,n,m,k)] \
- \frac{j+1}{\gamma_r p} E[X^{j+1}(r,n,m,k)].
\]

**iii) Pareto distribution**

\[
F(x) = 1 - \lambda^{-p} x^p, \quad \lambda \leq x \leq \infty
\]
We have \( a = -\lambda^{-p}, b = 1, c = 1 \)

and \( h(x) = x^{-p} \) and \( \xi(x) = x^{j+1} \)

Therefore, \( \psi(x) = \frac{\lambda^p(j + 1)x^{j+1}}{p} - \frac{j + 1}{p} x^{j+p+1} \)

\[ E[X^{j+1}(r, n, m, k)] - E[X^{j+1}(r - 1, n, m, k)] \]

\[ = \frac{j + 1}{\gamma_r^p} E[X^{j+1}(r, n, m, k)] \]

\[ - \frac{j + 1}{\gamma_r^p \lambda^p} E[X^{j+p+1}(r, n, m, k)] \]

iv) Burr type III

\( F(x) = (1 + \theta x^{-p})^{-\lambda}, 0 \leq x \leq \infty \)

here \( a = \theta, b = 1, c = -\lambda \)

and \( h(x) = x^{-p} \)

Let \( \xi(x) = x^{j+1} \) then,

\[ \psi(x) = \frac{\theta(j + 1)x^{j+1}}{p} - \frac{j + 1}{p} x^{j+p+1} \]

Thus from relation (4.2.7), we have

\[ E[X^{j+1}(r, n, m, k)] - E[\xi(X^{j+1}(r - 1, n, m, k))] \]

\[ = - \frac{j + 1}{\gamma_r^p \lambda^p} E[X^{j+1}(r, n, m, k)] \]

\[ - \frac{j + 1}{\gamma_r^p \lambda^p \theta} E[X^{j+p+1}(r, n, m, k)]. \]
Moments of dual generalized order statistics

Similarly several recurrence relation based on Theorem 4.2.1 and Theorem 4.2.2 and theorem 4.2.3 can be established with proper choice of $a$, $b$ and $h(x)$.

b) $F(x) = [ax + b]^c$, $\alpha < x < \beta$ \hspace{1cm} (4.2.8)

and

$$X_d(r) \sim \frac{1}{b} \left( \prod_{j=1}^{r} B_{j}^{\frac{1}{c}} - b \right)$$ \hspace{1cm} (4.2.9)

**Theorem 4.2.4:** (khan et al., 2009)

For the distribution given in (4.2.8) and any non-negative finite integer $p$

$$E[X_d^p(r, n, m, k)] = \frac{1}{a^p} \sum_{i=0}^{p} (-b)^{p-i} \binom{p}{i} \frac{C_{r-i}^{(k)}}{\Gamma(n+1) \Gamma(n+1+i-r)}$$ \hspace{1cm} (4.2.10)

**Proof:** see reference.

**Remark 4.2.5:** For order statistics $(m = 0, k = 1)$, we have

$$E(X_{n-r+1:n}^p) = \frac{1}{a^p} \sum_{i=0}^{p} (-b)^{p-i} \binom{p}{i} \frac{\Gamma(n+1) \Gamma(n+1+i-r)}{\Gamma(n-r+1) \Gamma(n+1+i)}$$

**Remark 4.2.6:** For lower records $(m = -1)$, we have

$$E[X_d^p(r, n-1, k)] = \frac{1}{a^p} \sum_{i=0}^{p} (-b)^{p-i} \binom{p}{i} \left( \frac{k}{k + \frac{i}{c}} \right)$$

**Examples**

i) **Power function distribution**

$$F(x) = \left( \frac{x - \alpha}{\beta - \alpha} \right)^\theta$$ \hspace{1cm} $\alpha < x < \beta$
Here \( a = \frac{1}{\beta - \alpha} \), \( b = -\frac{\alpha}{\beta - \alpha} \) and \( c = \theta \), then

\[
E[X_d^p(r, n, m, k)] = (\beta - \alpha)^p \sum_{i=0}^{p} \left( \frac{\alpha}{\beta - \alpha} \right)^i \binom{p}{i} \frac{C_r^{(k)} C_{r-1}^{(k+\frac{1}{\theta})}}{C_{r-1}^{(k+\frac{1}{\theta})}}
\]

Taking \( \alpha = 0 \), we get

\[
E[X_d^p(r, n, m, k)] = \beta^p \frac{C_r^{(k)} C_{r-1}^{(k+\frac{1}{\theta})}}{C_{r-1}^{(k+\frac{1}{\theta})}}
\]

and using (2.3.15) we have at \( \alpha = 0 \),

\[
E[X_d^p(r, n, m, k)] = \frac{\beta^p C_r^{(k)} C_{r-1}^{(k+\frac{1}{\theta})}}{(m+1)^{r-1}(r-1)!} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{(k+\frac{1}{\theta})^{r-u}}
\]

as obtained by Athar et al. (2007).

ii) Pareto distribution

\[
F(x) = \left( \frac{\delta - \beta}{\delta - x} \right)^\theta, \quad -\infty < x < \beta
\]

Here \( a = -\frac{1}{\delta - \beta} \), \( b = \frac{\delta}{\delta - \beta} \) and \( c = -\theta \)

Therefore

\[
E[X_d^p(r, n, m, k)] = (\delta - \beta)^p \sum_{i=0}^{p} (-1)^i \binom{\delta}{\delta - \beta}^{p-i} \binom{p}{i} \frac{C_r^{(k)} C_{r-1}^{(k+\frac{1}{\theta})}}{C_{r-1}^{(k+\frac{1}{\theta})}}
\]

iii) Reflected exponential distribution

\[
F(x) = e^{\lambda x}, \quad -\infty < x < 0
\]

Here \( a = \frac{\lambda}{c} \), \( b = 1 \) and \( c \to \infty \)

Following the steps of example (iii) in Section 2, we have
\[ E[X_d^p(r, n, m, k)] = \frac{(\lambda)^{-p} \cdot p! \cdot C_r^{(k)}}{(m + 1)^{r-1} (r - 1)!} \]
\[ \times \sum_{u=0}^{r-1} (-1)^{u+p} \binom{r - 1}{u} \frac{1}{\left(\frac{\gamma(k)}{r - u}\right)^{p+1}}, \quad m \neq -1 \]

and

\[ E[X_d^p(r, n, m, k)] = \frac{(-\lambda)^{-p} \cdot \Gamma(p + r)}{\Gamma(r) \cdot k^p}, \quad m = -1 \]

**Power function distribution**

The pdf of distribution is given as
\[
f(x) = pv^{-p}x^{p-1}; \quad 0 < x < \nu, \nu > 0 \tag{4.2.11}\]
\[
= 0, \quad \text{otherwise} \]

and the corresponding distribution function (df) is
\[
F(x) = \nu^{-p}x^{p}; \quad 0 < x < \nu, \nu > 0 \tag{4.2.12}\]

For Power function distribution, we have
\[
F(x) = \frac{x}{\nu}f(x) \tag{4.2.13}\]

**Lemma 4.2.2: (Athar et al., 2009)**

For power function distribution as given in (4.2.11) and any non negative finite integers \(a\) and \(b\).
\[
J_\alpha(a, b) = \frac{1}{(m + 1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} J_{\alpha}[a + (m + 1)i, 0] \tag{4.2.14}\]
\[
= \nu^\alpha \left( \frac{1}{(m + 1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} \frac{1}{t_{\alpha}[a + (m + 1)i]} \right), \quad m \neq -1 \tag{4.2.15}\]
\[
= \frac{b! \, p^b \, \nu^a}{[t_\alpha(a)]^{b+1}}, \quad m = -1
\] (4.2.16)

where
\[
J_\alpha(a, b) = \int_0^x x^{a-1} \left[ F(x) \right]^a g_m[F(x)] \, dx 
\] (4.2.17)
\[
J_\alpha(a, 0) = \frac{\nu^a}{t_\alpha(a)} 
\] (4.2.18)

and \( t_\alpha(a) = \alpha + a \, p \)

**Proof:** When \( m \neq -1 \)

Result (4.2.15) can be proved by expanding
\[
g_m[F(x)] = \left[ \frac{1}{m+1} \left( 1 - (F(x))^{m+1} \right) \right]^b
\]
binomially in (4.2.17) and then using the result (4.2.18).

when \( m = -1 \)
at \( m = -1 \) in (4.2.14)
\[
J_\alpha(a, b) = 0
\]
as \( \sum_{i=0}^b (-1)^i \binom{b}{i} = 0 \)

Since (4.2.15) is of the form \( 0/0 \) at \( m = -1 \), therefore after applying L-Hospital’s rule, we get
\[
\lim_{m \to -1} J_\alpha(a, b) = \frac{p^b \, \nu^a}{(\alpha + a \, p)^{b+1}} \sum_{i=0}^b (-1)^{i+b} \binom{b}{i} i^b, \quad b > 0
\]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)
\[
\sum_{i=0}^n (-1)^i \binom{n}{i} (x-i)^n = n!
\]

Therefore,
\[
\sum_{i=0}^b (-1)^{i+b} \binom{b}{i} i^b = b!.
\]

Hence
\[ \lim_{m \to -1} J_\alpha(a, b) = \frac{b! \cdot p^b \nu^\alpha}{(\alpha + a p)^{b+1}}. \]

**Theorem 4.2.5:** (Athar et al., 2009)

For power function distribution as given in (4.2.11) and \( \gamma_r \geq 1, \ k \geq 1, \ 1 \leq r \leq n, \ m \neq -1. \)

\[
E\left( X_{r,n,m,k}^\alpha \right) = \frac{p \cdot C_{r-1}}{(r-1)!} J_\alpha(\gamma_r, r-1) \]
\[
= \frac{p \nu^\alpha}{(m+1)^{r-1}} \frac{C_{r-1}}{(r-1)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha(\gamma_{r-i})} \]  
(4.2.19)

**Proof:** From (1.8.5), we have

\[
E\left( X_{r,n,m,k}^\alpha \right) = \frac{C_{r-1}}{(r-1)!} \int_0^\nu x^\alpha \left[ F(x) \right]^{r-1} f(x) g_m^{r-1} [F(x)] \, dx \]

Now on application of (4.2.13), we get

\[
E\left( X_{r,n,m,k}^\alpha \right) = \frac{p \nu^\alpha}{(m+1)^{r-1}} \frac{C_{r-1}}{(r-1)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha(\gamma_{r-i})} \]
(4.2.20)

and hence the theorem, in view of (4.2.17) and (4.2.18).

**Identity 4.2.1:** For \( \gamma_r \geq 1, \ k \geq 1, \ 1 \leq r \leq n \) and \( m \neq -1. \)

\[
\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}} = \frac{(m+1)^{r-1}(r-1)!}{\prod_{j=1}^r \gamma_j} \]
(4.2.21)

**Proof:** (4.2.21) can be proved by putting \( \alpha = 0 \) in (4.2.20).

**Remark 4.2.7:** If we put \( m = 0, \ k = 1 \) in (4.2.20), we get the result for order statistics

\[
E\left( X_{r,n,0,1}^\alpha \right) = E\left( X_{n-r+1:n}^\alpha \right) = p \nu^\alpha C_{n-r+1} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha(n+i-r+1)} \]
(4.2.22)

where,
\[ C_{n-r+l,n} = \frac{n!}{(n-r)!(r-1)!} \]

In view of Identity (4.2.1), (4.2.22) may also be expressed as

\[ E\left( X_{n-r+l,n}^{\alpha} \right) = \frac{\Gamma(n+1)\Gamma[(\alpha/p) + n - r + 1]v^\alpha}{\Gamma(n-r+1)\Gamma[n + (\alpha/p) + 1]}, \quad p > \alpha \]

as obtained by Malik (1967).

**Remark 4.2.8:** Moment of \( k \)-th lower record values from the power function distribution may be obtained in view of (4.2.16) and (4.2.19) at \( m = -1 \).

\[ E\left( X_{r,n,-1,k}^{\alpha} \right) = E(R_n^{(k)})^\alpha = \frac{(pk)^n v^\alpha}{[t_\alpha(k)]^n} \]

by noting \( \gamma = k \) and \( C_{r-1} = k^r \).

**Remark 4.2.9:** For \( m = 0 \) and \( k = \alpha' - n + 1, \alpha \in \mathbb{R}_+ \), we get the moment of order statistics with non-integral sample size

\[ E\left( X_{\alpha'-r+1,\alpha'}^{\alpha} \right) = p v^\alpha C_{\alpha'-r+1,\alpha'} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha(\alpha' + i - r + 1)} \]

**Exponential Pareto distribution**

The pdf of distribution is given as

\[ f(x) = \theta \lambda \left[ 1 - (1 + x)^{-\lambda} \right]^{\theta-1} (1 + x)^{-(\lambda+1)}, \quad x > 0, \quad \lambda, \theta > 0 \]

and the corresponding df is

\[ F(x) = \left[ 1 - (1 + x)^{-\lambda} \right]^{\theta}. \]

Therefore, in view of (4.2.23) and (4.2.24), we have

\[ F(x) = \frac{1}{\theta \lambda} \left[ \lambda x + \sum_{u=2}^{\lambda+1} \binom{\lambda+1}{u} x^u \right] f(x), \quad \lambda \text{ is positive integer.} \]
Here $\theta$ and $\lambda$ are two shape parameters. For $\theta = 1$, the above distribution corresponds to the standard Pareto distribution of second kind (Shawky and Abu-Zinadah, [10]).

**Theorem 4.2.6: (Khan and Kumar, 2010)**

For the distribution given in (4.2.24) and for $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$

$$E[X^{r,j}(r,n,m,k)] - E[X^{r,j}(r-1,n,m,k)]$$

$$= -\frac{j}{\theta \gamma_r} \left\{ E[X^{r,j}(r,n,m,k)] + \frac{1}{\lambda} \sum_{u=2}^{\lambda+1} \left( \frac{\lambda + 1}{u} \right) E[X^{r,j+u-1}(r,n,m,k)] \right\}$$

(4.2.26)

**Proof:** In view of (4.2.3) and (4.2.25), we have

$$E[X^{r,j}(r,n,m,k)] - E[X^{r,j}(r-1,n,m,k)]$$

$$= -\frac{j}{\gamma_r (r-1)!} \int_0^\infty x^{r-1}[F(x)]^{r-1} \left\{ \frac{1}{\theta} \left[ x + \frac{1}{\lambda} \sum_{u=2}^{\lambda+1} \left( \frac{\lambda + 1}{u} \right) x^u \right] \right\}$$

$$\times f(x) g_m^{r-1}(F(x)) dx$$

$$= -\frac{j}{\theta \gamma_r} \left\{ \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^{r-1} [F(x)]^{r-1} f(x) g_m^{r-1}(F(x)) dx$$

$$+ \frac{1}{\lambda} \sum_{u=2}^{\lambda+1} \left( \frac{\lambda + 1}{u} \right) \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^{j+u-1} [F(x)]^{r-1} f(x) g_m^{r-1}(F(x)) dx \right\}$$

and hence the result.

**Remark 4.2.10:** For $m = 0$, $k = 1$, the recurrence relations for lower generalized order statistics reduces to the recurrence relations of lower order statistics as
\[ E(X''_{n-r+1:n}) - E(X''_{n-r+2:n}) = - \frac{f}{\theta(n-r+1)} \left\{ E(X''_{n-r+1:n}) + \frac{\lambda+1}{\lambda u} E(X''_{n-r+1:n}) \right\} \]

**Theorem 4.2.7: (Khan and Kumar, 2010)**

For the distribution given in (4.2.24) and for \(2 < r < n,\) \(n > 2\) and \(k = 1, 2, \ldots\)

\[ E[X''_{(r,n,m,k)}] - E[X''_{(r-1,n-1,m,k)}] = - \frac{f}{\theta} \left\{ E[X''_{(r,n,m,k)}] + \frac{\lambda+1}{\lambda u} E[X''_{(r,n,m,k)}] \right\} \]

**Proof:** Proof is easy.

**Power function distribution [Generalized Uniform distribution]**

The pdf of the distribution is

\[ f(x) = \frac{\alpha+1}{\theta^{\alpha+1}} x^\alpha, \quad 0 < x < \theta \quad \text{(4.2.27)} \]

with df

\[ F(x) = \left( \frac{x}{\theta} \right)^{\alpha+1}, \quad 0 < x < \theta \quad \text{(4.2.28)} \]

where \(\alpha > -1\) is the shape parameter and \(\theta > 0\) is the threshold parameter.

Now in view of (4.2.27) and (4.2.28), we have

\[ F(x) = \frac{x}{\alpha+1} f(x) \quad \text{(4.2.29)} \]

The generalized uniform distribution is a uniform distribution at \(\alpha = 0\) and is a standard power distribution at \(\theta = 1\).
Case I: $\gamma_i \neq \gamma_j; i \neq j = 1, 2, \ldots, n-1$

Theorem 4.2.8: (Athar and Faizan, 2011)

For distribution as given in (4.2.27) and $n \in N, m \in R, k > 0, 1 \leq r \leq n$

$$E[X'(r, n, m, k)]^j = \frac{(\alpha + 1)\gamma_r}{\{j + \gamma_r(\alpha + 1)\}} E[X'(r - 1, n, m, k)]^j \quad (4.2.30)$$

Proof: We have Athar et al. (2008)

$$E[\xi\{X'(r, n, m, k)\}] - E[\xi\{X'(r - 1, n, m, k)\}]$$

$$= -C_{r-2} \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} dx \right\}$$

Let $\tilde{\xi}(x) = x^j$, then

$$E[\tilde{\xi}\{X'(r, n, m, k)\}]^j - E[\tilde{\xi}\{X'(r - 1, n, m, k)\}]^j$$

$$= -C_{r-2} \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} dx \right\}$$

Now in view of (4.2.29), we get

$$E[\tilde{\xi}\{X'(r, n, m, k)\}]^j - E[\tilde{\xi}\{X'(r - 1, n, m, k)\}]^j$$

$$= -\frac{j}{\gamma_r(\alpha + 1)} C_{r-1} \frac{\beta}{\alpha} \left\{ \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1} f(x) dx \right\}$$

and hence the result.

Case II. $m_i = m_j = m, i \neq j = 1, 2, \ldots, n - 1$.

Theorem 4.2.9: (Athar and Faizan, 2011)

For distribution as given in (4.2.27) and $n \in N, m \in R, k > 0, 1 \leq r \leq n$

$$E[X'(r, n, m, k)]^j = \frac{\gamma_r(\alpha + 1)}{\{j + \gamma_r(\alpha + 1)\}} E[X'(r - 1, n, m, k)]^j \quad (4.2.31)$$

$$E[X'(r, n, m, k)]^j = \theta^j (\alpha + 1)^r \prod_{i=1}^{r} \frac{\gamma_i}{\{j + \gamma_i(\alpha + 1)\}} \quad (4.2.32)$$
**Proof:** (4.2.31) can be established in view of Athar et al. (2008) and (4.2.27) on the lines of Theorem 4.2.8.

Since \( X'(0,n,m,k) = \theta \), the maximum of \( X \) in the power function distribution, we have

\[
E[X'(1,n,m,k)]^j = \frac{\gamma_1(\alpha + 1)}{\{j + \gamma_1(\alpha + 1)\}} \theta^j
\]  

(4.2.33)

(4.2.32) can be obtained by writing (4.2.31) recursively and using (4.2.33) as initial value.

**Remark 4.2.11:** Recurrence relation for single moments of order statistics (at \( m = 0, k = 1 \)) is

\[
E(X_{n-r+1}^j) = \frac{(\alpha + 1)(n-r+1)}{j + (\alpha + 1)(n-r+1)} E(X_{n-r+2}^j)
\]

Replacing \((n-r+1)\) by \((r-1)\), we have

\[
E(X_{r/n}^j) = \frac{j + (\alpha + 1)(r-1)}{(\alpha + 1)(r-1)} E(X_{r-1/n}^j)
\]

as obtained by Malik (1967)

or

\[
E(X_{r/n}^j) = \frac{n(\alpha + 1)}{n(\alpha + 1) + j} E(X_{r-1/n-1}^j)
\]

as obtained by Khan et al. (1983).

**Remark 4.2.12:** Recurrence relation for single moments of \( k-th \) lower record values will be

\[
E(X_r^{(k)})^j = \frac{(\alpha + 1)k}{\{j + 1 + k(\alpha + 1)\}} E(X_{r-1}^{(k)})^j
\]
as obtained by Bieniek and Szynal (2002) and

\[ E(X_r^{(k)})^j = \theta^j \left( \frac{(\alpha + 1)k}{j + k(\alpha + 1)} \right)^r \]

3. Product Moments of Dual Generalized Order Statistics

a) \( F(x) = [ah(x) + b]^\beta, \alpha \leq x \leq \beta \)

**Lemma 4.3.1: (Athar et al., 2008)**

For \( 1 \leq r < s \leq n - 1 \), \( n \geq 2 \) and \( k = 1, 2, \ldots \)

\[
E[\xi\{X^r(r,n,m,k)X^s(s,n,m,k)\}] - E[\xi\{X^r(r,n,m,k)X^{s-1}(n,m,k)\}]
\]

\[
= - \frac{C_{s-2}}{(r-1)!(s-r-1)!} \left[ \beta \frac{\partial}{\partial x} \xi(x,y) [F(x)]^m f(x) g_m^{r-1}(F(x)) \right]^{s-r-1} [1 - F(y)]^s d\bar{y} d\bar{x}, \quad \beta \geq x > y \geq \alpha
\]

(4.3.1)

where \( \xi(x,y) = \xi_1(x) \xi_2(y) \)

**Proof:** We have

\[
E[\xi\{X^r(r,n,m,k)X^s(s,n,m,k)\}] - E[\xi\{X^r(r,n,m,k)X^{s-1}(n,m,k)\}]
\]

\[
= \frac{C_{s-2}}{(r-1)!(s-r-1)!} \left[ \beta \frac{\partial}{\partial x} \xi(x,y) [F(x)]^m f(x) g_m^{r-1}(F(x)) \right]^{s-r-2} [F(y)]^{s-1} f(y)
\]

\[
\times \left[ \gamma_s \{h_m(F(y)) - h_m(F(x))\}^{s-r-2} (F(y))^{m+1} \right] d\bar{y} d\bar{x}
\]

(4.3.2)

Let
\[ v(x, y) = [h_m(F(y) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma s} \]

then

\[ \frac{\partial}{\partial y} v(x, y) = [h_m(F(y) - h_m(F(x))]^{s-r-2}[F(y)]^{\gamma s-1} f(y) \]

\[ \times [\gamma_s \{ h_m(F(y) - h_m(F(x)) - (s - r - 1)[F(y)]^{m+1} \} (4.3.3) \]

Putting the value of (4.3.3) in (4.3.2), we get

\[ E \{ \xi \{ X'(r, n, m, k), X'(s, n, m, k) \} \} \]

\[ - E \{ \xi \{ X'(r, n, m, k), X'(s-1, n, m, k) \} \} \]

\[ = \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int_x^y \int_x^y \xi(x, y)[F(x)]^m f(x) \]

\[ \times g_{m}^{r-1}(F(x)) \frac{\partial v(x, y)}{\partial y} dy dx \]

\[ = \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int_x^y \int_x^y \xi(x)[F(x)]^m f(x) \]

\[ \times g_{m}^{r-1}(F(x)) \left[ \int_x^y \frac{\partial v(x, y)}{\partial y} dy \right] dx. \] (4.3.4)

Now, we have

\[ \int_x^y \frac{\partial v(x, y)}{\partial y} dy = -\int_x^y \xi_1(x) [h_m(F(y) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma s} dy \]

(4.3.5)

After substituting (4.3.5) into (4.3.4) and noting that

\[ \frac{\partial}{\partial y} \xi(x, y) = \xi_1(x) \xi_2'(y) \] the required expression is obtained.
Theorem 4.3.1: (Athar et al., 2008)

For the distribution given in (4.2.1) and \(1 \leq r < s \leq n-1, n \in \mathbb{N}\)
\[
E \{\xi \{X'(r, n, m, k), X'(s, n, m, k)\}\} - E[\xi \{X'(r, n, m, k), X'(s-1, n, m, k)\}]
\]
\[
= -\frac{1}{ca'y_s} E[\psi \{X'(r, n, m, k), X'(s, n, m, k)\}] \quad (4.3.6)
\]

where \(\psi(x, y) = [ah(y) + b]^{x+y} \),

Proof: Proof is easy.

Remark 4.3.1: Under the assumption given in Theorem 4.3.1 with \(k = 1, m = 0\), we get the recurrence relation for product moments of order statistics (Ali and Khan, 1998) and at \(m = -1\), we have the recurrence relation for product moments of \(k\)-th record values.

Examples

i) Inverse weibull distribution

\[ F(x) = e^{-(\theta/x)^p}, \quad x > 0, p, \theta > 0. \]

We have \(a = 1, b = 0, c = 1\) and \(h(x) = e^{-(\theta/x)^p}\).

Let \(\xi(x, y) = x^iy^{j+1}\), then

\[
\psi(x, y) = [ah(y) + b]\frac{\partial \xi(x, y)}{h'(y)} = \frac{j + 1}{p\theta^p} x^iy^{j+p+1}.
\]

Thus from relation (4.3.6), we have
moments of dual generalized order statistics

\[ E[X^i(r,n,m,k).X'^i(s,n,m,k)] - E[X^i(r,n,m,k).X'^i(s-1,n,m,k)] \]

\[ = -\frac{j+1}{\gamma_s \rho \theta^p} E[X^i(r,n,m,k)X'^i+1(s,n,m,k)] \]

as obtained by Pawlas and Szyal (2001).

ii) Power function distribution

\[ F(x) = \lambda^{-p} x^p, \quad 0 \leq x \leq \lambda \]

here we have \( a = \lambda^{-p}, b = 0, c = 1 \) and \( h(x) = x^p \)

Let \( \xi(x,y) = x^i y^{j+1} \), then \( \psi(x,y) = \frac{\lambda^{-p}(j+1)}{p} x^i y^{j+1} \)

Therefore, from relation (4.3.6), we get

\[ E[X^i(r,n,m,k).X'^i(s,n,m,k)] - E[X^i(r,n,m,k).X'^i(s-1,n,m,k)] \]

\[ = -\frac{j+1}{\gamma_s \rho^p} E[X^i(r,n,m,k)X'^i+1(s,n,m,k)] \]

iii) Pareto distribution

\[ F(x) = 1 - \lambda^{-p} x^p, \quad \lambda \leq x \leq \infty \]

We have \( a = -\lambda^{-p}, b = 1, c = 1 \)

and \( h(x) = x^{-p} \) and \( \xi(x,y) = x^i y^{j+1} \)

\[ \psi(x,y) = \frac{(j+1)\lambda^p}{p} x^i y^{j+1} - \frac{j+1}{p} x^i y^{j+p+1} \]

Therefore, from (4.3.6) we have
\[ E[X^{i}(r,n,m,k)X^{j+1}(s,n,m,k)] - E[X^{i}(r,n,m,k)X^{j+1}(s-1,n,m,k)] \]
\[ = \frac{j+1}{\gamma_{s}} E[X^{i}(r,n,m,k)X^{j+1}(s,n,m,k)] \]
\[ - \frac{j+1}{\gamma_{s}p\lambda p} E[X^{i}(r,n,m,k)X^{j+p+1}(s,n,m,k)] \]

iv) Burr type III

\[ F(x) = (1 + \theta x^{-p})^{-\lambda}, \ 0 \leq x \leq \infty \]

Here \( a = \theta, b = 1, c = -\lambda \) and \( h(x) = x^{-p} \)

\[ \psi(x,y) = \frac{\theta(j+1)\lambda p}{p} x^{j+1} y^{j+1} - \frac{j+1}{p} x^{j+1} y^{j+p+1} \]

Thus from relation (4.3.6), we have

\[ E[X^{i}(r,n,m,k)X^{j+1}(s,n,m,k)] - E[X^{i}(r,n,m,k)X^{j+1}(s-1,n,m,k)] \]
\[ = - \frac{j+1}{\gamma_{s}p\lambda} E[X^{i}(r,n,m,k)X^{j+1}(s,n,m,k)] \]
\[ - \frac{j+1}{\gamma_{s}p\lambda\theta} E[X^{i}(r,n,m,k)X^{j+p+1}(r,n,m,k)] \]

b) \( F(x) = (ax + b)^{c}, \quad \alpha < x < \beta \)

**Theorem 4.3.2:** (Khan et al., 2009)

For the distribution given in (4.2.8)

\[ E[X^{p}(r,n,m,k)X^{q}(s,n,m,k)] \]
\[ (-1)^{p+q} \left( \frac{b}{a} \right)^{p+q} \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{u+v} \frac{1}{b^{u+v}} \binom{p}{u} \binom{q}{v} \frac{C_{s-1}^{(k)}}{C_{r-1}^{(k+\frac{v}{\theta})}} \frac{C_{s-1}^{(k+\frac{v}{\theta})}}{C_{r-1}^{(k+\frac{u+v}{\theta})}} \]

(4.3.7)

where \( p \) and \( q \) are non-negative integers.

**Proof:** See reference.

**Examples**

i) Power function distribution

\[ F(x) = \left( \frac{x - \alpha}{\beta - \alpha} \right)^{\theta}, \quad \alpha < x < \beta \]

Here \( a = \frac{1}{\beta - \alpha}, \ b = -\frac{\alpha}{\beta - \alpha} \) and \( c = \theta \), then

\[ E[X_d^p(r, n, m, k).X_d^q(s, n, m, k)] \]

\[ = \alpha^{p+q} \sum_{u=0}^{p} \sum_{v=0}^{q} \left( \frac{\beta - \alpha}{\alpha} \right)^{u+v} \binom{p}{u} \binom{q}{v} \frac{C_{s-1}^{(k)}}{C_{r-1}^{(k+\frac{v}{\theta})}} \frac{C_{s-1}^{(k+\frac{v}{\theta})}}{C_{r-1}^{(k+\frac{u+v}{\theta})}} \]

Taking \( \alpha = 0 \), we have

\[ E[X_d^p(r, n, m, k).X_d^q(s, n, m, k)] = \beta^{p+q} \frac{C_{s-1}^{(k)}}{C_{r-1}^{(k+\frac{q}{\theta})}} \frac{C_{s-1}^{(k+\frac{q}{\theta})}}{C_{r-1}^{(k+\frac{p+q}{\theta})}} \]

In view of (2.3.15) and (3.3.6), we have

\[ E[X_d^p(r, n, m, k).X_d^q(s, n, m, k)] = \frac{\beta^{p+q} C_{s-1}^{(k)}}{(m+1)^{r-2} (r-1)!(s-r-1)!} \]
as obtained by Athar et al. (2007).

(ii) Pareto distribution

\[ F(x) = \left( \frac{\beta - \delta}{\delta - x} \right)^\theta, \quad -\infty < x < \beta \]

Here \( a = -\frac{1}{\delta - \beta}, \quad b = \frac{\delta}{\delta - \beta} \) and \( c = -\theta \)

\[ E[X_d^p (r, n, m, k).X_d^q (s, n, m, k)] \]

\[ = \delta^{p+q} \sum_{u=0}^p \sum_{v=0}^q (-1)^{u+v} \left( \frac{\delta - \beta}{\delta} \right)^{u+v} \left( \begin{array}{c} p \\ u \end{array} \right) \left( \begin{array}{c} q \\ v \end{array} \right) \frac{C_s^{(k)}}{C_{s-1}^{(k-\frac{v}{\theta})}} \frac{C_r^{(k)}}{C_{r-1}^{(k-\frac{u+v}{\theta})}} \]

(iii) Reflected exponential distribution

\[ F(x) = e^{\lambda x}, \quad -\infty < x < 0 \]

Here \( a = \frac{\lambda}{c}, \quad b = 1 \) and \( c \to \infty \)

So we have,

\[ E[X_d^p (r, n, m, k).X_d^q (s, n, m, k)] = \left( \frac{1}{\lambda} \right)^{p+q} \frac{(-1)^{p+q} C_s^{(k)} p! q!}{(m+1)^{p+2} (s-r-1)! (r-1)!} \times \left[ \sum_{i=0}^{s-r-1} (-1)^i \left( \begin{array}{c} s-r-1 \\ i \end{array} \right) \frac{1}{\gamma_{s-i}^{(k)}} \right] \times \left[ \sum_{i=0}^{r-1} (-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \frac{1}{\gamma_{r-i}^{(k)}} \right], \quad m \neq -1 \]
and

\[ E[X^p_d(r, n, m, k)X^q_d(s, n, m, k)] = (-1)^{p+q} \left( \frac{1}{\lambda} \right)^{p+q} \frac{C_s^{(k)} \Gamma(p+r)\Gamma(q+s-r)}{(s-r-1)!(r-1)!} k^{-(p+q+s)}, \ m = -1 \]

### Power function Distribution

**Lemma 4.3.2: (Athar et al., 2009)**

For the power function distribution as given in (4.2.11) and non-negative integers \(a, b, c\) with \(m \neq -1\)

\[ J_{\alpha, \beta}(a, 0, c) = \frac{\nu^{\alpha+\beta}}{t_{\beta}(c)t_{\alpha+\beta}(a+c)} \]  

(4.3.8)

where

\[ J_{\alpha, \beta}(a, b, c) = \nu^\alpha \int_0^a x^{\alpha-1} y^{\beta-1} [F(x)]^a [h_m(F(y)) - h_m(F(x))]^b [F(y)]^c dy dx \]

(4.3.9)

**Proof:** From (4.2.12) and (4.3.9), we have

\[ J_{\alpha, \beta}(a, 0, c) = \nu^{-p(a+c)} \int_0^\nu x^{\alpha+ap-1} \int_0^{x^{\beta+cp-1}} dy dx \]

\[ = \frac{\nu^{\alpha+\beta}}{(\beta + cp)[\alpha + \beta + p(a + c)]} \]

and hence the lemma.

**Lemma 4.3.3: (Athar et al., 2009)**

For the power function distribution as given in (4.2.11) and any non-negative integers \(a, b, c\).
Moments of dual generalized order statistics

\[ J_{\alpha,\beta}(a, b, c) = \frac{\nu^{\alpha+\beta}}{(m+1)^b} \times \sum_{j=0}^{b} (-1)^j \binom{b}{j} \frac{1}{t_\beta [c + (m+1)j] t_{\alpha+\beta} [(a+c) + (m+1)b]} ; \quad m \neq -1 \]  

(4.3.10)

\[ J_{\alpha,\beta}(a, b, c) = \frac{b! p^b \nu^{\alpha+\beta}}{[t_\beta(c)]^{b+1} [t_{\alpha+\beta}(a+c)]} ; \quad m = -1 \]  

(4.3.11)

**Proof:** When \( m \neq -1 \)

(4.3.10) can be established by expanding \([h_m(F(y)) - h_m(F(x))]^b\) binomially in (4.3.9) and thereafter on application of Lemma 4.3.2.

When \( m = -1 \):

Since at \( m = -1 \) (4.3.10) is of the 0/0, so after applying L-Hospital’s rule (4.3.11) can be proved on the lines of (4.2.15).

**Theorem 4.3.3:** (Athar et al., 2009)

For power function distribution as given in (4.2.11) and \( \gamma_r, \gamma_s \geq 1, k \geq 1, 1 \leq r < s \leq n, m \neq -1 \).

\[ E\left( X_{r,n,m,k}^{*\alpha} X_{s,n,m,k}^{*\beta} \right) = \frac{p^2}{(m+1)^{r-1} (r-1)! (s-r-1)!} \times \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} J_{\alpha,\beta} [(m+1)(j+1), (s-r-1), \gamma_s] \]  

(4.3.12)
\[
\frac{p^2 v^{\alpha+\beta}}{(m+1)^{s-2}} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \\
\times \sum_{j=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{t_\beta(\gamma_{s-l}) t_{\alpha+\beta}(\gamma_{r-j})}
\]

and subsequently for \( s = r + 1 \)

\[
E\left( X_{r,n,m,k}^* X_{r+1,n,m,k}^* \right) = \frac{p^2 v^{\alpha+\beta}}{(m+1)^{r-1}} \frac{C_r}{(r-1)!}
\times \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{t_\beta(\gamma_{r+1}) t_{\alpha+\beta}(\gamma_{r-j})}
\]

\textbf{Proof:} From (1.8.2), we have

\[
E\left( X_{r,n,m,k}^* X_{s,n,m,k}^* \right) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \\
\times \int_0^x x^\alpha y^\beta \left[ F(x) \right]^m f(x) g_m^{-1}(F(x)) \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1}
\times [F(y)]^{s-1} f(y) dy dx
\]

Since

\[
g_m^{-1}(F(x)) = \left\{ \frac{1}{m+1} [1 - (F(x))^{m+1}] \right\}^{r-1} = \frac{1}{(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} [F(x)]^{(m+1)j}
\]

Therefore in view of (4.2.13), we get
Thus the theorem is proved by an application of lemma 4.3.2 and lemma 4.3.3.

Identity 4.3.1: (Athar et al., 2009)

For $\gamma_r, \gamma_s \geq 1$, $k \geq 1$, $1 \leq r < s \leq n$ and $m \neq -1$.

\[
\sum_{l=0}^{s-r-1} (-1)^l \binom{s-r-1}{l} \frac{1}{\gamma_{s-l}} \frac{(m+1)^{s-r-1} (s-r-1)!}{\prod_{i=r+1}^{s} \gamma_i} \tag{4.3.16}
\]

**Proof:** At $\alpha = \beta = 0$ in (4.3.13), we have

\[
1 = \frac{C_{s-1}}{(m+1)^{s-2} (r-1)! (s-r-1)!} \\
\times \left\{ \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{\gamma_{r-j}} \right\} \left\{ \sum_{l=0}^{s-r-1} (-1)^l \binom{s-r-1}{l} \frac{1}{\gamma_{s-l}} \right\}
\]

Now on application of (4.2.20), we get the required result.

At $r = 0$, (4.3.16) reduces to (4.2.20).

**Remark 4.3.2:** At $m = 0$ and $k = 1$, the product moment of order statistics is
\[ E\left(X_{r,n,0,1}^\alpha, X_{s,n,0,1}^\beta\right) = E\left(X_{n-r+1:n}^\alpha, X_{n-s+1:n}^\beta\right) \]

\[ = C_{n-s+1,n-r+1:n}P^2 \cdot \nu^{\alpha+\beta} \]

\[ \times \sum_{j=0}^{r-1} \sum_{l=0}^{s-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-1}{l} \left(\begin{array}{c} n-s+j \\hline j \end{array}\right) \]

\[ \times \frac{1}{t_\beta (n-s+l+1) t_{\alpha+\beta} (n-r+j+1)} \]  

(4.3.17)

where

\[ C_{r,s:n} = \frac{n!}{(r-1)! (s-r-1)! (n-s)!} = C_{n-s+1,n-r+1:n} \]

Using (4.2.20) and (4.3.16), (4.3.17) may be re-written as

\[ E\left(X_{n-r+1:n}^\alpha, X_{n-s+1:n}^\beta\right) = \frac{\Gamma(n+1)}{\Gamma(n-s+1)} \frac{\Gamma((\alpha/p) + n-s+1)}{\Gamma(n-r+1+(\alpha/p))} \]

\[ \times \frac{\Gamma[(\alpha+\beta)/p + n-r+1]}{\Gamma[n+(\alpha+\beta)/p+1]} \nu^{\alpha+\beta} \]  

(4.3.18)

At \( \alpha = \beta = 1 \), (4.3.18) reduces to

\[ E\left(X_{n-r+1:n}^\alpha, X_{n-s+1:n}^\beta\right) = \frac{\Gamma(n+1)}{\Gamma(n-s+1)} \frac{\Gamma(1/p + n-s+1)}{\Gamma(n-r+1+(1/p))} \frac{\Gamma(2/p + n-r+1)}{\Gamma(n+2/p+1)} \nu^2 \]

as obtained by Malik (1967).

**Remark 4.3.3:** At \( m \to -1 \) in (4.3.15), the moment of \( k \)-th record value is given by

\[ E\left(X_{r,n,-1,k}^\alpha, X_{r+1,n,-1,k}^\beta\right) = E\left((R_n^{(k)})^\alpha, (R_{n+1}^{(k)})^\beta\right) \]
\[
E\left( X_{r,n,m,k}^{\alpha} \right) = \frac{p^\alpha}{(m+1)^{r-1}} \frac{C_{r-1}}{(r-1)!} \sum_{j=0}^{r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{\gamma_{s-l} t_{\alpha}(\gamma_{r-l})}
\]

In view of (4.3.16), (4.3.20) becomes

\[
E\left( X_{r,n,m,k}^{\alpha} \right) = \frac{p^\alpha}{(m+1)^{r-1}} \frac{C_{r-1}}{(r-1)!} \sum_{j=0}^{r-1} (-1)^{j+l} \binom{r-1}{j} \frac{1}{\gamma_{s-l} t_{\alpha}(\gamma_{r-l})}
\]

as obtained in (4.2.19).

**Exponential Pareto distribution**

**Theorem 4.3.4:** (Khan and Kumar, 2010)

For the distribution given in (4.2.24) and for \(1 \leq r < s \leq n-1, n \geq 2\) and \(k = 1, 2, \ldots\)

\[
E\left[ X^{i} (r,n,m,k) X^{j} (s,n,m,k) \right] = E\left[ X^{i} (r,n,m,k) X^{j} (s-1,n,m,k) \right]
\]

\[
\begin{align*}
= & \frac{j}{\theta \gamma_s} \left\{ E\left[ X^{i} (r,n,m,k) X^{j} (s,n,m,k) \right] \right. \\
\quad & + \frac{1}{\lambda} \sum_{u=\lambda}^{\lambda+1} \left. \binom{\lambda + 1}{u} E\left[ X^{i} (r,n,m,k) X^{j+u-1} (s,n,m,k) \right] \right\}
\end{align*}
\]

**Proof:** In view of (4.3.1)
\[ E[X^{ri}(r,n,m,k) X^{sj}(s,n,m,k)] - E[X^{ri}(r,n,m,k) X^{sj}(s-1,n,m,k)] \]
\[ = - \frac{j C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} \int_0^x \int_0^y x^i y^{j-1} [F(x)]^m f(x) g_m^{r-1}(F(x)) \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1} dy dx, x > y. \]

(4.3.22)

and hence (4.3.21), using (4.2.25) and (4.3.23).

**Remark 4.3.5:** Under the assumption given in Theorem 4.3.4 with \( k = 1, m = 0 \), we get the recurrence relation for product moment of lower order statistics and at \( k = 1, m = -1 \), we deduce the recurrence relations for product moments of lower record values from exponentiated Pareto distribution, proved by Shawky and Abu-Zinadah [9].

**Power function distribution [Generalized Uniform distribution]**

**Case I:** \( m_i \neq m_j \) and \( \gamma_i \neq \gamma_j \); \( i \neq j = 1,2,..,n-1 \)

**Theorem 4.3.5:** (Athar and Faizan, 2011)

For distribution as given in (4.2.27). Fix a positive integer \( k \) and for \( n \in N, \tilde{m} \in R, 1 \leq r < s \leq n \)

\[ E[(X'(r,n,\tilde{m},k))^i (X'(s,n,\tilde{m},k))^j] \]
\[ = \frac{(\alpha + 1)\gamma_s}{j + (\alpha + 1)\gamma_s} E[(X'(r,n,\tilde{m},k))^i (X'(s-1,n,\tilde{m},k))^j] \]

(4.3.23)

**Proof:** We have Athar et al. (2008),

\[ E[\xi \{X'(r,n,\tilde{m},k), X'(s,n,\tilde{m},k)\}] \]
Moments of dual generalized order statistics

\[-E[\xi\{X'(r,n,\tilde{m},k),X'(s-1,n,\tilde{m},k)\}]\]

\[=-C_{s-2} \int_{\alpha<y<x<u} \frac{\partial}{\partial y} \xi(x,y) \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \left[ \frac{F(y)}{F(x)} \right]^{y_i} \]

\[\times \sum_{i=1}^{r} a_{i}(r)[F(x)]^{y_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} dy \, dx \quad (4.3.24)\]

Now consider \(\xi(x,y) = \xi_1(x), \xi_2(y) = x^i \cdot y^j\) in (4.3.24), then in view of (4.2.29), we get

\[-E[(X'(r,n,\tilde{m},k))^i \cdot (X'(s,n,\tilde{m},k))^j] \]

\[= -E[(X'(r,n,\tilde{m},k))^i \cdot (X'(s-1,n,\tilde{m},k))^j] \]

\[=- \int_{\alpha}^{\beta} \frac{C_{s-1}}{\gamma_s(\alpha+1)} \sum_{i=r+1}^{s} a_{i}^{(r)}(s) \left[ \frac{F(y)}{F(x)} \right]^{y_i} \]

\[\times \sum_{i=1}^{r} a_{i}(r)[F(x)]^{y_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} dy \, dx \]

which leads to (4.3.23).

**Case II.** \(m_i = m_j = m, i \neq j = 1,2,\ldots,n-1.\)

The joint pdf of \(X'(r,n,m,k)\) and \(X'(s,n,m,k)\), \(1 \leq r < s \leq n\) is given as

\[f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [F(x)]^m \frac{f(x)}{F(x)} g_{m}^{r-1}(F(x)) \]

\[\times [h_{m}(F(y)) - h_{m}(F(x))]^{s-r-1} \frac{f(y)}{F(y)} [F(y)]^{y_i}, \quad x > y \]

\[\quad (4.3.25)\]

**Theorem 4.3.6:** (Athar and Faizan, 2011)
For distribution as given in (4.2.27). Fix a positive integer \( k \) and for
\( n \in N, m \in R, 1 \leq r < s \leq n \)

\[
E[(X^r(r, n, m, k))^i (X^s(s, n, m, k))^j] = \frac{(\alpha + 1)\gamma_s^i}{j + \gamma_s^i(\alpha + 1)} E[(X^r(r, n, m, k))^i (X^{s-1}(s, n, m, k))^j]
\]
(4.3.25)

and

\[
E[(X^r(r, n, m, k))^i (X^s(s, n, m, k))^j] = (\alpha + 1)^s \theta^{i+j} \left( \prod_{u=1}^{r} \frac{\gamma_u}{\gamma_u(\alpha + 1) + i + j} \right) \left( \prod_{v=r+1}^{s} \frac{\gamma_v}{\gamma_v(\alpha + 1) + j} \right)
\]
(4.3.26)

**Proof:** (4.3.25) can be proved on the lines on Theorem 4.3.5. To obtain (4.3.26) we write (4.3.25) recursively.

**Remark 4.3.6:** Recurrence relation for product moments of order statistics (at \( m = 0, k = 1 \)) is

\[
E[X_{n-r+1,n}^i X_{n-s+1,n}^j] = \frac{\alpha + 1)(n - s + 1)}{j + (\alpha + 1)(n - s + 1)} E[(X_{n-r+1,n}^i X_{n-s+2,n}^j)]
\]

\[
= (\alpha + 1)^s \theta^{i+j} \left( \prod_{u=1}^{r} \frac{(n - u + 1)}{(n - u + 1)(\alpha + 1) + i + j} \right)
\]
\[
\times \left( \prod_{v=r+1}^{s} \frac{(n - v + 1)}{(n - v + 1)(\alpha + 1) + j} \right)
\]

That is

\[
E[X_{r,n}^i X_{s,n}^j] = \frac{i + (\alpha + 1)(s - 1)}{\alpha + 1)(s - 1)} E[X_{r,n}^i X_{s-1,n}^j],
\]
\[ 1 \leq r < s \leq n, \ (s-r) \geq 1 \]

**Remark 4.3.7:** Recurrence relation for product moments of \( k-th \) record values will be

\[
E[(X_r^{(k)})^i (X_s^{(k)})^j] = \frac{(\alpha + 1)k}{j + k(\alpha + 1)} E[(X_r^{(k)})^i (X_{s-1}^{(k)})^j]
\]

\[
= \theta^{i+j} \left( \frac{(\alpha + 1)k}{k(\alpha + 1) + i + j} \right)^r \left( \frac{(\alpha + 1)k}{k(\alpha + 1) + j} \right)^{s-r}
\]

**Remark 4.3.8:** At \( i = 0 \), we obtain recurrence relation for single moments are given in (4.2.31) and (4.2.32).
1. Introduction

In this Chapter, characterization of some specific distributions and general class of distributions through conditional variance of generalized and lower (dual) generalized order statistics is presented.

2. The Characterization of Weibull distribution by conditional variance of generalized order statistics

Before coming to the main result, the following lemma is proved.

Lemma 5.2.1: (Haque and Faizan, 2009)

Let $F(x)$ be a df such that $F(0) = 0$ and has a continuous second order derivative on $(0, \infty)$ with $F'(x) > 0$ for all $x > 0$ (so that $F(x) < 1$ for all $x$, in particular). If it satisfies the differential equation

$$
\frac{\overline{F}''(x)}{\overline{F}(x)} + (\gamma_{r+1} - 1)\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]^2 - \frac{(p-1)}{x} \frac{\overline{F}'(x)}{\overline{F}(x)} - \gamma_{r+1} \theta^2 p^2 x^{2(p-1)} = 0 
$$

(5.2.1)

Then $\overline{F}(x) = e^{-\theta x^p}$ for all $x, \theta, p, \gamma_{r+1} > 0$.

Proof: Let $\frac{\overline{F}'(x)}{\overline{F}(x)} = p \gamma_{r+1} \frac{x^{p-1}}{t}$, then (2.1) reduces to

$$
\frac{dt}{dx} = p x^{p-1} \left[ \gamma_{r+1}^2 - \theta^2 t^2 \right] 
$$

(5.2.2)
Therefore,

\[ \frac{1}{2\gamma_{r+1}} \int \left[ \frac{1}{(\gamma_{r+1} - \theta t)} + \frac{1}{(\gamma_{r+1} + \theta t)} \right] dt = p \int x^{p-1} dx \]

implying that

\[ \frac{\gamma_{r+1} + \theta t}{\gamma_{r+1} - \theta t} = A e^{2\gamma_{r+1} \theta x^p}, \]

where \( A \) is the constant of integration.

Thus,

\[ \frac{F'(x)}{F(x)} = \frac{1}{2\gamma_{r+1}} \left[ \frac{2A}{Au - 1} - \frac{1}{u} \right] du, \quad \text{where} \quad u = e^{2\gamma_{r+1} \theta x^p}, \]

and

\[ \overline{F}(x) = B \left[ A e^{\gamma_{r+1} \theta x^p} - e^{-\gamma_{r+1} \theta x^p} \right]^{1/\gamma_{r+1}} \]

(5.2.3)

where \( A \) and \( B \) are constants to be determined. Since \( F(x) \) is bounded, hence \( \overline{F}(x) = e^{-\theta x^p} \), in view of the initial conditions on \( x \).

**Theorem 5.2.1: (Haque and Faizan, 2009)**

Let \( x \) be a continuous random variable with the df \( F(x) \) and the pdf \( f(x) \) over the support \((0, \infty)\). Let \( 0 < p < \infty \) and \( F(x) \) has moment of order \( 2p \) then for \( 0 < r < n \),

\[ V[X^p(r + 1,n,m,k) | X(r,n,m,k) = x] = \frac{1}{\gamma_{r+1}^2 \theta^2} \]

if and only if

\[ \overline{F}(x) = e^{-\theta x^p} \text{ for } x \geq 0 \text{ and } \theta > 0. \]
**Proof:** It is easy to see that

\[ E[X^{pk}(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \int_y y^{pk} [F(y)]^{\gamma_{r+1}-1} f(y) dy \quad (5.2.4) \]

Thus, for the Weibull distribution

\[ F(x) = e^{-\theta x^p}, x \geq 0, \theta > 0 \]

\[ E[X^{pk}(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{\gamma_{r+1} \theta p}{e^{-\gamma_{r+1} \theta x^p}} \int_y y^{pk} y^{p-1} e^{-\gamma_{r+1} \theta y^p} dy = \sum_{m=0}^{k} \frac{k!}{m!} x^{pm} a^{p(k-m)}, \]

where \( a^{-p} = \gamma_{r+1} \theta \) and therefore,

\[ V[X^{p}(r+1,n,m,k) | X(r,n,m,k) = x] = a^2 = \frac{1}{\gamma_{r+1}^2 \theta^2} = c \]

This proves the necessary part.

For sufficiency part, we have

\[ \frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \int_y y^{2p} [F(y)]^{\gamma_{r+1}-1} f(y) dy - \frac{\gamma_{r+1}^2}{[F(x)]^{2\gamma_{r+1}}} \int_y [y^{2p} [F(y)]^{\gamma_{r+1}-1} f(y) dy]^2 = c \]

That is,

\[ \gamma_{r+1}[F(x)]^{\gamma_{r+1}} \int_y y^{2p} [F(y)]^{\gamma_{r+1}-1} f(y) dy \]
Differentiating (5.2.5) \(w.r.t. x\) and solving, we get

\[
\gamma_{r+1} p \int_{x}^{\infty} y^p [F(y)]^{\gamma_{r+1}-1} f(y) dy = c \gamma_{r+1} x^{1-p} [F(x)]^{\gamma_{r+1}-1} f(x) + p x^p [F(x)]^{\gamma_{r+1}} \tag{5.2.6}
\]

Differentiating (5.2.6) again \(w.r.t. x\), to get

\[
c \gamma_{r+1} x^{1-p} [F(x)]^{\gamma_{r+1}-1} \left[ -\frac{F''(x)}{F(x)} + \frac{1}{c \gamma_{r+1}} x^{2(p-1)} (F(x)) p^2 \right] = 0
\]

That is,

\[
\frac{F''(x)}{F(x)} + (\gamma_{r+1} - 1) \left[ \frac{F'(x)}{F(x)} \right]^2 - \frac{(p-1)}{x} \left[ \frac{F'(x)}{F(x)} \right] - \gamma_{r+1} \theta^2 p^2 x^{2(p-1)} = 0 \tag{5.2.7}
\]

Hence \(F(x) = e^{-\theta x^p}\) in view of the Lemma 5.2.1.

At \(p = 1\), this theorem gives the result for exponential distribution.

**Remark 5.2.1:** At \(m = 0, k = 1\) and \(\gamma_r = n - r + 1\), Theorem 5.2.1 reduces for order statistics as obtained by Beg and Kirmani (1978) at \(p = 1\) and Khan and Beg (1987).
Remark 5.2.2: At $m = -1$ and $\gamma_r = k$, Theorem 2.1 reduces for $k$-th record statistics.

3. Characterization of general class of distributions by conditional variance of generalized order statistics

We first state and prove a Lemma:

**Lemma 5.3.1: (Khan et al., 2009)**

Let the df $F(x)$ be twice differentiable on $(\alpha, \beta)$ and let $h(x)$ be a non-decreasing and a twice differentiable function of $x$ such that $h(x) \to 0$ as $x \to \alpha$. Then the solution of the differential equation

$$
\frac{F''(x)}{F(x)} + (\gamma_r + 1) \left[ \frac{F'(x)}{F(x)} \right]^2 - \frac{h''(x)}{h'(x)} \left[ \frac{F'(x)}{F(x)} \right] - a^2 \gamma_{r+1} \left[ h'(x) \right]^2 = 0
$$

is

$$
\bar{F}(x) = e^{-ah(x)} \quad \text{for all} \quad x \in (\alpha, \beta),
$$

where $a > 0$ is a constant.

**Proof:** Let $\frac{\bar{F}'(x)}{\bar{F}(x)} = \gamma_{r+1} \frac{h'(x)}{t}$,

Then using (5.3.1), we have

$$
\frac{dt}{dx} = h'(x) [\gamma_{r+1}^2 - a^2 t^2]
$$

Therefore,

$$
\frac{1}{2 \gamma_{r+1}} \int \left[ \frac{1}{(\gamma_{r+1} - at)} + \frac{1}{(\gamma_{r+1} + at)} \right] dt = \int h'(x) \, dx
$$

implying that
\[
\frac{\gamma_{r+1} + at}{\gamma_{r+1} - at} = A e^{2a \gamma_{r+1} h(x)},
\]

where \(A\) is the constant of integration.

Hence,

\[
\frac{F'(x)}{F(x)} = \frac{1}{2\gamma_{r+1}} \left[ \frac{2A}{Au - 1} - \frac{1}{u} \right] du, \text{ where } u = e^{2a \gamma_{r+1} h(x)},
\]

Implying that

\[
F(x) = B \left[ A e^{a \gamma_{r+1} h(x)} - e^{-a \gamma_{r+1} h(x)} \right]^{1/\gamma_{r+1}}
\]

(5.3.4)

where \(A\) and \(B\) are constants of integration. Since \(F\) is bounded, hence \(F(x) = e^{-a h(x)}\), in view of the initial conditions on \(h(x)\).

**Theorem 5.3.1: (Khan et al., 2009)**

Let \(X\) be a continuous random variable with the df \(F(x)\) and the pdf \(f(x)\) over the support \((\alpha, \beta)\). Let \(E[h(X)]^2\) exist, then for some \(0 < r < n,\)

\[
V[h(X(r+1,n,m,k)|X(r,n,m,k) = x] = \frac{1}{a^2 \gamma_{r+1}^2}
\]

if and only if

\[
F(x) = e^{-a h(x)}
\]

(5.3.5)

where \(a > 0, h(x)\) is a non-decreasing and a twice differentiable of \(x\) such that \(h(x) \to 0\) as \(x \to \alpha\) and \(h(x) F(x) \to 0\) and \(x \to \infty\).

**Proof:** It is easy to see that

\[
E[h(X(r+1,n,m,k))|X(r,n,m,k) = x]
\]
Characterization........

\[
\frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \beta \int_{h(y)} [F(y)]^{\gamma_{r+1} - 1} f(y) \, dy \quad (5.3.6)
\]

For \( F(x) = e^{-ah(x)} \)

\[
E[h\{X(r+1,n,m,k)\} | X(r,n,m,k) = x] = h(x) + \frac{1}{a \gamma_{r+1}}
\]

and

\[
E[h^2\{X(r+1,n,m,k)\} | X(r,n,m,k) = x] = h^2(x) + \frac{2h(x)}{a \gamma_{r+1}} + \frac{2}{a^2 \gamma_{r+1}^2}
\]

Thus

\[
V[X(r+1,n,m,k) | X(r,n,m,k) = x] = \frac{1}{a^2 \gamma_{r+1}^2}
\]

This proves the necessary part.

For sufficiency part, we have

\[
\frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \beta \int_{h(y)} [F(y)]^{\gamma_{r+1} - 1} f(y) \, dy - \frac{\gamma_{r+1}^2}{[F(x)]^{2\gamma_{r+1}}}
\]

\[
\times \left[ \int_{h(y)} [F(y)]^{\gamma_{r+1} - 1} f(y) \, dy \right]^2
\]

\[
= \frac{1}{a^2 \gamma_{r+1}^2}
\]

That is,
Differentiating (5.3.7) \( w.r.t. x \) and solving, we get

\[
\gamma_{r+1} h'(x) \int_0^\beta h(y) \left[ F(x) \right]^{\gamma_{r+1}-1} f(y) dy = \frac{1}{a^2 \gamma_{r+1}} \left[ F(x) \right]^{\gamma_{r+1}+1} f(x)
\]

That is,

\[
h'(x) \int_0^\beta h'(y) \left[ F(x) \right]^{\gamma_{r+1}-1} dy = \frac{1}{a^2 \gamma_{r+1}} \left[ F(x) \right]^{\gamma_{r+1}+1} f(x)
\]

(5.3.9)

Now differentiate (5.3.9) again \( w.r.t. x \), to get

\[
h''(x) \int_0^\beta h''(y) \left[ F(x) \right]^{\gamma_{r+1}+1} dy - \left[ F(x) \right]^{\gamma_{r+1}+1} \left[ h'(x) \right]^2 = \frac{1}{a^2 \gamma_{r+1}} \left[ F(x) \right]^{\gamma_{r+1}+2} f(x)
\]

\[
= \frac{1}{a^2 \gamma_{r+1}} \left[ - (\gamma_{r+1}+1) \right] \left[ F(x) \right]^{\gamma_{r+1}-2} \left[ f(x) \right]^2
\]

\[
+ \left[ F(x) \right]^{\gamma_{r+1}-1} f'(x)
\]

Therefore,
\[
\frac{\ddot{F}(x)}{F(x)} + (\gamma_r + 1 - 1) \left[ \frac{F'(x)}{F(x)} \right]^2 - \frac{h''(x)}{h'(x)} \left[ \frac{F'(x)}{F(x)} \right] = -a^2 \gamma_{r+1} \left[ h'(x) \right]^2 = 0
\]

Hence

\[ F(x) = e^{-a h(x)} \] in view of the Lemma 5.3.1.

**Remark 5.3.1:** At \( m = 0 \), \( k + 1 \) and \( \gamma_r = n - r \), Theorem 5.3.1 reduces for order statistics as obtained by Khan *et al.* (2008).

**Remark 3.2:** At \( m = -1 \) and \( \gamma_r = k \), Theorem 3.1 reduces for \( k^{th} \) order statistics.

### 4. Characterization of distributions through dual generalized order statistics

**Lemma 5.4.1:** (Khan *et al.*, 2009)

Let the df \( F(x) \) be twice differentiable on \((\alpha, \beta)\), and let \( h(x) \) be a non-increasing and a twice differentiable function of \( x \) such that \( h(x) \rightarrow 0 \) as \( x \rightarrow \beta \). Then the solution of the differential equation

\[
\frac{\ddot{F}(x)}{F(x)} + (\gamma_{r+1} - 1) \left[ \frac{F'(x)}{F(x)} \right]^2 - \frac{h''(x)}{h'(x)} \left[ \frac{F'(x)}{F(x)} \right] = -a^2 \gamma_{r+1} \left[ h'(x) \right]^2 = 0 \quad (5.4.1)
\]

is

\[ F(x) = e^{-a h(x)} \] for all \( x \in (\alpha, \beta) \), \( (5.4.2) \)

where \( a > 0 \) is a constant.

**Proof:** Let \( \frac{F'(x)}{F(x)} = \gamma_{r+1} \frac{h'(x)}{t} \), to get
\[
\frac{dt}{dx} = h'(x)[\gamma_{r+1}^2 - a^2t^2]
\]

(5.4.3)

Therefore,

\[
\frac{1}{2\gamma_{r+1}} \int \left[ \frac{1}{(\gamma_{r+1} - at)} + \frac{1}{(\gamma_{r+1} + at)} \right] dt = \int h'(x) \, dx
\]

implying that

\[
\frac{\gamma_{r+1} + at}{\gamma_{r+1} - at} = A e^{2a\gamma_{r+1} h(x)},
\]

where \( A \) is the constant of integration.

Hence,

\[
\frac{F'(x)}{F(x)} = \frac{1}{2\gamma_{r+1}} \left[ \frac{2A}{Au - 1} - \frac{1}{u} \right] du,
\]

where \( u = e^{2a\gamma_{r+1} h(x)} \),

implying that

\[
F(x) = B \left[ A e^{a\gamma_{r+1} h(x)} - e^{-a\gamma_{r+1} h(x)} \right]^{1/\gamma_{r+1}}
\]

(5.4.4)

where \( A \) and \( B \) are constants of integration. Since \( F \) is bounded, hence \( F(x) = e^{-a h(x)} \), in view of the initial conditions on \( h(x) \).

**Theorem 4.1:** (Khan et al., 2009)

Let \( X \) be a continuous random variable with the \( df F(x) \) and the \( pdf f(x) \) over the support \((\alpha, \beta)\). Let \( E[ h(X) ]^2 \) exist, then for some \( 0 < r < n \),

\[
V[ h \{ X' (r+1, n, m, k) \} | X'(r, n, m, k) = y ] = \frac{1}{a^2 \gamma_{r+1}^2}
\]

(5.4.5)

if and only if
Characterization

$$\bar{F}(y) = e^{-ah(y)} \quad (5.4.6)$$

where that \( a > 0 \), \( h(y) \) be a non-increasing and a twice differentiable function of \( y \) such that \( h(y) \to 0 \) as \( y \to \beta \) and \( h(y) F(y) \to 0 \) and \( y \to \alpha \).

**Proof:** It is easy to see that

$$E[h\{X(r+1,n,m,k)\} | X(r,n,m,k) = y]$$

$$= \frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \int_{\alpha}^{x} \frac{h(y)}{[\bar{F}(y)]^{\gamma_{r+1}}} f(y)dy \quad (5.4.7)$$

For \( F(y) = e^{-ah(y)} \)

$$E[h\{X(r+1,n,m,k)\} | X(r,n,m,k) = y] = h(y) + \frac{1}{a \gamma_{r+1}}$$

and

$$E[h^2\{X(r+1,n,m,k)\} | X(r,n,m,k) = y] = h^2(x)$$

$$+ \frac{2h(x)}{a \gamma_{r+1}} + \frac{2}{a^2 \gamma_{r+1}^2}$$

Now in view of (5.4.5) and (5.4.6),

$$V\{X(r+1,n,m,k) | X(r,n,m,k) = y\} = \frac{1}{a^2 \gamma_{r+1}^2}$$

This proves the necessary part.

For sufficiency part, we have

$$\frac{\gamma_{r+1}}{[F(x)]^{\gamma_{r+1}}} \int_{\alpha}^{x} \frac{h^2(y)}{[\bar{F}(y)]^{\gamma_{r+1}}} f(y)dy - \frac{\gamma_{r+1}^2}{[F(x)]^{2\gamma_{r+1}}}$$
\[ \frac{1}{a^{2} \gamma^{2} r + 1} \]

That is

\begin{align*}
\gamma_{r+1} [F(x)]^{\gamma_{r+1}} \int_{0}^{x} h^{2}(y) \left[ F(y) \right]^{\gamma_{r+1}-1} f(y) dy \\
- \left[ \gamma_{r+1} \int_{0}^{x} h(y) \left[ F(y) \right]^{\gamma_{r+1}-1} f(y) dy \right]^{2} \\
= \frac{1}{a^{2} \gamma^{2} r + 1} [F(y)]^{2 \gamma_{r+1}}
\end{align*}

(5.4.8)

Differentiating (5.4.8) w.r.t. \( y \) and solving, we get

\begin{align*}
\gamma_{r+1} h^{\prime}(y) \int_{0}^{x} h(y)[F(y)]^{\gamma_{r+1}-1} f(y) dy \\
= \frac{1}{a^{2} \gamma^{2} r + 1} [F(y)]^{\gamma_{r+1}-1} f(y) + [F(y)]^{\gamma_{r+1}} h(y) h^{\prime}(y)
\end{align*}

(5.4.9)

That is,

\[ h^{\prime}(x) \int_{0}^{x} h^{\prime}(y)[F(y)]^{\gamma_{r+1}-1} dy = \frac{1}{a^{2} \gamma^{2} r + 1} [F(y)]^{\gamma_{r+1}-1} f(y) \]

(5.4.10)

Now differentiate (5.4.10) again w.r.t. \( y \), to get
\begin{align*}
  & h^*(y) \left[ h'(y) [F(y)]^{\gamma_r+1} \right] dy - [F(y)]^{\gamma_r+1} [h'(y)]^2 \\
  & \alpha \\
  & = \frac{1}{\alpha^2 \gamma_r + 1} \left[ -(\gamma_r + 1 - 1) [F(y)]^{\gamma_r+1} - 2 [f(y)]^2 \\
  & + [F(y)]^{\gamma_r+1-1} f'(y) \right]
\end{align*}

Therefore,

\begin{align*}
  & \frac{F'(y)}{F(y)} + (\gamma_r+1 - 1) \left[ \frac{F'(y)}{F(y)} \right]^2 - \frac{h'(y)}{h'(y)} \left[ \frac{F'(y)}{F(y)} \right] \\
  & - a^2 \gamma_r [h'(y)]^2 = 0
\end{align*}

Hence

\[ F(y) = e^{-\alpha h(y)} \quad \text{in view of the Lemma 5.4.1.} \]

**Remark 5.4.1:** At \( m = 0, k + 1 \) and \( \gamma_r = n - r - 1 \), Theorem 5.4.1 reduces for lower order statistic.

**Remark 5.4.2:** At \( m = -1 \) and \( \lambda_r = k \), Theorem 5.3.1 reduces for \( k-th \) lower record statistic.
REFERENCES


References


