ABSTRACT OF THE THESIS ENTITLED

ON DERIVATIONS AND RELATED MAPPINGS IN RINGS AND NEAR-RINGS

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The present thesis is a part of the research work carried out by the author concerning derivations and its various generalizations in the setting of some special classes of rings and near-rings. This exposition comprises five chapters and each chapter is subdivided into various sections.

Chapter 1 contains preliminary notions; basic definitions, examples and some important well known results related to our study which are required for the development of the subject in the forthcoming chapters. This chapter is an attempt to make this thesis as self contained as possible. However, the basic knowledge of groups, rings, fields, ideals, modules and homomorphisms etc. has been pre assumed.

Let \( N \) be a non empty set equipped with two binary operations say ‘+’ and ‘.’. \( N \) is called a left near-ring (resp. a right near-ring) if (i) \((N,+)\) is a group (not necessarily abelian), (ii) \((N,.)\) is a semigroup and (iii) \(x.(y+z) = x.y + x.z\) for all \(x,y,z \in N\) (resp. \((y+z).x = y.x + z.x\) for all \(x,y,z \in N\)). For examples of such structures:

(i) let \((\mathbb{C},+)\) be usual group of complex numbers with regard to ordinary addition of complex numbers. Let us define ‘*’ in \(\mathbb{C}\) as following \(a*b = |a|b\) for all \(a,b \in \mathbb{C}\). Then \((\mathbb{C},+,*)\) is a left near-ring which is not a right near-ring. (ii) let \((G,+)\) be a non abelian group. Consider \(S\), the set of all functions from \(G\) to \(G\). Then \((S,+,\cdot)\) is a right near-ring with regard to the operation ‘+’ and ‘.’ defined as below.

\[
(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in G,
\]
\[
(fg)(x) = f(g(x)) \quad \text{for all } x \in G.
\]

where \(f, g \in S\). This is to be noted that it is not a left near-ring. A left near-ring \(N\) is called zero symmetric if \(0x = 0\) for all \(x \in N\). Throughout the discussion \(N\) will denote a zerosymmetric left near-ring with center \(Z(N)\) unless otherwise mentioned.

Chapter 2 is devoted to the study of \(n\)-derivations in near-rings and its various generalizations, where \(n\) is a positive integer. The study of derivation in near-rings was initiated by H.E. Bell and G. Mason [Near-rings and near-fields(Tübingen, 1985), 31 – 35, North-Holland Math. Stud., 137, North-Holland, Amsterdam, 1987] in 1987 and obtained various results regarding the behavior of near-ring \(N\) as a commutative ring. Later many authors viz. Ashraf, Golbasi, Maksa, Park etc. generalized the
notion of derivation in different directions viz. \((\alpha, \beta)\)-derivation, left generalized derivation, right generalized derivation, generalized derivation, symmetric-bi-derivation and permuting-tri-derivation etc. Motivated by the notion of permuting-\(n\)-derivation given by Park [J. Chungcheong Math. Soc., 22(2009), No.3, 451 – 458] in rings, we have introduced the notion of \(n\)-derivation and permuting-\(n\)-derivation in near-rings in the Section 2.1.

A map \(D : \bigotimes_{i=1}^{n} N \to N\) is said to be permuting if the equation
\[
\text{n-times}
D(x_1, x_2, \cdots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})
\]
holds for all \(x_1, x_2, \cdots, x_n \in N\) and for every permutation \(\pi \in S_n\), where \(S_n\) is the permutation group on \(\{1, 2, \cdots, n\}\).

A map \(d : N \to N\) defined by \(d(x) = D(x, x, \cdots, x)\) for all \(x \in N\) where \(D : \bigotimes_{i=1}^{n} N \to N\) is a permuting map, is called the trace of \(D\).

Let \(n\) be any fixed positive integer. An \(n\)-additive (i.e.; additive in each argument) mapping \(D : \bigotimes_{i=1}^{n} N \to N\) is called an \(n\)-derivation if the relations
\[
\begin{align*}
D(x_1', x_2, \cdots, x_n) &= D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n) \\
D(x_1, x_2', \cdots, x_n) &= D(x_1, x_2, \cdots, x_n)x_2' + x_2D(x_1, x_2', \cdots, x_n) \\
& \vdots \\
D(x_1, x_2, \cdots, x_n') &= D(x_1, x_2, \cdots, x_n)x_n' + x_nD(x_1, x_2, \cdots, x_n')
\end{align*}
\]
hold for all \(x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in N\). If in addition \(D\) is a permuting map, then all the above conditions are equivalent and in this case \(D\) is called a permuting \(n\)-derivation of \(N\). The main result of this section states that under certain constraints, a permuting \(n\)-additive mapping \(D\) on a \(n!\)-torsion free prime near-ring \(N\) is zero if the trace \(d\) of \(D\) is zero.

In Section 2.3, we have generalized the concept of \(n\)-derivation by introducing the notion of \((\sigma, \tau)\)-\(n\)-derivation in near-rings as follows: Let \(n\) be a fixed positive integer. An \(n\)-additive (i.e.; additive in each argument) mapping \(D : \bigotimes_{i=1}^{n} N \to N\) is called a \((\sigma, \tau)\)-\(n\)-derivation of \(N\) if there exist functions \(\sigma, \tau : N \to N\) such that the relations
\[
\begin{align*}
D(x_1', x_2, \cdots, x_n) &= D(x_1, x_2, \cdots, x_n)\sigma(x_1) + \tau(x_1)D(x_1', x_2, \cdots, x_n) \\
D(x_1, x_2', \cdots, x_n) &= D(x_1, x_2, \cdots, x_n)\sigma(x_2') + \tau(x_2)D(x_1, x_2', \cdots, x_n)
\end{align*}
\]

In this document, we have discussed the notion of derivation in different directions and introduced the concept of \(n\)-derivation and permuting-\(n\)-derivation in near-rings. We have also provided a detailed explanation of the conditions under which a permuting \(n\)-additive mapping is zero, and we have generalized the concept of \(n\)-derivation to \((\sigma, \tau)\)-\(n\)-derivation.
\[ D(x_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) + \tau(x_n)D(x_1, x_2, \ldots, x_n) \]

hold for all \( x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \in N \). Further in addition if \( D \) is a permuting map then all the above conditions are equivalent and in this case \( D \) is called a permuting \((\sigma, \tau)\)-\(n\)-derivation of \( N \). It is trivial to observe that a permuting \((\sigma, \tau)\)-\(n\)-derivation of \( N \) is a \((\sigma, \tau)\)-\(n\)-derivation of \( N \) but its converse is not true. We have constructed an example to justify this fact. Further some properties involving \((\sigma, \tau)\)-\(n\)-derivations of a prime near-ring \( N \) which force \( N \) to be a commutative ring have also been investigated. Additive commutativity of near-ring \( N \) satisfying certain identities involving \((\sigma, \tau)\)-\(n\)-derivations of a prime near-ring \( N \) has also been obtained. Some of the main results of this section are as follows:

(i) Let \( N \) be a prime near-ring and \( D \) a nonzero \((\sigma, \tau)\)-\(n\)-derivation of \( N \). If \( D(N, N, \ldots, N) \subseteq Z \), then \( N \) is a commutative ring.

(ii) Let \( N \) be a prime near-ring admitting a \((\sigma, \tau)\)-\(n\)-derivation \( D \) and a \((\sigma, \tau)\)-derivation \( d \) such that \( dD = 0 \), then one of the following will hold:

(i) \( D = 0 \)
(ii) \( d = 0 \)
(iii) \( (N, +) \) is abelian.

(iii) Let \( N \) be a prime near-ring admitting a \((\sigma, \tau)\)-\(n\)-derivation \( D \) and a derivation \( d \) such that \( (N, +) \) is non abelian. If \( dD \) is a \((\sigma, \tau)\)-\(n\)-derivation of \( N \), then either \( D = 0 \) or \( d = 0 \).

Section 2.4 is devoted to the study of generalized \(n\)-derivation in near-rings. We introduce the notion of generalized \(n\)-derivation in near-ring \( N \) and investigate several identities involving generalized \(n\)-derivations of a prime near-ring \( N \) which force \( N \) to be a commutative ring.

An \(n\)-additive mapping \( F : N \times N \times \cdots \times N \rightarrow N \) is called right generalized \(n\)-derivation of \( N \) with associated \(n\)-derivation \( D \) if the relations

\[ F(x_1x'_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x'_1 + x_1D(x_1', x_2, \ldots, x_n) \]
\[ F(x_1, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n) x'_2 + x_2 D(x_1, x_2, \cdots, x_n) \]

\[ F(x_1, x_2, \cdots, x_n x'_n) = F(x_1, x_2, \cdots, x_n) x'_n + x_n D(x_1, x_2, \cdots, x'_n) \]

hold for all \( x_i, x'_i, x_2, x'_2, \cdots, x_n, x'_n \in N \). If in addition both \( F \) and \( D \) are permuting maps then \( F \) is called a permuting right generalized \( n \)-derivation of \( N \) with associated permuting \( n \)-derivation \( D \). An \( n \)-additive mapping \( F : N \times N \times \cdots \times N \rightarrow N \) is called a left generalized \( n \)-derivation of \( N \) with associated \( n \)-derivation \( D \) if the relations

\[ F(x^i_1 x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n) x^i_1 + x_1 F(x_1, x_2, \cdots, x_n) \]

\[ F(x_1, x^i_2 x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n) x^i_2 + x_2 F(x_1, x^i_2, \cdots, x_n) \]

\[ \vdots \]

\[ F(x_1, x_2, \cdots, x_n x'_n) = D(x_1, x_2, \cdots, x_n) x'_n + x_n F(x_1, x_2, \cdots, x'_n) \]

hold for all \( x_1, x'_1, x_2, x'_2, \cdots, x_n, x'_n \in N \). If in addition both \( F \) and \( D \) are permuting maps then \( F \) is called a permuting left generalized \( n \)-derivation of \( N \) with associated permuting \( n \)-derivation \( D \). An \( n \)-additive mapping \( F : N \times N \times \cdots \times N \rightarrow N \) is called a generalized \( n \)-derivation of \( N \) with associated \( n \)-derivation \( D \) if it is both a right generalized \( n \)-derivation as well as a left generalized \( n \)-derivation of \( N \) with associated \( n \)-derivation \( D \). If in addition both \( F \) and \( D \) are permuting maps then \( F \) is called a permuting generalized \( n \)-derivation of \( N \) with associated permuting \( n \)-derivation \( D \).

In this section we have also constructed examples to justify these notions. We have improved a result of Gölbaşı [Theorem 2.6, Southeast Asian Bull. Math., 30(2006), 49—54], by proving the following theorem for generalized \( n \)-derivation in the setting of prime near-rings as follows: Let \( N \) be a prime near-ring admitting a nonzero generalized \( n \)-derivation \( F \) with associated \( n \)-derivation \( D \) of \( N \). If \( F(N, N, \cdots, N) \subseteq Z \), then \( N \) is a commutative ring. Some of the interesting results proved here are as follows:

(i) Let \( N \) be a prime near-ring admitting a generalized \( n \)-derivation \( F \) with associated nonzero \( n \)-derivation \( D \) of \( N \). If \( F([x, y], r_2, r_3, \cdots, r_n) \in Z \) for all \( x, y, r_2, r_3, \cdots, r_n \in N \), then \( N \) is commutative ring or \( D(Z, N, N, \cdots, N) = \{0\} \).
(ii) Let $N$ be a 2-torsion free prime near-ring admitting a generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F(x_{1}y_{1}, r_{2}, r_{3}, \ldots, r_{n}) \in Z$ for all $x_{1}, y_{1}, r_{2}, r_{3}, \ldots, r_{n} \in N$, then $N$ is a commutative ring if $D(Z, N, N, \ldots, N) = \{0\}$.

(iii) Let $F_{1}$ and $F_{2}$ be generalized $n$-derivations of prime near-ring $N$ with associated nonzero $n$-derivations $D_{1}$ and $D_{2}$ of $N$ respectively such that $[F_{1}(N, N, \ldots, N), F_{2}(N, N, \ldots, N)] = \{0\}$. Then $(N, +)$ is an abelian group.

Chapter 3 deals with the study of derivation on semigroup ideals in prime near-rings. Different identities on ideals which insure the ring behavior of prime near-ring have been obtained. A nonempty subset $U$ of $N$ is called semigroup left ideal (resp. semigroup right ideal) if $NU \subseteq U$ (resp. $UN \subseteq U$) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. Let $I$ be a nonempty subset of $N$ then a normal subgroup $(I, +)$ of $(N, +)$ is called a right ideal (resp. a left ideal) of $N$ if $(x + i)y - xy \in I$ for all $x, y \in N$ and for all $i \in I$ (resp. $xi \in I$ for all $i \in I$ and $x \in N$). $I$ is called an ideal of $N$ if it is both a left ideal as well as a right ideal of $N$.

In the Section 3.1, commutativity of addition and multiplication of prime near-rings satisfying certain identities involving $n$-derivations on semigroup ideals and ideals have been investigated. Some identities which have been studied under this section are as following:

(i) If $U_{1}, U_{2}, \ldots, U_{n}$ are nonzero semigroup ideals of $N$ such that at least one of the following holds:

   (i) $D([x, y], u_{2}, \ldots, u_{n}) = 0$, for all $x, y \in U_{1}, u_{2} \in U_{2}, \ldots, u_{n} \in U_{n}$,

   (ii) $D([x, y], u_{2}, \ldots, u_{n}) = \pm [x, y]$ for all $x, y \in U_{1}, u_{2} \in U_{2}, \ldots, u_{n} \in U_{n}$, for all $x, y \in U_{1}, u_{2} \in U_{2}, \ldots, u_{n} \in U_{n}$, then $N$ is a commutative ring.

(ii) Let $I_{1}, I_{2}, \ldots, I_{n}$ be nonzero ideals of $N$ such that at least one of the following holds:

   (i) $D(x_{1}y_{1}, i_{2}, \ldots, i_{n}) = \pm (x_{1}y_{1})$ for all $x_{1}, y_{1} \in I_{1}, i_{2} \in I_{2}, \ldots, i_{n} \in I_{n}$,

   (ii) $D(x_{1}y_{1}, x_{2}, \ldots, x_{n}) = \pm (x_{1}y_{1})$ for all $x_{1}, y_{1} \in I_{1}, i_{2} \in I_{2}, \ldots, i_{n} \in I_{n}$, then $N$ is a commutative ring.
Furthermore, we study the conditions with semigroup ideals for \( n \)-derivations \( D_1 \) and \( D_2 \) of \( N \) which imply that \( D_1 = D_2 \).

Section 3.2 is devoted to the study of the commutativity of prime near-rings satisfying certain identities involving generalized derivations on semigroup ideals or ideals. Following interesting results have been obtained under this section:

(i) Let \( N \) be a prime near-ring and \( U \) a nonzero semigroup ideal of \( N \). If \( N \) admits a generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \( f(U) \subseteq Z \), then \( N \) is a commutative ring.

(ii) Let \( N \) be a prime near-ring and \( U \) a nonzero semigroup ideal of \( N \). If \( N \) admits a generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \( d(Z) \neq \{0\} \) and \( f([x, y]) \in Z \) for all \( x, y \in U \), then \( N \) is a commutative ring.

(iii) Let \( N \) be a prime near-ring and \( U \) a nonzero semigroup ideal of \( N \). If \( N \) admits a generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \( [f(x), y] \in Z \) for all \( x, y \in U \), then \( N \) is a commutative ring.

Furthermore, we provide some examples to show that the restrictions imposed on the hypothesis of the various theorems are not superfluous.

The notion of involution is known in rings and algebras for a long time. The last section of this chapter deals with the notion of "involution" in near-rings. An additive mapping \( x \mapsto x^* \) of \( N \) into itself is called an involution on \( N \) if it satisfies the conditions; (i) \( (x^*)^* = x \), (ii) \( (xy)^* = y^*x^* \) for all \( x, y \in N \). A ring \( N \) equipped with an involution ‘*’ is called a *-ring. A near-ring \( N \) with involution ‘*’ is said to be *-prime if \( aNb = aNb^* = \{0\} \), where \( a, b \in N \) (equivalently \( aNb = a^*Nb = \{0\} \), where \( a, b \in N \) ) implies that either \( a = 0 \) or \( b = 0 \). Besides other results, it has been shown that under certain restrictions every near-ring with involution is a ring.

The remaining two chapters are based on the study of *-n-derivation and ring of quotients of a *-prime ring. In the remaining part, by \( R \) we mean an associative ring with center \( Z(R) \) unless otherwise stated:

In Chapter 4, we study the notion of *-n-derivation in the setting of prime and semiprime ring with involutions with their properties. We have also obtained an
extension of Posner's first theorem in the setting of *-prime rings. A ring $R$ with involution $\ast$ is said to be *-prime if $aRb = aRb^\ast = \{0\}$, where $a, b \in R$ (equivalently $aRb = a^\ast Rb = \{0\}$, where $a, b \in R$) implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution $\ast$ is *-prime but the converse is not true in general. Of course, if $R^\circ$ denotes the opposite ring of a prime ring $R$ then $R \times R^\circ$ equipped with the exchange involution $\ast_{ex}$, defined by $\ast_{ex}(x, y) = (y, x)$, is *-ex-prime but not prime. An ideal $I$ of $R$ is called a *-ideal of $R$ if $I^\ast = I$. In Section 4.2, we introduce the notion of *-$n$-derivation and reverse *-$n$-derivation in the *-ring $R$, where $n$ is a positive integer, and also investigate its various properties. Let $n$ be any fixed positive integer. An $n$-additive (i.e.; additive in each argument) mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called an *-$n$-derivation of $R$ if the relations

$$D(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x_1) + x_1 D(x_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2 x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x_2) + x_2 D(x_1, x_2, \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_n x_n') = D(x_1, x_2, \cdots, x_n)(x_n') + x_n D(x_1, x_2, \cdots, x_n')$$

hold for all $x_1, x_1', x_2, \cdots, x_n, x_n' \in R$.

Similarly an $n$-additive mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a reverse *-$n$-derivation of $R$ if the relations

$$D(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1^\ast + x_1 D(x_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2 x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_2^\ast + x_2 D(x_1, x_2, \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_n x_n') = D(x_1, x_2, \cdots, x_n)x_n'^\ast + x_n' D(x_1, x_2, \cdots, x_n')$$

hold for all $x_1, x_1', x_2, \cdots, x_n, x_n' \in R$. For an example of *-$n$-derivation, consider $\mathbb{C}$ the ring of complex numbers with involution $\ast$ defined by $z^\ast = \bar{z}$, where $\bar{z}$ denotes the conjugate of the complex number $z$. Now define $D : \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \rightarrow \mathbb{C}$ such that $D(z_1, z_2, \cdots, z_n) = \lambda(z_1 - \bar{z}_1)(z_2 - \bar{z}_2) \cdots (z_n - \bar{z}_n)$ where $\lambda$ is any fixed complex number. One can easily verify that $D$ is a *-$n$-derivation of $\mathbb{C}$.

In fact, it is shown that if a prime *-ring $R$ admits a nonzero *-$n$-derivation (resp.
reverse *-n-derivation) $D$, then $R$ is commutative. Further, some related properties of *-n-derivation in semiprime *-ring have also been investigated. It is shown that if $R$ is a semiprime *-ring, admitting a *-n-derivation $D$, then $D(R, R, \cdots, R) \subseteq Z$. Finally a structure theorem for *-n-derivation has also been obtained. In fact it is proved that if $R$ is a commutative *-ring admitting a *-derivation $d$, and $I$ is a nonzero ideal of $R$ such that it is invariant under both $*$ and $d$ i.e.; $I^* \subseteq I$ and $d(I) \subseteq I$, then $d$ induces a *-n-derivation $D$ on the quotient ring $R/I$ where $*$ is an involution on quotient ring $R/I$ induced by the involution $*$ of $R$.

Section 4.3 is devoted to the extension of Posner's first theorem in the setting of *-prime rings of characteristic different from 2. It is shown that if $R$ is a *-prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation of $R$ and at least one of $d_1$ and $d_2$ commutes with $*'$, then $d_1 = 0$ or $d_2 = 0$. From this theorem, we have also deduced Posner's first theorem for prime rings of characteristic different from 2.

In Section 5.2, we have investigated commutativity of *-prime ring $R$, which satisfies certain differential identities on *-ideal $I$; viz.;

1. $d(xoy) = d(x)oy$ for all $x, y \in I$,
2. $d(x)oy = xoy$ for all $x, y \in I$,
3. $d([x, y]) = \pm(xoy)$ for all $x, y \in I$,
4. $d(xoy) = \pm[x, y]$ for all $x, y \in I$,
5. $d(x)oy \in Z$ for all $x, y \in I$,
6. $d[x, y] \pm (xoy) \in Z$ for all $x, y \in I$,
7. $d(xoy) \pm [x, y] \in Z$ for all $x, y \in I$,
8. $d(x)od(y) = xoy$ for all $x, y \in I$ and
9. $(d(x)oy) - (xod(y)) \in Z$ for all $x, y \in I$.

We have also shown that there exists no nonzero derivation $d$ satisfying any of the following differential identities on *-ideal $I$ in a *-prime ring $R$;
(i) \( d(xoy) = d(x)oy \) for all \( x, y \in I \),

(ii) \( d(x)oy = xoy \) for all \( x, y \in I \),

(iii) \( d(x)oy = xod(y) \) for all \( x, y \in I \)

(iv) \( d(x)oy = d(x)od(y) \) for all \( x, y \in I \) and

(v) \( xod(y) = d(x)od(y) \) for all \( x, y \in I \).

Some results already known for prime rings on ideals have also been deduced. Finally, we provide several examples to justify that various restrictions imposed in the hypotheses of our theorems are not superfluous.

For a semiprime ring \( R \), \( Q_{mr} \) and \( Q_s \) will represent its Utumi right ring of quotients and right symmetric Martindale ring of quotients respectively.

Section 5.3 gives a glimpse of some extension problems in the setting of ring of quotients of a \(*\)-prime ring. Let \( R \) be a semiprime ring with an involution \( '\ast' \). Let \( Q_{mr} \) and \( Q_s \) denote its right Utumi quotient ring and right symmetric Martindale quotient ring respectively. In the present section the following extension theorems have been obtained:

(i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring,

(ii) if \( R \) is a \(*\)-prime ring, then so is its right symmetric Martindale quotient ring,

(iii) every \(*\)-derivation of a commutative semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring.

At the end of this section \( C \)-dependence of any two nonzero elements of right symmetric Martindale quotient ring of \(*\)-prime ring \( R \), where \( C \) is the extended centroid of \( R \), has also been discussed. We have proved the following: Let \( R \) be a \(*\)-prime ring, \( Q = Q_s \) and \( 0 \neq a, 0 \neq b \in Q \). Suppose that \( axb^* = bxa \) and \( a^*xb^* = b^*xa \) for all \( x \in R \). Then \( a \in Cb^* \) and hence \( a \) and \( b^* \) are \( C \)-dependent.
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DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)
2014
Dedicated

to my

Parents, wife and my

son and daughter
Certificate

This is to certify that the contents of this thesis entitled "ON DERIVATIONS AND RELATED Mappings IN RINGS AND NEAR-RINGS" is the original research work of Mr. Mohammad Aslam Siddeeque carried out under my supervision in the Department of Mathematics, Aligarh Muslim University, Aligarh. To the best of my knowledge, the work presented in the thesis is original and has not been submitted to any other university or institution for the award of a degree.

I further certify that Mr. Mohammad Aslam Siddeeque has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.

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Preface

Rings were first formalized as a common generalization of Dedekind domains that occur in number theory and of polynomial rings and rings of invariants that occur in algebraic geometry and invariant theory. They are also used in other branches of mathematics such as geometry and mathematical analysis. The formal definition of rings is relatively recent, dating from the 1920's. Where a ring is commutative or not has profound implication in the study of rings as abstract objects, the field called the ring theory. Realizing the importance of commutativity in rings, many algebraists have worked in this direction. The study of derivations in rings goes back to 1957 when Posner proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Many results in this direction were obtained by a number of authors in several ways. This thesis "ON DERIVATION AND RELATED MAPPINGS IN RINGS AND NEAR-RINGS" contains the research work carried out by the author on commutativity of certain classes of rings and near-rings possessing different kind of derivations.

The present exposition comprises five chapters and each chapter is further divided into sections. The definitions, examples, remarks, theorems, corollaries etcetera have been specified with the double decimal numbers. The first figure denotes the number of the chapter, the second represents the section in a chapter and the third points out the number of the definition, the example, or the theorem as the case may be in a particular chapter. For example, Theorem 2.3.4 refers to the fourth theorem appearing in the third section of the second chapter.

Chapter 1 contains preliminary notions, basic definitions, examples and some important well known results related to our study which are required for the development of the subject in the forthcoming chapters. This chapter is an attempt to make this thesis as
Chapter 2 deals with the study of $n$-derivations in near-rings and its various generalizations, where $n$ is a positive integer. The study of derivation in near-rings was initiated by H.E. Bell and G. Mason [24] in 1987. They obtained various results regarding the behavior of near-ring $N$. Later many authors [13], generalized the above notion in different directions namely $(\alpha, \beta)$-derivation, left generalized derivation, right generalized derivation, generalized derivation, symmetric-bi-derivation and permuting-tri-derivation etc. Motivated by the notion of permuting-$n$-derivation given by Park [72] in rings, we have introduced the notion of $n$-derivation and permuting-$n$-derivation in near-rings in the Section 2.1. The main result of this section states that under certain constraints, a permuting $n$-additive mapping $D$ on a $n!$-torsion free prime near-ring $N$ is zero if the trace $d$ of $D$ is zero.

In Section 2.3, we have generalized the concept of $n$-derivation by introducing the notion of $(\sigma, \tau)$-$n$-derivation in near-rings. Further some properties involving $(\sigma, \tau)$-$n$-derivations of a prime near-ring $N$ which force $N$ to be a commutative ring have been investigated. Additive commutativity of near-ring $N$ satisfying certain identities involving $(\sigma, \tau)$-$n$-derivations of a prime near-ring $N$ has also been obtained. Section 2.4 is devoted to the study of generalized $n$-derivation in near-rings. We have introduced the notion of generalized $n$-derivation and permuting generalized $n$-derivation in near-ring $N$ and investigated several identities involving generalized $n$-derivations of a prime near-ring $N$ which force $N$ to be a commutative ring. An example has also been constructed to justify that every generalized $n$-derivation can’t be permuting generalized $n$-derivation in near-ring $N$. The main result of this section states that if $N$ is a prime near-ring admitting a nonzero generalized $n$-derivation $F$ such that $F(N, N, \cdots, N) \subseteq Z$, then $N$ is a commutative ring.

Chapter 3 opens with the study of derivations on semigroup ideals in prime near-rings. A non empty subset $U$ of $N$ is said to be a semigroup left (resp. right)ideal of $N$ if $NU \subseteq U$ (resp. $UN \subseteq U$ ) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $N$. In Section 3.1, the commutativity of addition and multiplication of prime near-rings satisfying certain identities involving $n$-derivations on semigroup ideals and ideals have been investigated. Furthermore, we study the conditions on a near-ring which admits $n$-derivations $D_1$ and $D_2$ of $N$ which
imply that $D_1 = D_2$.

Section 3.2 is devoted to the study of commutativity of prime near-rings satisfying certain identities involving generalized derivations on semigroup ideals or ideals. Furthermore, we provide some examples to show that the restrictions imposed on the hypothesis of the various theorems are not superfluous. The last section of this chapter deals with the notion of "involution" in near-rings. Besides other results, it has been shown that under certain restrictions every near-ring with involution is a ring.

In Chapter 4, the notion of $*$-$n$-derivation in the setting of prime and semiprime ring $R$ with involution `$*$' has been studied. An extension of Posner's first theorem has also been obtained in the setting of $*$-prime rings. A ring $R$ with involution `$*$' is said to be $*$-prime if $aRh = aR^*b = \{0\}$, where $a, b \in R$ (equivalently $aRh = a^*Rb = \{0\}$, where $a, b \in R$) implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution `$*$' is $*$-prime but the converse is not true in general. Of course, if $R^*$ denotes the opposite ring of a prime ring $R$, then $R \times R^*$ equipped with the exchange involution $*_ex$, defined by $*_ex(x, y) = (y, x)$, is $*_ex$-prime but not prime. An ideal $I$ of $R$ is called a $*$-ideal of $R$ if $I^* = I$.

In Section 4.2, we have introduced the notion of $*$-$n$-derivation and reverse $*$-$n$-derivation in the $*$-ring $R$, where $n$ is a positive integer, and investigated its various properties. In fact, it is shown that if a prime $*$-ring $R$ admits a nonzero $*$-$n$-derivation (resp. reverse $*$-$n$-derivation) $D$, then $R$ is commutative. Further, some related properties of $*$-$n$-derivation in semiprime $*$-ring have been investigated. Finally a structure theorem for $*$-$n$-derivation has also been obtained. Section 4.3 is devoted to the extension of Posner's first theorem in the setting of $*$-prime rings of characteristic different from 2. It is shown that if $R$ is a $*$-prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation of $R$ and at least one of $d_1$ and $d_2$ commutes with `$*$', then $d_1 = 0$ or $d_2 = 0$. From this theorem, we have also deduced Posner's first theorem for prime rings of characteristic different from 2.

In Section 5.2, we have investigated commutativity of $*$-prime ring $R$, which satisfies certain differential identities on $*$-ideal $I$ of $R$ viz.; (i) $d(xoy) = d(x)oy$ (ii) $d(x)oy = xoy$, (iii) $d([x, y]) = \pm(xoy)$ (iv) $d(xoy) = \pm[x, y]$, (v) $d(x)oy \in Z$, (vi) $d(x, y) \pm (xoy) \in Z$, (vii) $d(xoy) \pm [x, y] \in Z$, (viii) $d(x)od(y) = xoy$ and (ix) $d(x)oy - (xod(y)) \in Z$.
for all $x, y \in I$. We have also shown that there exists no nonzero derivation $d$ satisfying any of the following differential identities on $*$-ideal $I$ in a $*$-prime ring $R$:

(i) $d(xoy) = d(x)oy$
(ii) $d(x)oy = xoy$
(iii) $d(x)oy = xod(y)$
(iv) $d(x)oy = d(x)od(y)$
(v) $xod(y) = d(x)od(y)$

for all $x, y \in I$. Some results already known for prime rings on ideals have also been deduced. Finally, we provide several examples to justify that various restrictions imposed in the hypotheses of our theorems are not altogether superfluous.

Section 5.3 gives a glimpse of some extension problems in the setting of ring of quotients of a $*$-prime ring. In fact, the following extension theorems have been obtained: (i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. (ii) if $R$ is a $*$-prime ring, then so is its right symmetric Martindale quotient ring. (iii) every $*$-derivation of a commutative semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. At the end of this section $C$-dependence of any two nonzero elements of right symmetric Martindale quotient ring of $*$-prime ring $R$, where $C$ is the extended centroid of $R$, has also been discussed.

At the end an extensive bibliography of the existing literature related to the subject matter is included.

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Chapter 1

Preliminaries

1.1 Introduction

In the present chapter we give a first overview on the subject stating more frequently used definitions, preliminary notions, more exciting examples and some elementary results required for the development of the subject matter in the subsequent chapters of the present thesis. The elementary knowledge of groups, rings, ideals, fields, modules, homomorphisms etc. have been pre assumed. For most of the material included in this chapter, we refer to Beidar et al. [20], Herstein [50,51], McCoy [62], Lam [56], Pilz [74], Clay [37] and Ferrero [42] etc. We include the basics related with near-rings in the Sections 1.2 – 1.4, whereas elements of ring theoretic notions, definitions and basic results have been discussed in the Sections 1.5 and 1.6. Throughout the present thesis \( N \) and \( R \) will represent a zero symmetric left near-ring and an associative ring respectively, while the multiplicative center of near-ring \( N \) (resp. center of \( R \)) will be denoted by \( Z \) unless otherwise mentioned.

1.2 Near-rings and related concepts

Near-fields were the first near-rings considered in the literature. In the year 1905, Dickson [40] changed the multiplication in the field in order to get examples of “one-sided distributive field” (near-fields) showing that the second distributive law does not follow from remaining axioms for a (skew) field. In the year 1936, Zassenhaus [89] determined all finite near-fields, which have order \( p^n \). Ore [63], Furtwangler-Taussky [44] and Taussky [80] started axiomatic study in the thirties of the last century which we now call near-ring. The first ones to use the name near-ring were Zassenhauss [89], Blackett [25] and P.Jordan [55]. The late fifties of the last century brought the start of a rapid development of the theory of near-rings.
Definition 1.2.1. Let $N$ be a non empty set equipped with two binary operations say ‘$+$’ and ‘$.$’. $N$ is called a left near-ring (resp. right near-ring) if (i) $(N, +)$ is a group (not necessarily abelian), (ii) $(N, .)$ is a semigroup and (iii) $x(y + z) = x.y + x.z$ for all $x, y, z \in N$ (resp. $(y + z).x = y.x + z.x$ for all $x, y, z \in N$).

Example 1.2.1. (i) Let $(\mathbb{C}, +)$ be usual group of complex numbers with regard to ordinary addition of complex numbers. Let us define ‘$*$’ in $\mathbb{C}$ as following $a*b = |a|.b$ for all $a, b \in \mathbb{C}$. Then $(\mathbb{C}, +, *)$ is a left near-ring which is not a right near-ring.

(ii) Let $(G, +)$ be a non abelian group. Consider $S$, the set of all functions from $G$ to $G$. Then $(S, +, .)$ is a right near-ring with regard to the operation ‘$+$’ and ‘$.$’ defined as below.

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in G,$$

and

$$(fg)(x) = f(g(x)) \text{ for all } x \in G$$

where $f, g \in S$. This is to be noted that it is not a left near-ring.

Definition 1.2.2. A left near-ring $N$ is called zero symmetric if $0x = 0$ for all $x \in N$.

Remark 1.2.1. The near-rings $\mathbb{C}$ and $S$ discussed in the above Examples 1.2.1 (i)& (ii) are zero symmetric.

Definition 1.2.3. A left near-ring $N$ is called prime near-ring if $xNy = \{0\}$, where $x, y \in N$, implies $x = 0$ or $y = 0$. It is called semiprime near-ring if $xNx = \{0\}$, where $x \in N$, implies $x = 0$.

Definition 1.2.4. The multiplicative center of near-ring $N$, usually denoted by $Z$ is defined as; $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. The additive center of $N$ is defined as; $\mathcal{X} = \{x \in N \mid x + y = y + x \text{ for all } y \in N\}$.

Definition 1.2.5. Let $N$ be a near-ring. Then $N$ is called a distributive near-ring if $(y + z)x = yx + zx$ for all $x, y, z \in N$.

Definition 1.2.6. Let $N$ be a near-ring. Then $N$ is called a pseudo-abelian near-ring if $xy + zt = zt + xy$ for all $x, y, z, t \in N$.

Definition 1.2.7. Let $N$ be a near-ring. Then $N$ is called a commutative near-ring if $xy = yx$ for all $x, y \in N$. 

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Remark 1.2.2. It is obvious to see that every commutative near-ring is a distributive near-ring but the converse is not true. For justification, consider a non abelian group \((G, +)\), a noncommutative ring \((R, +, \cdot)\) and \(N = G \times R\). Define componentwise addition ‘+’ in \(N\) and multiplication ∗ in \(N\) by \((g, r) ∗ (g', r') = (0, rr')\), where \((g, r), (g', r') \in N\). It can be easily verified that \((N, +, ∗)\) is a distributive near-ring but not commutative.

Definition 1.2.8. Let \(N\) be a near-ring and \(K\) a nonempty subset of \(N\). Then a normal subgroup \((K, +)\) of \((N, +)\) is called a left ideal (resp. a right ideal) of \(N\) if \(xk \in K\) (resp. \((x + k)y - xy \in K\)) holds for all \(x, y \in N\) and for all \(k \in K\). \(K\) is called an ideal of \(N\) if it is both a left ideal as well as a right ideal of \(N\).

Definition 1.2.9. Let \(N\) be a near-ring. Then a non empty subset \(U\) of \(N\) is said to be a semigroup left (resp. right) ideal of \(N\) if \(NU \subseteq U\) (resp. \(UN \subseteq U\)) and if \(U\) is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of \(N\).

Remark 1.2.3. For any \(x, y \in N\), the symbol \([x, y]\) will denote the multiplicative commutator \(xy - yx\), while \((x, y)\) will indicate the additive commutator \(x + y - x - y\) and \(x \circ y\) will represent the anti-commutator \(xy + yx\). If \(\sigma\) and \(\tau\) are automorphisms of \(N\), then the symbol \([x, y]_{\sigma, \tau}\) will denote the \((\sigma, \tau)\)-commutator \(x\sigma(y) - \tau(y)x\).

1.3 Derivation in near-rings

The notion of derivation in rings is quite old and plays a significant role in the integration analysis, algebraic geometry and algebra. It has got a tremendous development after Posner [75] established two very striking results on derivations in prime rings. Also there has been considerable interest in investigating commutativity of rings, more often that of prime ring and semiprime rings admitting suitably constrained derivations. Derivations in prime and semiprime rings have been studied by Bell, Bresar, Chuang, Hvala, Lanski, Martindale, Vukman etc. in several directions. Motivated by the concept of derivation in rings, Bell and Mason [24] introduced the concept of derivation in near rings as following.

Definition 1.3.1. A derivation \(d\) on \(N\) is defined to be an additive endomorphism satisfying the product rule \(d(xy) = xd(y) + d(x)y\) for all \(x, y \in N\).

Example 1.3.1. Let \(N = N_1 \oplus N_2\), where \(N_1\) is a zero symmetric left near-ring and \(N_2\) is a ring having a nonzero derivation \(\delta\). Then \(d : N \rightarrow N\) defined by \(d(x, y) = (0, \delta(y))\) for all \(x, y \in N\) is a nonzero derivation of the left near ring \(N\).
Example 1.3.2. Let us consider \((\mathbb{C}, +, *)\) where \('*'\) is defined as \(x \ast y = |x|y\) for all \(x, y \in \mathbb{C}\), then it can be easily seen that \((\mathbb{C}, +, *)\) is a zero symmetric left near-ring which is not a right near-ring. Assume \(N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}\), then \(N\) is a zero symmetric left near-ring which is not a right near-ring. Define \(d : N \rightarrow N\) as following
\[
d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.
\]
Then \(d\) is a non zero derivation on \(N\).

The notion of derivation in near-rings has been generalized by introducing the notions of \((\sigma, \tau)\)-derivation and generalized derivation in near-rings by Ashraf et al. [13] and Gölbasi [47] respectively.

Definition 1.3.2. An additive mapping \(d : N \rightarrow N\) is called a \((\sigma, \tau)\)-derivation of \(N\) if there exist functions \(\sigma, \tau : N \rightarrow N\) such that the relation \(d(xy) = d(x)\sigma(y) + \tau(x)d(y)\) hold for all \(x, y \in N\).

Example 1.3.3. Let \(C_1 = (\mathbb{C}, +, \cdot)\), the ring of complex numbers with regard to usual addition + and multiplication \(\cdot\) of complex numbers. Next suppose that \(C_2 = (\mathbb{C}, +, \ast)\), where \(\mathbb{C}\) is the set of complex numbers, + is the usual addition of complex numbers, \(\ast\) is defined as \(x \ast y = |x|y\), for all \(x, y \in \mathbb{C}\), where \(\cdot\) is the usual multiplication of complex numbers and \(|x|\) denotes the modulus of the complex number \(x\). Then \(C_2\) is a zero symmetric left near-ring. Further it can be easily verified that the set \(S = C_1 \times C_2\) is a zero symmetric left near-ring with regard to componentwise addition and multiplication.

Now suppose that \(N = \left\{ \begin{pmatrix} (x, x') & (y, y') \\ (0, 0) & (0, 0) \end{pmatrix} \mid (x, x'), (y, y'), (0, 0) \in S \right\}\). It can be easily checked that \(N\) is a non-commutative zero symmetric left near-ring with respect to matrix addition and matrix multiplication. Define \(d : N \rightarrow N\) and \(\sigma, \tau : N \rightarrow N\) such that
\[
d\begin{pmatrix} x_1, x'_1 \\ 0, 0 \end{pmatrix} \begin{pmatrix} y_1, y'_1 \\ 0, 0 \end{pmatrix} = \begin{pmatrix} (0, 0) & (\bar{x}_1, 0) \\ (0, 0) & (0, 0) \end{pmatrix},
\]
\[
\sigma\begin{pmatrix} x, x' \\ 0, 0 \end{pmatrix} \begin{pmatrix} y, y' \\ 0, 0 \end{pmatrix} = \begin{pmatrix} (x, x') & (-y, -y') \\ (0, 0) & (0, 0) \end{pmatrix},
\]
and
\[
\tau\begin{pmatrix} x, x' \\ 0, 0 \end{pmatrix} \begin{pmatrix} y, y' \\ 0, 0 \end{pmatrix} = \begin{pmatrix} (\bar{x}, \bar{x'}) & (\bar{y}, \bar{y'}) \\ (0, 0) & (0, 0) \end{pmatrix},
\]
where \(\bar{x}_1, \bar{x}, \bar{x'}, \bar{y} \) and \(\bar{y}'\) denote the conjugates of the complex numbers \(x_1, x, x', y\) and
respectively. It can be easily seen that $d$ is a $(\sigma, \tau)$-derivation of $N$, where $\sigma$ and $\tau$ are automorphisms of $N$.

**Definition 1.3.3.** Let $N$ be a near-ring. Then

(i) An additive mapping $f : N \rightarrow N$ is called a **right generalized derivation** of $N$ if there exists a derivation $d$ of $N$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in N$.

(ii) An additive mapping $f : N \rightarrow N$ is called a **left generalized derivation** of $N$ if there exists a derivation $d$ of $N$ such that $f(xy) = d(x)y + xf(y)$ for all $x, y \in N$.

(iii) An additive mapping $f : N \rightarrow N$ is called a **generalized derivation** of $N$ if there exists a derivation $d$ of $N$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in N$ and $f(xy) = d(x)y + xf(y)$ for all $x, y \in N$.

**Example 1.3.4.** Let $S$ be any zero symmetric left near-ring. Consider

$$N_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0, a, b \in S \right\}.$$ Then $N_1$ is a zero symmetric left near-ring with regard to the matrix addition and multiplication. Define $d : N_1 \rightarrow N_1$ and $f : N_1 \rightarrow N_1$ as $d \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $f \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It can be easily seen that $f$ is a right generalized derivation of $N_1$ with associated derivation $d$ of $N_1$ but it is not a left generalized derivation of $N_1$ with associated derivation $d$ of $N_1$.

**Example 1.3.5.** Consider $N_2 = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \mid 0, a, b \in S \right\}$. Then $N_2$ is a zero symmetric left near-ring with regard to the matrix addition and multiplication. Define $d : N_2 \rightarrow N_2$ and $f : N_2 \rightarrow N_2$ as $d \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$ and $f \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$. It can be noted that $f$ is a left generalized derivation of $N_2$ with associated derivation $d$ of $N_2$ but $f$ is not a right generalized derivation of $N_2$ with associated derivation $d$ of $N_2$.

**Example 1.3.6.** Consider $N_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in S \right\}$. Then $N_3$ is a zero symmetric left near-ring with regard to the matrix addition and multiplication. Define $d, f : N_3 \rightarrow N_3$ as $d \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then it can be easily seen that $f$ is a generalized derivation of $N_3$ with associated derivation $d$ of $N_3$. 

5
1.4 Some basic results of near-rings

In this section we discuss some well known results of near-rings which will be used frequently in the forthcoming Chapters.

Lemma 1.4.1 ([24, Lemma 3]). Let $N$ be a prime near-ring.

(i) If $z \in Z \setminus \{0\}$, then $z$ is not a zero divisor.

(ii) If $Z \setminus \{0\}$ contains an element $z$ for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 1.4.2 ([21, Lemma 1.2]). Let $N$ be a prime near-ring. If $z \in Z \setminus \{0\}$ and $x$ is an element of $N$ such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.

Lemma 1.4.3 ([21, Lemma 1.3]). Let $N$ be a prime near-ring.

(i) If $U$ is a nonzero semigroup right ideal (resp. semigroup left ideal) and $x$ is an element of $N$ such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.

(ii) If $U$ is a nonzero semigroup right ideal and $x$ is an element of $N$ which centralizes $U$, then $x \in Z$.

Lemma 1.4.4 ([21, Lemma 1.4]). Let $N$ be a prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

Lemma 1.4.5 ([21, Lemma 1.5]). Let $N$ be a prime near-ring, and $Z$ contains a nonzero semigroup left ideal or semigroup right ideal, then $N$ is a commutative ring.

Since in a left near ring, right distributive property does not hold in general, the following lemma provides us limited distributive property in near-ring.

Lemma 1.4.6 ([24, Lemma 1]). Let $d$ be an arbitrary derivation on the near-ring $N$. Then $N$ satisfies the following partial distributive law $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ for all $x, y, z \in N$.

Lemma 1.4.7 ([86, Proposition 1]). Let $d$ be an arbitrary additive endomorphism of $N$. Then $d$ is a derivation on $N$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$.

Lemma 1.4.8 ([86, Lemma 1]). Let $d$ be a derivation on $N$. Then $N$ satisfies the following partial distributive law $(d(x)y + xd(y))z = d(x)yz + xd(y)z$ for all $x, y, z \in N$.

Lemma 1.4.9 ([86, Lemma 2]). If $N$ admits a derivation $d$, then $d(Z) \subseteq Z$. 

6
1.5 Ring-theoretic notions

Let $R$ be an associative ring. For all $x, y \in R$, the symbol $[x, y]$ stands for Lie product $xy - yx$ and $x \circ y$ stands for Jordan product $xy + yx$ throughout the exposition.

**Definition 1.5.1.** The center of a ring $R$ is defined to be the set of all those elements of $R$ which commute with every element of $R$ and is denoted as $Z$ i.e., $Z = \{x \in R \mid xr = rx \text{ for all } r \in R\}$.

**Definition 1.5.2.** Let $R$ be a ring. If there exists a positive integer $n$ such that $nx = 0$ for all $x \in R$, then in this case the smallest positive integer with this property is called the characteristic of the ring $R$ and is denoted by $\text{char}(R)$. If no such positive integer exists, then $R$ is said to be of characteristic zero.

**Definition 1.5.3.** An element $x \in R$ is called $n$-torsion free if $nx = 0$ implies $x = 0$. Further if $nx = 0$ implies $x = 0$ for all $x \in R$, then $R$ is called an $n$-torsion free ring.

**Definition 1.5.4.** A ring $R$ is said to be a prime ring if zero ideal is a prime ideal of $R$.

**Remark 1.5.1.** A ring $R$ is a prime ring if and only if any one of the following holds:

(i) If $A$ and $B$ are ideals of $R$ such that $AB = \{0\}$, then $A = \{0\}$ or $B = \{0\}$.

(ii) If $a, b \in R$ such that $aRb = \{0\}$, then $a = 0$ or $b = 0$.

**Definition 1.5.5.** A ring $R$ which has no nonzero nilpotent ideal is said to be a semiprime ring.

**Remark 1.5.2.** A ring $R$ is semiprime if and only if for any $a \in R$ such that $aRa = \{0\}$ implies that $a = 0$.

**Remark 1.5.3.** Every prime ring is a semiprime ring but its converse is not true. The ring $\mathbb{Z}_6$ of residue classes modulo 6 is a semiprime ring but not a prime ring.

**Definition 1.5.6.** An additive mapping $x \mapsto x^*$ of $R$ into itself is called an involution on $R$ if it satisfies the conditions: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring $R$ equipped with an involution $'*$ is called a ring with involution or a $*$-ring.

**Example 1.5.1.** Let $Q = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in R\}$ be the ring of real quaternions. Define $q \mapsto q^*$ of $Q$ into itself as $q^* = \alpha - \beta i - \gamma j - \delta k$, where $q = \alpha + \beta i + \gamma j + \delta k \in Q$. It can be easily seen that $'*$ is an involution of $Q$. Therefore $Q$ is a $*$-ring.
Definition 1.5.7. Let $R$ be a $*$-ring. Then an element $x \in R$ is called a symmetric element of $R$ if $x^* = x$ and an element $y \in R$ is called a skew symmetric element of $R$ if $y^* = -y$. The set of all symmetric and skew symmetric elements of $R$ is denoted by $S_{a}(R)$.

Definition 1.5.8. Let $R$ be a $*$-ring. Then an ideal $I$ of $R$ is called an $*$-ideal if $I^* = I$.

Example 1.5.2. Let $R = \mathbb{R}[x] \times Q$, where $\mathbb{R}[x]$ is the polynomial ring over the ring $\mathbb{R}$ of real numbers and $Q$ is the ring of real quaternions. Define $* : R \rightarrow R$ as $*(f(x), q) = (f(-x), \bar{q})$, where $f(x) \in \mathbb{R}[x]$ and $\bar{q} = \alpha - \beta i - \gamma j - \delta k$, where $q = \alpha + \beta i + \gamma j + \delta k \in Q$. It can be easily shown that $*$ is an involution of $R$ and the set $I = \mathbb{R}[x] \times \{0\}$ is an $*$-ideal of $R$.

Definition 1.5.9. A ring $R$ with involution $'*$' is said to be $*$-prime if $aRb = aRb^* = \{0\}$, where $a, b \in R$ (equivalently $aRb = a^*Rb = \{0\}$) implies that either $a = 0$ or $b = 0$.

Example 1.5.3. Let $\mathbb{Z} \times \mathbb{Z}$ be an $*_{ex}$-ring, where $*_{ex}$ is the exchange involution defined on $\mathbb{Z} \times \mathbb{Z}$ by $*_{ex}(x, y) = (y, x)$. It can be easily proved that $\mathbb{Z} \times \mathbb{Z}$ is a $*_{ex}$-prime ring.

Definition 1.5.10. A right (resp. left) ideal $I$ of $R$ is said to be dense right (resp. left) ideal if for any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $r_1r \neq 0$ and $rr_2 \in I$ (resp. $rr_1 \neq 0$ and $r_2r \in I$). The collection of all dense right ideals of $R$ will be denoted by $D(R)$.

Definition 1.5.11. Let $R$ be a semiprime ring. Consider the set

$\mathcal{H} = \{(f; J) \mid J \in \mathcal{D}(R), f : J_R \rightarrow R_R\}$.

We let $J_R \& R_R$ denote right $R$-modules $J \& R$ respectively. Here $f$ is a homomorphism of right $R$-modules. We define a relation $'\sim'$ on $\mathcal{H}$ i.e.; $(f; J) \sim (g; K)$ if there exists $L \subseteq J \cap K$ such that $L \in \mathcal{D}$ and $f = g$ on $L$. It can be easily checked that $'\sim'$ is an equivalence relation on $\mathcal{H}$. Let $Q_{mr}$ be the set of equivalence classes of different elements of $\mathcal{H}$ relative to the relation $'\sim'$. Denote the equivalence class determined by $(f; J) \in \mathcal{D}$ as $[f; J]$. Define addition and multiplication on $Q_{mr}$ as follows: $[f; J] + [g; K] = [f + g; J \cap K]$ and $[f; J][g; K] = [fg; g^{-1}(J)]$. It can be verified that $Q_{mr}$ forms an associative ring with identity relative to above defined operations and is known as maximal right ring of quotients or right Utumi quotient ring of $R$.

Remark 1.5.4. Let $R$ be a semiprime ring. Then $Q_{mr}$ satisfies the following:
(i) $R$ is a subring of $Q_{mr}$.

(ii) For all $q \in Q_{mr}$ there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$.

(iii) For all $q \in Q_{mr}$ and $J \in \mathcal{D}$, $qJ = \{0\}$ if and only if $q = 0$.

(iv) If $J \in \mathcal{D}$ and $f : J_R \rightarrow R_R$ is a homomorphism of right $R$-modules, then there exists $q \in Q_{mr}$ such that $f(x) = qx$ for all $x \in J$.

Furthermore, properties (i) – (iv) characterize ring $Q_{mr}$ up to isomorphism.

Example 1.5.4. The ring $R = \mathbb{Z} \times \mathbb{Z}$ is clearly a semiprime ring. It can be seen that $Q_{mr}(R) = \mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the ring of rational numbers.

Definition 1.5.12. Let $R$ be a semiprime ring. Consider $\mathcal{I} = \mathcal{I}(R) = \{I \mid I$ is an ideal of $R$ and left annihilator of $I$ in $R$ i.e.; $l(I) = \{0\}\}$. Next we set $Q_s = \{q \in Q_{mr} \mid qJ \cup Jq \subseteq R$ for some $J \in \mathcal{I}\}$. It can be easily verified that $Q_s$ is a subring of $Q_{mr}$. $Q_s$ is called right symmetric Martindale quotient ring or right symmetric quotient ring of $R$.

Remark 1.5.5. Let $R$ be a semiprime ring. Then $Q_s$ satisfies the following:

(i) $R$ is a subring of $Q_s$.

(ii) For all $q \in Q_s$ there exists $J \in \mathcal{I}$ such that $qJ \cup Jq \subseteq R$.

(iii) For all $q \in Q_s$ and $J \in \mathcal{I}$, $qJ = \{0\}$ (or $Jq = \{0\}$) if and only if $q = 0$.

(iv) If $J \in \mathcal{I}$, $f : J_R \rightarrow R_R$ and $g : R_J \rightarrow R_R$ are homomorphism of right $R$-modules and homomorphism of left $R$-modules respectively such that $xf(y) = g(x)y$ for all $x, y \in J$, then there exists $q \in Q_s$ such that $f(x) = qx$ and $g(x) = xq$ for all $x \in J$.

Furthermore, properties (i) – (iv) characterize ring $Q_s$ up to isomorphism.

Definition 1.5.13. The center of the ring $Q_{mr}$ is known as the extended centroid of $R$. It is denoted by $C$.

Remark 1.5.6. (i) Let $R$ be a semiprime ring. Then

$$Z(Q_{mr}) = Z(Q_s) = C = \{q \in Q_{mr} \mid qr = rq \text{ for all } r \in R\}.$$ 

(ii) If $R$ is a prime ring, then $C$ is a field.
(iii) If $R$ is a semiprime ring and $C$ is a field, then $R$ must be a prime ring.

**Definition 1.5.14.** A mapping $d : R \rightarrow R$ is said to be a derivation of $R$ if it satisfies the following properties:

(i) $d(x + y) = d(x) + d(y)$

(ii) $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

**Example 1.5.5.** The most natural example of a non trivial derivation is the usual differentiation on the ring $F[x]$ of polynomials defined over a field $F$.

**Definition 1.5.15.** For a fixed $a \in R$, define $d_a : R \rightarrow R$ such that $d_a(x) = [a, x]$ for all $x \in R$. Then it can be shown that $d_a$ is a derivation of $R$. This $d_a$ is called an inner derivation of $R$ determined by ‘$a$’ and usually it is denoted by $I_a$. It is obvious to see that every inner derivation of a ring $R$ is a derivation. But the converse need not be true in general.

**Example 1.5.6.** Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ w.r.t. matrix addition and matrix multiplication. Define a mapping $d : R \rightarrow R$ as follows:

$$d \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

It is easy to show that $d$ is a derivation of $R$ which is not an inner derivation of $R$.

**1.6 Some basic results of rings**

In this section, we shall include some well known results which will be used for developing the subject matter in the subsequent Chapters 4 and 5.

**Lemma 1.6.1** ([75, Theorem 1]). Let $R$ be a prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation, then one at least of $d_1, d_2$ is zero. This is known as Posner’s first theorem.

**Lemma 1.6.2** ([66, Lemmas 1]). Let $R$ be a $*$-prime ring and $I$ be a nonzero $*$-ideal of $R$. If $x, y \in R$ satisfy $xIy = xIy^* = \{0\}$, then $x = 0$ or $y = 0$. 

Lemma 1.6.3 ([66, Lemmas 2]). Let $R$ be a $*$-prime ring admitting a nonzero derivation $d$ which commutes with $*'$. If $I$ is a nonzero $*$-ideal of $R$ and $[x, R]I d(x) = \{0\}$ for all $x \in I$, then $R$ is commutative.

Lemma 1.6.4 ([66, Lemmas 3]). Let $R$ be a $*$-prime ring admitting a nonzero derivation $d$ which commutes with $*'$. If $I$ is a nonzero $*$-ideal of $R$ and $[d(x), x] = 0$ for all $x \in I$, then $R$ is commutative.

Lemma 1.6.5 ([65, Theorem 3.2]). Let $d$ be a nonzero derivation of a 2-torsion free $*$-prime ring $R$ and $I$ a nonzero $*$-ideal of $R$. If $r \in S_{\text{a}}(R)$ satisfies $[d(x), r] = 0$ for all $x \in I$, then $r \in Z$. Furthermore, if $d(I) \subseteq Z$, then $R$ is commutative.

Lemma 1.6.6. Every $*$-prime ring is a semiprime ring.

Proof. Let $R$ be a $*$-prime ring and $a \in R$ such that $aRa = \{0\}$. This implies that $aRaRa^* = \{0\}$ also. Now $*$-primeness of $R$ insures that $a = 0$ or $aRa^* = \{0\}$. $aRa^* = \{0\}$ together with $aRa = \{0\}$ gives us $a = 0$. Thus we conclude that every $*$-prime ring is a semiprime ring.

Lemma 1.6.7. If $R$ is a $*$-prime ring of characteristic different from 2, then $R$ is 2-torsion free.

Proof. Suppose that $x \in R$ such that $2x = 0$. This implies that $2xrs = 0$ for all $r, s \in R$ i.e.; $xR(2s) = \{0\}$ for all $s \in R$. Since characteristic of $R$ is different from 2 and $R \neq \{0\}$, this provides us a nonzero element $l \in R$ such that $2l \neq 0$. Now we conclude that $xR(2l) = \{0\} = xR(2l)^*$. Finally $*$-primeness of $R$ provides us $x = 0$ and hence $R$ is 2-torsion free.

Lemma 1.6.8. If $R$ is a $*$-prime ring admitting a nonzero central $*$-ideal $I$ i.e.; $I \subseteq Z$, then $R$ is commutative.

Proof. Let $r, s \in R$ and $x \in I$. Using hypothesis we get $rsx = rxr = r sx$. This implies that $[r, s]I = \{0\}$ and hence $[r, s]I l = [r, s]I I^* = \{0\}$, where $0 \neq l \in R$. In view of Lemma 5.2.1, we get the required result.
Chapter 2

$n$-Derivation and its generalizations in near-rings

2.1 Introduction

The concepts of symmetric bi-derivation, permuting tri-derivation and permuting $n$-derivation have already been introduced in rings by G. Maksa, M.A. Özürk and K.H. Park in [58, 59], [69] and [72] respectively. Symmetric bi-derivations and permuting tri-derivations have been studied in near-rings by M. A. Özürk and K. H. Park in [70] and [73] respectively. In this chapter, motivated by these concepts, we define $n$-derivation and permuting $n$-derivation in near-rings, where $n$ is a positive integer and study some properties involved there. We have also generalized the notion of $n$-derivation in two ways by introducing the notion of $(\sigma, \tau)$-$n$-derivation and generalized $n$-derivation in the forthcoming sections of this chapter.

Section 2.2 deals with the study of $n$-derivation and permuting $n$-derivation in near-rings. The main result of this section states that under certain constraints, a permuting $n$-additive mapping $D$ on a $n!$-torsion free prime near-ring $N$ is zero if the trace $d$ of $D$ is zero.

In the Section 2.3, we have discussed the concept of $(\sigma, \tau)$-$n$-derivation in near-rings. Further we investigate some properties involving $(\sigma, \tau)$-$n$-derivations of a prime near-ring $N$ which force $N$ to be a commutative ring. Additive commutativity of near-ring $N$ satisfying certain identities involving $(\sigma, \tau)$-$n$-derivations of a prime near-ring $N$ has also been obtained. Related examples to justify the hypotheses in various theorems have also been provided.
The last Section 2.4 is devoted to the study of generalized $n$-derivation in near-rings. We introduce the notion of generalized $n$-derivation in near-ring $N$ and investigate several identities involving generalized $n$-derivations of a prime near-ring $N$ which force $N$ to be a commutative ring. Some more related results are also obtained.

2.2 Permuting $n$-derivation in near-rings

Let $N$ be a near-ring. An additive mapping $d : N \rightarrow N$ is called a derivation if $d(xy) = xd(y) + d(x)y$ (equivalently $d(xy) = d(x)y + xd(y)$) holds for all $x, y \in N$. The notion of derivation in near-ring was generalized by M. A. Öztiirk [70] by introducing the concept of symmetric-bi-derivation in near-rings as follows. A mapping $D : N \times N \rightarrow N$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in N$. A symmetric bi-additive (i.e., additive in both arguments) mapping is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$ is fulfilled for all $x, y, z \in N$. Later K.H. Park [73] introduced the notion of permuting tri-derivation as follows: A mapping $D : N \times N \times N \rightarrow N$ is said to be permuting if $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(z, y, x) = D(y, z, x)$ for all $x, y, z \in N$. A permuting tri-additive (i.e., additive in all three arguments) mapping is called a permuting tri-derivation if $D(xw, y, z) = D(x, y, z)w + xD(w, y, z)$ holds for all $x, y, z, w \in N$. The notions of bi-derivation and tri-derivation were generalized by Park [72] who introduced the notion of $n$-derivation in rings. Motivated by these notions, we define the concept of $n$-derivation and permuting $n$-derivation in near-rings in the present section as following:

Definition 2.2.1. A map $D : N \times N \times \cdots \times N \rightarrow N$ is said to be permuting if the $n$-times equation $D(x_1, x_2, \cdots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_1, x_2, \cdots, x_n \in N$ and for every permutation $\pi \in S_n$, where $S_n$ is the permutation group on $\{1, 2, \cdots, n\}$. A map $d : N \rightarrow N$ defined by $d(x) = D(x, x, \cdots, x)$ for all $x \in N$ where $D : N \times N \times \cdots \times N \rightarrow N$ is a permuting map, is called the trace of $D$.

Definition 2.2.2. Let $n$ be any fixed positive integer. An $n$-additive (i.e., additive in each argument) mapping $D : N \times N \times \cdots \times N \rightarrow N$ is called an $n$-derivation if the relations

$$D(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n)$$
\[ D(x_1, x_2' \cdots, x_n') = D(x_1, x_2', \cdots, x_n)x_2' + x_2D(x_1, x_2', \cdots, x_n) \]

\[ : \]

\[ D(x_1, x_2', \cdots, x_n') = D(x_1, x_2', \cdots, x_n)x_n' + x_nD(x_1, x_2', \cdots, x_n') \]

holds for all \( x_1, x_2', x_2, \cdots, x_n, x_n' \in N \).

If in addition \( D \) is a permuting map then all the above conditions are equivalent and in this case \( D \) is called a \textit{permuting n-derivation} of \( N \) i.e.; a permuting n-derivation of \( N \) can also be defined as below.

An \( n \)-additive permuting mapping \( D : N \times N \times \cdots \times N \rightarrow N \) is called a \textit{permuting n-derivation} of \( N \) if \( D(x_1, x_2, \cdots, x_ix_i', \cdots, x_n) = D(x_1, x_2, \cdots, x_i, \cdots, x_n)x_i' - x_iD(x_1, x_2, \cdots, x_i', \cdots, x_n) \) holds for all \( x_1, x_2, \cdots, x_i, \cdots, x_n \in N \). For an example of a permuting \( n \)-derivation, suppose that \( N' \) is a commutative near-ring. Then \( N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b, 0 \in N' \right\} \) is a noncommutative near-ring with regard to matrix addition and matrix multiplication. Define \( D : N \times N \times \cdots \times N \rightarrow N \) such that

\[
D\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1a_2\cdots a_n \\ 0 & 0 \end{pmatrix}.
\]

It is easy to see that \( D \) is a permuting \( n \)-derivation of near-ring \( N \). By definition it is clear that a permuting \( n \)-derivation of \( N \) is also an \( n \)-derivation but the converse is not true. The following example justifies this fact:

**Example 2.2.1.** Let \( R \) be a noncommutative ring and \( N_1 \) a zero symmetric left near-ring. Consider \( S = R \times N_1 \). Then it is clear that \( S \) is a zero symmetric left near-ring with regard to componentwise addition and multiplication. Now suppose that

\[
N = \left\{ \begin{pmatrix} (a, b) & (a', b') \\ (0, 0) & (0, 0) \end{pmatrix} | (a, b), (a', b'), (0, 0) \in S \right\}.
\]

It can be easily checked that \( N \) is a non-commutative zero symmetric left near-ring with respect to matrix addition and matrix multiplication. Define \( D : N \times N \times \cdots \times N \rightarrow N \) such that

\[
D\begin{pmatrix} (a_1, b_1) & (a'_1, b'_1) \\ (0, 0) & (0, 0) \end{pmatrix}, \begin{pmatrix} (a_2, b_2) & (a'_2, b'_2) \\ (0, 0) & (0, 0) \end{pmatrix}, \cdots, \begin{pmatrix} (a_n, b_n) & (a'_n, b'_n) \\ (0, 0) & (0, 0) \end{pmatrix} = \begin{pmatrix} (0, 0) & (a_1a_2\cdots a_n, 0) \\ (0, 0) & (0, 0) \end{pmatrix}.
\]

It is easy to see that \( D \) is an \( n \)-derivation of \( N \), however it is not a permuting \( n \)-derivation of \( N \).
Remark 2.2.1. In the above example, if we take $R$ as a commutative ring, then $D$ becomes a permuting $n$-derivation of $N$ also.

Now let $D$ be a permuting $n$-derivation of a near-ring $N$. Then it can be easily seen that 
\[ D(0, x_2, \ldots, x_n) = D(0 + 0, x_2, \ldots, x_n) = D(0, x_2, \ldots, x_n) + D(0, x_2, \ldots, x_n). \]
Therefore 
\[ D(0, x_2, \ldots, x_n) = 0 \] for all $x_2, \ldots, x_n \in N$. We also observe that 
\[ D(-x_1, x_2, \ldots, x_n) = -D(x_1, x_2, \ldots, x_n) \]
for all $x_i \in N; i = 1, 2, \ldots, n$.

We begin with the following lemmas which are essential for developing the proofs of our main results of this section. Proofs of Lemmas 2.2.1 & 2.2.2 can be seen in [24, Lemma 3] and [21, Lemma 1.2] respectively, while Lemmas 2.2.3 & 2.2.4 have essentially been proved in [15].

**Lemma 2.2.1.** Let $N$ be a prime near-ring.

(i) If $z \in Z \setminus \{0\}$, then $z$ is not a zero divisor.

(ii) If $Z \setminus \{0\}$ contains an element $z$ for which $z + z \in Z$, then $(N, +)$ is abelian.

**Lemma 2.2.2.** Let $N$ be a prime near-ring. If $z \in Z \setminus \{0\}$ and $x$ is an element of $N$ such that $xz \in Z$ or $zx \in Z$ then $x \in Z$.

**Lemma 2.2.3.** Let $N$ be a near-ring. Then $D$ is a permuting $n$-derivation of $N$ iff 
\[ D(x_1, x_2, \ldots, x_n) = x_1 D(x_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n) x_1 \]
for all $x_1, x_2, \ldots, x_n \in N$.

**Lemma 2.2.4.** Let $N$ be prime near-ring and $D$ be a nonzero permuting $n$-derivation of $N$.

(i) If $D(N, N, \ldots, N)x = \{0\}$ where $x \in N$ then $x = 0$,

(ii) If $xD(N, N, \ldots, N) = \{0\}$ where $x \in N$ then $x = 0$.

In a left near-ring $N$, right distributive law does not hold in general. However, we can prove the following partial distributive properties in $N$.

**Lemma 2.2.5.** Let $N$ be a near-ring and $D$ be a permuting $n$-derivation of $N$ with the trace $d$. Then for every $x_1, x_2, \ldots, x_n, y \in N$,

(i) \[ \{D(x_1, x_2, \ldots, x_n)x_1y + x_1D(x_1, x_2, \ldots, x_n)\}y = D(x_1, x_2, \ldots, x_n)x_1y + x_1D(x_1, x_2, \ldots, x_n)y, \]
(ii) \( \{x_1D(x'_1, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x'_1\}y \)
\[= x_1D(x'_1, x_2, \cdots, x_n)y + D(x_1, x_2, \cdots, x_n)x'_1y, \]

(iii) \( \{d(x)x_1 + xD(x, x, \cdots, x, x_1)\}y = d(x)x_1y + xD(x, x, \cdots, x, x_1)y, \)

(iv) \( \{xD(x, x, \cdots, x, x_1) + d(x)x_1\}y = xD(x, x, \cdots, x, x_1)y + d(x)x_1y. \)

Proof. (i) For all \( x_1, x_1', x_1'', x_2, \cdots, x_n \in N \)

\[
D((x_1x_1')x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1' + (x_1x_1')D(x_1, x_2, \cdots, x_n)
\]
\[= \{D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1, x_2, \cdots, x_n)\}x_1''
+ x_1x_1'D(x_1, x_2, \cdots, x_n).
\]

Also

\[
D(x_1(x_1'x_1''), x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1x_1'' + x_1D(x_1'x_1'', x_2, \cdots, x_n)
\]
\[= D(x_1, x_2, \cdots, x_n)x_1x_1'' + x_1\{D(x_1', x_2, \cdots, x_n)\}
+ x_1'D(x_1''', x_2, \cdots, x_n)
+ x_1x_1'D(x_1', x_2, \cdots, x_n).
\]

Combining the above two relations, we get

\[
\{D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n)\}x_1''
\]
\[= D(x_1, x_2, \cdots, x_n)x_1x_1'' + x_1D(x_1', x_2, \cdots, x_n)x_1''
+ x_1'D(x_1', x_2, \cdots, x_n).
\]

Putting \( y \) in the place of \( x_1'' \), we find that

\[
\{D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n)\}y
\]
\[= D(x_1, x_2, \cdots, x_n)x_1'y + x_1D(x_1', x_2, \cdots, x_n)y.
\]

(ii) It can be proved in a similar way as above, with the help of Lemma 2.2.3.

(iii) Putting \( x_1 = x_2 = x_3 = \cdots = x_n = x \) in (i), we find that

\[
\{d(x)x_1' + xD(x_1, x, \cdots, x)\}y = d(x)x_1'y + xD(x_1', x, \cdots, x)y.
\]
In particular for \( x_1' = x_1 \) we get

\[
\{d(x)x_1 + xD(x, x, \cdots, x)\}y = d(x)x_1y + xD(x, x, \cdots, x_1)y.
\]

(iv) It can be proved in a similar way as above. \( \square \)
Remark 2.2.2. It is obvious to observe that above Lemmas 2.2.3-2.2.5 also hold if $D$ is an $n$-derivation (not necessarily permuting) of prime near-ring $N$.

Lemma 2.2.6. Let $N$ be a $m!$-torsion free near-ring, where $(N, +)$ is an abelian group. Suppose $y_1, y_2, \cdots, y_m \in N$ satisfy $\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^m y_m = 0$ for $\alpha = 1, 2, \cdots, m$. Then $y_i = 0$ for all $i$.

Proof. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots \\ m & m^2 & \cdots & m^m \end{pmatrix}$ be any $m \times m$ matrix. Then by our assumption $A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Now pre multiplying by $\text{Adj } A$ yields $\text{Det } A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Since $\text{Det } A$ as a Vondermonde determinant, is equal to a product of positive integers, each of which is less than or equal to $m$ and as $N$ is a $m!$-torsion free near-ring, it follows immediately that $y_i = 0$ for all $i$. \qed 

Recently M.A. Öztürk and Y.B. Jun [70, Lemma 3.1] proved that in a 2-torsion free near-ring which admits a symmetric bi-additive mapping $D$ if the trace $d$ of $D$ is zero, then $D = 0$. In the year 2010, this result was further generalized by K.H. Park and Y.S.Jung [73, Lemma 2.2] for permuting tri-additive mapping in $3!$-torsion free near-ring. We have extended this result, as below, for permuting $n$-additive mapping in a $n!$-torsion free prime near-ring.

Theorem 2.2.1. Let $N$ be a $n!$-torsion free prime near-ring and $D$ be a permuting $n$-additive mapping of $N$ such that $D(N, N, \cdots, N) \subseteq Z$. If $d(x) = 0$, for all $x \in N$, then $D = 0$.

Proof. If $D = 0$, then we have nothing to do, if not then $D$ is a nonzero permuting $n$-additive mapping of prime near-ring $N$ such that $D(N, N, \cdots, N) \subseteq Z$. Hence there exist $x_1, x_2, \cdots, x_n \in N$, all nonzero such that $D(x_1, x_2, \cdots, x_n) \neq 0$ and $D(x_1, x_2, \cdots, x_n) \in Z$. Since $D(x_1 + x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n) \in Z$, by Lemma
2.2.1, \((N,+)\) is an abelian group. Hence the trace \(d(x) = D(x, x, \cdots, x)\) of permuting \(n\)-additive mapping \(D\) can be expressed as

\[
d(x + y) = d(x) + d(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(x, y)
\]

(2.2.1)

where \(x, y \in N\) and \(h_k(x, y) = D(x, x, \cdots, x, y, y, \cdots, y)\). In particular by our hypothesis \(d(\mu x + x_n) = 0\) where \(1 \leq \mu \leq n - 1\). With the help of equation (2.2.1), we get

\[
0 = d(\mu x) + d(x_n) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(\mu x, x_n)
= \sum_{k=1}^{n-1} \binom{n}{k} h_k(\mu x, x_n).
\]

This yields that

\[
\mu y_1 + \mu^2 y_2 + \cdots + \mu^{n-2} y_{n-2} + \mu^{n-1} n D(x, x, \cdots, x, x_n) = 0,
\]

where \(y_1, y_2, \cdots, y_{n-2} \in N\). By our hypothesis and Lemma 2.2.6, we deduce that

\[
D(x, x, \cdots, x, x_n) = 0
\]

(2.2.2)

for all \(x, x_n \in N\). Let \(\nu(1 \leq \nu \leq n - 2)\) be any integer. By equation (2.2.2), we find that

\[
D(\nu x + x_{n-1}, \nu x + x_{n-1}, \cdots, \nu x + x_{n-1}, x_n) = 0.
\]

Expanding the above relation and using equation (2.2.2) again we obtain

\[
\nu z_1 + \nu^2 z_2 + \cdots + \nu^{n-3} z_{n-3} + \nu^{n-2} \binom{n}{2} D(x, x, \cdots, x, x_{n-1}, x_n) = 0
\]

where \(z_1, z_2, \cdots, z_{n-3} \in N\). By our hypothesis and Lemma 2.2.6, we conclude that \(D(x, x, \cdots, x, x_{n-1}, x_n) = 0\) for all \(x, x_{n-1}, x_n \in N\). Now if we continue the above process inductively, then we finally arrive at \(D(x_1, x_2, \cdots, x_{n-1}, x_n) = 0\). This gives that \(D = 0\), a contradiction. \(\Box\)

In the theorem given below the symbol \(C\) will represent the set of all additive commutators of \(N\) i.e.; \(C = \{(x, y) \mid x, y \in N\}\).
Theorem 2.2.2. Let $D$ be a nonzero permuting $n$-derivation of a prime near-ring $N$. If $D(C, N, N, \cdots, N) = \{0\}$, then $(N, +)$ is an abelian group.

Proof. Since $D(c, r_2, \cdots, r_n) = 0$ for all $c \in C$ and for all $r_2, \cdots, r_n \in N$, $D(wc, r_2, \cdots, r_n) = 0$ where $w \in N$ i.e.; $wD(c, r_2, \cdots, r_n) + D(w, r_2, \cdots, r_n)c = 0$. In turn we get $D(w, r_2, \cdots, r_n)c = 0$ but $D \neq 0$, and therefore by Lemma 2.2.4, $c = 0$. Hence $(N, +)$ is an abelian group. \qed

Theorem 2.2.3. Let $N$ be a prime near-ring and $D$ a nonzero permuting $n$-derivation of $N$. If $K = \{a \in N \mid [D(N, N, \cdots, N), a] = \{0\}\}$, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$,

(ii) $d(K) \subseteq Z$,

(iii) $K$ is a semigroup under multiplication,

(iv) If there exists an element $a \in K$ for which $d(a) \neq 0$ and $D(a^2, a, \cdots, a) \in Z$, then $(N, +)$ is an abelian group.

Proof. (i) We have

$$D(x_1, x_2, \cdots, x_n)a = aD(x_1, x_2, \cdots, x_n). \quad \text{(2.2.3)}$$

for all $x_1, x_2, \cdots, x_n \in N$. Putting $ax_1$ in place of $x_1$ in the above equation and using Lemma 2.2.5 we get

$$D(a, x_2, \cdots, x_n)x_1a + aD(x_1, x_2, \cdots, x_n)a = aD(a, x_2, \cdots, x_n)x_1 + aaD(x_1, x_2, \cdots, x_n).$$

Using the equation (2.2.3), we get $D(a, x_2, \cdots, x_n)x_1a = aD(a, x_2, \cdots, x_n)x_1$. Now putting $x_1y_1$ for $x_1$ in the latter relation and using it again, we have $D(a, x_2, \cdots, x_n)x_1[y_1, a] = 0$ where $y_1 \in N$. This gives us $D(a, x_2, \cdots, x_n)[a, y_1] = \{0\}$. Since $N$ is a prime near-ring, either $[a, y_1] = 0$ for all $y_1 \in N$ or $D(a, x_2, \cdots, x_n) = 0$ for all $x_2, \cdots, x_n \in N$. If the first property holds then $a \in Z$, if not then $D(a, x_2, \cdots, x_n) = 0$, and hence in particular, $D(a, a, \cdots, a) = 0$ or $d(a) = 0$.

(ii) From the above proof we observe that if $a \in K$ then either $a \in Z$ or $d(a) = 0$. But $d(a) = 0$ implies $d(a) \in Z$. If $d(a) \neq 0$ then we have $a \in Z$. In this case we have $D(xa, a \cdots, a) = D(ax, a, \cdots, a)$ for all $x \in N$. This yields that $xD(a, a, \cdots, a) + D(x, a, \cdots, a)a = D(a, a, \cdots, a)x + aD(x, a, \cdots, a)$. This reduces to $xD(a, a, \cdots, a) = 0$.
$D(a, a, \cdots, a)x$, which shows that $d(a) \in Z$ and thus $d(K) \subseteq Z$.

(iii) Let $a, b \in K$. Hence $abD(r_1, r_2, \cdots, r_n) = D(r_1, r_2, \cdots, r_n)ab$ holds trivially. Associativity of $N$ shows that $K$ is a semigroup.

(iv) Consider $D(a^2, a, \cdots, a) = aD(a, a, \cdots, a) + D(a, a, \cdots, a)a \in Z$. Since $d(a) = D(a, a, \cdots, a) \neq 0$, we find that $a \in Z$ by (i). Hence $D(a^2, a, \cdots, a) = D(a, a, \cdots, a)$ $(a + a)$. By above proof (ii) we find that $D(a, a, \cdots, a) \in Z \setminus \{0\}$ and hence using Lemma 2.2.2, $a + a \in Z$. By Lemma 2.2.1 we conclude that $(N, +)$ is an abelian group. \(\square\)

2.3 \((\sigma, \tau)-n\)-derivation in near-rings

The concept of derivation has been generalized in several ways by various authors. The notions of $(\sigma, \tau)$-derivation, symmetric bi-$(\sigma, \tau)$-derivation and permuting tri-$(\sigma, \tau)$-derivation have already been introduced and studied in near-rings by Ashraf et. al. [13], Yılmaz Ceven [35] and Öztürk [71] respectively, where $\sigma$ and $\tau$ are any functions from $N$ to $N$. These notions are given as following:

An additive mapping $d : N \rightarrow N$ is called a $(\sigma, \tau)$-derivation if there exists automorphisms $\sigma, \tau : N \rightarrow N$ such that $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in N$.

A symmetric bi-additive (i.e.; additive in both arguments) mapping is called a symmetric bi-$(\sigma, \tau)$-derivation if there exist automorphisms $\sigma, \tau : N \rightarrow N$ such that $D(xy, z) = D(x, z)\sigma(y) + \tau(x)D(y, z)$ is fulfilled for all $x, y, z \in N$.

A permuting tri-additive (i.e.; additive in all three arguments) mapping is called a permuting tri-$(\sigma, \tau)$-derivation if there exist functions $\sigma, \tau : N \rightarrow N$ such that $D(xw, y, z) = D(x, y, z)\sigma(w) + \tau(x)D(w, y, z)$ holds for all $x, y, z, w \in N$.

Inspired by these concepts, in the present section, we define $(\sigma, \tau)$-$n$-derivation in the setting of near-rings and study its various properties.

Definition 2.3.1. Let $n$ be a fixed positive integer. An $n$-additive (i.e.; additive in each argument) mapping $D : N \times N \times \cdots \times N \rightarrow N$ is called a $(\sigma, \tau)$-$n$-derivation of $N$ if there exist functions $\sigma, \tau : N \rightarrow N$ such that the relations

\[
D(x_1x'_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)\sigma(x'_1) + \tau(x_1)D(x_1', x_2, \cdots, x_n)
\]

\[
D(x_1, x_2x'_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)\sigma(x'_2) + \tau(x_2)D(x_1, x_2', \cdots, x_n)
\]

\vdots
\[ D(x_1, x_2, \ldots, x_n, x'_n) = D(x_1, x_2, \ldots, x_n) \sigma(x'_n) + \tau(x_n)D(x_1, x_2, \ldots, x'_n) \]

hold for all \( x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \in N \).

Further in addition if \( D \) is a permuting map then all the above conditions are equivalent and in this case \( D \) is called a permuting \((\sigma, \tau)-n\)-derivation of \( N \). For such an example, let \( n \) be a fixed positive integer and \( C_1 = (\mathbb{C}, +, \cdot) \), the ring of complex numbers with regard to usual addition ‘+’ and multiplication ‘.’. Next suppose that \( C_2 = (\mathbb{C}, +, \ast) \), where \( \mathbb{C} \) is the set of complex numbers, + is the usual addition of complex numbers, ‘∗’ is defined as following \( x \ast y = |x|y \), for all \( x, y \in \mathbb{C} \), where ‘.’ is the usual multiplication of complex numbers and \(|x|\) denotes the modulus of the complex number \( x \). Then \( C_2 \) is a zero symmetric left near-ring. Further, it can be easily verified that the set \( S = C_1 \times C_2 \) is a zero symmetric left near-ring with respect to componentwise addition and multiplication. Now suppose that \( N = \left\{ \begin{pmatrix} (x_1, x'_1) & (y_1, y'_1) \\ (0, 0) & (0, 0) \end{pmatrix}, \ldots, \begin{pmatrix} (x_n, x'_n) & (y_n, y'_n) \\ (0, 0) & (0, 0) \end{pmatrix} \right\} \). It can be easily checked that \( N \) is a non-commutative zero symmetric left near-ring with respect to matrix addition and matrix multiplication. Define \( D : N \times N \times \cdots \times N \to N \) and \( \sigma, \tau : N \to N \) such that

\[
D \left( \begin{pmatrix} (x_1, x'_1) & (y_1, y'_1) \\ (0, 0) & (0, 0) \end{pmatrix}, \ldots, \begin{pmatrix} (x_n, x'_n) & (y_n, y'_n) \\ (0, 0) & (0, 0) \end{pmatrix} \right) = \begin{pmatrix} (0, 0) & (\bar{x}_1, \bar{x}'_1, \ldots, \bar{x}_n, 0) \\ (0, 0) & (0, 0) \end{pmatrix}, \quad \sigma \begin{pmatrix} (x, x') & (y, y') \\ (0, 0) & (0, 0) \end{pmatrix} = \begin{pmatrix} (x, x') & (-y, -y') \\ (0, 0) & (0, 0) \end{pmatrix}
\]

and

\[
\tau \begin{pmatrix} (x, x') & (y, y') \\ (0, 0) & (0, 0) \end{pmatrix} = \begin{pmatrix} (\bar{x}, \bar{x}') & (\bar{y}, \bar{y}') \\ (0, 0) & (0, 0) \end{pmatrix}, \quad \text{where} \quad \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{x}', \bar{y} \text{ and } \bar{y}' \text{ denote the conjugates of the complex numbers } x_1, x_2, \ldots, x_n, x, y \text{ and } y' \text{ respectively. It can be easily seen that } D \text{ is a permuting } (\sigma, \tau)-n\text{-derivation of } N, \text{ where } \sigma \text{ and } \tau \text{ are automorphisms of } N.
\]

It is to be noticed that a \((\sigma, \tau)-n\)-derivation of \( N \) need not be a permuting \((\sigma, \tau)-n\)-derivation of \( N \). For justification, let \( R \) be a noncommutative ring and \( N_1 \) a zero symmetric left near-ring. Then \( N_2 = R \times N_1 \), forms a zero symmetric left near-ring with respect to component wise addition and multiplication. Now set

\[
N = \left\{ \begin{pmatrix} (0, 0) & (x, y) & (x', y') \\ (0, 0) & (0, 0) & (0, 0) \end{pmatrix}, \begin{pmatrix} (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) \end{pmatrix} \right\} \). It can be easily seen that \( N \) is a zero symmetric left near-ring with respect to matrix addition and matrix multiplication. Now define \( D : N \times N \times \cdots \times N \to N \) and \( \sigma, \tau : N \to N \).
such that $D \left( \begin{pmatrix} (0,0) & (x_1, y_1) \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} \right) \left( \begin{pmatrix} (0,0) & (x_2, y_2) \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} \right), \ldots$, \\
$\begin{pmatrix} (0,0) & (x_n, y_n) \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} \right) = \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}$,
\\
$\begin{pmatrix} (0,0) & (x, y) \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}$, \\
$\begin{pmatrix} (0,0) & (x', y') \\ (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}$

It can be seen that $D$ is a $(\sigma, \tau)$-$n$-derivation of $N$, where $\sigma$ and $\tau$ are automorphisms of $N$, however it is not a permuting $(\sigma, \tau)$-$n$-derivation.

Throughout this section, $\sigma$ and $\tau$ will represent automorphisms of $N$. We facilitate our discussion with the following lemmas, which play crucial role in proving the main results of this section:

**Lemma 2.3.1.** Let $N$ be a near-ring. Then $D$ is a $(\sigma, \tau)$-$n$-derivation of $N$ if and only
if
\[
D(x_1, x_2, \ldots, x_n) = \tau(x_1)D(x_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\sigma(x_1),
\]
\[
D(x_1, x_2, x', \ldots, x_n) = \tau(x_2)D(x_1, x', \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\sigma(x_2),
\]
\[
\vdots
\]
\[
D(x_1, x_2, \ldots, x_n x'_n) = \tau(x_n)D(x_1, x_2, \ldots, x_n') + D(x_1, x_2, \ldots, x_n)\sigma(x_n);
\]
for all \(x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N.\)

**Proof.** Let \(D\) be a \((\sigma, \tau)\)-\(n\)-derivation of \(N\). Consider
\[
D(x_1 x_1', x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\sigma(x_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)
\]
\[
= D(x_1, x_2, \ldots, x_n x_1, x_1', \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\sigma(x_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)
\]
and
\[
D(x_1 x_1' + x_1 x_1', x_2, \ldots, x_n) = D(x_1 x_1', x_2, \ldots, x_n) + D(x_1 x_1', x_2, \ldots, x_n)
\]
\[
= D(x_1, x_2, \ldots, x_n)\sigma(x_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)
\]
Combining above two equalities we obtain that
\[
D(x_1, x_2, \ldots, x_n)\sigma(x_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n) = \tau(x_1)D(x_1', x_2, \ldots, x_n) +
\]
\[
D(x_1, x_2, \ldots, x_n)\sigma(x_1'). \text{ Therefore, } D(x_1 x_1', x_2, \ldots, x_n) = \tau(x_1)D(x_1', x_2, \ldots, x_n) +
\]
\[
D(x_1, x_2, \ldots, x_n)\sigma(x_1'). \text{ Similarly other relations can be also proved. Converse can be}
\]
\[
\text{shown in a similar manner.} \quad \square
\]

In a left near-ring \(N\), right distributive law does not hold in general. However, we can prove the following partial distributive properties in \(N.\)

**Lemma 2.3.2.** Let \(N\) be a near-ring and \(D\) be a \((\sigma, \tau)\)-\(n\)-derivation of \(N.\) Then
\[
\{D(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)D(x_1', x_2, \ldots, x_n)\}y
\]
\[
= D(x_1, x_2, \ldots, x_n)\sigma(x_1')y + \tau(x_1)D(x_1', x_2, \ldots, x_n)y,
\]
\[
\{D(x_1, x_2, \ldots, x_n)\sigma(x_2') + \tau(x_2)D(x_1, x_2', \ldots, x_n)\}y
\]
\[
= D(x_1, x_2, \ldots, x_n)\sigma(x_2')y + \tau(x_2)D(x_1, x_2', \ldots, x_n)y,
\]
\[
\vdots
\]
24
\begin{align*}
\{D(x_1, x_2, \ldots, x_n)\sigma(x'_n) + \tau(x_n)D(x_1, x_2, \ldots, x'_n)\}y \\
= D(x_1, x_2, \ldots, x_n)\sigma(x'_n)y + \tau(x_n)D(x_1, x_2, \ldots, x'_n)y,
\end{align*}
for all \(x_1, x'_1, x_2, \ldots, x_n, y \in N\).

**Proof.** For all \(x_1, x'_1, x_2, \ldots, x_n \in N\)

\[
D((x_1x'_1)x''_1, x_2, \ldots, x_n) = D(x_1x'_1, x_2, \ldots, x_n)\sigma(x''_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)\sigma(x''_1)
\]

Also

\[
D(x_1(x'_1x''_1), x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)\sigma(x'_1)\sigma(x''_1) + \tau(x_1)\{D(x_1, x_2, \ldots, x_n)\sigma(x''_1)
\] 

\[
\sigma(x'_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)\}
\]

Combining the above two equalities, we find that

\[
\{D(x_1, x_2, \ldots, x_n)\sigma(x'_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)\}y
\]

Since \(\sigma\) is an automorphism of \(N\), replacing \((x''_1)\) by \(\sigma^{-1}(y)\), where \(y\) is an arbitrary element of \(N\), we find that

\[
\{D(x_1, x_2, \ldots, x_n)\sigma(x'_1) + \tau(x_1)D(x_1, x_2, \ldots, x_n)\}y
\]

Similarly other relations can be also proved. \(\square\)

Taking the help of Lemma 2.3.1 and using similar arguments with necessary variations as used to prove Lemma 2.3.2, one can easily obtain the following:

**Lemma 2.3.3.** Let \(N\) be a near-ring and \(D\) be a \((\sigma, \tau)-n\)-derivation of \(N\). Then

\[
\{\tau(x_1)D(x'_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\sigma(x'_1)\}y
\]

\[
= \tau(x_1)D(x'_1, x_2, \ldots, x_n)y + D(x_1, x_2, \ldots, x_n)\sigma(x'_1)y.
\]

\[
\{\tau(x_2)D(x_1, x'_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)\sigma(x'_2)\}y
\]

\[
= \tau(x_2)D(x_1, x'_2, \ldots, x_n)y + D(x_1, x_2, \ldots, x_n)\sigma(x'_2)y.
\]

\vdots

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Lemma 2.3.4. Let \( N \) be a prime near-ring, \( D \) a nonzero \((\sigma, \tau)\)-\( n \)-derivation of \( N \) and \( x \in N \).

(i) If \( D(N, N, \ldots, N)x = \{0\} \) then \( x = 0 \).

(ii) If \( xD(N, N, \ldots, N) = \{0\} \) then \( x = 0 \).

Proof. (i) For all \( x_1, x_2, \ldots, x_n \in N \), we have \( D(x_1, x_2, \ldots, x_n)x = 0 \). Taking \( x_1x_1' \) instead of \( x_1 \) and using hypothesis and Lemma 2.3.2 we get \( D(x_1, x_2, \ldots, x_n)\sigma(x_1')x = 0 \). Since \( \sigma \) is an automorphism of \( N \), we have \( D(x_1, x_2, \ldots, x_n)N \subseteq \{0\} \). But since \( D \neq 0 \) and \( N \) is a prime near-ring, we conclude that \( x = 0 \).

(ii) It can be proved in a similar way. 

\[ \square \]

In the year 2004, Ashraf et. al. [13, Theorem 3.1] proved that if a prime near-ring \( N \) admits a non-trivial \((\sigma, \tau)\)-derivation \( d \) for which \( d(N) \subseteq Z \), then \((N, +)\) is abelian. Moreover, if \( N \) is 2-torsion free and \( \sigma, \tau \) commute with \( d \), then \( N \) is a commutative ring. Later this result was generalized for symmetric bi-(\( \sigma, \tau \))-derivations in 2007 by Yılmaz Ceven [35, Theorem 1] who proved that if \( N \) is a 2-torsion free prime near-ring which admits a nonzero symmetric bi-(\( \sigma, \tau \))-derivation \( D \) such that \( D(N, N) \subseteq Z \), then \( N \) is a commutative ring. Very recently this result was generalized by Öztürk [71, Theorem 1] in the setting of permuting tri-(\( \sigma, \tau \))-derivation. In fact he proved that if \( N \) is a prime near-ring, \( D \) a nonzero permuting tri-(\( \sigma, \tau \))-derivation of \( N \) such that \( D(N, N, N) \subseteq Z \), then \( N \) is a commutative ring. We have obtained its analogue in the setting of \((\sigma, \tau)\)-\( n \)-derivation. It is also shown that symmetric and permuting properties used by above authors are redundant. In fact, we have proved the following:

Theorem 2.3.1. Let \( N \) be a prime near-ring and \( D \) a nonzero \((\sigma, \tau)\)-\( n \)-derivation of \( N \). If \( D(N, N, \ldots, N) \subseteq Z \), then \( N \) is a commutative ring.

Proof. Since \( D(N, N, \ldots, N) \subseteq Z \) and \( D \) is a nonzero \((\sigma, \tau)\)-\( n \)-derivation of \( N \), there exist nonzero elements \( x_1, x_2, \ldots, x_n \in N \), such that \( D(x_1, x_2, \ldots, x_n) \in Z \setminus \{0\} \). We have \( D(x_1 + x_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n) \in Z \). By Lemma 2.2.1(ii) we obtain that \((N, +)\) is abelian. By hypothesis we get \( D(y_1, y_2, \ldots, y_n)y = \ldots \)
yD(y_1, y_2, \cdots, y_n) for all y, y_1, y_2, \cdots, y_n \in N. Now replacing y_1 by y_1y_1' where y_1' \in N in the previous relation we have

\[ \{D(y_1, y_2, \cdots, y_n)\sigma(y_1') + \tau(y_1)D(y_1, y_2, \cdots, y_n)\}y \]

(2.3.1)

for all y, y_1', y_2, \cdots, y_n \in N. Now replacing y by \(\sigma(y_1')\) in the relation (2.3.1) and using Lemma 2.3.2 we find that \(D(y_1, y_2, \cdots, y_n)\sigma(y_1')\sigma(y_1') + \tau(y_1)D(y_1, y_2, \cdots, y_n)\sigma(y_1') = \sigma(y_1')D(y_1, y_2, \cdots, y_n)\sigma(y_1') + \sigma(y_1')\tau(y_1)D(y_1, y_2, \cdots, y_n).\) By using hypothesis again, the preceding relation reduces to \(D(y_1, y_2, \cdots, y_n)N[\sigma(y_1'), \tau(y_1)] = \{0\}\) for all \(y_1, y_1, y_2, \cdots, y_n \in N.\) Since \(N\) is a prime near-ring, we see that for each \(y_1' \in N,\) either \(D(y_1, y_2, \cdots, y_n) = 0\) or \([\sigma(y_1'), \tau(y_1)] = 0\) for all \(y_1, y_2, \cdots, y_n \in N.\) If \(D(y_1, y_2, \cdots, y_n) = 0,\) then using hypothesis, relation (2.3.1) takes the form \(D(y_1, y_2, \cdots, y_n)N[\sigma(y_1'), \tau(y_1)] = \{0\}.\) Since \(D \neq 0,\) primeness of \(N\) insures that \([y, \sigma(y_1')] = 0\) for all \(y.\) But since \(\sigma\) is an automorphism, \(y_1' \in Z.\) On the other hand if \([\sigma(y_1'), \tau(y_1)] = 0,\) then again \(y_1' \in Z\) and hence we find that \(N = Z,\) and \(N\) is a commutative ring.

The following example demonstrates that \(N\) to be prime is essential in the hypothesis of the above theorem.

**Example 2.3.1.** Suppose \(N = R[x] \times N',\) i.e.; the cartesian product of \(R[x]\) and \(N',\) where \(R[x]\) is the polynomial ring in \(x\) over the field of real numbers \(R\) and \(N'\) is a zero symmetric noncommutative prime left near-ring. It is obvious that \(N\) forms a zero symmetric left near-ring with regard to component wise addition and multiplication. Define \(D : N \times N \times \cdots \times N \rightarrow N\) such that \(D((f_1(x), a_1), (f_2(x), a_2), \cdots, (f_n(x), a_n)) = ((d(f_1(x))d(f_2(x)) \cdots d(f_n(x)), 0),\) where \(d(f_i(x)), 1 \leq i \leq n\) is the usual differentiation of \(f_i(x) \in R[x], 1 \leq i \leq n.\) It can be easily verified that \(D\) is a nonzero \((I, I)-n\)-derivation of \(N,\) where \(I\) is the identity automorphism of \(N.\) Further it can be easily shown that \(N\) is a semiprime near-ring which is not a prime and \(D(N, N, \cdots, N) \subseteq Z.\) However, \(N\) is not a commutative ring.

Let \(X\) and \(Y\) be nonempty subsets of \(N.\) The notation \([X, Y],\) used onward in this section, denotes a subset of \(N\) defined by \([X, Y] = \{[x, y] | x \in X, y \in Y\}.\)

**Theorem 2.3.2.** Let \(N\) be a prime near-ring and \(D_1, D_2\) be any two nonzero \((\sigma, \tau)-n\)-derivations of \(N.\) If \([D_1(N, N, \cdots, N), D_2(N, N, \cdots, N)] = \{0\},\) then \((N, +)\) is abelian.
Proof. Assume that $[D_1(N, N, \cdots, N), D_2(N, N, \cdots, N)] = \{0\}$. If both $z$ and $z + z$ commute element wise with $D_2(N, N, \cdots, N)$, then

$$zD_2(x_1, x_2, \cdots, x_n) = D_2(x_1, x_2, \cdots, x_n)z$$

and

$$(z + z)D_2(x_1, x_2, \cdots, x_n) = D_2(x_1, x_2, \cdots, x_n)(z + z)$$

for all $x_1, x_2, \cdots, x_n \in N$. In particular, $(z + z)D_2(x_1 + x'_1, x_2, \cdots, x_n) = D_2(x_1 + x'_1, x_2, \cdots, x_n)(z + z)$ for all $x_1, x'_1, \cdots, x_n \in N$. From the previous equalities we get $zD_2(x_1 + x'_1 - x_1 - x'_1, x_2, \cdots, x_n) = 0$ i.e.; $zD_2((x_1, x'_1), x_2, \cdots, x_n) = 0$. Putting $z = D_1(y_1, y_2, \cdots, y_n)$ we get $D_1(y_1, y_2, \cdots, y_n)D_2((x_1, x'_1), x_2, \cdots, x_n) = 0$. By Lemma 2.3.4(i) we conclude that $D_2((x_1, x'_1), x_2, \cdots, x_n) = 0$. Since we know that for each $w \in N$, $w(x_1, x'_1) = w(x_1 + x'_1 - x_1) = wx_1 + wx'_1 - wx_1 - wx'_1 = (wx_1, wx'_1)$ which is again an additive commutator of near-ring $N$, putting $w(x_1, x'_1)$ in place of additive commutator $(x_1, x'_1)$ in the relation $D_2((x_1, x'_1), x_2, \cdots, x_n) = 0$, we get $D_2(w(x_1, x'_1), x_2, \cdots, x_n) = 0$ i.e.; $D_2(w, x_2, \cdots, x_n)\sigma(x_1, x'_1) + \tau(w)D_2((x_1, x'_1), x_2, \cdots, x_n) = 0$. Previous equality yields $D_2(w, x_2, \cdots, x_n)\sigma(x_1, x'_1) = 0$. Since $\sigma$ is an automorphism, using Lemma 2.3.4(i) again we conclude that $(x_1, x'_1) = 0$. Hence $(N, +)$ is abelian. 

Theorem 2.3.3. Let $N$ be a prime near-ring with nonzero $(\sigma, \tau)$-n-derivations $D_1$ and $D_2$ such that

$$D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) = -D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. Then $(N, +)$ is abelian.

Proof. By our hypothesis we have,

$$D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n) = 0$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. Replacing $y_1$ by $y_1 + y'_1$ in the previous relation we get

$$D_1(x_1, x_2, \cdots, x_n)D_2(y_1 + y'_1, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n)D_1(y_1 + y'_1, y_2, \cdots, y_n) = 0.$$ 

Using our hypothesis again we get,

$$D_1(x_1, x_2, \cdots, x_n)D_2(-y_1, y_2, \cdots, y_n) + D_1(x_1, x_2, \cdots, x_n)D_2(y'_1, y_2, \cdots, y_n) + D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) + D_1(x_1, x_2, \cdots, x_n)D_2(-y'_1, y_2, \cdots, y_n) = 0$$

i.e.;

$$D_1(x_1, x_2, \cdots, x_n)D_2((y_1, y'_1), y_2, \cdots, y_n) = 0.$$ 

Now using Lemma 2.3.4(i) we conclude that $D_2((y_1, y'_1), y_2, \cdots, y_n) = 0$. Putting $w(y_1, y'_1)$ in place of the additive commutator $(y_1, y'_1)$ where $w \in N$ in the previous equality and using Lemma 2.3.4(i); as used in the
previous theorem, we conclude that \((N, +)\) is abelian.

**Theorem 2.3.4.** Let \(N\) be a prime near-ring with nonzero \((\sigma, \tau)\)-\(n\)-derivations \(D_1\) and \(D_2\) such that

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) + \tau D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N\). Then \((N, +)\) is abelian.

**Proof.** By assumption, we have

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) + \tau D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0
\]  \hspace{1cm} (2.3.2)

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N\). Substituting \(x + y\), where \(x, y \in N\) for \(y_1\) in the relation (2.3.2) and using it again we obtain that,

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(x, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)\sigma D_2(-y, y_2, \ldots, y_n) = 0.
\]

Now using Lemma 2.3.4(i) we conclude that \(\sigma D_2((x, y), y_2, \ldots, y_n) = 0\). Since \(\sigma\) is an automorphism of \(N\), we conclude that \(D_2((x, y), y_2, \ldots, y_n) = 0\). Now using similar arguments as used in the end of the proof of Theorem 2.3.2, we conclude that \((N, +)\) is abelian.

**Theorem 2.3.5.** Let \(N\) be a prime near-ring admitting a nonzero \((\sigma, \tau)\)-\(n\)-derivation \(D_1\) and a nonzero \(n\)-derivation \(D_2\) such that

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) + \tau D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N\). Replacing \(y_1\) by \(y_1 + y_1'\) in the previous equation we get,

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1 + y_1', y_2, \ldots, y_n) + \tau D_2(x_1, x_2, \ldots, x_n)D_1(y_1 + y_1', y_2, \ldots, y_n) = 0.
\]

Using our hypothesis again we arrive at,

\[
D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1', y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)\sigma D_2(-y_1, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)\sigma D_2(-y_1', y_2, \ldots, y_n) = 0.
\]

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i.e.; \( D_1(x_1, x_2, \ldots, x_n)\sigma D_2(y_1, y_2, \ldots, y_n) = 0 \). Now using Lemma 2.3.5(i) we find that \( \sigma D_2((y_1, y_1'), y_2, \ldots, y_n) = 0 \). But \( \sigma \) is an automorphism of \( N \), we conclude that \( D_2((y_1, y_1'), y_2, \ldots, y_n) = 0 \). Treating \( D_2 \) as \((I, I)\)-\( n \)-derivation of \( N \) where \( I \) is the identity automorphism of \( N \) and arguing on similar lines as in case of Theorem 2.3.2; we conclude that \((N, +)\) is abelian.

\[ \text{Corollary 2.3.1 (}[15, \text{Theorem 3.4}].) \text{ Let } N \text{ be a prime near-ring with nonzero permuting } n \text{-derivations } D_1 \text{ and } D_2 \text{ such that } \]
\[ D_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) + D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0 \]
\[ \text{for all } x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N. \text{ Then } (N, +) \text{ is abelian.} \]

\[ \text{Theorem 2.3.6. Let } N \text{ be a prime near-ring admitting a } (\sigma, \tau)\text{-}n\text{-derivation } D \text{ and a } (\sigma, \tau)\text{-derivation } d \text{ such that } dd = 0. \text{ Then one of the following holds: (i) } D = 0 (\text{ii) } d = 0 (\text{iii) } (N, +) \text{ is abelian.} \]

\[ \text{Proof. By our hypothesis we have, } dD(x_1, x_2, \ldots, x_n) = 0 \text{ for all } x_1, x_2, \ldots, x_n \in N. \]
Replacing \( x_1 \) by \( x_1' \) where \( x_1' \in N \) we have \( d\{D(x_1, x_2, \ldots, x_n)\sigma(x_1') + \tau(x_1)D(x_1', x_2, \ldots, x_n)\} = 0 \) i.e.; \( dD(x_1, x_2, \ldots, x_n)\sigma^2(x_1') + \tau D(x_1, x_2, \ldots, x_n)d\sigma(x_1') + d\tau(x_1)\sigma D(x_1', x_2, \ldots, x_n) + \tau^2(x_1)dD(x_1', x_2, \ldots, x_n) = 0 \). Using the hypothesis again we get

\[ \tau D(x_1, x_2, \ldots, x_n)d\sigma(x_1') + d\tau(x_1)\sigma D(x_1', x_2, \ldots, x_n) = 0 \quad (2.3.3) \]
for all \( x_1', x_1, x_2, \ldots, x_n \in N \). Replacing \( x_1' \) by \( x + y \) where \( x, y \in N \) in the above relation (2.3.3) and using it again we get \( \tau D(x_1, x_2, \ldots, x_n)d\sigma(x) + \tau D(x_1, x_2, \ldots, x_n)d\sigma(y) + \tau D(x_1, x_2, \ldots, x_n)d\sigma(-x) + \tau D(x_1, x_2, \ldots, x_n)d\sigma(-y) = 0 \) i.e.; \( \tau D(x_1, x_2, \ldots, x_n)d\sigma(x + y - x - y) = 0 \). This implies that \( D(x_1, x_2, \ldots, x_n)\tau^{-1}d\sigma(x + y - x - y) = 0 \). If \( D = 0 \), then nothing to do. Suppose that \( D \neq 0 \), hence by Lemma 2.3.4(i) we have \( \tau^{-1}d\sigma(x + y - x - y) = 0 \) i.e.; \( d\sigma(x, y) = 0 \) for all \( x, y \in N \). Since \( \sigma \) is an automorphism of \( N \). Replacing \( x \) by \( \sigma^{-1}(x) \) and \( y \) by \( \sigma^{-1}(y) \), we conclude that \( d(x, y) = 0 \) for all \( x, y \in N \). Now substituting \( w(x, y) \) where \( w \in N \), for \( (x, y) \) in the relation \( d(x, y) = 0 \) and using it again we get \( d(w)\sigma(x, y) = 0 \). Replacing \( (x, y) \) by \( v(x, y) \) where \( v \in N \) in the relation \( d(w)\sigma(x, y) = 0 \) we conclude that \( d(w)\sigma(v)\sigma(x, y) = 0 \) i.e.; \( \sigma^{-1}(d(w))N(x, y) = \{0\} \). If \( d = 0 \), then nothing to do. If \( d \neq 0 \), then primeness of \( N \) provides us \( (x, y) = 0 \) for all \( x, y \in N \). Hence \((N, +)\) is abelian.
Theorem 2.3.7. Let $N$ be a prime near-ring admitting a $(\sigma, \tau)$-n-derivation $D$ and a derivation $d$ such that $(N, +)$ is non-abelian. If $dD$ is a $(\sigma, \tau)$-n-derivation of $N$, then either $D = 0$ or $d = 0$.

Proof. Since $dD$ is a $(\sigma, \tau)$-n-derivation of $N$, $dD(x_1', x_2, \cdots, x_n) = dD(x_1, x_2, \cdots, x_n)$ for all $x_1', x_1, x_2, \cdots, x_n \in N$. On the other hand, we also have

$$dD(x_1x_1', x_2, \cdots, x_n) = d\{D(x_1, x_2, \cdots, x_n)\sigma(x_1') + \tau(x_1)dD(x_1', x_2, \cdots, x_n)\}$$

for all $x_1', x_1, x_2, \cdots, x_n \in N$. Now comparing the above two values of $dD(x_1x_1', x_2, \cdots, x_n)$ we obtain that

$$D(x_1, x_2, \cdots, x_n)\sigma(x_1') + \tau(x_1)dD(x_1', x_2, \cdots, x_n) = 0 \quad (2.3.4)$$

for all $x_1', x_1, x_2, \cdots, x_n \in N$. Replacing $x_1'$ by $x + y$ where $x, y \in N$ in the above relation (2.3.4) and using it again we get $D(x_1, x_2, \cdots, x_n)d\sigma(x) + D(x_1, x_2, \cdots, x_n)d\sigma(y) + D(x_1, x_2, \cdots, x_n)d\sigma(-x) + D(x_1, x_2, \cdots, x_n)d\sigma(-y) = 0$. i.e.; $D(x_1, x_2, \cdots, x_n)d\sigma(x + y - x - y) = 0$. If $D = 0$, then nothing to do. Suppose that $D \neq 0$. Hence by Lemma 2.3.4(i) we have $d\sigma(x + y - x - y) = 0$ i.e.; $d\sigma(x, y) = 0$ for all $x, y \in N$. Since $\sigma$ is an automorphism of $N$, we conclude that $d(x, y) = 0$ for all $x, y \in N$. Now substituting $w(x, y)$ where $w \in N$, for $(x, y)$ in the relation $d(x, y) = 0$ and using it again we get $d(w)(x, y) = 0$. Replacing $(x, y)$ by $v(x, y)$ where $v \in N$ in the relation $d(w)(x, y) = 0$ we conclude that $d(w)v(x, y) = 0$ i.e.; $d(w)N(x, y) = \{0\}$. Then primeness of $N$ provides us either (i) $d = 0$ or (ii) $(N, +)$ is abelian, a contradiction to the assumption. The proof is now complete.

Theorem 2.3.8. Let $N$ be a semiprime near-ring and $D$ a $(\sigma, \tau)$-n-derivation of $N$. If $D(x_1, x_2, \cdots, x_n)\sigma(y_1) = \tau(x_1)D(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$, then $D = 0$.

Proof. We have

$$D(x_1, x_2, \cdots, x_n)\sigma(y_1) = \tau(x_1)D(y_1, y_2, \cdots, y_n). \quad (2.3.5)$$

Putting $y_1z_1$ in place of $y_1$ in the above equation, where $z_1 \in N$, we get
\[ D(x_1, x_2, \cdots, x_n)\sigma(y_1 z_1) = \tau(x_1)D(y_1 z_1, y_2, \cdots, y_n) \\
= \tau(x_1)D(y_1, y_2, \cdots, y_n)\sigma(z_1) + \tau(x_1)\tau(y_1)D(z_1, y_2, \cdots, y_n). \]

By equation (2.3.5) we get \[ D(x_1, x_2, \cdots, x_n)\sigma(y_1)\sigma(z_1) = D(x_1, x_2, \cdots, x_n)\sigma(y_1)\sigma(z_1) + \tau(x_1)\tau(y_1)D(z_1, y_2, \cdots, y_n). \] This yields that \[ \tau(x_1)\tau(y_1)D(z_1, y_2, \cdots, y_n) = 0. \] Since \( \tau \) is an automorphism of \( N \), we get \( uvD(z_1, y_2, \cdots, y_n) = 0 \) where \( u, v \in N \). Now replacing \( u \) by \( D(z_1, y_2, \cdots, y_n) \) we infer that \( D(z_1, y_2, \cdots, y_n)ND(z_1, y_2, \cdots, y_n) = \{0\} \). Finally by semiprimeness of \( N \), we conclude that \( D = 0. \)

**Corollary 2.3.2 ([15, Theorem 3.6]).** Let \( N \) be a semiprime near-ring and \( D \) be a permuting \( n \)-derivation of \( N \). If \( D(x_1, x_2, \cdots, x_n)y_1 = x_1D(y_1, y_2, \cdots, y_n) \), for all \( x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n \in N \), then \( D = 0 \).

The following example demonstrates that \( N \) to be prime and semiprime is essential in the hypothesis of the Theorems 2.3.2-2.3.8.

**Example 2.3.2.** Let \( S \) be a commutative near-ring, and let

\[ N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}. \]

Define \( D_1, D_2, D : N \times N \times \cdots \times N \to N \) such that

\[
D_1 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2 \cdots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
D_2 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & y_1y_2 \cdots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
D \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & x_1x_2 \cdots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Also define $d_1, d_2, \tau : N \rightarrow N$ such that

\[
\begin{pmatrix}
0 & x & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & x \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & x & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and

\[
\tau \begin{pmatrix}
0 & x & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & y & x \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

It can be easily seen that $\tau$ is an automorphism of near-ring $N$ which is not semiprime, having $d_1$ and $d_2$ as nonzero $(\sigma, \tau)$-derivation and nonzero derivation respectively. Further it can be easily shown that $D_1, D_2$ are nonzero $(\sigma, \tau)$-n-derivations and $D$ a nonzero n-derivation of $N$ where $\sigma = I$, the identity automorphism of $N$. We also have (i) $[D_1(N, N, \cdots, N), D_2(N, N, \cdots, N)] = \{0\}$, (ii) $D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) = -D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)$, (iii) $D_1(x_1, x_2, \cdots, x_n)\sigma D_2(y_1, y_2, \cdots, y_n) + \tau D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n) = 0$, (iv) $D_1(x_1, x_2, \cdots, x_n)\sigma D(y_1, y_2, \cdots, y_n) + \tau D(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n) = 0$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. However $(N, +)$ is not abelian. It can be also noted that $N$ satisfies (v) $d_1d_2 = 0$ (vi) $d_2d_2$ is a $(\sigma, \tau)$-n-derivation of $N$ (vii) $D_1(x_1, x_2, \cdots, x_n)\sigma(y_1) = \tau(x_1)D_1(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$.

**Theorem 2.3.9.** Let $N$ be a prime near-ring and $D$ a nonzero $(\sigma, \tau)$-n-derivation of $N$. If $K = \{a \in N \mid [D(N, N, \cdots, N), \tau(a)] = \{0\}\}$, then $a \in K$ implies either $a \in Z$ or $D(a, a, \cdots, a) = 0$.

**Proof.** We have

\[
D(x_1, x_2, \cdots, x_n)\tau(a) = \tau(a)D(x_1, x_2, \cdots, x_n)
\]

for all $x_1, x_2, \cdots, x_n \in N$. Putting $ax_1$ in place of $x_1$ in the above equation and using Lemma 2.3.2, we get $D(a, x_2, \cdots, x_n)\sigma(x_1)\tau(a) + \tau(a)D(a, x_2, \cdots, x_n)\sigma(x_1) = \tau(a)D(a, x_2, \cdots, x_n)\sigma(x_1) + \tau(a)\tau(a)D(x_1, x_2, \cdots, x_n)$. Using the equation (2.3.6), we get $D(a, x_2, \cdots, x_n)\sigma(x_1)\tau(a) = \tau(a)D(a, x_2, \cdots, x_n)\sigma(x_1)$. Now putting $x_1y_1$ for $x_1$ in the latter relation and using it again, we have $D(a, x_2, \cdots, x_n)\sigma(x_1)[\sigma(y_1), \tau(a)] = 0$, where $y_1 \in N$. Since $\sigma$ is an automorphism, we have $D(a, x_2, \cdots, x_n)N[\sigma(y_1), \tau(a)] = \{0\}$ and the primeness of $N$ implies that either $[\sigma(y_1), \tau(a)] = 0$ for all $y_1 \in N$ or $D(a, x_2, \cdots, x_n) = 0$ for all $x_2, \cdots, x_n \in N$. If first holds then $\tau(a) \in Z$ due to the fact that $\sigma$ is an
automorphism. This implies that \(a \in \mathbb{Z}\). If second case holds then \(D(a, x_2, \cdots, x_n) = 0\), for all \(x_2, x_3, \cdots, x_n \in N\) and hence in particular, \(D(a, a, \cdots, a) = 0\).

**Theorem 2.3.10.** Let \(N\) be a prime near-ring admitting a \((\sigma, \tau)\)-\(n\)-derivation \(D\) and \(a \in N\). If \([D(N, N, \cdots, N), a]_{\sigma, \tau} = \{0\}\) then \(D(a, x_2, \cdots, x_n) = 0\) for all \(x_2, x_3, \cdots, x_n \in N\) or \(a \in \mathbb{Z}\).

**Proof.** By hypothesis, we have \(D(ax, x_2, \cdots, x_n)\sigma(a) = \tau(a)D(ax, x_2, \cdots, x_n)\) for all \(x \in N\) and so by Lemma 2.3.2,

\[
D(a, x_2, \cdots, x_n)\sigma(x)\sigma(a) + \tau(a)D(x, x_2, \cdots, x_n)\sigma(a) \\
= \tau(a)D(a, x_2, \cdots, x_n)\sigma(x) + \tau(a)\tau(a)D(x, x_2, \cdots, x_n)
\]

Using hypothesis again we have

\[
D(a, x_2, \cdots, x_n)\sigma(x)\sigma(a) = \tau(a)D(a, x_2, \cdots, x_n)\sigma(x)
\]

i.e.,

\[
D(a, x_2, \cdots, x_n)[\sigma(x), \sigma(a)] = 0 \quad (2.3.7)
\]

for all \(x \in N\). Substituting \(xy\), where \(y \in N\) for \(x\) in the relation (2.3.7) and using it again we get \(D(a, x_2, \cdots, x_n)\sigma(x)[\sigma(y), \sigma(a)] = 0\) i.e., \(D(a, x_2, \cdots, x_n)N[\sigma(y), \sigma(a)] = \{0\}\) for all \(y, x_2, x_3, \cdots, x_n \in N\). Since \(\sigma\) is an automorphism of prime near-ring \(N\), we get \(D(a, x_2, \cdots, x_n) = 0\) for all \(x_2, x_3, \cdots, x_n \in N\) or \(a \in \mathbb{Z}\). This completes the proof.

**Corollary 2.3.3** ([46, Theorem 6]). Let \(D\) be a nonzero \((\sigma, \tau)\)-derivation of a prime near-ring \(N\) and \(a \in N\). If \([D(N), a]_{\sigma, \tau} = 0\), then \(D(a) = 0\) or \(a \in \mathbb{Z}\).

### 2.4 Generalized \(n\)-derivation in near-rings

Let \(N\) be a near-ring with derivation \(d\). An additive mapping \(f : N \rightarrow N\) is called a **right generalized derivation** (resp. **left generalized derivation**) of \(N\) if there exists a derivation \(d\) of \(N\) such that \(f(xy) = f(x)y + xd(y)\) (resp. \(f(xy) = d(x)y + xf(y)\)) for all \(x, y \in N\). Finally an additive mapping \(f : N \rightarrow N\) is called a **generalized derivation** of \(N\) if there exists a derivation \(d\) of \(N\) such that \(f(xy) = f(x)y + xd(y)\) for all \(x, y \in N\) and \(f(xy) = d(x)y + xf(y)\) for all \(x, y \in N\).

Very recently M. A. Öztürk gave the notion of permuting tri-generalized derivation in near-rings as follows: A permuting tri-additive (i.e; additive in all three arguments) mapping \(F : N \times N \times N \rightarrow N\) is called a **permuting tri-right** (resp. **left**) generalized
derivation of $N$ associated with a permuting tri-derivation $D : N \times N \times N \rightarrow N$ if

$$F(xw, y, z) = F(x, y, z)w + xD(w, y, z)$$

(resp. $F(xw, y, z) = D(x, y, z)w + xF(w, y, z)$) holds for all $x, y, z, w \in N$. Finally $F$ is said to be a permuting tri-generalized derivation of $N$ associated with a permuting tri-derivation $D : N \times N \times N \rightarrow N$ if it is both permuting tri-right and permuting tri-left generalized derivation of $N$ associated with $D$.

Now motivated by above notions and that of $n$-derivation and permuting $n$-derivation of a near-ring in the present section we introduce and study the notion of generalized $n$-derivation in near-rings as following.

Definition 2.4.1. An $n$-additive mapping $F : N \times N \times \cdots \times N \rightarrow N$ is called a right generalized $n$-derivation of $N$ with associated $n$-derivation $D$ if the relations

$$F(x_1, x_1', x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1, x_2, \cdots, x_n)$$

$$F(x_1, x_2, x_2', \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x_2' + x_2D(x_1, x_2', \cdots, x_n)$$

$$\vdots$$

$$F(x_1, x_2, \cdots, x_n x_n') = F(x_1, x_2, \cdots, x_n)x_n' + x_nD(x_1, x_2, \cdots, x_n')$$

hold for all $x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in N$.

If in addition both $F$ and $D$ are permuting maps then $F$ is called a permuting right generalized $n$-derivation of $N$ with associated permuting $n$-derivation $D$.

An $n$-additive mapping $F : N \times N \times \cdots \times N \rightarrow N$ is called a left generalized $n$-derivation of $N$ with associated $n$-derivation $D$ if the relations

$$F(x_1, x_1', x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1' + x_1F(x_1', x_2, \cdots, x_n)$$

$$F(x_1, x_2, x_2', \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_2' + x_2F(x_1, x_2', \cdots, x_n)$$

$$\vdots$$

$$F(x_1, x_2, \cdots, x_n x_n') = D(x_1, x_2, \cdots, x_n)x_n' + x_nF(x_1, x_2, \cdots, x_n')$$

hold for all $x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in N$.

If in addition both $F$ and $D$ are permuting maps then $F$ is called a permuting left generalized $n$-derivation of $N$ with associated permuting $n$-derivation $D$.

An $n$-additive mapping $F : N \times N \times \cdots \times N \rightarrow N$ is called a generalized $n$-derivation of $N$ with associated $n$-derivation $D$ if it is both a right generalized $n$-derivation as well
as a left generalized n-derivation of N with associated n-derivation D. If in addition both F and D are permuting maps then F is called a permuting generalized n-derivation of N with associated permuting n-derivation D.

Example 2.4.1. Let n be a fixed positive integer, $S$ a commutative left near-ring.

(i) Consider $N_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}$. Obviously it is a noncommutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_1 : N_1 \times N_1 \times \cdots \times N_1 \rightarrow N_1$ such that

$$D_1 \left( \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & a_1a_2 \cdots a_n \\ 0 & 0 \end{pmatrix}.$$ 

It is easy to see that $D_1$ is an n-derivation of $N_1$. Define $F_1 : N_1 \times N_1 \times \cdots \times N_1 \rightarrow N_1$ such that

$$F_1 \left( \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & b_1b_2 \cdots b_n \\ 0 & 0 \end{pmatrix}.$$ 

It can be easily verified that $F_1$ is a left generalized n-derivation of $N_1$ with associated n-derivation $D_1$ but not a right generalized n-derivation of $N_1$ with associated n-derivation $D_1$. It can be also seen that $F_1$ is a permuting left generalized n-derivation of $N_1$ with associated permuting n-derivation $D_1$ but not a permuting right generalized n-derivation of $N_1$ with associated permuting n-derivation $D_1$.

(ii) Consider $N_2 = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \mid c, d, 0 \in S \right\}$. It can be easily shown that $N_2$ is a noncommutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_2 : N_2 \times N_2 \times \cdots \times N_2 \rightarrow N_2$ such that

$$D_2 \left( \begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix} \right) = \begin{pmatrix} 0 & c_1c_2 \cdots c_n \\ 0 & 0 \end{pmatrix}.$$ 

It is easy to see that $D_2$ is an n-derivation of $N_2$. Define $F_2 : N_2 \times N_2 \times \cdots \times N_2 \rightarrow N_2$ such that

$$F_2 \left( \begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & d_1d_2 \cdots d_n \end{pmatrix}.$$ 

It can be easily verified that $F_2$ is a right generalized n-derivation of $N_2$ with associated n-derivation $D_2$ but not a left generalized n-derivation of $N_2$ with associated n-derivation
D2. It can be also seen that $F_2$ is a permuting right generalized $n$-derivation of $N_2$ with associated permuting $n$-derivation $D_2$ but not a permuting left generalized $n$-derivation of $N_2$ with associated permuting $n$-derivation $D_2$.

(iii) Consider $N_3 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in S \right\}$. It can be easily seen that $N_3$ is a noncommutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_3 : N_3 \times N_3 \times \cdots \times N_3 \to N_3$ such that

$$D_3 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1 x_2 \cdots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $D_3$ is an $n$-derivation of $N_3$. Define $F_3 : N_3 \times N_3 \times \cdots \times N_3 \to N_3$ such that

$$F_3 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that $F_3$ is a generalized $n$-derivation (i.e.; both left generalized $n$-derivation and right generalized $n$-derivation) of $N_3$ with associated $n$-derivation $D_3$. It can be also easily seen that $F_3$ is permuting generalized $n$-derivation with associated permuting $n$-derivation $D_3$.

It is to be noted that if in the above examples we take $S$ to be a distributive near-ring, then $F_1$, $F_2$ and $F_3$ become left generalized $n$-derivation, right generalized $n$-derivation and generalized $n$-derivation associated with $n$-derivations $D_1$, $D_2$ and $D_3$ respectively. However these are not permuting left generalized $n$-derivation, permuting right generalized $n$-derivation and permuting generalized $n$-derivation respectively.

Recently many authors have studied commutativity of rings satisfying certain properties and identities involving derivations, generalized derivations, permuting $n$-derivations etc. (see for detail reference [2], [9], [28], [43], [54], [57], [72], [77]). Motivated by these results commutativity behavior of prime near-rings satisfying certain identities involving derivations, generalized derivations, permuting tri-generalized derivations, permuting $n$-derivations etc. have also been investigated by several authors (see [15], [24], [21], [47], [48], [49], [71] where further references can be found). Now our objective is to
study the commutativity behavior of prime near-rings which admit suitably constrained generalized n-derivations.

Now we state the following lemmas which will be used in proving our main results of this section. The proof of the Lemma 2.4.1 can be seen in [15].

**Lemma 2.4.1.** Let $D$ be a nonzero permuting n-derivation of prime near-ring $N$ such that $D(N, N, \ldots, N) \subseteq Z$. Then $N$ is a commutative ring.

**Remark 2.4.1.** It can be easily shown that above Lemma 2.4.1 also holds if $D$ is a nonzero n-derivation of prime near-ring $N$.

**Lemma 2.4.2.** $F$ is a right generalized n-derivation of $N$ with associated n-derivation $D$ if and only if

\[
F(x_1, x_2', \ldots, x_n) = x_1 D(x_1, x_2', \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_1'
\]

\[
F(x_1, x_2, \ldots, x_n) = x_2 D(x_1, x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_2
\]

\[
F(x_1, x_2, \ldots, x_n, x_n') = x_n D(x_1, x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_n'
\]

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

**Proof.** Let $F$ be a right generalized n-derivation of $N$ with associated n-derivation $D$. Then $F(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1' + x_1 D(x_1, x_2, \ldots, x_n)$, for all $x_1, x_1', x_2, \ldots, x_n \in N$.

Consider

\[
F(x_1(x_1' + x_1'), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(x_1' + x_1') + x_1 D(x_1', x_1', x_1, x_2, \ldots, x_n)
\]

\[
= F(x_1, x_2, \ldots, x_n) x_1' + F(x_1, x_2, \ldots, x_n)x_1' + x_1 D(x_1', x_1, x_2, \ldots, x_n) + x_1 D(x_1', x_1, x_2, \ldots, x_n).
\]

Also

\[
F(x_1(x_1' + x_1'), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1' + F(x_1, x_2, \ldots, x_n) x_1' + x_1 D(x_1', x_1, x_2, \ldots, x_n)
\]

Combining the above two equalities we find that $F(x_1, x_2, \ldots, x_n)x_1' + x_1 D(x_1', x_2, \ldots, x_n) = x_1 D(x_1', x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_1'$, for all $x_1, x_1'$.
Lemma 2.4.3. Let $N$ be a near-ring admitting a right generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$. Then,

$$
\{ F(x_1, x_2, \cdots, x_n)x'_1 + x_1D(x_1', x_2, \cdots, x_n) \}y = F(x_1, x_2, \cdots, x_n)x'_1y + x_1D(x_1', x_2, \cdots, x_n)y.
$$

$$
\{ F(x_1, x_2, \cdots, x_n)x'_2 + x_2D(x_1, x_2', \cdots, x_n) \}y = F(x_1, x_2, \cdots, x_n)x'_2y + x_2D(x_1, x_2', \cdots, x_n)y.
$$

$$\vdots$$

$$
\{ F(x_1, x_2, \cdots, x_n)x'_n + x_nD(x_1, x_2, \cdots, x_n') \}y = F(x_1, x_2, \cdots, x_n)x'_ny + x_nD(x_1, x_2, \cdots, x_n')y,
$$

hold for all $x_1, x_2, x_2', \cdots, x_n, x'_n, y \in N$.

Proof. For all $x_1, x'_1, x_2, \cdots, x_n \in N$,

$$
F((x_1x'_1)x_1')x_1'' + x_1D(x_1', x_2, \cdots, x_n)x_1'' = F(x_1, x_2, \cdots, x_n)x'_1 + (x_1x'_1)D(x_1'', x_2, \cdots, x_n)
$$

$$
= \{ F(x_1, x_2, \cdots, x_n)x'_1 + x_1D(x_1', x_2, \cdots, x_n) \}x_1'' + (x_1x'_1)D(x_1'', x_2, \cdots, x_n).
$$

Also

$$
F(x_1(x'_1x_1''), x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x'_1x'' + x_1D(x'_1, x'_2, \cdots, x_n)
$$

$$
= F(x_1, x_2, \cdots, x_n)x'_1x'' + x_1 \{ D(x'_1, x'_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n) \}
$$

$$
= F(x_1, x_2, \cdots, x_n)x'_1x'' + x_1D(x'_1, x_2, \cdots, x_n)x_1'' + x_1x_1'D(x'_1', x_2, \cdots, x_n).
$$

Combining the above two relations, we get

$$
\{ F(x_1, x_2, \cdots, x_n)x'_1 + x_1D(x_1', x_2, \cdots, x_n) \}x_1'' = F(x_1, x_2, \cdots, x_n)x'_1x'' + x_1D(x_1', x_2, \cdots, x_n)x_1'.'
$$
Putting $y$ in place of $x_1''$, we find that
\[
\{F(x_1, x_2, \cdots, x_n)x_1' + x_1 D(x_1', x_2, \cdots, x_n)\}y = F(x_1, x_2, \cdots, x_n)x_1'y
\]
\[+ x_1 D(x_1', x_2, \cdots, x_n)y.
\]

Similarly other $(n - 1)$ relations can be proved.

Using Lemma 2.4.2 and similar techniques as used to prove the above lemma, one can easily get the following:

**Lemma 2.4.4.** Let $N$ be a near-ring admitting a right generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$. Then,
\[
\{x_1 D(x_1', x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_1'\}y = x_1 D(x_1', x_2, \cdots, x_n)y
\]
\[+ F(x_1, x_2, \cdots, x_n)x_1'y,
\]
\[
\{x_2 D(x_1', x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_2'\}y = x_2 D(x_1', x_2, \cdots, x_n)y
\]
\[+ F(x_1, x_2, \cdots, x_n)x_2'y,
\]
\[
\vdots
\]
\[
\{x_n D(x_1', x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_n'\}y = x_n D(x_1', x_2, \cdots, x_n)y
\]
\[+ F(x_1, x_2, \cdots, x_n)x_n'y,
\]
hold for all $x_1, x_1', x_2, x_2', \cdots, x_n, x_n', y \in N$.

**Lemma 2.4.5.** $F$ is a left generalized $n$-derivation of $N$ with associated $n$-derivation $D$ if and only if
\[
F(x_1' x_1, x_2, \cdots, x_n) = x_1 F(x_1', x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_1',
\]
\[
F(x_1 x_2', x_2, \cdots, x_n) = x_2 F(x_1', x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_2',
\]
\[
\vdots
\]
\[
F(x_1 x_2 \cdots x_n') = x_n F(x_1, x_2, \cdots, x_n') + D(x_1, x_2, \cdots, x_n)x_n'
\]
hold for all $x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in N$.

**Proof.** Use same arguments as used in the proof of Lemma 2.4.2. \qed
Lemma 2.4.6. Let $N$ be a near-ring admitting a generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$. Then,

\[
\{D(x_1, x_2, \cdots, x_n)x'_1 + x_1 F(x'_1, x_2, \cdots, x_n)\}y = D(x_1, x_2, \cdots, x_n)x'_1y + x_1 F(x'_1, x_2, \cdots, x_n)y,
\]

\[
\{D(x_1, x_2, \cdots, x_n)x'_2 + x_2 F(x'_1, x_2', \cdots, x_n)\}y = D(x_1, x_2, \cdots, x_n)x'_2y + x_2 F(x'_1, x_2, \cdots, x_n)y,
\]

\[
\{D(x_1, x_2, \cdots, x_n)x'_n + x_n F(x_1, x_2, \cdots, x'_n)\}y = D(x_1, x_2, \cdots, x_n)x'_ny + x_n F(x_1, x_2, \cdots, x'_n)y,
\]

hold for all $x_1, x_1', x_2, x_2', \cdots, x_n, x_n', y \in N$.

Proof. For all $x_1, x_1', x_2, x_2', \cdots, x_n \in N$,

\[
F((x_1 x_1')x''_1, x_2, \cdots, x_n) = F(x_1 x'_1, x_2, \cdots, x_n) x''_1 + (x_1 x'_1) D(x''_1, x_2, \cdots, x_n)
\]

\[
= \{D(x_1, x_2, \cdots, x_n)x'_1 + x_1 F(x'_1, x_2, \cdots, x_n)\}x''_1
\]

\[
+ (x_1 x'_1) D(x''_1, x_2, \cdots, x_n).
\]

Also

\[
F(x_1(x'_1) x''_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x'_1 x''_1 + x_1 F(x'_1, x''_1, x_2, \cdots, x_n)
\]

\[
= D(x_1, x_2, \cdots, x_n)x'_1 x''_1 + x_1 \{F(x'_1, x''_1, x_2, \cdots, x_n) x''_1
\]

\[
+ x'_1 D(x''_1, x_2, \cdots, x_n)\}
\]

\[
= D(x_1, x_2, \cdots, x_n)x'_1 x''_1 + x_1 F(x'_1, x''_1, x_2, \cdots, x_n) x''_1
\]

\[
+ x'_1 D(x''_1, x_2, \cdots, x_n).
\]

Combining the above two relations, we get

\[
\{D(x_1, x_2, \cdots, x_n)x'_1 + x_1 F(x'_1, x_2, \cdots, x_n)\} x''_1 = D(x_1, x_2, \cdots, x_n)x'_1 x''_1
\]

\[
+ x_1 F(x'_1, x_2, \cdots, x_n)x''_1.
\]

Putting $y$ in place of $x''_1$, we find that

\[
\{D(x_1, x_2, \cdots, x_n)x'_1 + x_1 F(x'_1, x_2, \cdots, x_n)\}y = D(x_1, x_2, \cdots, x_n)x'_1 y
\]

\[
+ x_1 F(x'_1, x_2, \cdots, x_n)y.
\]
Similarly other \((n - 1)\) relations can be shown.

Lemma 2.4.7. Let \(N\) be a near-ring admitting a generalized \(n\)-derivation \(F\) with associated \(n\)-derivation \(D\) of \(N\). Then,

\[
\{x_1 F(x_1', x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_1\}y = x_1 F(x_1', x_2, \ldots, x_n)y + D(x_1, x_2, \ldots, x_n)x_1'y,
\]

\[
\{x_2 F(x_1, x_2', \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_2\}y = x_2 F(x_1, x_2', \ldots, x_n)y + D(x_1, x_2, \ldots, x_n)x_2'y,
\]

\[\vdots\]

\[
\{x_n F(x_1, x_2, \ldots, x_n') + D(x_1, x_2, \ldots, x_n)x_n\}y = x_n F(x_1, x_2, \ldots, x_n')y + D(x_1, x_2, \ldots, x_n)x_n'y,
\]

hold for all \(x_1, x_1', x_2, \ldots, x_n, x_n', y \in N\).

Proof. Using Lemmas 2.4.2, 2.4.5 and the same trick as used in the proof of above lemma, one can get its proof easily.

Lemma 2.4.8. Let \(N\) be prime near-ring admitting a generalized \(n\)-derivation \(F\) with associated nonzero \(n\)-derivation \(D\) of \(N\) and \(x \in N\).

(i) If \(xF(N, N, \ldots, N) = \{0\}\), then \(x = 0\).

(ii) If \(F(N, N, \ldots, N)x = \{0\}\), then \(x = 0\).

Proof. (i) Given that \(xF(x_1', x_2, \ldots, x_n) = 0\) for all \(x_1, x_1', \ldots, x_n \in N\). This yields that \(x\{F(x_1, x_2, \ldots, x_n)x_1 + D(x_1', x_2, \ldots, x_n)\} = 0\). By hypothesis we have \(xND(x_1', x_2, \ldots, x_n) = \{0\}\). But since \(N\) is a prime near ring and \(D \neq 0\), we have \(x = 0\).

(ii) It can be proved in a similar way by using Lemma 2.4.6.

Lemma 2.4.9. Let \(N\) be near-ring admitting a generalized \(n\)-derivation \(F\) with associated \(n\)-derivation \(D\) of \(N\). Then \(F(Z, N, N, \ldots, N) \subseteq Z\).

Proof. Let \(z \in Z\), then \(F(zr_1, r_2, \ldots, r_n) = F(r_1z, r_2, \ldots, r_n)\) for all \(r_1, r_2, \ldots, r_n \in N\). Using Lemma 2.4.5 we have \(F(z, r_2, \ldots, r_n)r_1 + zD(r_1, r_2, \ldots, r_n) = r_1 F(z, r_2, \ldots, r_n) + D(r_1, r_2, \ldots, r_n)z\). Which in turn gives us \(F(z, r_2, \ldots, r_n)r_1 = r_1 F(z, r_2, \ldots, r_n)\) i.e.; \(F(Z, N, N, \ldots, N) \subseteq Z\).
In the year 2006, Öznur Gölbaşı [47, Theorem 2.6] proved that if \( N \) is a prime near-ring with a nonzero generalized derivation \( f \) such that \( f(N) \subseteq Z \), then \( (N, +) \) is abelian. Moreover if \( N \) is 2-torsion free, then \( N \) is a commutative ring. The following result shows that "2-torsion free restriction" in the above result used by Öznur Gölbaşı is superfluous. In fact, for generalized \( n \)-derivation in a prime near-ring \( N \) we have obtained the following:

**Theorem 2.4.1.** Let \( N \) be a prime near-ring admitting a nonzero generalized \( n \)-derivation \( F \) with associated \( n \)-derivation \( D \) of \( N \). If \( F(N, N, \cdots, N) \subseteq Z \), then \( N \) is a commutative ring.

**Proof.** For all \( x_1, x_1', \cdots, x_n \in N \)

\[
F(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n) x_1' + x_1 F(x_1', x_2, \cdots, x_n) \in Z. \tag{2.4.1}
\]

Hence \( \{D(x_1, x_2, \cdots, x_n)x_1' + x_1 F(x_1', x_2, \cdots, x_n)\} x_1 = x_1 \{D(x_1, x_2, \cdots, x_n)x_1' + x_1 F(x_1', x_2, \cdots, x_n)\} \). By hypothesis and Lemma 2.4.6 we obtain \( D(x_1, x_2, \cdots, x_n)x_1' = x_1 D(x_1, x_2, \cdots, x_n)x_1' \), putting \( x_1' y \) where \( y \in N \) for \( x_1' \) in the preceding relation and using it again we get \( D(x_1, x_2, \cdots, x_n)x_1'(yx_1 - x_1y) = 0 \) i.e.; \( D(x_1, x_2, \cdots, x_n)N(yx_1 - x_1y) = \{0\} \). But primeness of \( N \) yields that for each fixed \( x_1 \) either \( x_1 \in Z \) or \( D(x_1, x_2, \cdots, x_n) = 0 \) for all \( x_2, x_3, \cdots, x_n \in N \). If first case holds then \( D(x_1 t, x_2, \cdots, x_n) = D(t x_1, x_2, \cdots, x_n) \) for all \( t, x_2, \cdots, x_n \in N \). Using Lemma 2.2.3 and Remark 2.2.2 we obtain that

\[
D(x_1, x_2, \cdots, x_n)t + x_1 D(t, x_2, \cdots, x_n) = tD(x_1, x_2, \cdots, x_n) + D(t, x_2, \cdots, x_n)x_1
\]

for all \( t, x_2, \cdots, x_n \in N \) i.e.; \( D(x_1, x_2, \cdots, x_n) \in Z \) and second case implies \( D(x_1, x_2, \cdots, x_n) = 0 \) i.e.; \( 0 = D(x_1, x_2, \cdots, x_n) \in Z \). Including both the cases we get \( D(x_1, x_2, \cdots, x_n) \in Z \) for all \( x_1, x_2, \cdots, x_n \in N \) i.e.; \( D(N, N, \cdots, N) \subseteq Z \). If \( D \neq 0 \), then by Lemma 2.4.1 and Remark 2.4.1, \( N \) is a commutative ring. On the other hand if \( D = 0 \), then equation (2.4.1) takes the form \( F(x_1 x_1', x_2, \cdots, x_n) = x_1 F(x_1', x_2, \cdots, x_n) \) for all \( x_1, x_1', \cdots, x_n \in N \). By hypothesis and Lemma 2.2.2, \( x_1 \in Z \) i.e.; \( N = Z \). Thus we conclude that \( N \) is a commutative near-ring. Since \( N \neq \{0\} \), there exists \( 0 \neq p \in N = Z \) such that \( p + p \in N = Z \). By Lemma 2.2.1(ii) we find that \( N \) is a commutative ring. □

**Corollary 2.4.1 (15, Theorem 3.2).** Let \( N \) be a prime near-ring admitting a nonzero permuting \( n \)-derivation \( D \) such that \( D(N, N, \cdots, N) \subseteq Z \). Then \( N \) is a commutative ring.
Recently Öznur Gölbasi [48, Theorems 3.1. & 3.2.] showed that if $f$ is a generalized derivation of a prime near-ring $N$ with associated nonzero derivation $d$ such that $f([x, y]) = 0$ for all $x, y \in N$ or $f([x, y]) = \pm [x, y]$ for all $x, y \in N$, then $N$ is a commutative ring. While proving the theorem it has been assumed that $f$ is a left generalized derivation with associated nonzero derivation $d$. We have extended these results in the setting of left generalized $n$-derivations in prime near-rings by establishing the following theorems.

**Theorem 2.4.2.** Let $N$ be a prime near-ring admitting a left generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F([x, y], r_2, r_3, \ldots, r_n) = 0$ for all $x, y, r_2, r_3, \ldots, r_n \in N$, then $N$ is commutative ring.

**Proof.** Since $F([x, y], r_2, \ldots, r_n) = 0$, substituting $xy$ for $y$ we obtain $F(x[x, y], r_2, \ldots, r_n) = 0$ i.e.; $D(x, r_2, \ldots, r_n)[x, y] + xF([x, y], r_2, \ldots, r_n) = 0$. By hypothesis we get $D(x, r_2, \ldots, r_n)[x, y] = 0$ that is,

$$D(x, r_2, \ldots, r_n)xy = D(x, r_2, \ldots, r_n)yx. \quad (2.4.2)$$

Putting $yz$ for $y$ in $(2.4.2)$ and using it again we have $D(x, r_2, \ldots, r_n)y(xz - zx) = 0$ i.e.; $D(x, r_2, \ldots, r_n)N[x, z] = \{0\}$. For each fixed $x \in N$ primeness of $N$ yields either $x \in Z$ or $D(x, r_2, \ldots, r_n) = 0$ for all $r_2, \ldots, r_n \in N$. If the first case holds then $D(x, r_2, \ldots, r_n) = D(tx, r_2, \ldots, r_n)$ for all $t, r_2, \ldots, r_n \in N$. Using Lemma 2.2.3 and Remark 2.2.2, we obtain that $D(x, r_2, \ldots, r_n)t + xD(t, r_2, \ldots, r_n) = xD(x, r_2, \ldots, r_n) + D(t, r_2, \ldots, r_n)x$ for all $t, r_2, \ldots, r_n \in N$ i.e.; $D(x, r_2, \ldots, r_n) \in Z$ and second case implies $D(x, r_2, \ldots, r_n) = 0$ i.e.; $0 = D(x, r_2, \ldots, r_n) \in Z$. Including both the cases we get $D(x, r_2, \ldots, r_n) \in Z$ for all $x, r_2, \ldots, r_n \in N$ i.e.; $D(N, N, \ldots, N) \subseteq Z$. Hence by Lemma 2.4.1 and Remark 2.4.1, $N$ is a commutative ring. \qed

**Theorem 2.4.3.** Let $N$ be a prime near-ring admitting a left generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F([x, y], r_2, r_3, \ldots, r_n) = \pm [x, y]$ for all $x, y, r_2, r_3, \ldots, r_n \in N$, then $N$ is commutative ring.

**Proof.** Since $F([x, y], r_2, \ldots, r_n) = \pm [x, y]$, substituting $xy$ for $y$ we obtain $F(x[x, y], r_2, \ldots, r_n) = \pm x[x, y]$ i.e.; $D(x, r_2, \ldots, r_n)[x, y] + xF([x, y], r_2, \ldots, r_n) = \pm x[x, y]$. By hypothesis we get $D(x, r_2, \ldots, r_n)[x, y] = 0$ that is,

$$D(x, r_2, \ldots, r_n)xy = D(x, r_2, \ldots, r_n)yx.$$

This is identical with $(2.4.2)$ of Theorem 2.4.2. Now arguing in the same way as in Theorem 2.4.2, we conclude that $N$ is a commutative ring. \qed
The conclusion of Theorems 2.4.2 and 2.4.3 remain valid if we replace the Lie product $[x, y]$ by the Jordan product $xoy$. In fact, we obtained the following results.

**Theorem 2.4.4.** Let $N$ be a prime near-ring admitting a left generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F(xoy, r_2, r_3, \cdots, r_n) = 0$ for all $x, y, r_2, r_3, \cdots, r_n \in N$, then $N$ is commutative ring.

**Proof.** Given that $F(xoy, r_2, \cdots, r_n) = 0$. Substituting $xy$ for $y$ we obtain $F(x(xoy), r_2, \cdots, r_n)(xoy) + xF(xoy, r_2, \cdots, r_n) = 0$. By hypothesis we get $D(x, r_2, \cdots, r_n)(xoy) = 0$, that is,

$$D(x, r_2, \cdots, r_n)xy = -D(x, r_2, \cdots, r_n)yx. \quad (2.4.3)$$

Putting $yz$ for $y$ in (2.4.3) we have $D(x, r_2, \cdots, r_n)xyz = -D(x, r_2, \cdots, r_n)yxz$ i.e.;

$$D(x, r_2, \cdots, r_n)xyz + D(x, r_2, \cdots, r_n)yzx = 0. \quad \text{(2.4.4)}$$

Now substituting the values from (2.4.3) in the preceding relation we get

$$\{-D(x, r_2, \cdots, r_n)yzx + D(x, r_2, \cdots, r_n)yxz = 0 \} \text{ that is } D(x, r_2, \cdots, r_n)y(zx-zx) = 0 \text{ or } D(-x, r_2, \cdots, r_n)N[x, z] = \{0\}. \quad \text{(2.4.5)}$$

For each fixed $x \in N$ primeness of $N$ yields either $x \in Z$ or $D(-x, r_2, \cdots, r_n) = 0$. If the first case holds then $D(x, r_2, \cdots, r_n) = D(t, r_2, \cdots, r_n)$ for all $t, r_2, \cdots, r_n \in N$. Using Lemma 2.2.3 and Remark 2.2.2, we obtain that $D(x, r_2, \cdots, r_n)t + xD(t, r_2, \cdots, r_n) = tD(x, r_2, \cdots, r_n) + D(t, r_2, \cdots, r_n)x$ for all $t, r_2, \cdots, r_n \in N$ i.e.; $D(x, r_2, \cdots, r_n) \in Z$ and the second case implies $-D(x, r_2, \cdots, r_n) = 0$ i.e.; $0 = D(x, r_2, \cdots, r_n) \in Z$. Combining both the cases we get $D(x, r_2, \cdots, r_n) \in Z$ for all $x, r_2, \cdots, r_n \in N$ i.e.; $D(N, N, \cdots, N) \subseteq Z$. Hence by Lemma 2.4.1 and Remark 2.4.1, $N$ is a commutative ring.

**Theorem 2.4.5.** Let $N$ be a prime near-ring admitting a left generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F(xoy, r_2, r_3, \cdots, r_n) = \pm(xoy)$ for all $x, y, r_2, r_3, \cdots, r_n \in N$, then $N$ is a commutative ring.

**Proof.** We have $F(xoy, r_2, \cdots, r_n) = \pm(xoy)$. Substituting $xy$ for $y$ we obtain $F(x(xoy), r_2, \cdots, r_n) = \pm x(xoy)$ i.e.;

$$D(x, r_2, \cdots, r_n)(xoy) + xF(xoy, r_2, \cdots, r_n) = \pm x(xoy). \quad \text{(2.4.6)}$$

By hypothesis we get

$$D(x, r_2, \cdots, r_n)(xoy) = 0 \quad \text{that is, } D(x, r_2, \cdots, r_n)xy = -D(x, r_2, \cdots, r_n)y. \quad \text{(2.4.7)}$$

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This is identical with equation (2.4.3) of Theorem 2.4.4 and hence using same arguments, as in the proof of Theorem 2.4.4, we conclude that \( N \) is a commutative ring.

**Theorem 2.4.6.** Let \( N \) be a prime near-ring admitting a left generalized \( n \)-derivation \( F \) with associated nonzero \( n \)-derivation \( D \) of \( N \). If \( F([x, y], r_2, r_3, \cdots, r_n) = \pm(xoy) \) for all \( x, y, r_2, r_3, \cdots, r_n \in N \), then \( N \) is a commutative ring.

**Proof.** We have \( F([x, y], r_2, \cdots, r_n) = \pm(xoy) \). Substituting \( xy \) for \( y \) we obtain \( F(x[x, y], r_2, \cdots, r_n) = \pm(xoy) \). By hypothesis we get \( D(x, r_2, \cdots, r_n)[x, y] = 0 \), that is, \( D(x, r_2, \cdots, r_n)xy = D(x, r_2, \cdots, r_n)yx \), which is identical with equation (2.4.2) of Theorem 2.4.4 and using similar arguments, we conclude that \( N \) is a commutative ring.

**Theorem 2.4.7.** Let \( N \) be a prime near-ring admitting a left generalized \( n \)-derivation \( F \) with associated nonzero \( n \)-derivation \( D \) of \( N \). If \( F(xoy, r_2, r_3, \cdots, r_n) = \pm[x, y] \) for all \( x, y, r_2, r_3, \cdots, r_n \in N \), then \( N \) is a commutative ring.

**Proof.** Since \( F(xoy, r_2, \cdots, r_n) = \pm[x, y] \). Substituting \( xy \) for \( y \) we obtain \( F(x(xoy), r_2, \cdots, r_n) = \pm[x, y] \). By hypothesis we get \( D(x, r_2, \cdots, r_n)(xoy) = 0 \) that is, \( D(x, r_2, \cdots, r_n)xy = -D(x, r_2, \cdots, r_n)yx \), which is identical with (2.4.3) of Theorem 2.4.4. Now using similar arguments, we conclude that \( N \) is a commutative ring.

**Theorem 2.4.8.** Let \( N \) be a prime near-ring admitting a generalized \( n \)-derivation \( F \) with associated nonzero \( n \)-derivation \( D \) of \( N \). If \( F([x, y], r_2, r_3, \cdots, r_n) \in Z \) for all \( x, y, r_2, r_3, \cdots, r_n \in N \), then \( N \) is commutative ring or \( D(Z, N, N, \cdots, N) = \{0\} \).

**Proof.** For all \( x, y, r_2, r_3, \cdots, r_n \in N \),

\[ F([x, y], r_2, \cdots, r_n) \in Z. \]  \hspace{1cm} (2.4.4)

Now we have two cases,

Case I: If \( Z = \{0\} \), it follows \( F([x, y], r_2, \cdots, r_n) = 0 \) for all \( x, y, r_2, r_3, \cdots, r_n \in N \). Now by Theorem 2.4.3 we conclude that \( N \) is a commutative ring.

Case II: If \( Z \neq \{0\} \), replacing \( y \) by \( yz \) in (2.4.4), where \( z \in Z \), we get \( D(z, r_2, \cdots, r_n)[x, y] + zF([x, y], r_2, \cdots, r_n) \in Z \) for all \( x, y, r_2, r_3, \cdots, r_n \in N, z \in Z \). Using (2.4.4) together with Lemma 2.4.6, preceding relation forces \( D(z, r_2, \cdots, r_n)[x, y] \in Z \). Since \( z \in Z \),

\[ D(z, r_2, \cdots, r_n) = D(tz, r_2, \cdots, r_n) \] for all \( t, r_2, \cdots, r_n \in N \). Using Lemma 2.2.3 and Remark 2.2.2, we obtain that \( D(z, r_2, \cdots, r_n)t + zD(t, r_2, \cdots, r_n) = tD(z, r_2, \cdots, r_n) + zD(t, r_2, \cdots, r_n) \).
Now we infer that $D(2, r_2, \ldots, r_n)z$ for all $t, r_2, \ldots, r_n \in N$. If $D(Z, N, N, \ldots, N) \neq \{0\}$ then by Lemma 2.2.1(i), we have $[x, y, t] = 0$ i.e.; $[x, y] \in Z$. Now replacing $y$ by $xy$ in the preceding relation $[x, y, t] = 0$, we have $[x, y][x, t] = 0$ which in turn gives us $[x, y]N[x, t] = \{0\}$. In particular, we have $[x, y]N[x, y] = \{0\}$. In light of primeness of $N$ we obtain that $[x, y] = 0$ and hence $N$ is a commutative near-ring i.e; $N = Z$. Since $N \neq \{0\}$, there exists $p \in N \setminus \{0\}$. Hence $p + p \in N = Z$ and by Lemma 2.2.1(ii), we conclude that $N$ is a commutative ring.

**Theorem 2.4.9.** Let $N$ be a 2-torsion free prime near-ring admitting a generalized $n$-derivation $F$ with associated nonzero $n$-derivation $D$ of $N$. If $F(xoy, r_2, r_3, \ldots, r_n) \in Z$ for all $x, y, r_2, r_3, \ldots, r_n \in N$, then $N$ is a commutative ring or $D(Z, N, N, \ldots, N) = \{0\}$.

**Proof.** For all $x, y, r_2, r_3, \ldots, r_n \in N$,

$$F(xoy, r_2, \ldots, r_n) \in Z. \quad (2.4.5)$$

Now we separate the proof in two cases,

Case I: If $Z = \{0\}$, it follows $F(xoy, r_2, \ldots, r_n) = 0$ for all $x, y, r_2, r_3, \ldots, r_n \in N$. Hence by Theorem 2.4.4 we conclude that $N$ is a commutative ring.

Case II: If $Z \neq \{0\}$, replacing $y$ by $y^2$ in (2.4.5), where $z \in Z$, we get $D(z, r_2, \ldots, r_n)(xoy) + zF(xoy, r_2, \ldots, r_n) \in Z$ for all $x, y, r_2, r_3, \ldots, r_n \in N, z \in Z$. Using (2.4.5) together with Lemma 2.4.6, preceding relation forces $D(z, r_2, \ldots, r_n)(xoy) \in Z$. Since $z \in Z$, $D(2, r_2, \ldots, r_n) = D(2, r_2, \ldots, r_n)$ for all $t, r_2, \ldots, r_n \in N$. Using Lemma 2.2.3 and Remark 2.2.2, we obtain that $D(z, r_2, \ldots, r_n)t + zD(t, r_2, \ldots, r_n) = tD(z, r_2, \ldots, r_n) + D(t, r_2, \ldots, r_n)z$ for all $t, r_2, \ldots, r_n \in N$ i.e.; $D(z, r_2, \ldots, r_n) \in Z$ and hence we infer that $D(z, r_2, \ldots, r_n)[xoy, t] = 0$ for all $t \in N$. But if $D(Z, N, N, \ldots, N) \neq \{0\}$ then by Lemma 2.2.1(i) we have $[xoy, t] = 0$ i.e., $(xoy) \in Z$. Let $0 \neq y \in Z$. Hence $xoy = y(x + x), x^2oy = y(x^2 + x^2)$, it follows by Lemma 2.2.2 that $x + x \in Z, x^2 + x^2 \in Z$ for all $x \in N$. Thus $(x + x)xt = x(x + x)t = (x^2 + x^2)t = tx(x + x) = (x + x)tx$ for all $x, t \in N$ and therefore $(x + x)N[x, t] = \{0\}$ for all $x, t \in N$. Once again using primeness, we get $x \in Z$ or $2x = 0$ in latter case 2-torsion freeness forces $x = 0$. Consequently, in both the cases we arrive at $x \in Z$ i.e.; $N = Z$ and therefore $N$ is a commutative near-ring. Since $N \neq \{0\}$, there exists $p \in N \setminus \{0\}$. Hence $p + p \in N = Z$ and by Lemma 2.2.1(ii), we conclude that $N$ is a commutative ring.

Very recently Öznur Gölbasi [49, Theorem 3.1] proved that if $N$ is a semi prime near-
ring and \( f \) is a nonzero generalized derivation on \( N \) with an associated derivation \( d \) such that \( f(x)y = xf(y) \) for all \( x, y \in N \), then \( d = 0 \). While proving the theorem it has been assumed that \( f \) is a right generalized derivation of \( N \) with associated derivation \( d \). We have extended this result in the setting of generalized \( n \)-derivation. In fact, we have proved the following:

**Theorem 2.4.10.** Let \( N \) be a semiprime near-ring admitting a generalized \( n \)-derivation \( F \) with associated \( n \)-derivation \( D \) of \( N \). If \( F(x_1, x_2, \ldots, x_n)y_i = x_1F(y_1, y_2, \ldots, y_n) \) for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N \), then \( D = 0 \).

**Proof.** We have

\[
F(x_1, x_2, \ldots, x_n)y_i = x_1F(y_1, y_2, \ldots, y_n). \tag{2.4.6}
\]

Putting \( x_1z_1 \) in place of \( x_1 \) in the above identity (2.4.6), where \( z_1 \in N \) and using Lemma 2.4.6, we get

\[
x_1z_1F(y_1, y_2, \ldots, y_n) = F(x_1z_1, x_2, \ldots, x_n)y_i.
\]

By (2.4.6) we find that

\[
x_1z_1F(y_1, y_2, \ldots, y_n) = D(x_1, x_2, \ldots, x_n)z_1y_i + x_1F(z_1, x_2, \ldots, x_n)y_i.
\]

This yields that

\[
D(x_1, x_2, \ldots, x_n)z_1y_i = 0.
\]

Now replacing \( y_i \) by \( D(x_1, x_2, \ldots, x_n) \) we get \( D(x_1, x_2, \ldots, x_n) \) \( ND(x_1, x_2, \ldots, x_n) = \{0\} \). But since \( N \) is a semiprime near-ring, we conclude that \( D = 0 \). \( \Box \)

**Corollary 2.4.2 ([15, Theorem 3.6]).** Let \( N \) be a semiprime near-ring and \( D \) a permuting \( n \)-derivation of \( N \). If \( D(x_1, x_2, \ldots, x_n)y_i = x_1D(y_1, y_2, \ldots, y_n) \), for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N \), then \( D = 0 \).

**Theorem 2.4.11.** Let \( N \) be a prime near-ring admitting a generalized \( n \)-derivation \( F \) with associated \( n \)-derivation \( D \) of \( N \). If \( d \) is the trace of \( D \) and \( K = \{a \in N \mid [F(N, N, \ldots, N), a] = \{0\}\} \), then

(i) \( a \in K \) implies either \( a \in Z \) or \( d(a) = 0 \),

(ii) \( d(K) \subseteq Z \).

**Proof.** (i) We have

\[
F(x_1, x_2, \ldots, x_n)a = aF(x_1, x_2, \ldots, x_n) \tag{2.4.7}
\]
for all $x_1, x_2, \cdots, x_n \in N$. Putting $ax_1$ in place of $x_1$ in the above equation and using Lemma 2.4.6 we get $D(a, x_2, \cdots, x_n) x_1 a + aF(x_1, x_2, \cdots, x_n) a = aD(a, x_2, \cdots, x_n)x_1 + aaF(x_1, x_2, \cdots, x_n)$. Using the identity (2.4.7), we get $D(a, x_2, \cdots, x_n) x_1 a = aD(a, x_2, \cdots, x_n)x_1$. Now putting $x_1 y_1$ for $x_1$ in the latter relation and using it again, we have $D(a, x_2, \cdots, x_n) x_1 [y_1, a] = 0$ where $y_1 \in N$. This gives us $D(a, x_2, \cdots, x_n) N[y_1, a] = \{0\}$. Since $N$ is a prime near-ring, either $[a, y_1] = 0$ for all $y_1 \in N$ or $D(a, x_2, \cdots, x_n) = 0$ for all $x_2, \cdots, x_n \in N$. If the first holds then $a \in Z$. If not then $D(a, x_2, \cdots, x_n) = 0$, and hence in particular, $D(a, a, \cdots, a) = 0$ or $d(a) = 0$.

(ii) From the above proof we observe that if $a \in K$ then either $a \in Z$ or $d(a) = 0$. But $d(a) = 0$ implies $d(a) \in Z$. If $d(a) \neq 0$ then we have $a \in Z$. In this case we have $D(xa, a, \cdots, a) = D(ax, a, \cdots, a)$ for all $x \in N$. Using Lemma 2.2.3 and Remark 2.2.2, we obtain that $xD(a, a, \cdots, a) + D(x, a, \cdots, a)a = D(a, a, \cdots, a)x + aD(x, a, \cdots, a)$. This reduces to $xD(a, a, \cdots, a) = D(a, a, \cdots, a)x$, which shows that $d(a) \in Z$ and thus $d(K) \subseteq Z$. □

**Corollary 2.4.3** ([15, Theorem 3.7]). Let $N$ be a prime near-ring and $D$ be a nonzero permuting $n$-derivation of $N$. If $K = \{a \in N \mid [D(N, N, \cdots, N), a] = \{0\}\}$ and $d$ stands for the trace of $D$, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$,

(ii) $d(K) \subseteq Z$.

**Corollary 2.4.4** ([48, Theorem 3.6]). If $f$ is a generalized derivation of prime near-ring $N$ with associated nonzero derivation $d$, $a \in N$ and $[f(x), a] = 0$ for all $x \in N$, then $d(a) \in Z$.

**Theorem 2.4.12.** Let $N$ be a prime near-ring admitting a generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$ such that $D(Z, N, \cdots, N) \neq \{0\}$ and $a \in N$. If $[F(N, N, \cdots, N), a] = \{0\}$ for all $x \in N$, then $a \in Z$.

**Proof.** Since $D(Z, N, \cdots, N) \neq \{0\}$, there exist $c \in Z, r_2, \cdots, r_n \in N$ all being nonzero such that $D(c, r_2, \cdots, r_n) \neq 0$. Furthermore, as $D$ is an $n$-derivation of $N$ and $c \in Z$, $D(ct, r_2, \cdots, r_n) = D(tc, r_2, \cdots, r_n)$ for all $t \in N$. By Lemma 2.2.3 and Remark 2.2.2, we infer that $D(c, r_2, \cdots, r_n)t + cD(t, r_2, \cdots, r_n) = tD(c, r_2, \cdots, r_n) + D(t, r_2, \cdots, r_n)c$ for all $t \in N$ i.e.; $D(c, r_2, \cdots, r_n) \in Z$. By hypothesis $F(cx, r_2, \cdots, r_n)a = aF(cx, r_2, \cdots, r_n)$ for all $x \in N$. Using Lemma 2.4.6 and Remark 2.2.2, we get $D(a, x_2, \cdots, x_n)x_1 a + aF(x_1, x_2, \cdots, x_n)a = aD(a, x_2, \cdots, x_n)x_1 + aaF(x_1, x_2, \cdots, x_n)$. Using the identity (2.4.7), we get $D(a, x_2, \cdots, x_n)x_1 a = aD(a, x_2, \cdots, x_n)x_1$. Now putting $x_1 y_1$ for $x_1$ in the latter relation and using it again, we have $D(a, x_2, \cdots, x_n)x_1 [y_1, a] = 0$ where $y_1 \in N$. This gives us $D(a, x_2, \cdots, x_n) N[y_1, a] = \{0\}$. Since $N$ is a prime near-ring, either $[a, y_1] = 0$ for all $y_1 \in N$ or $D(a, x_2, \cdots, x_n) = 0$ for all $x_2, \cdots, x_n \in N$. If the first holds then $a \in Z$. If not then $D(a, x_2, \cdots, x_n) = 0$, and hence in particular, $D(a, a, \cdots, a) = 0$ or $d(a) = 0$.

(ii) From the above proof we observe that if $a \in K$ then either $a \in Z$ or $d(a) = 0$. But $d(a) = 0$ implies $d(a) \in Z$. If $d(a) \neq 0$ then we have $a \in Z$. In this case we have $D(xa, a, \cdots, a) = D(ax, a, \cdots, a)$ for all $x \in N$. Using Lemma 2.2.3 and Remark 2.2.2, we obtain that $xD(a, a, \cdots, a) + D(x, a, \cdots, a)a = D(a, a, \cdots, a)x + aD(x, a, \cdots, a)$. This reduces to $xD(a, a, \cdots, a) = D(a, a, \cdots, a)x$, which shows that $d(a) \in Z$ and thus $d(K) \subseteq Z$. □

**Corollary 2.4.3** ([15, Theorem 3.7]). Let $N$ be a prime near-ring and $D$ be a nonzero permuting $n$-derivation of $N$. If $K = \{a \in N \mid [D(N, N, \cdots, N), a] = \{0\}\}$ and $d$ stands for the trace of $D$, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$,

(ii) $d(K) \subseteq Z$.

**Corollary 2.4.4** ([48, Theorem 3.6]). If $f$ is a generalized derivation of prime near-ring $N$ with associated nonzero derivation $d$, $a \in N$ and $[f(x), a] = 0$ for all $x \in N$, then $d(a) \in Z$.

**Theorem 2.4.12.** Let $N$ be a prime near-ring admitting a generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$ such that $D(Z, N, \cdots, N) \neq \{0\}$ and $a \in N$. If $[F(N, N, \cdots, N), a] = \{0\}$ for all $x \in N$, then $a \in Z$.

**Proof.** Since $D(Z, N, \cdots, N) \neq \{0\}$, there exist $c \in Z, r_2, \cdots, r_n \in N$ all being nonzero such that $D(c, r_2, \cdots, r_n) \neq 0$. Furthermore, as $D$ is an $n$-derivation of $N$ and $c \in Z$, $D(ct, r_2, \cdots, r_n) = D(tc, r_2, \cdots, r_n)$ for all $t \in N$. By Lemma 2.2.3 and Remark 2.2.2, we infer that $D(c, r_2, \cdots, r_n)t + cD(t, r_2, \cdots, r_n) = tD(c, r_2, \cdots, r_n) + D(t, r_2, \cdots, r_n)c$ for all $t \in N$ i.e.; $D(c, r_2, \cdots, r_n) \in Z$. By hypothesis $F(cx, r_2, \cdots, r_n)a = aF(cx, r_2, \cdots, r_n)$
for all $x \in N$ using Lemma 2.4.6 we have
\[ D(c, r_2, \ldots, r_n)x + cF(x, r_2, \ldots, r_n)a = aD(c, r_2, \ldots, r_n)x + acF(x, r_2, \ldots, r_n). \]
Since both $D(c, r_2, \ldots, r_n)\) and $c$ are elements of $Z$, using the hypothesis again previous equation takes the form
\[ D(c, r_2, \ldots, r_n)[x, a] = 0 \]
i.e., $D(c, r_2, \ldots, r_n)N[x, a] = \{0\}$. By primeness of $N$ and $0 \not= D(c, r_2, \ldots, r_n)$, we obtain that $a \in Z$.

**Corollary 2.4.5 ([48, Theorem 3.5])**. If $f$ is a generalized derivation of prime near-ring $N$ with associated nonzero derivation $d$ such that $d(Z) \not= \{0\}$, and $a \in N$, $[f(x), a] = 0$ for all $x \in N$, then $a \in Z$.

**Theorem 2.4.13**. Let $N$ be a prime near-ring admitting a generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$ such that $D(Z, N, \ldots, N) \not= \{0\}$. If $G : N \times N \times \cdots \times N \to N$ is a map such that $[F(N, N, \ldots, N), G(N, N, \ldots, N)] = \{0\}$, then $G(N, N, \ldots, N) \subseteq Z$.

**Proof.** Taking $G(N, N, \ldots, N)$ instead of $a$ in Theorem 2.4.12, we get the required result.

**Theorem 2.4.14**. Let $N$ be a prime near-ring admitting a generalized $n$-derivation $F$ with associated $n$-derivation $D$ of $N$ such that $D(Z, N, \ldots, N) \not= \{0\}$. If $G$ is a nonzero generalized $n$-derivation of $N$ such that $[F(N, N, \ldots, N), G(N, N, \ldots, N)] = \{0\}$, then $N$ is a commutative ring.

**Proof.** Since $G$, a nonzero generalized $n$-derivation is a map from $N \times N \times \cdots \times N$ to $N$.
Therefore by Theorem 2.4.13 we get $G(N, N, \ldots, N) \subseteq Z$. Thus $N$ is a commutative ring by Theorem 2.4.1.

**Theorem 2.4.15**. Let $F$ and $G$ be generalized $n$-derivations of prime near-ring $N$ with associated nonzero $n$-derivations $D$ and $H$ of $N$ respectively such that
\[ F(x_1, x_2, \ldots, x_n)H(y_1, y_2, \ldots, y_n) = -G(x_1, x_2, \ldots, x_n)D(y_1, y_2, \ldots, y_n) \]
for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$. Then $(N, +)$ is an abelian group.

**Proof.** For all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$ we have,
\[ F(x_1, x_2, \ldots, x_n)H(y_1, y_2, \ldots, y_n) = -G(x_1, x_2, \ldots, x_n)D(y_1, y_2, \ldots, y_n). \]
We substitute $y_1 + y'_1$ for $y_1$ in preceding relation thereby obtaining,
\[ F(x_1, x_2, \ldots, x_n)H(y_1 + y'_1, y_2, \ldots, y_n) = 0 \]
i.e.,
\[ F(x_1, x_2, \ldots, x_n)H(y_1, y_2, \ldots, y_n) + F(x_1, x_2, \ldots, x_n)H(y'_1, y_2, \ldots, y_n) + G(x_1, x_2, \ldots, x_n)D(y_1, y_2, \ldots, y_n) + G(x_1, x_2, \ldots, x_n)D(y'_1, y_2, \ldots, y_n) \]

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\[ D(y_1', y_2, \cdots, y_n) = 0. \] Using the hypothesis we get, \[ F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) + F(x_1, x_2, \cdots, x_n)H(y_1', y_2, \cdots, y_n) - F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) - F(x_1, x_2, \cdots, x_n)H((y_1, y_1'), y_2, \cdots, y_n) = 0. \] Now using Lemma 2.4.8(ii) we get \[ H((y_1, y_1'), y_2, \cdots, y_n) = 0. \] Replacing additive commutator \((y_1, y_1')\) by \(w(y_1, y_1')\) where \(w \in N\) in the previous relation and using it again we have \[ H(w, y_2, \cdots, y_n)(y_1, y_1') = 0 \] for all \(w, y_1, y_1', y_2, \cdots, y_n \in N\). Since \(H \neq 0\), by Lemma 2.2.4(i) and Remark 2.2.2, we conclude that \((y_1, y_1') = 0\), i.e.; \((N, +)\) is an abelian group.

**Corollary 2.4.6** ([15, Theorem 3.4]). Let \(N\) be a prime near-ring with nonzero permuting \(n\)-derivations \(D_1\) and \(D_2\) such that
\[
D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) = -D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)
\]
for all \(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N\). Then \((N, +)\) is an abelian group.

**Theorem 2.4.16.** Let \(F_1\) and \(F_2\) be generalized \(n\)-derivations of prime near-ring \(N\) with associated nonzero \(n\)-derivations \(D_1\) and \(D_2\) of \(N\) respectively such that \([F_1(N, N, \cdots, N), F_2(N, N, \cdots, N)] = \{0\}\). Then \((N, +)\) is an abelian group.

**Proof.** If both \(z\) and \(z + z\) commute element wise with \(F_2(N, N, \cdots, N)\), then \(zF_2(x_1, x_2, \cdots, x_n) = F_2(x_1, x_2, \cdots, x_n)z\) and \((z + z)F_2(x_1, x_2, \cdots, x_n) = F_2(x_1, x_2, \cdots, x_n)(z + z)\) for all \(x_1, x_2, \cdots, x_n \in N\). In particular, \((z + z)F_2(x_1 + x_1', x_2, \cdots, x_n) = F_2(x_1 + x_1', x_2, \cdots, x_n)(z + z)\) for all \(x_1, x_1', \cdots, x_n \in N\). From the previous equalities we get \(zF_2(x_1 + x_1' - x_1 - x_1', x_2, \cdots, x_n) = 0\), i.e.; \(zF_2((x_1, x_1'), x_2, \cdots, x_n) = 0\). Putting \(z = F_1(y_1, y_2, \cdots, y_n)\) we get \(F_1(y_1, y_2, \cdots, y_n)F_2((x_1, x_1'), x_2, \cdots, x_n) = 0\). By Lemma 2.4.8(ii) we conclude that \(F_2((x_1, x_1'), x_2, \cdots, x_n) = 0\). Putting \(w(x_1, x_1')\) in place of additive commutator \((x_1, x_1')\) where \(w \in N\) we have \(F_2(w(x_1, x_1'), x_2, \cdots, x_n) = 0\) i.e.; \(D_2(w, x_2, \cdots, x_n)(x_1, x_1') + wF_2((x_1, x_1'), x_2, \cdots, x_n) = 0\). Previous equality yields \(D_2(w, x_2, \cdots, x_n)(x_1, x_1') = 0\). By Lemma 2.2.4(i) and Remark 2.2.2, we conclude that \((N, +)\) is an abelian group.

**Corollary 2.4.7** ([15, Theorem 3.3]). Let \(N\) be a prime near-ring and \(D_1\) and \(D_2\) be any two nonzero permuting \(n\)-derivations of \(N\). If \([D_1(N, N, \cdots, N), D_2(N, N, \cdots, N)] = \{0\}\), then \((N, +)\) is an abelian group.
Chapter 3
Semigroup ideals and derivations in near-rings

3.1 Introduction

A nonempty subset $U$ of $N$ is called a semigroup left ideal (resp. semigroup right ideal) if $NU \subseteq U$ (resp. $UN \subseteq U$) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. Let $I$ be a nonempty subset of $N$ then a normal subgroup $(I,+)$ of $(N,+)$ is called a right ideal (resp. a left ideal) of $N$ if $(x + i)y - xy \in I$ for all $x, y \in N$ and for all $i \in I$ (resp. $xi \in I$ for all $i \in I$ and $x \in N$). $I$ is called an ideal of $N$ if it is both a left ideal as well as a right ideal of $N$.

In Section 3.1, we investigate the commutativity of addition and multiplication of prime near-rings satisfying certain identities involving $n$-derivations on semigroup ideals and ideals. Furthermore, we study the conditions with semigroup ideals for $n$-derivations $D_1$ and $D_2$ of $N$ which imply that $D_1 = D_2$.

Section 3.2 is devoted to the study of the commutativity of prime near-rings satisfying certain identities involving generalized derivations on semigroup ideals or ideals. Furthermore, we give examples to show that the restrictions imposed on the hypothesis of various theorems are not superfluous. The last section of this chapter deals with the notion of "involution" in near-rings. Besides other results, it has been shown that under certain restrictions every near-ring with involution is a ring.
3.2 Semigroup ideals and \( n \)-derivations in near-rings

We are well aware that there exist several results in the existing literature which assert that prime near-rings with certain constrained derivation have ring-like behavior. Recently several authors (see [15], [24], [21], [22], for reference where further references can be found) have investigated commutativity of prime near-rings satisfying certain identities on some appropriate subsets of \( N \). Motivated by these results now we shall consider \( n \)-derivation on a near-ring \( N \) and show that prime near-rings involving \( n \)-derivations and satisfying some identities on semigroup ideals or ideals are commutative rings.

We begin with the following lemmas which are necessary for developing the proofs of our main results of this section. Proofs of first three lemmas can be found in [21].

Lemma 3.2.1. Let \( N \) be a prime near-ring.

(i) If \( U \) is a nonzero semigroup right ideal (resp. semigroup left ideal) and \( x \) is an element of \( N \) such that \( Ux = \{0\} \) (resp. \( xU = \{0\} \)) then \( x = 0 \).

(ii) If \( U \) is a nonzero semigroup right ideal and \( x \) is an element of \( N \) which centralizes \( U \), then \( x \in Z \).

Lemma 3.2.2. Let \( N \) be a prime near-ring and \( U \) a nonzero semigroup ideal of \( N \). If \( x, y \in N \) and \( xUy = \{0\} \), then \( x = 0 \) or \( y = 0 \).

Lemma 3.2.3. Let \( N \) be a prime near-ring. If \( Z \) contains a nonzero semigroup left ideal or semigroup right ideal, then \( N \) is a commutative ring.

In the year 1994 it was proved by X.K.Wang [86, Lemma 2] that if a near-ring \( N \) admits a derivation \( d \) then \( d(Z) \subseteq Z \). We have extended this result in the setting of \( n \)-derivation in a near-ring \( N \) as given below. It can be obtained as a corollary of Lemma 2.4.9.

Lemma 3.2.4. Let \( D \) be a \( n \)-derivation of a near-ring \( N \). Then \( D(Z, N, ..., Z) \subseteq Z \).

Lemma 3.2.5. Let \( N \) be a prime near-ring and \( D \) a nonzero \( n \)-derivation of \( N \).

(i) If \( U_1, U_2, ..., U_n \) are nonzero semigroup right ideals (resp., semigroup left ideals) and \( (x_1, x_2, ..., x_n) \in (N, N, ..., N) \) such that \( (U_1, U_2, ..., U_n)(x_1, x_2, ..., x_n) = \{(0, 0, ..., 0)\} \) (resp. \( (x_1, x_2, ..., x_n)(U_1, U_2, ..., U_n) = \{(0, 0, ..., 0)\} \) ) then \( (x_1, x_2, ..., x_n) = (0, 0, ..., 0) \).
(ii) If \( U_1, U_2, \ldots, U_n \) are nonzero semigroup right ideals or nonzero semigroup left ideals then \( D(U_1, U_2, \ldots, U_n) \neq \{0\} \).

(iii) If \( U_1, U_2, \ldots, U_n \) are nonzero semigroup right ideals and \( (x_1, x_2, \ldots, x_n) \in (N, N, \cdots, N) \) which centralizes \( (U_1, U_2, \ldots, U_n) \), then \( (x_1, x_2, \ldots, x_n) \in (Z, Z, \cdots, Z) \).

(iv) Suppose that \( U_1, U_2, \ldots, U_n \) are nonzero semigroup ideals of \( N \). If \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in (N, N, \cdots, N) \) and \( (x_1, x_2, \ldots, x_n)(U_1, U_2, \ldots, U_n)(y_1, y_2, \ldots, y_n) = \{(0, 0, \ldots, 0)\} \), also if \( x_1 \neq 0, x_2 \neq 0, \ldots, x_n \neq 0 \), then \( (y_1, y_2, \ldots, y_n) = (0, 0, \ldots, 0) \).

Proof. (i) Let \( N \) be a prime near-ring. It is obvious that \( (N, N, \cdots, N) \) also forms a near-ring with respect to componentwise addition and component wise multiplication. If \( (U_1, U_2, \ldots, U_n)(x_1, x_2, \ldots, x_n) = \{(0, 0, \ldots, 0)\} \) then we obtain \( (u_1, u_2, \ldots, u_n)(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0) \) for all \( u_i \in U_i, 1 \leq i \leq n \). This implies that \( (u_1x_1, u_2x_2, \ldots, u_nx_n) = (0, 0, \ldots, 0) \) i.e; \( u_1x_1 = 0, u_2x_2 = 0, \ldots, u_nx_n = 0 \) for all \( u_i \in U_i, 1 \leq i \leq n \). Since \( u_1x_1 = 0 \) for all \( u_1 \in U_1 \) and \( x_1 \in N \), replacing \( u_1 \) in the preceding relation by \( u_1r \) where \( r \in N \) we have \( u_1nx_1 = \{0\} \). But \( N \) is a prime near-ring and \( U_1 \neq \{0\} \), we conclude that \( x_1 = 0 \). Similarly we can prove that \( x_2 = 0, x_3 = 0, \ldots, x_n = 0 \), so we lastly get the required result \( (x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0) \).

(ii) Assume \( D(U_1, U_2, \ldots, U_n) = \{0\} \). This gives us that

\[
D(u_1, u_2, \ldots, u_n) = 0 \tag{3.2.1}
\]

for all \( u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \). Putting \( u_1r_1 \), where \( r_1 \in N \), for \( u_1 \) in the relation(3.2.1) and using it again we have \( u_1D(r_1, u_2, \ldots, u_n) = 0 \). Now replacing \( u_1 \) by \( u_1r \) where \( r \in N \) in the preceding relation we have \( u_1rD(r_1, u_2, \ldots, u_n) = 0 \) i.e.; \( U_1ND(r_1, u_2, \ldots, u_n) = \{0\} \). But \( U_1 \neq \{0\} \) and \( N \) is a prime near-ring, we conclude that

\[
D(r_1, u_2, \ldots, u_n) = 0. \tag{3.2.2}
\]

Now putting \( u_2r_2 \in U_2 \) in place of \( u_2 \), where \( r_2 \in N \), in relation (3.2.2) and proceeding as above we get \( D(r_1, r_2, u_3, \ldots, u_n) = 0 \). Proceeding inductively as before we conclude that \( D(r_1, r_2, \ldots, r_n) = 0 \) for all \( r_1, r_2, \ldots, r_n \in N \). This shows that \( D(N, N, \cdots, N) = \{0\} \), leading to a contradiction as \( D \) is a nonzero n-derivation. Therefore \( D(U_1, U_2, \ldots, U_n) \neq \{0\} \). We can also say that \( D(U_1, U_2, \ldots, U_n) = \{0\} \) implies that \( D(N, N, \ldots, N) = \{0\} \).
Similar arguments can be given for semigroup left ideals also.

(iii) Using hypothesis we obtain

\[(x_1, x_2, ..., x_n)(u_1, u_2, ..., u_n) = (u_1, u_2, ..., u_n)(x_1, x_2, ..., x_n)\] for all \(u_i \in U_i; 1 \leq i \leq n\).

Hence \((x_1u_1, x_2u_2, ..., x_nu_n) = (u_1x_1, u_2x_2, ..., u_nx_n)\) for all \(u_i \in U_i; 1 \leq i \leq n\). This implies that \(x_iu_i = u_ix_i\) for all \(u_i \in U_i; 1 \leq i \leq n\). Now putting \(ur\) for \(u_i\) where \(r \in N\) in the preceding relation and using this relation again we have \(u_i[x_i, r] = 0\). Using \(u_is\) for \(u_i\) where \(s \in N\), in the relation \(u_i[x_i, r] = 0\) we get \(u_iN[x_i, r] = \{0\}\) for all \(u_i \in U_i; 1 \leq i \leq n\).

Since \(N\) is a prime near ring and \(U_i \neq \{0\}; 1 \leq i \leq n\), we obtain finally \([x_i, r] = 0\). In turn we get \(x_i \in Z; 1 \leq i \leq n\). Therefore \((x_1, x_2, x_3, ..., x_n) \in (Z, Z, ..., Z)\).

(iv) Since \((x_1, x_2, ..., x_n)(U_1, U_2, ..., U_n)(y_1, y_2, ..., y_n) = \{(0, 0, ..., 0)\}\), we find that \((x_1, x_2, ..., x_n)(u_1, u_2, ..., u_n)(y_1, y_2, ..., y_n) = \{(0, 0, ..., 0)\}\) for all \(u_i \in U_i; 1 \leq i \leq n\) i.e.; \(x_iU_iy_i = \{0\}; 1 \leq i \leq n\). Using the hypothesis and Lemma 3.2.2, we get \((y_1, y_2, ..., y_n) = (0, 0, ..., 0)\). □

Lemma 3.2.6. Let \(N\) be a prime near-ring, \(D\) a nonzero \(n\)-derivation of \(N\) and \(U_1, U_2, ..., U_n\) be nonzero semigroup ideals of \(N\).

(i) If \(x \in N\) and \(D(U_1, U_2, ..., U_n)x = \{0\}\), then \(x = 0\).

(ii) If \(x \in N\) and \(xD(U_1, U_2, ..., U_n) = \{0\}\), then \(x = 0\).

Proof. (i) By our hypothesis, \(D(U_1, U_2, ..., U_n)x = \{0\}\) i.e.;

\[D(u_1, u_2, ..., u_n)x = 0\] for all \(u_i \in U_i; 1 \leq i \leq n\). (3.2.3)

Putting \(r_1u_1\) in place of \(u_1\), where \(r_1 \in N\), in relation (3.2.3) we get \(D(r_1u_1, u_2, ..., u_n)x = 0\). Using Lemmas 2.2.3 & 2.2.5(ii) and Remark 2.2.2 previous relation takes the form \(r_1D(u_1, u_2, ..., u_n)x + D(r_1, u_2, ..., u_n)u_1x = 0\). Using the hypothesis again we get \(D(r_1, u_2, ..., u_n)u_1x = 0\). Replacing \(u_1\) by \(u_1s\) where \(s \in N\) in preceding relation we obtain \(D(r_1, u_2, ..., u_n)u_1sx = 0\) i.e., \(D(r_1, u_2, ..., u_n)u_1N_x = \{0\}\). Since \(N\) is a prime near ring, either \(D(r_1, u_2, ..., u_n)u_1 = 0\) or \(x = 0\). Our claim is that \(D(r_1, u_2, ..., u_n)u_1 \neq 0\), for some \(r_1 \in N, u_1 \in U_1, u_2 \in U_2, ..., u_n \in U_n\). For other wise if \(D(r_1, u_2, ..., u_n)u_1 = 0\) for all \(r_1 \in N, u_1 \in U_1, u_2 \in U_2, ..., u_n \in U_n\), then \(D(r_1, u_2, ..., u_n)tu_1 = 0\) where \(t \in N\) i.e., \(D(r_1, u_2, ..., u_n)N_u_1 = \{0\}\). As \(U_1 \neq \{0\}\), primeness of \(N\) yields \(D(r_1, u_2, ..., u_n) = 0\).
for all \( r_1 \in N, u_2 \in U_2, \ldots, u_n \in U_n \). Now using similar arguments as used in the proof of Lemma 3.2.5(ii), we can show that \( D(N, N, \ldots, N) = \{0\} \) leading to a contradiction. Therefore, we conclude that \( x = 0 \).

(ii) It can be proved in a similar way. \( \square \)

**Lemma 3.2.7.** Let \( N \) be a near-ring possessing right cancelation law. If \( N \) admits \( n \)-derivations \( D_1 \) and \( D_2 \) such that \( D_1(u_1, u_2, \ldots, u_n) = D_2(u_1, u_2, \ldots, u_n) \) for all \( u_i \in U_i; 1 \leq i \leq n \) where \( U_1, U_2, \ldots, U_n \) are nonzero semigroup left ideals of \( N \), then \( D_1 = D_2 \).

**Proof.** We have

\[
D_1(u_1, u_2, \ldots, u_n) = D_2(u_1, u_2, \ldots, u_n) \quad \text{for all} \quad u_i \in U_i; 1 \leq i \leq n. \tag{3.2.4}
\]

Putting \( r_1 u_1 \) for \( u_1 \) where \( r_1 \in N \) in above equation we get \( D_1(r_1 u_1, u_2, \ldots, u_n) = D_2(r_1 u_1, u_2, \ldots, u_n) \). Therefore,

\[
D_1(r_1, u_2, \ldots, u_n)u_1 + r_1 D_1(u_1, u_2, \ldots, u_n) = D_2(r_1, u_2, \ldots, u_n)u_1 + r_1 D_2(u_1, u_2, \ldots, u_n).
\]

By using relation (3.2.4) we have \( D_1(r_1, u_2, \ldots, u_n)u_1 = D_2(r_1, u_2, \ldots, u_n)u_1 \). Since \( U_1 \neq \{0\} \), using hypothesis again we obtain

\[
D_1(r_1, u_2, \ldots, u_n) = D_2(r_1, u_2, \ldots, u_n) \tag{3.2.5}
\]

for all \( r_1 \in N, u_2 \in U_2, \ldots, u_n \in U_n \). Now putting \( r_2 u_2 \) for \( u_2 \), where \( r_2 \in N \), in the above equation (3.2.5) and arguing in the same way as before, we obtain that \( D_1(r_1, r_2, u_3, \ldots, u_n) = D_2(r_1, r_2, u_3, \ldots, u_n) \). Now proceeding inductively in a similar manner as above, we conclude that \( D_1(r_1, r_2, \ldots, r_n) = D_2(r_1, r_2, \ldots, r_n) \) for all \( r_1, r_2, \ldots, r_n \in N \). Hence, \( D_1 = D_2 \). \( \square \)

**Lemma 3.2.8.** Let \( N \) be a prime near-ring admitting \( n \)-derivations \( D_1 \) and \( D_2 \) such that \( D_1(u_1, u_2, \ldots, u_n) = D_2(u_1, u_2, \ldots, u_n) \) for all \( u_i \in U_i; 1 \leq i \leq n \) where \( U_1, U_2, \ldots, U_n \) are nonzero semigroup right ideals of \( N \). Then \( D_1 = D_2 \).

**Proof.** By our hypothesis \( D_1(u_1, u_2, \ldots, u_n) = D_2(u_1, u_2, \ldots, u_n) \) for all \( u_i \in U_i; 1 \leq i \leq n \). Putting \( u_1 r_1 \) where \( r_1 \in N \) in place of \( u_1 \) in previous relation and using it again: we get \( u_1 \{D_1(r_1, u_2, \ldots, u_n) - D_2(r_1, u_2, \ldots, u_n)\} = 0 \). i.e.; \( u_1 t \{D_1(r_1, u_2, \ldots, u_n) - D_2(r_1, u_2, \ldots, u_n)\} = 0 \) for all \( t \in N \). This shows that \( u_1 N\{D_1(r_1, u_2, \ldots, u_n) - D_2(r_1, u_2, \ldots, u_n)\} = 0 \). Since \( U_1 \neq \{0\} \) and \( N \) is a prime
near-ring, we infer that \( D_1(r_1, u_2, ..., u_n) = D_2(r_1, u_2, ..., u_n) \). Similarly putting \( r_2u_2 \) in place of \( u_2 \), where \( r_2 \in N \), in the preceding equation and using the above trick we get \( D_1(r_1, r_2, u_3, ..., u_n) = D_2(r_1, r_2, u_3, ..., u_n) \). Proceeding inductively after \( n \) steps we get \( D_1(r_1, r_2, ..., r_n) = D_2(r_1, r_2, ..., r_n) \) and hence \( D_1 = D_2 \).

Let \( K = \{ a \in N \mid [a, d(u)] = 0, \text{ for all } u \in U \} \), where \( U \) is a nonzero semigroup ideal and \( d \) a nonzero derivation of a prime near-ring \( N \). In the year 1997, H.E.Bell [15] proved that (i) if \( a \in K \), then \( a \in Z \) or \( d(a) = 0 \) and (ii) \( d(K) \subseteq Z \). Inspired by this result we have proved the following theorem in the setting of \( n \)-derivation:

**Theorem 3.2.1.** Let \( N \) be a prime near-ring, \( D \) a nonzero \( n \)-derivation of \( N \) and \( U_1, U_2, ..., U_n \) be nonzero semigroup ideals of \( N \). Let \( K_n = \{ a \in N \mid [D(u_1, u_2, ..., u_n), a] = 0 \text{ for all } u_i \in U_i; 1 \leq i \leq n \} \).

(i) If \( a \in K_n \), then \( a \in Z \) or \( D(a, a, \cdots, a) = 0 \).

(ii) \( D(a, a, \cdots, a) \in Z \) for all \( a \in K_n \).

**Proof.** (i) Since \( a \in K_n \), \([D(u_1, u_2, ..., u_n), a] = 0 \text{ for all } u_i \in U_i; 1 \leq i \leq n \}. \) Therefore

\[
aD(u_1, u_2, ..., u_n) = D(u_1, u_2, ..., u_n)a
\]

for all \( u_i \in U_i; 1 \leq i \leq n \). Putting \( au_1 \) in place of \( u_1 \) in the relation (3.2.6) and using Lemma 2.2.5(i) together with Remark 2.2.2 we get

\[
aD(a, u_2, ..., u_n)u_1 = D(a, u_2, ..., u_n)u_1a
\]

for all \( u_i \in U_i; 1 \leq i \leq n \). Putting \( u_1r \) where \( r \in N \) in place of \( u_1 \) in relation (3.2.7) and using the same we get \( D(a, u_2, ..., u_n)u_1ar = D(a, u_2, ..., u_n)u_1ra \). This implies that \( D(a, u_2, ..., u_n)u_1N(ar - ra) = \{0 \} \). Since \( N \) is a prime near ring, for given \( a \in N \) either \( a \in Z \) or \( D(a, u_2, u_3, ..., u_n)u_1 = 0 \). If first case holds then nothing to do if not, then second case implies that \( D(a, u_2, u_3, ..., u_n)N u_1 = \{0 \} \). Since \( U_1 \neq \{0 \} \), primeness of \( N \) yields

\[
D(a, u_2, ..., u_n) = 0,
\]

for all \( u_i \in U_i; 2 \leq i \leq n \). Now putting \( au_2 \) in place of \( u_2 \) in relation (3.2.8) and using it again, we get \( D(a, a, u_3, u_4, ..., u_n)u_2 = 0 \) or \( D(a, a, u_3, u_4, ..., u_n)NU_2 = \{0 \} \). Now the primeness of \( N \) and \( U_2 \neq \{0 \} \) yield \( D(a, a, u_3, u_4, ..., u_n) = 0 \). Proceeding inductively as above we conclude that \( D(a, a, ..., a) = 0 \).
(ii) By preceding proof (i) it is clear that for any \( a \in K_n \), either \( D(a, a, \cdots, a) = 0 \) or \( a \in \mathbb{Z} \). First case implies that \( D(a, a, \cdots, a) \in \mathbb{Z} \), for the second case by using Lemma 3.2.4, we obtain that \( D(a, a, \cdots, a) \in \mathbb{Z} \). Lastly we conclude that \( D(a, a, \cdots, a) \in \mathbb{Z} \) for all \( a \in K_n \).

**Theorem 3.2.2.** Let \( N \) be a prime near-ring. Let \( D_1 \) and \( D_2 \) be any two nonzero \( n \)-derivations of \( N \). If \([D_1(U_1, U_2, \ldots, U_n), D_2(U_1, U_2, \ldots, U_n)] = \{0\}\) where \( U_1, U_2, \ldots, U_n \) are nonzero semigroup ideals of \( N \), then \((N, +)\) is an abelian group.

**Proof.** It is straight forward to show that if \( z \in N \) is such that \([z, D_2(U_1, U_2, \ldots, U_n)] = \{0\}\) and \( u_1, u'_1 \in U_1 \) are such that \( u_1 + u'_1 \in U_1 \), then \( zD_2(c, u_2, \ldots, u_n) = 0 \), where \( c \) is the additive commutator \((u_1 + u'_1 - u_1 - u'_1)\).\( u_2 \in U_2, \ldots, u_n \in U_n \). If \( r, s \in U_1 \) we have \( rs \in U_1 \) and \( rs + rs = r(s + s) \in U_1 \) and since \([D_1(U_1, U_2, \ldots, U_n), D_2(U_1, U_2, \ldots, U_n)] = \{0\}\), taking \( z = D_1(rs, u_2', \ldots, u_n') \) where \( r, s \in U_1, u_2' \in U_2, \ldots, u_n' \in U_n \) gives \( D_1(U_1^2, U_2, \ldots, U_n)D_2(c, u_2, \ldots, u_n) = \{0\}\) because for all \( r, s \in U_1 \) implies that \( rs \in U_1^2 \). But \( U_1^2 = \{pq \mid p, q \in U_1\} \) is a nonzero semigroup ideal, so by Lemma 3.2.6(i) we get

\[
D_2(u_1 + u'_1 - u_1 - u'_1, u_2, u_3, \ldots, u_n) = 0 \tag{3.2.9}
\]

for all \( u_1, u'_1 \in U_1 \) such that \( u_1 + u'_1 \in U_1 \). Now take \( u_1 = rx \) and \( u'_1 = ry \) where \( r \in U_1 \) and \( x, y \in N \), so that \( u_1, u'_1 \) and \( u_1 + u'_1 \) are all in \( U_1 \). It follows from relation (3.2.9) that \( D_2(rx + ry - rx - ry, u_2, u_3, \ldots, u_n) = 0 \) for all \( r \in U_1 \) and for all \( x, y \in N \). Replacing \( r \) by \( wr, w \in U_1 \), we get \( D_2(U_1, U_2, \ldots, U_n)(rx + ry - rx - ry) = \{0\}\) for all \( r \in U_1 \) and \( x, y \in N \) i.e.; \( D_2(U_1, U_2, \ldots, U_n)U_1(x + y - x - y) = \{0\}\) for all \( x, y \in N \) and by Lemmas 3.2.5(ii) and 3.2.2 we find that \( x + y - x - y = 0 \) for all \( x, y \in N \), and hence \((N, +)\) is an abelian group.

**Corollary 3.2.1** ([21, Theorem 3.3.]). Let \( N \) be a prime near-ring. Let \( U \) be nonzero semigroup ideal of \( N \) and \( d \) be a nonzero derivation on \( N \). If \([d(U), d(U)] = \{0\}\), then \((N, +)\) is an abelian group.

**Corollary 3.2.2** ([15, Theorem 3.3.]). Let \( N \) be a prime near-ring and \( D_1 \) and \( D_2 \) be any two nonzero permuting \( n \)-derivations of \( N \). If \([D_1(N, N, \ldots, N), D_2(N, N, \ldots, N)] = \{0\}\), then \((N, +)\) is an abelian group.

**Theorem 3.2.3.** Let \( N \) be a prime near-ring, \( U_1, U_2, \ldots, U_n \) be nonzero semigroup right ideals of \( N \) and let \( D \) be a nonzero \( n \)-derivation of \( N \). If \( D(U_1, U_2, \ldots, U_n) \subseteq Z \), then \( N \) is a commutative ring.
Proof. For all $u_i, n_i \in \mathbb{F}^i, u_2 \in \mathbb{F}^2, \ldots, u_n \in \mathbb{F}^n$ we get

$$D(u_1 u'_1, u_2, \ldots, u_n) = D(u_1, u_2, \ldots, u_n)u'_1 + u_1 D(u'_1, u_2, \ldots, u_n) \in \mathbb{Z}. \quad (3.2.10)$$

Now commuting the equation (3.2.10) with the element $u'_1$ we have

$$\{D(u_1, u_2, \ldots, u_n)u'_1 + u_1 D(u'_1, u_2, \ldots, u_n)\}u'_1 = u'_1 \{D(u_1, u_2, \ldots, u_n)u'_1 + u_1 D(u'_1, u_2, \ldots, u_n)\}.$$

Using the hypothesis and Lemma 2.2.5(i) together with Remark 2.2.2 we get

$$D(u'_1, u_2, \ldots, u_n)(u'_1 u_1 - u_1 u'_1) = 0$$

By Lemma 2.2.1(i), we see that for each $u'_1 \in U_1$, either $u'_1$ centralizes $U_1$ or $D(u'_1, u_2, \ldots, u_n) = 0$. If $u'_1$ centralizes $U_1$ then by Lemma 3.2.1(ii) we get $u'_1 \in \mathbb{Z}$. If $D(u'_1, u_2, \ldots, u_n) = 0$, then (3.2.10) takes the form

$$D(u'_1, u_2, \ldots, u_n) = D(u'_1, u_2, \ldots, u_n)u'_1 \in \mathbb{Z}$$

and by Lemmas 2.2.2 and 3.2.5(ii), we get $u'_1 \in \mathbb{Z}$ in this case also i.e.; we have shown that if for some $u'_1 \in U_1$,

$$D(u'_1, u_2, \ldots, u_n) = 0, \text{ for all } u_2 \in U_2, \ldots, u_n \in U_n \text{ then, } u'_1 \in \mathbb{Z}. \quad (3.2.11)$$

Now we conclude that $U_1 \subseteq \mathbb{Z}$ and $N$ is therefore a commutative ring by Lemma 3.2.3.

Theorem 3.2.4. Let $N$ be a prime near-ring, $D$ a nonzero $n$-derivation of $N$ and $U_1, U_2, \ldots, U_n$ be nonzero semigroup left ideals of $N$. If $D(U_1, U_2, \ldots, U_n) \subseteq \mathbb{Z}$, then $N$ is a commutative ring.

Proof. Using same arguments as used in the proof of Theorem 3.2.3, we conclude that all $U'_i$s are commutative. It follows that if at least one $U_i$ contains a nonzero central element $w$, then we have $xwui = u_ixw = wuix$, and therefore $w(xu_i - u_ix) = 0$ for all $x \in N, u_i \in U_i$. Thus $U_i \subseteq \mathbb{Z}$, by Lemma 2.2.1(i) and hence $N$ is commutative ring by Lemma 3.2.3.

We may now assume that $U_i \cap \mathbb{Z} = \{0\}$, for all $i = 1, 2, 3, \ldots, n$; and under this condition relation (3.2.11) of Theorem 3.2.3 shows that $D(u_1, u_2, \ldots, u_i, \ldots, u_n) \neq 0$, for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_i \in U_i \setminus \{0\}, \ldots, u_n \in U_n$. For each $u_i \in U_i \setminus \{0\}$; and for

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every $u_i \in U_i, u_2 \in U_2, \ldots, u_{i-1} \in U_{i-1}, u_{i+1} \in U_{i+1}, \ldots, u_n \in U_n; D(u_1, u_2, \ldots, u_i, \ldots, u_n) = D(u_1, u_2, \ldots, u_i, \ldots, u_n)(u_i + u_i)$ and hence by Lemma 2.2.2, $2u_i \in Z$. Suppose that $2u_i \neq 0$ for all $u_i \in U_i \setminus \{0\}$. Lemma 3.2.1(i) guarantees that for each $x \in N \setminus \{0\}$; there exists $u_i \in U_i$ such that $xu_i \neq 0$. Since $xu_i \in U_i$; we have $2xu_i^2 = x(2u_i) \in Z$; and by Lemma 2.2.2 we get $x \in Z$. Therefore $N = Z$, i.e.; $N$ is a commutative near-ring. Since $N \neq \{0\}$, there exists $0 \neq p \in N$ such that $p + p \in N = Z$. Hence by Lemma 2.1(ii), $N$ becomes a commutative ring.

The only remaining possibility is that $U_i \cap Z = \{0\}$ and there exists $u_i \in U_i \setminus \{0\}$ such that $2u_i = 0$ and we complete our proof by showing that this can not occur.

Suppose then, that $u_i \in U_i \setminus \{0\}$ and $2u_i = 0$. We have $D(u_1, u_2, \ldots, u_i, \ldots, u_n) = 3u_i^2D(u_1, u_2, \ldots, u_i, \ldots, u_n) \in Z$ and since $2u_i^2D(u_1, u_2, \ldots, u_i, \ldots, u_n) = 0$, we get $u_i^2D(u_1, u_2, \ldots, u_i, \ldots, u_n) \in Z$. This implies that $u_i^2 \in Z$. Since $U_i \cap Z = \{0\}$, $u_i^2 = 0$.

Now in view of Lemma 2.2.3 and Remark 2.2.2 we know that

$$D(u_1, u_2, \ldots, xu_i, \ldots, x_n) =xD(u_1, u_2, \ldots, u_i, \ldots, u_n) + D(u_1, u_2, \ldots, x, \ldots, u_n)u_i$$

for all $x \in N, u_i \in U_i, 1 \leq i \leq n$. Hence $u_i\{xD(u_1, u_2, \ldots, u_i, \ldots, u_n) + D(u_1, u_2, \ldots, x, \ldots, u_n)u_i\} = \{xD(u_1, u_2, \ldots, u_i, \ldots, u_n) + D(u_1, u_2, \ldots, x, \ldots, u_n)u_i\} u_i$. Using Lemma 2.2.5(ii) and Remark 2.2.2 the right hand side of previous relation takes the form $xD(u_1, u_2, \ldots, u_i, \ldots, u_n)u_i$. On left multiplying by $u_i$ in the previous relation we have $u_i^2\{xD(u_1, u_2, \ldots, u_i, \ldots, u_n) + D(u_1, u_2, \ldots, x, \ldots, u_n)u_i\} = u_ixD(u_1, u_2, \ldots, u_i, \ldots, u_n)u_i$, which implies that $u_ixD(u_1, u_2, \ldots, u_i, \ldots, u_n)u_i = 0$. i.e.; $u_iND(u_1, u_2, \ldots, u_i, \ldots, u_n)u_i = \{0\}$. Primeness of $N$ yields $D(u_1, u_2, \ldots, u_i, \ldots, u_n)u_i = 0$ and since $D(u_1, u_2, \ldots, u_i, \ldots, u_n) \in Z \setminus \{0\}$, we conclude that $u_i = 0$, a contradiction. \hfill \square

**Corollary 3.2.3** ([21, Theorem 2.1]). Let $N$ be a prime near-ring and $U$ be a nonzero semigroup right ideal of $N$ or a nonzero semigroup left ideal. If $N$ admits a nonzero derivation $d$ for which $d(U) \subseteq Z$, then $N$ is a commutative ring.

**Corollary 3.2.4** ([15, Theorem 3.2]). Let $N$ be a prime near-ring admitting a nonzero permuting $n$-derivation $D$ such that $D(N, N, \ldots, N) \subseteq Z$, then $N$ is a commutative ring.

**Corollary 3.2.5** ([15, Theorem 3.8]). Let $N$ be a prime near-ring which admits a nonzero permuting $n$-derivation $D$ such that $D(C, C, \ldots, C) \subseteq Z$. Then $N$ is a commutative ring, where $C \neq \{0\}$.

In the year 2001 H.E.Bell and N.Argac [22, Theorem 3.5] proved that if $N$ is a near-ring with no nonzero divisors of zero and $N$ admits a nonzero derivation $d$ such that
\[ d(xy) = d(yx) \] for all \( x, y \) in a nonzero semigroup right ideal \( U \) of \( N \), then \( N \) is a commutative ring. We have extended this result in the setting of \( n \)-derivations and semigroup right ideals in near-ring \( N \).

**Theorem 3.2.5.** Let \( N \) be a prime near-ring with no nonzero divisors of zero and \( U_1, U_2, \ldots, U_n \) be any \( n \) nonzero semigroup right ideals of \( N \). If \( N \) admits a nonzero \( n \)-derivation \( D \) such that \( D(u_1u'_1, u_2, \ldots, u_n) = D(u'_1u_1, u_2, \ldots, u_n) \) for all \( u_1, u'_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \), then \( N \) is a commutative ring.

**Proof.** Since \( D(u_1u'_1, u_2, \ldots, u_n) = D(u'_1u_1, u_2, \ldots, u_n) \) for all \( u_1, u'_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \), we find that
\[
D(u_1u'_1 - u'_1u_1, u_2, \ldots, u_n) = 0. \tag{3.2.12}
\]

Putting \( u_1u'_1 \) for \( u'_1 \) in relation (3.2.12) and using it again we have \( D(u_1, u_2, \ldots, u_n) \) for all \( u_1 \in U_1, \) either \( D(u_1, u_2, \ldots, u_n) = 0 \) or \( u_1 \) centralizes \( U_1 \). Applying Lemma 3.2.1 we see that either \( D(u_1, u_2, \ldots, u_n) = 0 \) or \( u_1 \in Z \). By Lemma 3.2.4 we conclude that \( D(u_1, u_2, \ldots, u_n) \in Z \) for all \( u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \) i.e., \( D(U_1, U_2, \ldots, U_n) \subseteq Z \). Hence by Theorem 3.2.3, \( N \) is a commutative ring. \( \square \)

**Corollary 3.2.6.** Let \( N \) be a prime near-ring with no nonzero divisors of zero. If \( N \) admits a nonzero \( n \)-derivation \( D \) such that \( D(x_1x'_1, x_2, \ldots, x_n) = D(x'_1x_1, x_2, \ldots, x_n) \) for all \( x_1, x'_1, x_2, \ldots, x_n \in N \), then \( N \) is a commutative ring.

Very recently Boua and Oukhtite [26] showed that if \( N \) is a prime near ring which admits a nonzero derivation \( d \) satisfying any one of the following conditions: (i) \( d[x, y] = 0 \), (ii) \( d[x, y] = \pm[x, y] \) and (iii) \( d(xoy) = \pm(xoy) \) for all \( x, y \in N \) then \( N \) is a commutative ring. We have extended these results in the context of \( n \)-derivations.

**Theorem 3.2.6.** Let \( N \) be a prime near-ring having a nonzero \( n \)-derivation \( D \). If \( U_1, U_2, \ldots, U_n \) are nonzero semigroup ideals of \( N \) such that \( D([x, y], u_2, \ldots, u_n) = 0 \) for all \( x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \), then \( N \) is a commutative ring.

**Proof.** Given that
\[
D([x, y], u_2, \ldots, u_n) = 0 \text{ for all } x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n. \tag{3.2.13}
\]

Replacing \( y \) by \( xy \) in (3.2.13) we get \( D([x, xy], u_2, \ldots, u_n) = 0 \) i.e., \( D([x, y], u_2, \ldots, u_n) = 0 \). This gives us that \( D(x, u_2, \ldots, u_n)[x, y] + xD([x, y], u_2, \ldots, u_n) = 0 \) and hence in view
of (3.2.13) we find that

\[ D(x, u_2, \ldots, u_n)xy = D(x, u_2, \ldots, u_n)yx. \quad (3.2.14) \]

Replacing \( y \) by \( yr \), where \( r \in N \), in (3.2.14) and using it again we get \( D(x, u_2, \ldots, u_n)y[x, r] = 0 \) i.e.,

\[ D(x, u_2, \ldots, u_n)U_1[x, r] = \{0\}. \quad (3.2.15) \]

By using Lemma 3.2.2 we conclude that for each \( x \in U_1 \) either \( x \in Z \) or \( D(x, u_2, \ldots, u_n) = 0 \). But using Lemma 3.2.4 lastly we get \( D(x, u_2, \ldots, u_n) \in Z \) for all \( x \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \) i.e.; \( D(U_1, U_2, \ldots, U_n) \subseteq Z \). Now by using Theorem 3.2.4, we find that \( N \) is a commutative ring.

\begin{corollary}
Let \( N \) be a prime near-ring having a nonzero \( n \)-derivation \( D \). If \( D([x, y], u_2, \ldots, u_n) = 0 \) for all \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a commutative ring.
\end{corollary}

\begin{theorem}
Let \( N \) be a prime near-ring having a nonzero \( n \)-derivation \( D \). If \( U_1, U_2, \ldots, U_n \) are nonzero semigroup ideals of \( N \) such that \( D([x, y], u_2, \ldots, u_n) = \pm [x, y] \) for all \( x, y \in U_1, u_2 \in U_2, \ldots, u_n \in U_n \), then \( N \) is a commutative ring.
\end{theorem}

\begin{proof}
Replacing \( y \) by \( xy \) in the given hypothesis and using it again we get

\[ D(x, u_2, \ldots, u_n)[x, y] = 0 \] i.e.; \( D(x, u_2, \ldots, u_n)xy = D(x, u_2, \ldots, u_n)yx. \) This is identical with the relation (3.2.14) in the Theorem 3.2.6. Now arguing in the same way as above, we infer that \( N \) is a commutative ring.
\end{proof}

\begin{corollary}
Let \( N \) be a prime near-ring having a nonzero \( n \)-derivation \( D \). If \( D([x, y], x_2, \ldots, x_n) = \pm [x, y] \) for all \( x, y, x_2, \ldots, x_n \in N \), then \( N \) is a commutative ring.
\end{corollary}

\begin{theorem}
Let \( N \) be a prime near-ring having a nonzero \( n \)-derivation \( D \). Let \( I_1, I_2, \ldots, I_n \) be nonzero ideals of \( N \) such that \( D(xoy, i_2, \ldots, i_n) = \pm (xoy) \) for all \( x, y \in I_1, i_2 \in I_2, \ldots, i_n \in I_n \), then \( N \) is a commutative ring.
\end{theorem}

\begin{proof}
Replacing \( y \) by \( xy \) in the given relation we get \( D(x, i_2, \ldots, i_n)(xoy) = 0 \) i.e.;

\[ D(x, i_2, \ldots, i_n)xy = -D(x, i_2, \ldots, i_n)yx. \quad (3.2.16) \]

Putting \( yz \) for \( y \) where \( z \in N \) in the relation (3.2.16) we have

\[ D(x, i_2, \ldots, i_n)xyz = -D(x, i_2, \ldots, i_n)yxz \] i.e.; \( D(x, i_2, \ldots, i_n)xyz + D(x, i_2, \ldots, i_n)yxz = 0 \). Now substituting the values from the relation (3.2.16) in the preceding relation we get \( \{-D(x, i_2, \ldots, i_n)yx\}z + D(x, i_2, \ldots, i_n)yzz = 0 \) i.e.; \( D(x, i_2, \ldots, i_n)y(-x)z = 0 \).
\end{proof}
$D(x, i_2, \cdots, i_n)yzzx = 0$. Hence replacing $x$ by $-x$ in the preceding relation we have $D(-x, i_2, \cdots, i_n)yzz + D(-x, i_2, \cdots, i_n)yzz(-x) = 0$, in turn we get $D(-x, i_2, \cdots, i_n)y(xz - xz) = 0$ or $D(-x, i_2, \cdots, i_n)I_1[x, z] = \{0\}$. Since $N$ is a zero symmetric left near-ring and $I_1$ is a nonzero ideal of $N$, We find that $I_1$ is a nonzero semigroup ideal of $N$. For each fixed $x \in I_1$ Lemma 3.2.2 yields either $x \in Z$ or $D(-x, i_2, \cdots, i_n) = 0$. If the first case holds then by Lemma 3.2.4 we have $D(x, i_2, \cdots, i_n) \in Z$ for all $i_2, i_3, \cdots, i_n \in I_n$ and the second case implies $-D(x, i_2, \cdots, i_n) = 0$ i.e.; $0 = D(x, i_2, \cdots, i_n) \in Z$. Including both the cases we get $D(x, i_2, \cdots, i_n) \in Z$ for all $i_2, i_3, \cdots, i_n \in I_n$. But since $x$ is an arbitrary element of $I_1$, we conclude that $D(I_1, I_2, \cdots, I_n) \subseteq Z$. Since $I_1, I_2, \cdots, I_n$ are nonzero ideals of $N$, $I_1, I_2, \cdots, I_n$ are also nonzero semigroup ideals of $N$. Therefore using Theorem 3.2.4, we infer that $N$ is a commutative ring.

**Corollary 3.2.9.** Let $N$ be a prime near-ring having a nonzero $n$-derivation $D$. If $D(xoy, x_2, \cdots, x_n) = \pm(xoy)$ for all $x, y, x_2, \cdots, x_n \in N$, then $N$ is a commutative ring.

**Remark 3.2.1.** All the results obtained above are also true if we replace semigroup left ideals by left ideals, semigroup right ideals by right ideals and semigroup ideals by ideals respectively.

### 3.3 Generalized derivations on semigroup ideals and commutativity of prime near-rings

The existing literature on prime near-rings contains a number of theorems concerning multiplicative commutativity of near-rings. H. E. Bell, G. Mason, A. Boua and L. Oukhtite have proved several results on commutativity of prime near-rings with derivations (for reference see [24], [21], [22], [26] etc.). The notion of generalized derivation in rings was introduced by Matej Brešar [28] in the year 1991 and subsequently a number of authors have studied generalized derivation in the setting of prime and semiprime rings (for reference see [2], [12], [9], [54], [77] where further references can be found). Motivated by the notion of generalized derivation in rings, Öznur Gölbaşı introduced generalized derivation in near-rings. Several commutativity theorems for prime near-rings with generalized derivations have also been proved by Öznur Gölbaşı (for reference see [47], [48], [49] etc.). It is natural to look for comparable results for prime near-rings having generalized derivations with semigroup ideals and ideals. Our aim in this section is to study the commutativity of prime near-rings satisfying certain identities involving
generalized derivations on ideals and semigroup ideals of a prime near-ring.

Now we begin with the following lemmas which will be used frequently. Proof of the first lemma can be seen in [24, Lemma 3(iv)] while those of next three can be found in [21]. Lemma 3.3.5 is essentially proved in [47, Lemma 2.3(i)].

**Lemma 3.3.1.** Let $N$ be a prime near-ring. If $N$ is 2-torsion-free and $d$ is a derivation on $N$ such that $d^2 = 0$, then $d = 0$.

**Lemma 3.3.2.** Let $N$ be a prime near-ring and $d$ a nonzero derivation on $N$. If $U$ is a nonzero semigroup right ideal or semigroup left ideal of $N$, then $d(U) \neq \{0\}$.

**Lemma 3.3.3.** Let $N$ be a prime near-ring and $U$ be nonzero semigroup right ideal or a nonzero semigroup left ideal of $N$. If $N$ admits a nonzero derivation $d$ for which $d(U) \subseteq Z$, then $N$ is a commutative ring.

**Lemma 3.3.4.** Let $N$ be prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $d$ is a nonzero derivation on $N$ such that $d^2(U) = 0$, then $d^2 = 0$.

**Lemma 3.3.5.** Let $N$ be a near-ring and $f$ be a right generalized derivation of $N$ with associated derivation $d$. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y, z \in N$.

**Lemma 3.3.6.** Let $N$ be a near-ring and $f$ be a generalized derivation of $N$ with associated derivation $d$. Then $(d(x)y + xf(y))z = d(x)yz + xf(y)z$ for all $x, y, z \in N$.

Proof. We have $f((xy)z) = f(xy)z + xyd(z) = (d(x)y + xf(y))z + xyd(z)$. On the other hand we have $f(x(yz)) = d(x)yz + xf(yz) = d(x)yz + xf(y)z + xyd(z)$. Comparing these two expressions we get our required result. □

**Lemma 3.3.7.** Let $N$ be a prime near-ring and $U$ a nonzero semigroup ideal of $N$.

(i) If $f$ is a right generalized derivation of $N$ with associated nonzero derivation $d$ of $N$ such that $af(U) = \{0\}$ where $a \in N$, then $a = 0$.

(ii) If $f$ is a generalized derivation of $N$ with associated nonzero derivation $d$ of $N$ such that $f(U)a = \{0\}$ where $a \in N$, then $a = 0$.

Proof. (i) Since $af(U) = \{0\}$, we find that $af(ur) = 0$ for all $u \in U, r \in N$ i.e., $af(u)r + aud(r) = 0$ for all $u \in U, r \in N$. By hypothesis we get $aud(r) = 0$, which shows that $aUd(r) = \{0\}$. As $d \neq 0$, using Lemma 3.2.2 we conclude that $a = 0$.

(ii) It can be proved in a similar way. □
Lemma 3.3.8. Let $N$ be a prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $f$ is a right (or left) generalized derivation of $N$ with associated nonzero derivation $d$ of $N$, then $f(U) \neq \{0\}$.

Proof. Suppose that $f$ is a right generalized derivation of $N$ and if possible let $f(U) = \{0\}$ i.e., $f(ur) = 0$ for all $u \in U, r \in N$. This shows that $f(u)r + ud(r) = 0$, and hence by hypothesis we obtain that $ud(r) = 0$ i.e., $usd(r) = 0$ for all $u \in U, s, r \in N$. Lastly we obtain that $uNd(r) = \{0\}$, as $U \neq \{0\}$, primeness of $N$ yields $d(r) = 0$ for all $r \in N$ i.e., $d = 0$ a contradiction. Similarly one can prove this result if $f$ is a left generalized derivation of $N$.

Lemma 3.3.9. Let $N$ be a prime near-ring. If $N$ admits left generalized derivations $f_1$ and $f_2$ with associated derivation $d$ of $N$ and $U$ is a nonzero semigroup ideal of $N$ such that $f_1(u) = f_2(u)$ for all $u \in U$, then $f_1 = f_2$.

Proof. By hypothesis we have $f_1(ur) = f_2(ur)$ for all $u \in U, r \in N$ i.e., $d(u)r + uf_1(r) = d(u)r + uf_2(r)$. This implies that $u(f_1(r) - f_2(r)) = 0$. Previous relation gives us that $uN(f_1(r) - f_2(r)) = \{0\}$. Since $U \neq \{0\}$, primeness of $N$ yields $f_1 = f_2$.

Lemma 3.3.10. Let $N$ be a prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated derivation $d$ of $N$ such that $f(u)v = uf(v)$ for all $u, v \in U$, then $d = 0$.

Proof. Given that $f(u)v = uf(v)$ for all $u, v \in U$. Now putting $vw$ for $v$, where $w \in U$ in the previous relation we have $f(u)vw = uf(vw)$, which implies that $f(u)vw = ud(v)w + uvf(w)$. By hypothesis we obtain $f(u)vw = ud(v)w + uf(v)w$ which also gives us that $f(u)vw = ud(v)w + f(u)vw$ i.e., $ud(v)w = 0$ for all $u, v, w \in U$. Lastly putting $wr$ where $r \in N$ for $u$ in the previous relation we have $uNd(v)w = \{0\}$. Since $U \neq \{0\}$ and $N$ is a prime near-ring, we conclude that $d(v)w = 0$ for all $v, w \in U$. Now putting $sw$ where $s \in N$ for $w$ in the relation $d(v)w = 0$ we have $d(v)Nw = \{0\}$. Since $U \neq \{0\}$ and $N$ is a prime near-ring, we conclude that $d(v) = 0$ for all $v \in U$ i.e., $d(U) = \{0\}$. We claim that $d = 0$, for otherwise Lemma 3.3.2 forces $d(U) \neq \{0\}$, leading to a contradiction. Hence our claim stands proved.

Lemma 3.3.11. Let $N$ be a 2-torsion free prime near-ring. If $N$ admits a generalized derivation $f$ with associated derivation $d$ of $N$ and $U$ is a nonzero semigroup ideal of $N$ such that $f^2(U) = \{0\}$, then $d = 0$.
Proof. Since \( f^2(U) = \{0\} \), we find that \( f^2(f(u)v) = 0 \) for all \( u, v \in U \) i.e. \( f(f^2(u)v + f(u)d(v)) = 0 \). Using hypothesis we get \( f(f(u)d(v)) = 0 \) i.e., \( f^2(u)d(v) + f(u)d^2(v) = 0 \) for all \( u, v \in U \). Hypothesis assures us \( f(u)d^2(v) = 0 \) for all \( u, v \in U \) i.e.; \( f(U)d^2(v) = \{0\} \). We claim that \( d = 0 \), for otherwise Lemma 3.3.7(ii) gives \( d^2(v) = 0 \). Under this situation \( d^2(U) = \{0\} \). By using Lemma 3.3.4 we have \( d^2 = 0 \) and by Lemma 3.3.1, we conclude that \( d = 0 \), leading to a contradiction.

Recently Öznur Gölbasi [48, Theorems 3.1&3.2] proved that if \( N \) is a prime near-ring admitting a left generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) and satisfying either of the following identities: (i) \( f([x,y]) = 0 \) for all \( x, y \in N \) or (ii) \( f([x,y]) = \pm[x,y] \) for all \( x, y \in N \), then \( N \) is a commutative ring. We have shown that these results are still true if both identities hold on some nonzero semigroup ideal \( U \) of \( N \). In fact, we proved the following.

**Theorem 3.3.1.** Let \( N \) be a prime near-ring and \( U \) be a nonzero semigroup ideal of \( N \). If \( N \) admits a left generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \( f([x,y]) = 0 \) for all \( x, y \in U \), then \( N \) is a commutative ring.

Proof. Given that \( f([x,y]) = 0 \) for all \( x, y \in U \). Putting \( xy \) in place of \( y \), obtaining \( f([x,xy]) = f(x[x,y]) = d(x)[x,y] + xf([x,y]) = 0 \). Since the second term is zero, it is clear that

\[
d(x)xy = d(x)yx
\]

(3.3.1)

for all \( x, y \in U \). Replacing \( y \) by \( yz \) where \( z \in N \) in (3.3.1) and using this relation again, we get \( d(x)U[x,z] = \{0\} \) for all \( x \in U, z \in N \). Hence by Lemma 3.2.2 for each \( x \in U \) either \( d(x) = 0 \) or \( x \in Z \). If \( x \in Z \) then \( xr = rx \) for all \( r \in N \). This gives us \( d(x)x + xd(r) = rd(x) + d(r)x \) for all \( r \in N \). Previous relation implies that \( d(x) \in Z \). Hence we conclude that \( d(U) \subseteq Z \). Now by Lemma 3.3.3 we infer that \( N \) is a commutative ring.

**Theorem 3.3.2.** Let \( N \) be a prime near-ring and \( U \) be a nonzero semigroup ideal of \( N \). If \( N \) admits a left generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \( f([x,y]) = \pm[x,y] \) for all \( x, y \in U \), then \( N \) is a commutative ring.

Proof. We have \( f([x,y]) = \pm[x,y] \) for all \( x, y \in U \). Putting \( xy \) in place of \( y \), obtaining \( f([x,xy]) = f(x[x,y]) = d(x)[x,y] + xf([x,y]) = \pm x[x,y] \). Using our hypothesis we get \( d(x)xy = d(x)yx \) for all \( x, y \in U \) which is identical with the relation (3.3.1). Now arguing in the similar way as in the Theorem 3.3.1 we conclude that \( N \) is a commutative ring.
Corollary 3.3.1 (Theorem 2.2. [26]). Let $N$ be a prime near-ring. If $N$ admits a nonzero derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in N$, then $N$ is a commutative ring.

Corollary 3.3.2 (Theorem 2.3. [26]). Let $N$ be a prime near-ring. If $N$ admits a nonzero derivation $d$ such that $d([x, y]) = -[x, y]$ for all $x, y \in N$, then $N$ is a commutative ring.

The conclusion of Theorems 3.3.1 and 3.3.2 remains valid if we replace the product $[x, y]$ by $xoy$ and nonzero semigroup ideals by nonzero ideals respectively. In fact, we obtain the following results:

Theorem 3.3.3. Let $N$ be a prime near-ring and $I$ be a nonzero ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f(xoy) = 0$ for all $x, y \in I$, then $N$ is a commutative ring.

Proof. Suppose that $f(xoy) = 0$ for all $x, y \in I$. Putting $xy$ in place of $y$, we obtain $f(xoxy) = f(x(xoy)) = d(x)(xoy) + xf(xoy) = 0$. Since the second term is zero, it is clear that

$$d(x)xy = -d(x)yx \text{ for all } x, y \in I.$$  \hspace{1cm} (3.3.2)

Replacing $y$ by $yz$ where $z \in N$ in (3.3.2) and using this relation again, we get

$$d(x)y(-x)z + d(x)yxz = 0 \text{ i.e., } d(x)I((-x)z + zx) = \{0\} \text{ for all } x, y \in I, z \in N.$$ 

Since $(I, +)$ is a normal subgroup of $(N, +)$, $x \in I$ implies that $-x \in I$. Now replacing $x$ by $-x$ in the preceding relation we find that $d(-x)I[x, z] = \{0\}$ for all $x \in I, z \in N$. As $I$ is an ideal of $N$, $I$ is also a semigroup ideal of $N$. Hence by Lemma 3.2.2 for each $x \in I$ either $d(-x) = 0$ i.e., $d(x) = 0$ or $x \in Z$. If $x \in Z$ then $xr = rx$ for all $r \in N$. This gives us $d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$. Previous relation implies that $d(x) \in Z$. Hence we conclude that $d(I) \subseteq Z$. Now by Lemma 3.3.3 we infer that $N$ is a commutative ring.

Using similar arguments as used in the proof of the above theorem, one can prove the following:

Theorem 3.3.4. Let $N$ be a prime near-ring and $I$ be a nonzero ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f(xoy) = \pm(xoy)$ for all $x, y \in I$, where $I$ is a nonzero ideal of $N$, then $N$ is a commutative ring.
Corollary 3.3.3 ([26, Theorem 2.4.]). Let $N$ be a prime near-ring. If $N$ admits a nonzero derivation $d$ such that $d(xoy) = xoy$ for all $x, y \in N$, then $N$ is a commutative ring.

Corollary 3.3.4 ([26, Theorem 2.5.]). Let $N$ be a prime near-ring. If $N$ admits a nonzero derivation $d$ such that $d(xoy) = -(xoy)$ for all $x, y \in N$, then $N$ is a commutative ring.

Theorem 3.3.5. Let $N$ be a prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f([x, y]) = \pm(xoy)$ for all $x, y \in U$, then $N$ is a commutative ring.

Proof. Assume that $f([x, y]) = \pm(xoy)$ for all $x, y \in U$. Putting $xy$ in place of $y$, we obtain $f([x, xy]) = f(x[x, y]) = d(x)[x, y] + xf([x, y]) = \pm x(xoy)$. Using our hypothesis we get $d(x)xy = d(x)yx$ for all $x, y \in U$ which is identical with the relation (3.3.1). Now arguing in the similar way as in the Theorem 3.3.1, we conclude that $N$ is a commutative ring. \qed

Theorem 3.3.6. Let $N$ be a prime near-ring and $I$ be a nonzero ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f(xoy) = \pm[x, y]$ for all $x, y \in I$, then $N$ is a commutative ring.

Proof. Use similar arguments as used in the proof of Theorem 3.3.3 \qed

The following example shows that the restriction of primeness imposed on the hypothesis of the above theorems is not superfluous.

Example 3.3.1. Let $S$ be a noncommutative zero symmetric left near-ring and let $N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}$. Then $N$ is a left near-ring and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b, 0 \in S \right\}$ is both a nonzero ideal and a nonzero semigroup ideal of $N$. Define $f : N \to N$ by $f \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that $f$ is a left generalized derivation of $N$ with associated nonzero derivation $d : N \to N$ defined by $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

If we set $p = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ with $0 \neq s$, then $pNp = \{0\}$ proving that $N$ is not a prime near-ring. It can be easily shown that $N$ satisfies the following properties: (i) $f([x, y]) = 0$, (ii) $f([x, y]) = \pm [x, y]$, (iii) $f(xoy) = 0$, (iv) $f(xoy) = \pm(xoy)$, (v) $f([x, y]) = \pm(xoy)$, (vi) $f(xoy) = \pm [x, y]$ for all $x, y \in I$. However, $N$ is not a commutative ring.
Recently Öznur Gölbasi [47, Theorem 2.6.] proved that if $A'$ is a prime near-ring with a nonzero generalized derivation $f$ associated with a derivation $d$ such that $f(N) \subseteq Z$ then $(N,+)$ is an abelian. Moreover if $N$ is 2-torsion free, then $N$ is a commutative ring. We have generalized this result for semigroup ideals. The following result shows that "2-torsion free restriction" in the above result used by Öznur Gölbasi is superfluous. In fact, we have obtained the following:

**Theorem 3.3.7.** Let $N$ be a prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $N$ admits a generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f(U) \subseteq Z$, then $N$ is a commutative ring.

**Proof.** For all $u_1, u'_1 \in U$, we have $f(u_1u'_1) = d(u_1)u'_1 + u_1f(u'_1) \in Z$. Hence $u_1\{d(u_1)u'_1 + u_1f(u'_1)\} = \{d(u_1)u'_1 + u_1f(u'_1)\}u_1$. Using the hypothesis and Lemma 3.3.6 we find that $u_1d(u_1)u'_1 = d(u_1)u'_1u_1$. Now replacing $u'_1$ by $u'_1r$ where $r \in N$ in the preceding identity and using it again we have $d(u_1)u'_1[u_1, r] = 0$ i.e., $d(u_1)U[u_1, r] = \{0\}$. By Lemma 3.2.2 we infer that for each fixed $u_1 \in U$ either $d(u_1) = 0$ or $u_1 \in Z$. If second condition holds then $u_1r = ru_1$ for all $r \in N$. This gives us that $d(u_1)r + u_1d(r) = rd(u_1) + d(r)u_1$ for all $r \in N$. Previous relation implies that $d(u_1) \in Z$. Lastly we conclude that $d(U) \subseteq Z$ and by Lemma 3.3.3, $N$ is a commutative ring. □

**Theorem 3.3.8.** Let $N$ be a prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $N$ admits a generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $d(Z) \neq \{0\}$ and $f([x, y]) \in Z$ for all $x, y \in U$, then $N$ is a commutative ring.

**Proof.** We are given that for all $x, y \in U$, $f([x, y]) \in Z$.

CaseI: If $Z = \{0\}$, it follows that $f([x, y]) = 0$ for all $x, y \in U$. This is identical with Theorem 3.3.1. Hence for this case, the proof of the Theorem 3.3.1 shows that $N$ is a commutative ring.

CaseII: If $Z \neq \{0\}$, replacing $y$ by $yz$ where $z \in Z$ in our hypothesis, we get $d(z)[x, y] + zf[x, y] \in Z$ for all $x, y \in U, z \in Z$. Using our hypothesis again together with Lemma 3.3.6 previous relation forces $d(z)[x, y] \in Z$ for all $x, y \in U, z \in Z$. Since $z \in Z$, $zx = rz$ for all $r \in N$, which gives us $d(z)r + zd(r) = rd(z) + d(r)z$ for all $r \in N$. Previous relation implies that $d(z) \in Z$. Therefore we find that $[d(z)[x, y], t] = d(z)[[x, y], t] = 0$ for all $t \in N$ and thus $d(z)N[x, y], t] = \{0\}$ for all $x, y \in U, t \in N, z \in Z$. Now primeness of $N$ yields $d(Z) = \{0\}$ or $[[x, y], t] = 0$ for all $x, y \in U$ and $t \in N$. By our hypothesis $d(Z) \neq \{0\}$ therefore $[[x, y], t] = 0$ for all $x, y \in U, t \in N$. Substituting $xy$ for $y$ in preceding relation we get $[x][x, y], t] = 0$ but $[x, y] \in Z$ and therefore $[x, y][x, t] = 0$ for all $x, y \in U, t \in N$. As $[x, y] \in Z$, $[x, y]N[x, y] = \{0\}$ for all $x, y \in U$. In the light
of primeness of \( N \), we obtain that \([x, y] = 0\) for all \( x, y \in U \). Putting \( yr \) for \( y \) where \( r \in N \) in the previous relation and using the same again we get \( y[x, r] = 0 \). Now again replacing \( y \) by \( ys \), where \( s \in N \) in the relation \( y[x, r] = 0 \), we have \( yN[x, r] = \{0\} \). Since \( N \) is a prime near-ring and \( U \neq \{0\} \), we conclude that \( U \subseteq Z \). Now by Lemma 3.2.3, \( N \) is a commutative ring.

**Theorem 3.3.9.** Let \( N \) be a prime near-ring and \( U \) a nonzero semigroup ideal of \( N \). If \( N \) admits a generalized derivation \( f \) with associated nonzero derivation \( d \) of \( N \) such that \([f(x), y] \in Z\) for all \( x, y \in U \), then \( N \) is a commutative ring.

**Proof.** Assume that \([f(x), y] \in Z\) for all \( x, y \in U \). Hence \([[[f(x), y], t] = 0\) for all \( x, y \in U, t \in N \). Replacing \( y \) by \( f(x)y \) in the previous relation we find that \([f(x)][f(x), y], t] = 0\) for all \( x, y \in U, t \in N \). In view of hypothesis, we get \([f(x), y][f(x), t] = 0\) i.e., \([f(x), y]N[f(x), y] = \{0\}\) for all \( x, y \in U \). Primeness of \( N \) yields \([f(x), y] = 0\) i.e., \( f(x)y = yf(x) \) for all \( x, y \in U \). Putting \( yr \) for \( y \) where \( r \in N \) in the preceding relation and using the same again we arrive at \( y[f(x), r] = 0\) for all \( x, y \in U, r \in N \). Now substituting \( ys \) for \( y \) where \( s \in N \) we get \( yN[f(x), r] = \{0\}\). Since \( U \neq \{0\} \), primeness of \( N \) yields \( f(U) \subseteq Z \). By application of Theorem 3.3.7 we conclude that \( N \) is a commutative ring. \( \square \)

The following example shows that the restriction of primeness imposed on the hypothesis of the above Theorems 3.3.8 and 3.3.9 is not superfluous.

**Example 3.3.2.** Let \( S \) be a noncommutative zero symmetric left near-ring and let

\[
N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in S \right\}.
\]

Then \( N \) is a left near-ring and \( U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\} \) is a nonzero semigroup ideal of \( N \). Define \( f : N \to N \) by \( f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \). Then it is easy to see that \( f \) is a generalized derivation of \( N \) with associated nonzero derivation.
Theorem 3.3.10. Let $N$ be a prime near-ring with no nonzero divisors of zero, and $U$ a nonzero semigroup right ideal of $N$. If $N$ admits a left generalized derivation $f$ with associated nonzero derivation $d$ of $N$ such that $f([x,y]) = 0$ for all $x,y \in U$, then $N$ is a commutative ring.

Proof. Assume that $f([x,y]) = 0$ for all $x,y \in U$. Putting $xy$ in place of $y$, we obtain $f([x,xy]) = f(x,y) = d(x)[x,y] + xf([x,y]) = 0$. Since the second term is zero, it is clear that $d(x)[x,y] = 0$ for all $x,y \in U$. Thus by hypothesis for each $x \in U$, either $d(x) = 0$ or $x$ centralizes $U$. Applying Lemma 3.2.1(ii), we see that either $d(x) = 0$ or $x \in Z$. If $x \in Z$ then $rx = xr$ for all $r \in N$, which gives us $d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$. Previous relation implies that $d(x) \in Z$. Therefore we conclude that $d(U) \subseteq Z$. Lastly result follows by Lemma 3.3.3.

Corollary 3.3.5 ([22, Theorem 3.5.]). Let $N$ be a prime near-ring with no nonzero divisors of zero, and $U$ a nonzero semigroup right ideal of $N$. If $N$ admits a nonzero derivation $d$ such that $d(xy) = d(yx)$ for all $x,y \in U$, then $N$ is a commutative ring.

3.4 Near-rings with involution

The involution in rings has been studied by several authors in different directions and it has got tremendous applications in various areas of mathematics (see [51], for further reference). Motivated by this concept we introduce the notion of involution in near-rings.

We have shown that certain near-rings with involutions are rings:

Definition 3.4.1. Let $N$ be a left near-ring. An additive mapping $x \mapsto x^*$ on $N$ is said to be an involution on $N$ if (i) $(x^*)^* = x$ and (ii) $(xy)^* = y^*x^*$ hold for all $x,y \in N$.

In this case we call that $N$ is a near-ring with involution or $*$-near-ring. It is trivial to see that involution $^*$ satisfies the following properties, (i) $0^* = 0$, (ii) $(-x)^* = -x^*$ and (iii) $^*$ is a bijective map. Finally we can say that $^*$ is a near-ring anti-automorphism of $N$. 

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Example 3.4.1. Let $S$ be a zero symmetric left near-ring. Suppose

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}.$$ 

Define $*: N \rightarrow N$ such that

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then, it is straightforward to check that $N$ is a zero symmetric left near-ring and $'*'$ is an involution of $N$.

Example 3.4.2. Suppose $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}$, where $S$ is a commutative near-ring. Define $*: N \rightarrow N$ such that

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$ 

It is straightforward to check that $N$ is a $*$-near-ring.

Now we state the following lemma which plays a key role in proving our main result.

Lemma 3.4.1. Let $N$ be a near-ring with involution $'*'$. Then

(i) $N$ is a distributive near-ring.

(ii) $N$ is a pseudo-abelian near-ring i.e.; $xy + zt = zt + xy$ for all $x, y, z, t \in N$.

Proof. (i) For all $x, y, z \in N$ we have $(y + z)x^* = x^*y^* + x^*z^*$, now taking the image of both the sides under $'*'$ we get $(y + z)x = yx + zx$. This means that $N$ is a distributive near-ring.

(ii) Since $N$ has both distributive properties, expanding $(x + z)(t + y)$ for all $x, y, z, t \in N$, we have $xt + xy + zt + zy = xt + zt + xy + zy$. This implies our required result. \qed

We are aware of the notions of prime rings with involution, semiprime rings with involution and $*$-prime rings in ring theory earlier with their nice properties. Motivated by
these concepts, we introduce involution ‘*’ on prime near-ring and semiprime near-rings and prove that prime near-rings with involution, semiprime near-rings with involution and *-prime near-rings are prime rings, semiprime rings and *-prime rings respectively.

**Definition 3.4.2.** Let \( N \) be a near-ring with involution ‘*’. Near ring \( N \) is called *-prime near-ring if \( a, b \in N, aNb = \{0\} \) and \( aNb^* = \{0\} \) (equivalently \( a, b \in N, aNb = \{0\} \) and \( a^*Nb = \{0\} \)) implies that \( a = 0 \) or \( b = 0 \).

Now we prove our main results of this section as given below:

**Theorem 3.4.1.** Let \( N \) be a semiprime near-ring with involution. Then \( N \) is a ring.

**Proof.** Since \( N \) is a semiprime near-ring with involution ‘*’, by above lemma we obtain that \( N \) is a distributive near-ring and for all \( x, y, z, t \in N \) we have \( xy + zt = zt + xy \). Now replacing \( y \) by \( t \) in the latter relation we obtain that \( xt + zt - xt - zt = 0 \) for all \( x, z, t \in N \). This implies that \( (x + z - x - z)N = \{0\} \) i.e.; \( (x + z - x - z)N(x + z - x - z) = \{0\} \).

Now semiprimeness of \( N \) provides that \( x + z = z + x \) for all \( x, z \in N \). Therefore \( (N, +) \) is abelian. Finally we conclude that \( N \) is a ring. \( \square \)

**Corollary 3.4.1.** Let \( N \) be a prime near-ring with involution. Then \( N \) is a ring.

**Theorem 3.4.2.** Let \( N \) be a *-prime near-ring. Then \( N \) is a *-prime ring.

**Proof.** Since \( N \) is *-prime near-ring, by above lemma we obtain that \( N \) is a distributive near-ring and for all \( x, y, z, t \in N \) we have \( xy + zt = zt + xy \). Now replacing \( y \) by \( t \) in the previous relation we obtain that \( xt + zt - xt - zt = 0 \) for all \( x, z, t \in N \). This implies that \( (x + z - x - z)N = \{0\} \). In turn we obtain that \( (x + z - x - z)Nl = \{0\} = (x + z - x - z)Nl^* \), where \( 0 \neq l \in N \). Now *-primeness of \( N \) provides that \( x + z = z + x \) for all \( x, z \in N \). Therefore \( (N, +) \) is abelian. Finally we conclude that \( N \) is a *-prime ring. \( \square \)
Chapter 4

*-n-Derivation in rings with involution

4.1 Introduction

An additive mapping \( x \mapsto x^* \) of a ring \( R \) into itself is called an involution on \( R \) if it satisfies the conditions; (i) \((x^*)^* = x\), (ii) \((xy)^* = y^*x^*\) for all \( x, y \in R \). A ring \( R \) equipped with an involution \('*'\) is called a \;++-ring. A ring \( R \) with involution \('*'\) is said to be \;++-prime if \( aRb = aRb^* = \{0\} \), where \( a, b \in R \) (equivalently \( aRb = a^*Rb = \{0\} \), where \( a, b \in R \) ) implies that either \( a = 0 \) or \( b = 0 \). It is to be noted that every prime ring having an involution \('*'\) is \;++-prime but the converse is not true in general. Of course, if \( R^o \) denotes the opposite ring of a prime ring \( R \), then \( R \times R^o \) equipped with the exchange involution \(*_{ex} \), defined by \(*_{ex}(x, y) = (y, x) \), is \;++-prime but not prime.

An ideal \( I \) of \( R \) is called a \;++-ideal of \( R \) if \( I^* = I \). Let \( R \) be a \;++-prime ring, \( a \in R \) such that \( aRa = \{0\} \). This implies that \( aRa^* = \{0\} \) also. Now \;++-primeness of \( R \) insures that \( a = 0 \) or \( aRa^* = \{0\} \). Now \( aRa^* = \{0\} \) together with \( aRa = \{0\} \) gives us \( a = 0 \). Thus we conclude that every \;++-prime ring is semiprime.

We introduce the notion of \;++-n-derivation in the \;++-ring \( R \), where \( n \) is a positive integer, and also investigate its various properties in Section 4.2. In fact, it is shown that if a prime \;++-ring \( R \) admits a nonzero \;++-n-derivation (resp. reverse \;++-n-derivation), then \( R \) is commutative. Further, some related properties of \;++-n-derivation in semiprime \;++-ring have also been investigated. Finally, a structure theorem for \;++-n-derivation has also been obtained.

Section 4.3 is devoted to the extension of Posner's first theorem in the setting of \;++-prime rings of characteristic different from 2. It is shown that if \( R \) is a \;++-prime ring of characteristic not 2 and \( d_1, d_2 \) derivations of \( R \) such that the iterate \( d_1d_2 \) is also a
derivation of $R$ and at least one of $d_1$ and $d_2$ commutes with '$\ast$', then $d_1 = 0$ or $d_2 = 0$.

### 4.2 $\ast$-n-derivation in ring with involution

An additive mapping $d : R \rightarrow R$ is said to be a derivation (resp. reverse derivation) on $R$ if $d(xy) = d(x)y + xd(y)$ (resp. $d(xy) = d(y)x + yd(x)$) holds for all $x, y \in R$. Let $R$ be a $\ast$-ring. An additive mapping $d : R \rightarrow R$ is said to be a $\ast$-derivation (resp. $\ast$-reverse derivation) on $R$ if $d(xy) = d(x)y^\ast + xd(y)$ (resp. $d(xy) = d(y)x^* + yd(x)$) holds for all $x, y \in R$. If $R$ is a commutative $\ast$-ring then $d : R \rightarrow R$ defined by $d(x) = a(x - x^*)$, where $a \in R$, is a $\ast$-derivation on $R$ (for reference see [32]). An additive map $T : R \rightarrow R$ is called a left (resp. right) $\ast$-multiplier if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. There has been a great deal of work concerning commutativity of prime and semiprime rings admitting certain types of derivations (for reference see [10] - [23], [33], [75], [83] etc., where further references can be found). Very recently Ali [4] defined symmetric $\ast$-biderivation, symmetric left (resp. right) $\ast$-bimultiplier and studied some properties of prime $\ast$-rings and semiprime $\ast$-rings, admitting symmetric $\ast$-biderivation and symmetric left (resp. right) $\ast$-bimultiplier. Motivated by these concepts and the notion of $n$-derivation given by Park (see [72]) we introduce the concept of $\ast$-n-derivation (reverse $\ast$-n-derivation) and $\ast$-n-multiplier in the setting of $\ast$-rings.

Let $n$ be any fixed positive integer. An $n$-additive (i.e., additive in each argument) mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a $\ast$-n-derivation of $R$ if the relations

$$D(x_1 x'_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x'_1) + x_1 D(x'_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2 x'_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x'_2) + x_2 D(x'_1, x_2, \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_n x'_n) = D(x_1, x_2, \cdots, x_n)(x'_n) + x_n D(x_1, x_2, \cdots, x'_n)$$

hold for all $x_1, x'_1, x_2, x'_2, \cdots, x_n, x'_n \in R$.

Similarly an $n$-additive mapping $D : R \times R \times \cdots \times R \rightarrow R$ is called a reverse $\ast$-n-derivation of $R$ if the relations

$$D(x_1 x'_1, x_2, \cdots, x_n) = D(x'_1, x_2, \cdots, x_n)x_1 + x'_1 D(x_1, x_2, \cdots, x_n)$$

$$D(x_1, x_2 x'_2, \cdots, x_n) = D(x_1, x'_2, \cdots, x_n)x_2 + x'_2 D(x_1, x_2, \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_n x'_n) = D(x_1, x_2, \cdots, x'_n)x_n + x'_n D(x_1, x_2, \cdots, x_n)$$

hold for all $x_1, x'_1, x_2, x'_2, \cdots, x_n, x'_n \in R$. 

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\[ D(x_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n') x_n^* + x_n D(x_1, x_2, \cdots, x_n) \]

hold for all \( x_1, x_2, \cdots, x_n, x_n' \in R \).

For an example of \(*\)-\( n \)-derivation, consider \( C \) the ring of complex numbers with involution \(*\) defined by \( z^* = \bar{z} \), where \( \bar{z} \) denotes the conjugate of the complex number \( z \).

Now define \( D : C \times C \times \cdots \times C \to C \) such that

\[ D(z_1, z_2, \cdots, z_n) = \lambda (z_1 - \bar{z}_1)(z_2 - \bar{z}_2) \cdots (z_n - \bar{z}_n) \]

where \( \lambda \) is any fixed complex number. One can easily verify that \( D \) is a \(*\)-\( n \)-derivation of \( C \).

An \( n \)-additive mapping \( T_1 : R \times R \times \cdots \times R \to R \) is called a left \(*\)-\( n \)-multiplier of \( R \) if

\[ T(x_1 x_1, x_2, \cdots, x_n) = T(x_1, x_2, \cdots, x_n)(x_1^*) \]
\[ T(x_1, x_2 x_2, \cdots, x_n) = T(x_1, x_2, \cdots, x_n)(x_2^*) \]
\[ \vdots \]
\[ T(x_1, x_2, \cdots, x_n x_n') = T(x_1, x_2, \cdots, x_n)(x_n^*) \]

hold for all \( x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in R \).

An \( n \)-additive mapping \( T_2 : R \times R \times \cdots \times R \to R \) is called a right \(*\)-\( n \)-multiplier of \( R \) if

\[ T(x_1 x_1, x_2, \cdots, x_n) = x_1^* T(x_1, x_2, \cdots, x_n) \]
\[ T(x_1, x_2 x_2, \cdots, x_n) = x_2^* T(x_1, x_2, \cdots, x_n) \]
\[ \vdots \]
\[ T(x_1, x_2, \cdots, x_n x_n') = x_n^* T(x_1, x_2, \cdots, x_n') \]

hold for all \( x_1, x_1', x_2, x_2', \cdots, x_n, x_n' \in R \).

For examples of left \(*\)-\( n \)-multiplier and right \(*\)-\( n \)-multiplier, consider \( S \) to be a commutative ring which is not a zero ring and \( R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\} \). Define

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Let \( D \) be a \(*\)-\( n \)-derivation of \(*\)-ring \( R \). If \( D \) is also a permuting map, then all the above \( n \)-conditions used in the definition of \(*\)-\( n \)-derivation are equivalent and in this case \( D \) is called permuting \(*\)-\( n \)-derivation of \(*\)-ring \( R \) i.e.; an \( n \)-additive permuting map \( D : R \times R \times \cdots \times R \longrightarrow R \) is called a permuting \(*\)-\( n \)-derivation of \(*\)-ring \( R \) if

\[
D(x_1, x_2, \cdots, x_n) = D(x_1^{'}, x_2, \cdots, x_n)(x_1^{'}) + x_1 D(x_1^{'}, x_2, \cdots, x_n)
\]

hold for all \( x_1, x_2, \cdots, x_n \in R \). It is obvious that every permuting \(*\)-\( n \)-derivation of \(*\)-ring \( R \) is also a \(*\)-\( n \)-derivation but its converse is not true. For justification, let us consider the following example: Let \( S \) be a noncommutative ring. Set

\[
T_1, T_2 : R \times R \times \cdots \times R \longrightarrow R \text{ and } r \mapsto r^* \text{ of } R \text{ into itself, where } r \in R \text{ such that}
\]

\[
T_1 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_1x_2 \cdots x_n & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
T_2 \left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_1z_2 \cdots z_n & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}
\]

One can easily verify that \('*'\) is an involution on \( R \). Also it is straightforward to check that \( T_1 \) is a nonzero left \(*\)-\( n \)-multiplier but not a right \(*\)-\( n \)-multiplier of the \(*\)-ring \( R \) and \( T_2 \) is a nonzero right \(*\)-\( n \)-multiplier but not a left \(*\)-\( n \)-multiplier of the \(*\)-ring \( R \). Finally an \( n \)-additive mapping \( T : R \times R \times \cdots \times R \longrightarrow R \) is called an \(*\)-\( n \)-multiplier of \( R \) if it is both a left \(*\)-\( n \)-multiplier and a right \(*\)-\( n \)-multiplier of \( R \). For an example of \(*\)-\( n \)-multiplier, consider \( C \) the ring of complex numbers with involution \('*'\) defined by \( z^* = \bar{z} \), where \( \bar{z} \) denotes the conjugate of the complex number \( z \). Now define \( T : C \times C \times \cdots \times C \longrightarrow C \) such that \( T(z_1, z_2, \cdots, z_n) = \mu \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n \), where \( \mu \) is any fixed complex number. One can easily verify that \( T \) is a \(*\)-\( n \)-multiplier of \( C \).
Define $D : R \times R \times \cdots \times R \rightarrow R$ and $r \mapsto r^*$ of $R$ into itself, where $r \in R$ such that

$$D\left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & x_1x_2\cdots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily show that $R$ is a $*$-ring and $D$ is a $*$-$n$-derivation of $R$. However $D$ is not a permuting $*$-$n$-derivation of $R$. Similarly the notions of permuting left $*$-$n$-multiplier, permuting right $*$-$n$-multiplier and permuting $*$-$n$-multiplier can be defined. It is obvious to observe that the map $D$ just discussed above is also a $*$-$n$-multiplier of $*$-ring $R$ but not a permuting $*$-$n$-multiplier of $R$.

In 1989 Brešar and Vukman (see Proposition 1 of [32]) proved that if a prime $*$-ring $R$ admits a nonzero $*$-derivation (resp. reverse $*$-derivation), then $R$ is commutative. We have proved its analogue in the setting of $*$-$n$-derivation for prime $*$-rings. Some properties of $*$-$n$-derivation in semiprime $*$-rings have also been discussed. Some results related with $*$-$n$-multipliers in prime $*$-rings and semiprime $*$-rings have also been obtained. In the beginning of this section, we have shown that $*$-derivations generate different $*$-$n$-derivations in prime $*$-rings or commutative $*$-rings. In the end of present section a structure theorem for $*$-$n$-derivation in commutative $*$-rings has also been investigated.

We facilitate our discussion with the following lemmas;

**Lemma 4.2.1.** Let $R$ be a prime $*$-ring having $*$-derivations $d_1, d_2, \ldots, d_n$. If $D : R \times R \times \cdots \times R \rightarrow R$ such that $D(x_1, x_2, \cdots, x_n) = \{d_1(x_1)\}^*\{d_2(x_2)\}^*\cdots\{d_n(x_n)\}^*$, then $D$ is a $*$-$n$-derivation of $R$.

**Proof.** If at least one among $d_1, d_2, \ldots, d_n$ is a zero map then we are done. Now suppose that none of the given $*$-derivations of $R$ is zero. Then by Proposition 1 of [32] one
conclude that $R$ is commutative. Consider

$$D(x_1 + x'_1, x_2, \cdots, x_n) = \{d_1(x_1 + x'_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\}$$

$$= \{d_1(x_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\}$$

$$+ \{d_1(x'_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\}$$

$$= D(x_1, x_2, \cdots, x_n) + D(x'_1, x_2, \cdots, x_n).$$

Thus $D$ is additive in the first argument. Similarly we can prove that it is additive in all arguments. Therefore $D$ is an $n$-additive map.

Consider

$$D(x_1x'_1, x_2, \cdots, x_n) = \{d_1(x'_1x_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\}$$

$$= \{d_1(x'_1)x_1 + x'_1d_1(x_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\}$$

$$+ \{d_1(x'_1)\} \cdot \{d_2(x_2)\} \cdots \{d_n(x_n)\} \cdot (x'_1)^*$$

$$= D(x_1, x_2, \cdots, x_n)(x'_1)^* + x_1D(x'_1, x_2, \cdots, x_n).$$

Similarly we can prove that the above property holds in all arguments. Therefore, $D$ is a $*-$n-derivation of $R$.

---

**Lemma 4.2.2.** Let $R$ be a prime $*$-ring having $*$-derivations $d_1, d_2, \cdots, d_n$. If $D : R \times R \times \cdots \times R \rightarrow R$ such that $D(x_1, x_2, \cdots, x_n) = d_1(x_1)d_2(x_2) \cdots d_n(x_n)$, then $D$ is a $*-$n-derivation of $R$.

**Proof.** If at least one among $d_1, d_2, \cdots, d_n$ is a zero map then we are done. Now suppose that none of the given $*$-derivations of $R$ is zero. Then by Proposition 1 of [32] one conclude that $R$ is commutative. It can be seen that $D$ is an $n$-additive map, and

$$D(x_1x'_1, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)(x'_1)^* + x_1D(x'_1, x_2, \cdots, x_n).$$

Similarly we can prove that the above property holds in all arguments. Therefore $D$ is a $*-$n-derivation of $R$. □

**Lemma 4.2.3.** Let $R$ be a prime $*$-ring having $*$-n-derivations $D_1$ and $D_2$. Further assume that $I_1, I_2, \cdots, I_n$ are nonzero right ideals of $R$ such that $D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n)$ for all $i_r \in I_r, 1 \leq r \leq n$. Then $D_1 = D_2$.

**Proof.** We have

$$D_1(i_1, i_2, \cdots, i_n) = D_2(i_1, i_2, \cdots, i_n) \quad (4.2.1)$$
for all $i_r \in I_r, 1 \leq r \leq n$. Now putting $i_1 r_1$, where $r_1 \in R$, for $i_1$ in the relation (4.2.1) we have $D_1(i_1 r_1, i_2, \cdots , i_n) = D_2(i_1 r_1, i_2, \cdots , i_n)$ i.e.; $D_1(i_1, i_2, \cdots , i_n) r_1^* + i_1 D_1(r_1, i_2, \cdots , i_n) = D_2(i_1, i_2, \cdots , i_n) r_1^* + i_1 D_2(r_1, i_2, \cdots , i_n)$. Using the relation (4.2.1) we get $i_1 D_1(r_1, i_2, \cdots , i_n) = i_1 D_2(r_1, i_2, \cdots , i_n)$ i.e.; $i_1 \{ D_1(r_1, i_2, \cdots , i_n) - D_2(r_1, i_2, \cdots , i_n) \} = 0$. This shows that $i_1 R \{ D_1(r_1, i_2, \cdots , i_n) - D_2(r_1, i_2, \cdots , i_n) \} = \{ 0 \}$. Since $I_1 \neq \{ 0 \}$, primeness of $R$ implies that

$$D_1(r_1, i_2, \cdots , i_n) = D_2(r_1, i_2, \cdots , i_n) \quad (4.2.2)$$

for all $r_1 \in R, i_r \in I_r, 2 \leq r \leq n$. Now putting $i_2 r_2$, where $r_2 \in R$, for $i_2$ in the relation (4.2.2) and using the similar arguments we find that $D_1(r_1, r_2, i_3, \cdots , i_n) = D_2(r_1, r_2, i_3, \cdots , i_n)$. Now proceeding inductively in the same way as above we conclude that $D_1 = D_2$.

**Remark 4.2.1.** In 1989 Brešar and Vukman [32, Proposition 1] proved that if a prime $*$-ring $R$ admits a nonzero $*$-derivation (resp. reverse $*$-derivation) then $R$ is commutative. Recently Ali [4, Theorems 3.3 & 3.4] proved its analogue in the setting of symmetric $*$-bi-derivation for prime $*$-rings. We have shown that the restriction of symmetry of $*$-bi-derivation used by Ali is redundant. In fact, for $*$-$n$-derivation in a prime $*$-ring, we have obtained the following.

**Theorem 4.2.1.** Let $R$ be a prime $*$-ring. If it admits a nonzero $*$-$n$-derivation (resp. reverse $*$-$n$-derivation) $D$, then $R$ is commutative.

**Proof.** By hypothesis we have, for all $x_1, y, z, x_2, \cdots , x_n \in R$

$$D((x_1 y)z, x_2, \cdots , x_n) = D(x_1 y, x_2, \cdots , x_n) z^* + x_1 y D(z, x_2, \cdots , x_n)$$

$$= \{ D(x_1, x_2, \cdots , x_n) y^* + x_1 D(y, x_2, \cdots , x_n) \} z^*$$

$$+ x_1 y D(z, x_2, \cdots , x_n)$$

$$= D(x_1, x_2, \cdots , x_n) y^* z^* + x_1 D(y, x_2, \cdots , x_n) z^*$$

$$+ x_1 y D(z, x_2, \cdots , x_n).$$

Also

$$D(x_1(y z), x_2, \cdots , x_n) = D(x_1, x_2, \cdots , x_n)(y z)^* + x_1 D(y z, x_2, \cdots , x_n)$$

$$= D(x_1, x_2, \cdots , x_n) z^* y^* + x_1 \{ D(y, x_2, \cdots , x_n) z^*$$

$$+ y D(z, x_2, \cdots , x_n) \}$$

$$= D(x_1, x_2, \cdots , x_n) z^* y^* + x_1 D(y, x_2, \cdots , x_n) z^*$$

$$+ x_1 y D(z, x_2, \cdots , x_n).$$

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Combining the above two relations, we get
\[ D(x_1, x_2, \cdots, x_n)y^*z^* = D(x_1, x_2, \cdots, x_n)z^*y^* \text{ for all } x_1, x_2, \cdots, x_n, y, z \in R. \]
Putting \( y^* \) and \( z^* \) in the places of \( y \) and \( z \) respectively, we find that
\[ D(x_1, x_2, \cdots, x_n)yz = D(x_1, x_2, \cdots, x_n)zy. \quad (4.2.3) \]

Now replacing \( y \) by \( yr \) where \( r \in R \), in the relation (4.2.3) and using it again we arrive at
\[ D(x_1, x_2, \cdots, x_n)yrz = D(x_1, x_2, \cdots, x_n)yvr \text{ i.e.; } D(x_1, x_2, \cdots, x_n)R[r, z] = \{0\}. \]
Since \( D \neq 0 \), primeness of \( R \) implies that \( rz = zr \) for all \( z, r \in R \). Therefore, we conclude that \( R \) is commutative. \( \square \)

**Corollary 4.2.1.** Let \( R \) be a noncommutative prime ring with involution `*`. If it admits a \(*\)-\( n \)-derivation (resp. reverse \(*\)-\( n \)-derivation) \( D \), then \( D = 0 \).

Following example demonstrates that the primeness in the hypothesis of the above theorem can not be omitted.

**Example 4.2.1.** Let \( Q \) and \( C \) be the ring of real quaternions and complex numbers respectively. Assume \( R = Q \times C \) is the ring of cartesian product of \( Q \) and \( C \) with regard to componentwise addition and multiplication. Let \(*_1, *_2 \) and \(* \) denote the involutions of rings \( Q, C \) and \( R \) respectively, defined by \( q^*_1 = \alpha - \beta i - \gamma j - \delta k \), where \( q = \alpha + \beta i + \gamma j + \delta k \in Q; \) \( z^*_2 = x - iy \), where \( z = x + iy \in C \) and \( (q, z)^* = (q^*_1, z^*_2) \) for all \( (q, z) \in R \). Let \( d \) be \(*_2\)-derivation of \( C \) defined by \( d(z) = \eta(z - z^*) \) where \( \eta \) is any fixed complex number. Define \( D : R \times R \times \cdots \times R \rightarrow R \) such that \( D((q_1, z_1), (q_2, z_2), \cdots, (q_n, z_n)) = (0, d(z_1)d(z_2) \cdots d(z_n)). \) It can be easily verified that \( R \) is a semiprime ring but not a prime ring and \( D \) is a nonzero \(*\)-\( n \)-derivation of \( R \). However, \( R \) is not commutative.

**Remark 4.2.2.** Lemma 4.2.3 also holds good for left ideals. In fact, in Lemma 4.2.3, if both \( D_1 \) and \( D_2 \) are zero then result holds trivially. On the other hand if at least one out of \( D_1 \) and \( D_2 \) is nonzero, then by Theorem 4.2.1, \( R \) is commutative and therefore each right ideal is also a left ideal.

**Theorem 4.2.2.** Let \( R \) be a prime ring with involution `*`. If \( F : R^n \rightarrow R \) is a nonzero \( n \)-additive mapping such that \( F(x_1y, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)y^* \) for all \( x_1, y, x_2, \cdots, x_n \in R \), then \( R \) is commutative.

**Proof.** By hypothesis, for all \( x_1, y, z, x_2, \cdots, x_n \in R \) we have \( F(x_1(yz), x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)z^*y^* \). On the other hand we also have \( F((x_1y)z, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)z^*y^* \).
Combining the preceding two relations we have $F(x_1, x_2, \ldots, x_n)z^* y^* = F(x_1, x_2, \ldots, x_n)y^* z^*$. Replacing $y, z$ by $y^*$ and $z^*$ respectively we arrive at

$$F(x_1, x_2, \ldots, x_n)zy = F(x_1, x_2, \ldots, x_n)y z.$$  \hspace{1cm} (4.2.4)

Now putting $zr$ for $z$ where $r \in R$, in the relation (4.2.4) and using it again we have $F(x_1, x_2, \ldots, x_n)z[r, y] = 0$ i.e., $F(x_1, x_2, \ldots, x_n)R[r, y] = \{0\}$. Since $F \neq 0$, primeness of $R$ yields $[r, y] = 0$ for all $r, y \in R$. Finally, we conclude that the ring $R$ is commutative.

**Corollary 4.2.2.** Let $R$ be a prime $*$-ring. If it admits a nonzero left $*$-n-multiplier $T$, then $R$ is commutative.

**Corollary 4.2.3** ([4, Theorem 2.4]). Let $R$ be a prime $*$-ring. If $M : R \times R \rightarrow R$ is a nonzero biadditive mapping such that $M(xy, z) = M(x, z)y^*$ for all $x, y, z \in R$, then $R$ is commutative.

**Theorem 4.2.3.** Let $R$ be a prime $*$-ring. If $F : R^n \rightarrow R$ is a nonzero $n$-additive mapping such that $F(yx_1, x_2, \ldots, x_n) = y^* F(x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in R$, then $R$ is commutative.

**Proof.** Computing $F((yz)x_1, x_2, \ldots, x_n)$ and $F(y(zx_1), x_2, \ldots, x_n)$ where $x_1, y, z, x_2, \ldots, x_n \in R$, in two different ways according to the given hypothesis and using the similar techniques as used to prove Theorem 4.2.2, one can easily get the required result. \hfill \Box

**Corollary 4.2.4.** Let $R$ be a prime $*$-ring. If $R$ admits a nonzero right $*$-n-multiplier $T$, then $R$ is commutative.

**Corollary 4.2.5** ([4, Theorem 2.5]). Let $R$ be a prime $*$-ring. If $M : R \times R \rightarrow R$ is a nonzero bi-additive mapping such that $M(xy, z) = x^* M(y, z)$ for all $x, y, z \in R$, then $R$ is commutative.

The following example shows that the restriction of primeness imposed on the hypotheses of Theorems 4.2.2 and 4.2.3 is not superfluous.

**Example 4.2.2.** Consider the $*$-ring $R$ given in Example 4.2.1. Now define $D_1 : R \times R \times \cdots \times R \rightarrow R$ such that $D_1((q_1, z_1), (q_2, z_2), \ldots, (q_n, z_n)) = (0, z_1^{*2}z_2^{*2} \cdots z_n^{*2})$.

It can be easily verified that $R$ is a semiprime ring but not a prime ring and $D_1$ is a nonzero $n$-additive mapping of $R$ such that $D_1((q_1, z_1)(q_1', z_1'), (q_2, z_2), \ldots, (q_n, z_n)) = D_1((q_1, z_1), (q_2, z_2), \ldots, (q_n, z_n))(q_1', z_1'^*)$ and $D_1((q_1, z_1)(q_1', z_1'), (q_2, z_2), \ldots, (q_n, z_n)) = D_1((q_1, z_1), (q_2, z_2), \ldots, (q_n, z_n))$.
(q₁, z₁)D₁((q₁', z₁'), (q₂, z₂), · · · , (qₙ, zₙ)) hold for all (q₁, z₁), (q₁', z₁'), (q₂, z₂), · · · , (qₙ, zₙ) ∈ R. However R is not commutative.

Remark 4.2.3. The above Theorems 4.2.2 & 4.2.3 also hold if the relations in the hypotheses hold in any argument.

Theorem 4.2.4. Let R be a prime ring with involution ‘*’. If there exists 0 ≠ a ∈ R such that a*x = ±x*a for all x ∈ R. Then R is commutative.

Proof. We have a*x = ±x*a for all x ∈ R. Putting xy, where y ∈ R in place of x in the preceding relation and using it again we get a*xy = y'a*x for all x, y ∈ R. Now replacing x by xt, where t ∈ R in previous relation and using it again we obtain a*R[t, y] = {0} for all y, t ∈ R. Primeness of R yields either a* = 0 or [t, y] = 0. If first case holds then we obtain (a*)* = 0 i.e.; a = 0 which is contrary to our hypothesis. Thus we conclude that [t, y] = 0 for all t, y ∈ R. Hence R is commutative.

Corollary 4.2.6. Let R be a prime ring with involution ‘*’. If x'y = ±y'x for all x, y ∈ R. Then R is commutative.

Theorem 4.2.5. Let R be a 2-torsion free prime *-ring possessing *-n-derivations D₁ and D₂. Then

\[ D₁(x₁, x₂, · · · , xₙ)D₂(y₁, y₂, · · · , yₙ) + D₂(x₁, x₂, · · · , xₙ)D₁(y₁, y₂, · · · , yₙ) = 0 \]

for all x₁, x₂, · · · , xₙ, y₁, y₂, · · · , yₙ ∈ R iff either D₁ = 0 or D₂ = 0.

Proof. Given that

\[ D₁(x₁, x₂, · · · , xₙ)D₂(y₁, y₂, · · · , yₙ) + D₂(x₁, x₂, · · · , xₙ)D₁(y₁, y₂, · · · , yₙ) = 0 \] (4.2.5)

for all x₁, x₂, · · · , xₙ, y₁, y₂, · · · , yₙ ∈ R. Then we have to show that either D₁ = 0 or D₂ = 0. Now putting yₙz where z ∈ R in place of yₙ in identity (4.2.5) we arrive at D₁(x₁, x₂, · · · , xₙ)D₂(y₁z, y₂, · · · , yₙ) + D₂(x₁, x₂, · · · , xₙ)D₁(y₁z, y₂, · · · , yₙ) = 0 for all x₁, x₂, · · · , xₙ, y₁, y₂, · · · , yₙ, z ∈ R i.e.; D₁(x₁, x₂, · · · , xₙ){D₂(y₁, y₂, · · · , yₙ)z* + y₁D₂(x₁, x₂, · · · , xₙ)} + D₂(x₁, x₂, · · · , xₙ){D₁(y₁, y₂, · · · , yₙ)z* + y₁D₁(x₁, x₂, · · · , xₙ)} = 0. Now the previous relation takes the form

\[ \{D₁(x₁, x₂, · · · , xₙ)D₂(y₁, y₂, · · · , yₙ) + D₂(x₁, x₂, · · · , xₙ)D₁(y₁, y₂, · · · , yₙ)\}z* + D₁(x₁, x₂, · · · , xₙ)y₁D₂(x₁, x₂, · · · , yₙ) + D₂(x₁, x₂, · · · , xₙ)y₁D₁(x, y₂, · · · , yₙ) = 0. \]
Using relation (4.2.5) we have,

\[ D_1(x_1, x_2, \cdots, x_n) y_1 D_2(z, y_2, \cdots, y_n) + D_2(x_1, x_2, \cdots, x_n) y_1 D_1(z, y_2, \cdots, y_n) = 0. \]  

(4.2.6)

Multiplying by \( p D_1(r_1, r_2, \cdots, r_n) \) where \( r_1, r_2, \cdots, r_n; p \in R \) from right in the relation (4.2.6) we arrive at

\[ D_1(x_1, x_2, \cdots, x_n) y_1 D_2(z, y_2, \cdots, y_n) p D_1(r_1, r_2, \cdots, r_n) + D_2(x_1, x_2, \cdots, x_n) (y_1 D_1(z, y_2, \cdots, y_n) p) D_1(r_1, r_2, \cdots, r_n) = 0. \]

Relation (4.2.6) and 2-torsion freeness of \( R \) provide us \( D_1(x_1, x_2, \cdots, x_n) y_1 D_1(z, y_2, \cdots, y_n) p D_2(r_1, r_2, \cdots, r_n) = 0 \) i.e.,

\[ D_1(x_1, x_2, \cdots, x_n) y_1 D_1(z, y_2, \cdots, y_n) R D_2(r_1, r_2, \cdots, r_n) = \{0\}. \]

Now primeness of \( R \) forces either

\[ D_1(x_1, x_2, \cdots, x_n) y_1 D_1(z, y_2, \cdots, y_n) = 0 \]

or \( D_2 = 0 \). But in first case we have \( D_1(x_1, x_2, \cdots, x_n) R D_1(z, y_2, \cdots, y_n) = \{0\} \). Now again using the primeness of \( R \) we conclude that \( D_1 = 0 \). Finally we have shown that either \( D_1 = 0 \) or \( D_2 = 0 \). Converse is a trivial fact. \( \square \)

**Corollary 4.2.7.** Let \( R \) be a 2-torsion free prime \(*\)-ring, admitting \(*\)-derivations \( D_1 \) and \( D_2 \). Then \( D_1(x) D_2(y) + D_2(x) D_1(y) = 0 \) for all \( x, y \in R \) iff either \( D_1 = 0 \) or \( D_2 = 0 \).

**Theorem 4.2.6.** Let \( R \) be a semiprime \(*\)-ring, admitting a \(*\)-\( n \)-derivation \( D \). Then \( D(R, R, \cdots, R) \subseteq Z \).

**Proof.** Since \( R \) is a \(*\)-ring having a \(*\)-\( n \)-derivation \( D \), we have relation (4.2.3). Now putting \( y D(x_1, x_2, \cdots, x_n) \) in place of \( y \) in the relation (4.2.3) and using it again we get

\[ D(x_1, x_2, \cdots, x_n) y[D(x_1, x_2, \cdots, x_n), z] = 0 \]

for all \( x_1, x_2, \cdots, x_n, y, z \in R \). This relation provides us

\[ z D(x_1, x_2, \cdots, x_n) y[D(x_1, x_2, \cdots, x_n), z] = 0. \]  

(4.2.7)

Replacing \( y \) by \( z y \) in the relation \( D(x_1, x_2, \cdots, x_n) y[D(x_1, x_2, \cdots, x_n), z] = 0 \) we obtain that

\[ D(x_1, x_2, \cdots, x_n) z y[D(x_1, x_2, \cdots, x_n), z] = 0. \]  

(4.2.8)

Now comparing the identities (4.2.7) and (4.2.8) we arrive at

\[ D(x_1, x_2, \cdots, x_n) z y[D(x_1, x_2, \cdots, x_n), z] = z D(x_1, x_2, \cdots, x_n) y[D(x_1, x_2, \cdots, x_n), z] \]
i.e.; \([D(x_1, x_2, \cdots, x_n), z]y[D(x_1, x_2, \cdots, x_n), z] = 0\). This relation provides us

\[ [D(x_1, x_2, \cdots, x_n), z]R[D(x_1, x_2, \cdots, x_n), z] = \{0\} \]

Now semiprimeness of \(R\) yields that \([D(x_1, x_2, \cdots, x_n), z] = 0\) i.e.; \(D(R, R, \cdots, R) \subseteq Z\).

**Corollary 4.2.8.** Let \(R\) be a semiprime \(*\)-ring. If \(R\) admits a \(*\)-derivation \(d\), then \(d\) maps \(R\) into \(Z\).

**Theorem 4.2.7.** Let \(R\) be a semiprime ring with involution \(*\). If \(R\) admits an \(n\)-additive mapping \(F : R^n \to R\) such that \(F(x_1y, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)y^*\) for all \(x_1, x_2, \cdots, x_n, y \in R\). Then \(F(R, R, \cdots, R) \subseteq Z\).

**Proof.** By hypothesis \(R\) is a \(*\)-ring having an \(n\)-additive mapping \(F : R^n \to R\) such that \(F(x_1y, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)y^*\) for all \(x_1, x_2, \cdots, x_n, y \in R\), hence we have relation (4.2.4). Now substituting \(yF(x_1, x_2, \cdots, x_n)\) in place of \(y\) in the relation (4.2.4) and using it again we arrive at \(F(x_1, x_2, \cdots, x_n)y[F(x_1, x_2, \cdots, x_n), z] = 0\) for all \(x_1, x_2, \cdots, x_n, y, z \in R\). This relation provides us

\[ zF(x_1, x_2, \cdots, x_n)y[F(x_1, x_2, \cdots, x_n), z] = 0. \]  \hspace{1cm} (4.2.9)

Replacing \(y\) by \(zy\) in the relation \(F(x_1, x_2, \cdots, x_n)y[F(x_1, x_2, \cdots, x_n), z] = 0\) we obtain that

\[ F(x_1, x_2, \cdots, x_n)zy[F(x_1, x_2, \cdots, x_n), z] = 0. \]  \hspace{1cm} (4.2.10)

Now comparing the identities (4.2.9) and (4.2.10) we arrive at

\[ [F(x_1, x_2, \cdots, x_n), z]y[F(x_1, x_2, \cdots, x_n), z] = 0\] for all \(x_1, x_2, \cdots, x_n; y, z \in R\). This implies that \([F(x_1, x_2, \cdots, x_n), z]R[F(x_1, x_2, \cdots, x_n), z] = \{0\}\). Now semiprimeness of \(R\) yields that \([F(x_1, x_2, \cdots, x_n), z] = 0\) i.e.; \(F(R, R, \cdots, R) \subseteq Z\).

**Corollary 4.2.9.** Let \(R\) be a semiprime \(*\)-ring. If it admits a left \(*\)-\(n\)-multiplier \(T\), then \(T(R, R, \cdots, R) \subseteq Z\).

**Corollary 4.2.10 ( [4, Theorem 2.1]).** Let \(R\) be a semiprime \(*\)-ring. If \(M : R \times R \to R\) is a nonzero biadditive mapping such that \(M(xy, z) = M(x, z)y^*\) for all \(x, y, z \in R\), then \(M\) maps \(R \times R\) into \(Z\).

**Corollary 4.2.11 ( [3, Theorem 2.2]).** Let \(R\) be a semiprime \(*\)-ring. If \(T : R \to R\) is an additive mapping such that \(T(xy) = T(x)y^*\) for all \(x, y \in R\), then \(T\) maps \(R\) into \(Z\).
Theorem 4.2.8. Let \( R \) be a semiprime \(*\)-ring. If \( R \) admits an \( n \)-additive mapping \( F : R^n \rightarrow R \) such that \( F(yx_1, x_2, \cdots , x_n) = y^*F(x_1, x_2, \cdots , x_n) \) for all \( x_1, x_2, \cdots , x_n; y \in R \), then \( F(R, R, \cdots , R) \subseteq Z \).

Proof. Using similar arguments with necessary variations as used to prove Theorem 4.2.7, one can easily obtain the required result.

Remark 4.2.4. Focussing on the examples of left (resp. right) \(*\)-\( n \)-multipliers given in the beginning of this section, it is obvious to see that the restriction of semiprimeness imposed on the hypotheses of Theorems 4.2.7 & 4.2.8 is not superfluous.

Corollary 4.2.12. Let \( R \) be a semiprime \(*\)-ring. If \( R \) admits a right \(*\)-\( n \)-multiplier \( T \), then \( T(R, R, \cdots , R) \subseteq Z \).

Corollary 4.2.13. ([4, Theorem 2.2]). Let \( R \) be a semiprime \(*\)-ring. If \( M : R \times R \rightarrow R \) is a nonzero biadditive mapping such that \( M(xy, z) = x^*M(y, z) \) for all \( x, y, z \in R \), then \( M \) maps \( R \times R \) into \( Z \).

Remark 4.2.5. The above Theorems 4.2.7 & 4.2.8 also hold if the relations in the hypotheses hold in any argument.

Theorem 4.2.9. Let \( R \) be a semiprime ring with involution \('*'\). If \( D \) is a \(*\)-\( n \)-derivation of \( R \) such that \( D(x_1, x_2, \cdots , x_n)y_1 = x_1D(y_1, y_2, \cdots , y_n) \) for all \( x_1, x_2, \cdots , x_n, y_1, y_2, \cdots , y_n \in R \), then \( D = 0 \).

Proof. By hypothesis we have \( D(x_1, x_2, \cdots , x_n)y_1 = x_1D(y_1, y_2, \cdots , y_n) \) for all \( x_1, x_2, \cdots , x_n, y_1, y_2, \cdots , y_n \in R \). Substituting \( x_1z \) where \( z \in R \) in the place of \( x_1 \) in the previous relation we obtain

\[
D(x_1, x_2, \cdots , x_n)z^*y_1 + x_1D(z, x_2, \cdots , x_n)y_1 = x_1zD(y_1, y_2, \cdots , y_n).
\]

Using hypothesis again we have

\[
D(x_1, x_2, \cdots , x_n)z^*y_1 + x_1zD(y_1, y_2, \cdots , y_n) = x_1zD(y_1, y_2, \cdots , y_n)
\]

i.e.; \( D(x_1, x_2, \cdots , x_n)z^*y_1 = 0 \). Now replacing \( z \) in the preceding relation by \( z^* \) we get \( D(x_1, x_2, \cdots , x_n)zy_1 = 0 \) for all \( x_1, x_2, \cdots , x_n, y_1, z \in R \). As \( y_1 \) is an arbitrary element of \( R \), using \( D(x_1, x_2, \cdots , x_n) \) for \( y_1 \) in the relation \( D(x_1, x_2, \cdots , x_n)zy_1 = 0 \) we have \( D(x_1, x_2, \cdots , x_n)zD(x_1, x_2, \cdots , x_n) = 0 \) i.e.; \( D(x_1, x_2, \cdots , x_n)RD(x_1, x_2, \cdots , x_n) = \{0\} \). Finally semiprimeness of \( R \) forces \( D = 0 \).
Corollary 4.2.14. Let $R$ be a semiprime ring with involution $\ast$. If $D$ is a $\ast$-derivation of $R$ such that $D(x)\ast y = xD(\ast y)$ for all $x, y \in R$, then $D = 0$.

The following example shows that the restriction of semiprimeness imposed in the hypothesis of Theorem 4.2.9 is not superfluous.

Example 4.2.3. Consider the $\ast$-ring $R$ used in the beginning of this section while constructing the examples of left (resp. right) $\ast$-multiplier. Define $D : R \times R \times \cdots \times R \rightarrow R$ such that

$$D\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x_1x_2 \cdots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $R$ is not a semiprime $\ast$-ring and $D$ is a $\ast$-n-derivation of $R$ such that $D(r_1, r_2, \cdots, r_n) = r_1D(r_1', r_2', \cdots, r_n')$ for all $r_1, r_2, \cdots, r_n; r_1', r_2', \cdots, r_n' \in R$. However $D \neq 0$.

Theorem 4.2.10. Let $R$ be a commutative $\ast$-ring admitting a $\ast$-derivation $d$. Suppose $I$ is a nonzero ideal of $R$ such that it is invariant under both $\ast$ and $d$ i.e.; $I^* \subseteq I$ and $d(I) \subseteq I$. Then $d$ induces an $\ast$-n-derivation $D$ on the quotient ring $R/I$ where $\ast$ is an involution on quotient ring $R/I$ induced by the involution $\ast$ of $R$.

Proof. Define a map $x + I \mapsto (x + I)^\ast$ of $R/I$ into itself such that $(x + I)^\ast = x^\ast + I$ for all $(x + I) \in R/I$. Let $x + I = y + I$. This implies that $x - y \in I$. Hence by hypothesis $(x - y)^\ast \in I$ i.e.; $x^\ast - y^\ast \in I$. Therefore $x^\ast + I = y^\ast + I$ i.e.; $(x + I)^\ast = (y + I)^\ast$. Thus $\ast$ is a well defined map on quotient ring $R/I$. By using addition and product of quotient ring $R/I$ we see that (i) $((x + I) + (y + I))^\ast = ((x + y) + I)^\ast = (x + y)^\ast + I = (x^\ast + y^\ast) + I = (x^\ast + I) + (y^\ast + I) = (x + I)^\ast + (y + I)^\ast$, (ii) $((x + I)^\ast)^\ast = (x^\ast + I)^\ast = (x^\ast)^\ast + I = x + I$ and (iii) $((x + I)(y + I))^\ast = (xy + I)^\ast = (xy)^\ast + I = y^\ast x^\ast + I = (y^\ast + I)(x^\ast + I) = (y + I)^\ast(x + I)^\ast$ for all $(x + I), (y + I) \in R/I$. All previous facts (i), (ii) and (iii) insure that $\ast$ is an involution on quotient ring $R/I$. Now define $D : R/I \times R/I \times \cdots \times R/I \rightarrow R/I$ as below $D((x_1 + I), (x_2 + I), \cdots, (x_n + I)) = d(x_1)d(x_2)\cdots d(x_n) + I$. Let $((x_1 + I), (x_2 + I), \cdots, (x_n + I)) = ((y_1 + I), (y_2 + I), \cdots, (y_n + I))$. This implies that $(x_1 - y_1) \in I, (x_2 - y_2) \in I, \cdots, (x_n - y_n) \in I$. By hypothesis $d(x_1) + I = d(y_1) + I, d(x_2) + I = d(y_2) + I, \cdots, d(x_n) + I = d(y_n) + I$ i.e.; $d((x_1 + I)(x_2 + I)\cdots d(x_n) + I) = (d(y_1) + I)(d(y_2) + I)\cdots (d(y_n) + I)$. Now we obtain that $d(x_1)d(x_2)\cdots d(x_n) + I = d(y_1)d(y_2)\cdots d(y_n) + I$ i.e.; $D((x_1 + I), (x_2 + I), \cdots, (x_n + I)) = D((y_1 + I), (y_2 + I), \cdots, (y_n + I))$.
Thus \( D \) is a well defined map. Consider \( D((x_1 + I) + (x'_1 + I), (x_2 + I), \ldots, (x_n + I)) = d(x_1 + x'_1)d(x_2) \cdots d(x_n) + I = (d(x_1)d(x_2) \cdots d(x_n) + I) + (d(x'_1)d(x_2) \cdots d(x_n) + I) = D((x_1 + I), (x_2 + I), \ldots, (x_n + I)) + D((x'_1 + I), (x_2 + I), \ldots, (x_n + I)). \) The previous relation insures that \( D \) is additive in the first argument. Similarly one can show that \( D \) is additive in all arguments.

Thus \( D \) is an \( n \)-additive map. Consider \( D((x_1 + I)(x'_1 + I), (x_2 + I), \ldots, (x_n + I)) = D((x_1x'_1 + I), (x_2 + I), \ldots, (x_n + I)) = d(x_1x'_1)d(x_2) \cdots d(x_n) + I = \{(d(x_1))(x'_1)^* + x_1d(x'_1)d(x_2) \cdots d(x_n)) \} + I = \{(d(x_1)d(x_2) \cdots d(x_n) + I)((x'_1)^* + 1)\} + \{(x_1 + I)(d(x'_1)d(x_2) \cdots d(x_n) + I)\} = D((x_1 + I), (x_2 + I), \ldots, (x_n + I))(x'_1 + I) + (x_1 + I)D((x'_1 + I), (x_2 + I), \ldots, (x_n + I)). \) Similarly we can prove that the previous relation holds in each argument. Finally we conclude that \( D \) is a \( \ast \)-\( n \)-derivation on quotient ring \( R/I \).

**Remark 4.2.6.** By above proof it is clear that for \( n = 1 \) the commutativity in the hypothesis of Theorem 4.2.10 becomes redundant. Thus we can say, if \( R \) is a \( \ast \)-ring having \( \ast \)-derivation \( d \) and \( I \) is a nonzero ideal of \( R \) such that it is invariant under both \( \ast \) and \( d \) i.e., \( I^* \subseteq I \) and \( d(I) \subseteq I \), then \( d \) induces an \( \ast \)-derivation \( D \) on the quotient ring \( R/I \) where \( \ast \) is an involution on quotient ring \( R/I \) induced by the involution \( \ast \) of \( R \).

### 4.3 Posner’s first theorem for \( \ast \)-prime rings

An additive mapping \( d : R \rightarrow R \) is said to be a derivation on \( R \) if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in R \). Let \( I \) be a nonzero ideal of \( R \). Then an additive mapping \( d : I \rightarrow R \) is called a derivation from \( I \) to \( R \) if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in I \). In the year 1957, E. C. Posner initiated the study of derivations in rings and proved two very striking theorems. These results have been generalized by several authors in different directions (see for reference [29], [61], & [64] for reference where further references can be found). Posner’s first theorem [75, Theorem 1] states that if \( R \) is a prime ring of characteristic not 2 and the iterate of two derivations on \( R \) is also a derivation, then at least one of them is zero. In this section we extend this result to \( \ast \)-prime rings of characteristic different from 2.

**Theorem 4.3.1.** Let \( R \) be a \( \ast \)-prime ring of characteristic not 2, \( I \) a nonzero \( \ast \)-ideal and \( d_1, d_2 : I \rightarrow R \) are derivations such that the product map \( d_1d_2 : I \rightarrow R \) is also a derivation. If at least one of \( d_1 \) and \( d_2 \) commutes with \( \ast \), then \( d_1 = 0 \) or \( d_2 = 0 \).

For developing the proof of the above theorem we begin with the following lemmas:
Lemma 4.3.1. If $R$ is a $*$-prime ring of characteristic different from 2, then $R$ is 2-torsion free.

Proof. Suppose that $x \in R$ such that $2x = 0$. This implies that $2xrs = 0$ for all $r, s \in R$ i.e.; $xR(2s) = \{0\}$ for all $s \in R$. Since characteristic of $R$ is different from 2 and $R \neq \{0\}$, this provides us a nonzero element $l \in R$ such that $2l \neq 0$. Now we conclude that $xR(2l) = \{0\} = xR(2l)^*$. Finally $*$-primeness of $R$ provides us $x = 0$ and hence $R$ is 2-torsion free. □

Lemma 4.3.2. Let $R$ be a $*$-prime ring and $I$ a nonzero $*$-ideal of $R$. If $d : I \longrightarrow R$ is a derivation such that $d$ commutes with $'*'$, If $a$ is an element of $R$ and ad$(x) = 0$ (resp. $d(x)a = 0$) for all $x \in I$, then either $a = 0$ or $d = 0$.

Proof. Replacing $x$ by $xy$, where $y \in I$ in the relation $ad(x) = 0$, we obtain that $ad(x)y + axd(y) = 0$ i.e.; $axd(y) = 0$ for all $x, y \in I$. Replacing $x$ by $xs$ where $s \in R$ in the latter relation, we arrive at $axsd(y) = 0$ i.e.; $axRd(y) = \{0\}$ for all $x, y \in I$. Since $d$ commutes with $'*'$ and $I$ is a $*$-ideal, we obtain that $axRd(y) = \{0\} = axR\{d(y)\}^*$ for all $x, y \in I$. Now $*$-primeness of $R$ provides us $d = 0$ or $ax = 0$ for all $x \in I$. Putting $tx$ where $t \in R$ for $x$ in the latter relation, we arrive at $atx = 0$ i.e.; $aRx = \{0\}$ for all $x \in I$. Since $I$ is a $*$-ideal of $R$, we also have $aRx = aR^*x = \{0\}$. Now $*$-primeness of $R$ and $I \neq \{0\}$ imply that $a = 0$. Similarly we can also show that $d(x)a = 0$ for all $x \in I$ implies that $a = 0$ or $d = 0$. □

Proof of the Theorem 4.3.1. We divide the proof in following two cases:

Case I: Let us suppose that $d_1$ commutes with $'*'$. Since the map $d_1d_2 : I \longrightarrow R$ is a derivation, it is obvious that $d_2(I) \subseteq I$ and $d_1d_2(xy) = d_1d_2(x)y + xd_1d_2(y)$ for all $x, y \in I$. As $d_1, d_2 : I \longrightarrow R$ are derivations, we obtain that

$$d_1d_2(xy) = d_1(d_2(xy)) = d_1d_2(x)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1d_2(y).$$

By above relations we conclude that

$$d_2(x)d_1(y) + d_1(x)d_2(y) = 0 \text{ for all } x, y \in I. \tag{4.3.1}$$

Now replacing $x$ by $xd_2(z)$, where $z \in I$ in the relation (4.3.1) we obtain that

$$d_2(xd_2(z))d_1(y) + d_1(xd_2(z))d_2(y) = 0$$

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for all \(x, y, z \in I\). This gives us:

\[
d_2(x)d_2(z)d_1(y) + x_2^2(z)d_1(y) + d_1(x)d_2(z)d_2(y) +
\]

\[
x_1d_2(z)d_2(y) = 0.
\]

In view of equation (4.3.1) and using the fact that \(d_2(I) \subseteq I\), we find

\[
(d_2(d_2(z))d_1(y) + d_1(d_2(z))d_2(y)) = 0.
\]

Hence we arrive at

\[
d_2(x)d_2(z)d_1(y) + d_1(x)d_2(z)d_2(y) = 0
\]

(4.3.2)

for all \(x, y, z \in I\). Using the relation (4.3.1) and Lemma 4.3.1, the relation (4.3.2) reduces to \(d_1(x)d_2(z)d_2(y) = 0\) for all \(x, y, z \in I\). Now Lemma 4.3.2 provides us either \(d_1 = 0\) or \(d_2(z)d_2(y) = 0\) for all \(y, z \in I\). If the first case holds then nothing to do, if not we have \(d_2(z)d_2(y) = 0\) for all \(y, z \in I\). Replacing \(y\) by \(yz\) in the latter relation and using the same again we arrive at \(d_2(z)y_2d_2(z) = 0\) for all \(y, z \in I\). Replacing \(y\) by \(sy\) where \(s \in R\) in the latter relation we arrive at \(d_2(z)y_2d_2(z) = \{0\}\) for all \(y, z \in I\). Since \(R\) is a \(*\)-prime ring, it is semiprime also and hence we obtain that \(y_2d_2(z) = 0\) for all \(y, z \in I\). Replacing \(y\) by \(yt\) where \(t \in R\) in the latter relation we arrive at \(ytd_2(z) = 0\) i.e.; \(yRd_2(z) = \{0\}\) for all \(y, z \in I\). But since \(I\) is a \(*\)-ideal of \(R\), also get \(yd_2(z) = \{0\}\) for all \(y, z \in I\). Finally \(*\)-primeness of \(R\) and \(I \neq \{0\}\) imply that \(d_2 = 0\).

Case II: Let us suppose that \(d_2\) commutes with \('*'\). From Case I, we have \(d_1(x)d_2(z)d_2(y) = 0\) for all \(x, y, z \in I\). Now Lemma 4.3.2 provides us either \(d_1 = 0\) or \(d_1(x)d_2(z) = 0\) for all \(x, z \in I\). If the first case holds then nothing to do, if not we have \(d_1(x)d_2(z) = 0\) for all \(x, z \in I\). Again using Lemma 4.3.2 we conclude that either \(d_1 = 0\) or \(d_2 = 0\).

The following example shows that the hypothesis of \(*\)-primeness is crucial in the above theorem.

\textbf{Example 4.3.1.} Let \(R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}\), where \(\mathbb{Z}\) is the set of integers.

Consider the map

\[
\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^*
\]

of \(R\) into itself such that

\[
\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^* = \begin{pmatrix} z & 0 \\ -y & x \end{pmatrix}.
\]

It is easy to verify that \('*'\) is an involution of the ring \(R\), where characteristic of \(R\) is different from 2. Further if we set \(I = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y, 0 \in \mathbb{Z} \right\}\), then \(I\) is a nonzero
ideal of $R$. Now consider the maps $d_1, d_2 : I \rightarrow R$ defined by

$$d_1 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad d_2 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}.$$ 

Then it is obvious to observe that $d_1$ and $d_2$ are derivations and `$*$' commutes with $d_1$.

Further it can be also shown that the map $d_1d_2 : I \rightarrow R$ is a derivation and $R$ is not a $*$-prime ring. However neither $d_1 = 0$ nor $d_2 = 0$.

The following example shows that the hypothesis of "characteristic different from 2" is crucial in the above theorem.

**Example 4.3.2.** Suppose that $R = \mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$, where $\mathbb{Z}_2[x]$ is the polynomial ring over $\mathbb{Z}_2$. Let us consider the map $(f(x), g(x)) \mapsto (f(x), g(x))^*$ of $R$ into itself such that $(f(x), g(x))^* = (g(x), f(x))$. It is easy to check that `$*$' is an involution of $R$, known as exchange involution denoted by $*_{ex}$ and $R$ is a $*_{ex}$-prime ring. Further assume that $I = [x^2]$ is the ideal of $\mathbb{Z}_2[x]$ generated by $x^2 \in \mathbb{Z}_2[x]$. Then it can be easily shown that $\mathcal{I} = I \times I$ is a nonzero $*_{ex}$-ideal of $R$. Next consider $D_1, D_2 : \mathcal{I} \rightarrow R$ such that $D_1(f(x), g(x)) = (d(f(x)), d(g(x)))$ and $D_2(f(x), g(x)) = (d(f(x))^*, 0)$, where $d$ is the usual differentiation in $\mathbb{Z}_2[x]$. It is obvious to see that $D_1, D_2$ and $D_1D_2 : \mathcal{I} \rightarrow R$ are derivations. Moreover, $R$ is a ring of characteristic 2 and $D_1*_{ex} = *_{ex}D_1$. However $D_1 \neq 0$ and $D_2 \neq 0$.

Now taking $I = R$ in the above theorem we obtain the following:

**Corollary 4.3.1.** Let $R$ be a $*$-prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation of $R$. If at least one of $d_1$ and $d_2$ commutes with `$*$', then $d_1 = 0$ or $d_2 = 0$.

Now using the above theorem we can obtain Posner's first theorem.

**Corollary 4.3.2 (\cite[Theorem 1]{75}).** Let $R$ be a prime ring of characteristic not 2 and $d_1, d_2$ derivations of $R$ such that the iterate $d_1d_2$ is also a derivation, then one at least of $d_1, d_2$ is zero.

**Proof.** Since $R$ is a prime ring of characteristic not 2, $\mathcal{R} = R \times R^e$ is clearly a $*_{ex}$-prime ring of characteristic not 2. Set $I = \mathcal{R}$, which is a nonzero $*_{ex}$-ideal of $\mathcal{R}$. Now define $D_1, D_2 : I \rightarrow \mathcal{R}$ by $D_1(x, y) = (d_1(x), d_1(y))$ and $D_2(x, y) = (d_2(x), d_2(y))$. Using hypothesis it can be easily seen that $D_1, D_2 : I \rightarrow \mathcal{R}$ are derivations and the product map $D_1D_2 : I \rightarrow \mathcal{R}$ is also a derivation. Moreover $D_1*_{ex} = *_{ex}D_1$. In view of the
Theorem 4.3.1 we deduce that either $D_1 = 0$ or $D_2 = 0$, in turn we obtain that either $d_1 = 0$ or $d_2 = 0$. 

Chapter 5

Derivation in *-prime rings and its ring of quotients

5.1 Introduction

Let $R$ be a ring with involution '$\ast$'. We shall denote the set of all symmetric and skew symmetric elements of $R$ by $S_{\ast}(R)$ i.e.; $S_{\ast}(R) = \{x \in R \mid x^{\ast} = \pm x\}$. For a semiprime ring $R$, $Q_{mr}$ and $Q_{s}$ will represent its Utumi right ring of quotients and right symmetric Martindale ring of quotients respectively.

In Section 5.2, we investigate commutativity of *-prime ring $R$, which satisfies certain differential identities on *-ideals of $R$. Some results already known for prime rings on ideals have also been deduced. Finally, we provide several examples to justify that various restrictions imposed in the hypotheses of our theorems are not superfluous.

Section 5.3 gives a glimpse of some extension problems in the setting of ring of quotients of a *-prime ring. Let $R$ be a semiprime ring with an involution '$\ast$'. Let $Q_{mr}$ and $Q_{s}$ denote its right Utumi quotient ring and right symmetric Martindale quotient ring respectively. In the present section the following extension problems have been studied:

(i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. (ii) if $R$ is a *-prime ring, then so is its right symmetric Martindale quotient ring. (iii) every *-derivation of a commutative semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. At the end of this section $C$-dependence of any two nonzero elements of right symmetric Martindale quotient ring of *-prime ring $R$, where $C$ is the extended centroid of $R$, has also been discussed.
5.2 Certain differential identities in prime rings with involution

Several authors have studied the commutativity of prime and semiprime rings, satisfying certain differential identities on some appropriate subsets of \( R \) (see for reference [11], [23], [52] & [53] etc., where further references can be found). Hence it is natural to question that what can we say about the commutativity of \(*\)-prime rings in which derivations satisfy certain identities on \(*\)-ideals. In this direction Oukhtite et al. ([64], [65], [66], [68]) have already investigated several differential identities on \(*\)-ideals.

In this chapter we have obtained the commutativity of \(*\)-prime rings satisfying certain differential identities on \(*\)-ideal \( I \); viz.;

(i) \( d(xoy) = d(x)oy \)
(ii) \( d(x)oy = xoy \)
(iii) \( d([x,y]) = \pm(xoy) \)
(iv) \( d(xoy) = \pm[x,y] \)
(v) \( d(x)oy \in Z \)
(vi) \( d[x,y] \pm (xoy) \in Z \)
(vii) \( d(xoy) \pm |x,y| \in Z \)
(viii) \( d(x)od(y) = xoy \)
(ix) \( d(x)oy - (xod(y)) \in Z \)

for all \( x, y \in I \). We have also shown that there exists no nonzero derivation \( d \) satisfying any of the following differential identities on \(*\)-ideal \( I \) in a \(*\)-prime ring \( R \);

(i) \( d(xoy) = d(x)oy \)
(ii) \( d(x)oy = xoy \)
(iii) \( d(x)oy = xod(y) \)
(iv) \( d(x)oy = d(x)od(y) \)
(v) \( d(x)oy = d(x)od(y) \)

We facilitate our discussion with the following lemmas which are essential for developing the proof of our results of the present section. The proofs of Lemmas 5.2.1-5.2.3 can be seen in [66, Lemmas 1—3] while Lemma 5.2.4 can be found in [65, Theorem 3.2].

**Lemma 5.2.1.** Let \( R \) be a \(*\)-prime ring and \( I \) be a nonzero \(*\)-ideal of \( R \). If \( x, y \in R \) satisfy \( xly = xly^* = \{0\} \), then \( x = 0 \) or \( y = 0 \).

**Lemma 5.2.2.** Let \( R \) be a \(*\)-prime ring admitting a nonzero derivation \( d \) which commutes with \(*\). If \( I \) is a nonzero \(*\)-ideal of \( R \) and \( [x, R]Id(x) = \{0\} \) for all \( x \in I \), then \( R \) is commutative.

**Lemma 5.2.3.** Let \( R \) be a \(*\)-prime ring admitting a nonzero derivation \( d \) which commutes with \(*\). If \( I \) is a nonzero \(*\)-ideal of \( R \) and \( [d(x), x] = 0 \) for all \( x \in I \), then \( R \) is commutative.

**Lemma 5.2.4.** Let \( d \) be a nonzero derivation of a 2-torsion free \(*\)-prime ring \( R \) and \( I \) a nonzero \(*\)-ideal of \( R \). If \( r \in S_{\ast}(R) \) satisfies \( [d(x), r] = 0 \) for all \( x \in I \), then \( r \in Z \). Furthermore, if \( d(I) \subseteq Z \), then \( R \) is commutative.

Now we prove the following:

**Lemma 5.2.5.** If \( R \) is a \(*\)-prime ring admitting a nonzero central \(*\)-ideal \( I \) i.e.; \( I \subseteq Z \), then \( R \) is commutative.
Proof. Let \( r, s \in R \) and \( x \in I \). Using hypothesis we get \( rsx = rxs = srx \). This implies that \( [r, s]I = \{0\} \) and hence \( [r, s]II = [r, s]II^* = \{0\} \), where \( 0 \neq l \in R \). In view of Lemma 5.2.1, we get the required result. \( \square \)

Lemma 5.2.6. Let \( R \) be a \(*\)-prime ring admitting a nonzero derivation \( d \) which commutes with \('\ast'\). If \( I \) is a nonzero \(*\)-ideal of \( R \) and \( d(x)I[x, R] = \{0\} \) for all \( x \in I \), then \( R \) is commutative.

Proof. For the proof, first we show that \( d(x)I[x, R] = \{0\} \) for all \( x \in I \) if and only if \( [x, R]Id(x) = \{0\} \) for all \( x \in I \). Suppose that \( d(x)I[x, R] = \{0\} \) for all \( x \in I \). This implies that \( [x, R]^*i\{d(x)\}^* = \{0\} \) for all \( x \in I \). Since \( I \) is a \(*\)-ideal of \( R \) and \( d \) commutes with \('\ast'\), we conclude that \( [x^*, R]Id(x^*) = \{0\} \) for all \( x \in I \). Now replacing \( x \) by \( x^* \) in the last relation we obtain that \( [x, R]Id(x) = \{0\} \) for all \( x \in I \). Converse can be proved in similar way. Finally using Lemma 5.2.2, we get the required result. \( \square \)

Now we prove the main results of this section:

Theorem 5.2.1. Let \( R \) be a \(*\)-prime ring, \( I \) be a nonzero \(*\)-ideal of \( R \) and \( d \) a nonzero derivation of \( R \) such that \( d \) commutes with \('\ast'\). If \( d(xoy) = d(x)oy \) for all \( x, y \in I \) or \( d(x)oy = xoy \) for all \( x, y \in I \), then \( R \) is commutative.

Proof. Assume that \( d(xoy) = d(x)oy \) for all \( x, y \in I \). Now replacing \( y \) by \( xy \) we arrive at \( d(xoy) = d(x)oxy \) i.e.; \( d(x)(xoy) + xd(xoy) = d(x)xy + xyd(x) \). Using hypothesis we obtain that \( d(x)(xoy) + xd(x)oy = d(x)xy + xyd(x) \). This implies that \( d(x)xy + d(x)yx + xd(x)y + yxd(x) = d(x)xy + yxd(x) \) i.e.,

\[
d(x)yx = -xd(x)y \quad \text{for all } x, y \in I.
\]

Putting \( yr \), where \( r \in R \), for \( y \) in the relation (5.2.1) and using it again we conclude that \( d(x)yrx = -xd(x)yr = d(x)yxr \) i.e.; \( d(x)I[x, R] = \{0\} \) for all \( x \in I \) and Lemma 5.2.6 forces that \( R \) is commutative.

Now suppose that \( d(x)oy = xoy \) for all \( x, y \in I \). Replacing \( x \) by \( yx \), we get \( d(yx)oy = yxoy \) i.e.; \( d(yx)oy = y(xoy) \) for all \( x, y \in I \). Using our hypothesis we obtain that \( d(yx)y + yd(yx) = y(d(x)oy) \) i.e.; \( d(yx)y + yd(x)y + yd(y)x + y^2d(x) = yd(x)y + y^2d(x) \) and therefore \( d(y)yx = -yd(y)x \) for all \( x, y \in I \). In view of the latter relation we arrive at \( d(x)yx = -xd(x)y \) for all \( x, y \in I \). This is identical with the relation (5.2.1). Arguing as in above we conclude that \( R \) is commutative. \( \square \)
Theorem 5.2.2. Let $R$ be a $*$-prime ring of characteristic different from 2, $I$ be a nonzero $*$-ideal of $R$ and $d$ a derivation of $R$ such that $d$ commutes with $*$. If $d(xy) = d(x)y$ for all $x,y \in I$ or $d(x)y = xy$ for all $x,y \in I$, then $d = 0$.

Proof. Suppose that $d(xy) = d(x)y$ for all $x,y \in I$. Then we have to show that $d = 0$. Suppose on contrary that $d \neq 0$. Therefore by Theorem 5.2.1 we conclude that $R$ is commutative. By hypothesis given we have $2d(xy) = 2d(x)y$ for all $x,y \in I$ and hence $d(xy) = d(x)y$ for all $x,y \in I$. This yields that $d(y) = 0$ for all $x,y \in I$ and since $R$ is commutative we arrive at $d(xy) = 0$ for all $x,y \in I$. Replacing $y$ by $sy$ where $s \in R$ in the last relation we obtain that $d(xy) = 0$ i.e.; $d(x)Ry = \{0\}$ for all $x,y \in I$. Since $I$ is a $*$-ideal of $R$, we conclude that $d(x)Ry = \{0\}$ for all $x,y \in I$ also. $I \neq \{0\}$ and $*$-primeness of $R$ provide us $d(x) = 0$ for all $x \in I$. Now putting $xt$ where $t \in R$ in place of $x$ in the last relation and using the same again we arrive at $d(t) = 0$ i.e.; $Id(t) = \{0\}$. Using hypothesis this relation provides us $Id(t) = Id(t)^* = \{0\}$ where $0 \neq t \in R$. Finally Lemma 5.2.1 assures that $d = 0$, leading to a contradiction.

Now assume that $d(xy) = xy$ for all $x,y \in I$. Then we have to show that $d = 0$. If $d \neq 0$, then by Theorem 5.2.1 we conclude that $R$ is commutative. By hypothesis given we have $2d(xy) = 2xy$ for all $x,y \in I$, then $d(xy) = xy$ for all $x,y \in I$. Replacing $x$ by $rx$, where $r \in R$ in the last relation and using the same again we infer that $d(rx) = rx$ i.e.; $d(r)xy + rd(x)y = rxy$. This implies that $d(r)xy = 0$ for all $r \in R$ and $x,y \in I$. Finally we conclude that $d(r)Iy = \{0\}$ for all $r \in R$ and $y \in I$. In particular we also obtain that $d(r)Iy^* = \{0\}$. Lemma 5.2.1 and $I \neq \{0\}$ assure that $d = 0$, leading to a contradiction. $\square$

The following example shows that the existence of "characteristic different from 2" in the hypothesis of the above theorem is not superfluous.

Example 5.2.1. Suppose that $R = \mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$, where $\mathbb{Z}_2[x]$ is the polynomial ring over $\mathbb{Z}_2$. Let us consider $D, * : R \rightarrow R$ such that $D(f(x), g(x)) = (d(f(x)), d(g(x)))$ and $(f(x), g(x))^* = (g(x), f(x))$, where $d$ is the usual differentiation in $\mathbb{Z}_2[x]$. It is easy to check that $R$ is a $*_{ex}$-prime ring since $*$ is an involution of $R$, known as exchange involution denoted by $*_{ex}$ and $D$ is a derivation on $R$. Moreover, it is obvious that $R$ is a ring of characteristic 2 and $D*_{ex} = *_{ex}D$. Further assume that $I = [x^2]$ is the ideal of $\mathbb{Z}_2[x]$ generated by $x^2 \in \mathbb{Z}_2[x]$. Then it can be easily shown that $I = I \times I$ is a nonzero $*_{ex}$-ideal of $R$ such that $D(xy) = D(x)oy$ for all $x,y \in I$ and $D(xy) = xy$ for all $x,y \in I$. However $D \neq 0$. 98
In the year 2007, Oukhtite and Salhi ([68], Theorem 1.3) obtained the commutativity of *
prime ring $R$ having "characteristic different from 2" and admitting a nonzero derivation $d$ which commutes with '*', such that $d([x,y]) = 0$ for all $x,y$ in a nonzero *-ideal of $R$. We have improved this result and showed that the restriction of "characteristic different from 2" on $R$ used in the above theorem is redundant. In addition we have also investigated similar other differential identities which insure the commutativity of *
prime rings. In fact we have obtained the following.

**Theorem 5.2.3.** Let $R$ be a *-prime ring and $I$ a nonzero *-ideal of $R$. If $R$ admits a nonzero derivation $d$ which commutes with '*' and satisfies any one of the following differential identities: (i) $d([x,y]) = 0$ for all $x,y \in I$, (ii) $d([x,y]) = \pm [x,y]$ for all $x,y \in I$, (iii) $d([x,y]) = \pm (xoy)$ for all $x,y \in I$, (iv) $d(xoy) = 0$ for all $x,y \in I$, (v) $d(xoy) = \pm (xoy)$ for all $x,y \in I$ and (vi) $d(xoy) = \pm [x,y]$ for all $x,y \in I$, then $R$ is commutative.

**Proof.** (i) By hypothesis we have $d([x,y]) = 0$, for all $x,y \in I$. Now replacing $y$ by $yx$ and using the hypothesis, we obtain that $[x,y]d(x) = 0$ for all $x,y \in I$ i.e.;

$$xyd(x) = yxd(x)$$ (5.2.2)

for all $x,y \in I$. Replacing $y$ by $ry$, where $r \in R$ in the relation (5.2.2) and using it again, we arrive at $[x,r]yd(x) = 0$ for all $x,y \in I$ and for all $r \in R$ i.e.; $[x,R]Id(x) = \{0\}$ for all $x \in I$. Now by Lemma 5.2.2, the result follows.

(ii) We have $d([x,y]) = \pm [x,y]$, for all $x,y \in I$. Now replacing $y$ by $yx$ and using the hypothesis, we infer that $[x,y]d(x) = 0$ for all $x,y \in I$ i.e.; $xyd(x) = yxd(x)$ for all $x,y \in I$. This is identical with the relation (5.2.2). Now arguing in the similar way as above (i), we get our required result.

(iii) Using the same trick as used in (ii), result follows.

(iv) By hypothesis we have $d(xoy) = 0$, for all $x,y \in I$. Now replacing $y$ by $yx$ and using the hypothesis, we obtain that $xoyd(x) = 0$ for all $x,y \in I$ i.e.;

$$xyd(x) = -yxd(x)$$ (5.2.3)

for all $x,y \in I$. Replacing $y$ by $ry$, where $r \in R$ in the relation (5.2.3) and using it again,
we arrive at \([x, r]yd(x) = 0\) for all \(x, y \in I\) and for all \(r \in R\) i.e.; \([x, R]Id(x) = \{0\}\) for all \(x \in I\). Now by Lemma 5.2.2, the result follows.

(v) By hypothesis we have \(d(xoy) = \pm (xoy)\), for all \(x, y \in I\). Now replacing \(y\) by \(yx\) and using the hypothesis, we conclude that \((xoy)d(x) = 0\) for all \(x, y \in I\) i.e.; \(xyd(x) = -yxd(x)\) for all \(x, y \in I\). This is identical with the relation (5.2.3). Now using similar arguments as used in (iv), we get our required result.

(vi) Using the same arguments as used in (v), result follows. \(\square\)

**Corollary 5.2.1.** Let \(R\) be a prime ring and \(I\) a nonzero ideal of \(R\). If \(R\) admits a nonzero derivation \(d\) and satisfying any one of the following differential identities: (i) \(d([x, y]) = 0\) for all \(x, y \in I\), (ii) \(d([x, y]) = \pm [x, y]\) for all \(x, y \in I\), (iii) \(d([x, y]) = \pm (xoy)\) for all \(x, y \in I\), (iv) \(d(xoy) = 0\) for all \(x, y \in I\), (v) \(d(xoy) = \pm (xoy)\) for all \(x, y \in I\) and (vi) \(d(xoy) = \pm [x, y]\) for all \(x, y \in I\), then \(R\) is commutative.

**Proof.** Let \(d\) be a nonzero derivation of \(R\) satisfying any one of above differential identities. Since \(R\) is a prime ring, consider \(\mathcal{R} = R \times R^c\), which is clearly a \(*_{ex}\)-prime ring. Set \(I = I \times I\) is a nonzero \(*_{ex}\)-ideal of \(\mathcal{R}\). Now define \(D : \mathcal{R} \rightarrow \mathcal{R}\) by \(D(x, y) = (d(x), d(y))\). Using hypothesis it can be easily proved that \(D\) is a nonzero derivation of \(\mathcal{R}\). Moreover \(D_{*ex} = *_{ex}D\) and (i) \(D([x, y]) = 0\) for all \(x, y \in I\), (ii) \(D([x, y]) = \pm [x, y]\) for all \(x, y \in I\), (iii) \(D([x, y]) = \pm (xoy)\) for all \(x, y \in I\), (iv) \(D(xoy) = 0\) for all \(x, y \in I\), (v) \(D(xoy) = \pm (xoy)\) for all \(x, y \in I\) and (vi) \(D(xoy) = \pm [x, y]\) for all \(x, y \in I\). Using the Theorem 5.2.3, we deduce that \(\mathcal{R}\) is commutative and in turn we obtain that \(R\) is also commutative. \(\square\)

**Theorem 5.2.4.** Let \(R\) be a \(*\)-prime ring of characteristic different from 2 and \(I\) a nonzero \(*\)-ideal of \(R\). If \(R\) admits a nonzero derivation \(d\) such that \(d(x)oy \in Z\) for all \(x, y \in I\), then \(R\) is commutative.

**Proof.** Assume that

\[
d(x)oy \in Z
\] (5.2.4)

for all \(x, y \in I\). The relation (5.2.4) implies that \(d(x)y + yd(x) \in Z\) for all \(x, y \in I\). Since \(I\) is a nonzero ideal of \(R\), \(d(x)y + yd(x) \in I\) for all \(x, y \in I\) also. Now we conclude that \(d(x)y + yd(x) \in Z \cap I\) for all \(x, y \in I\). Now we break the proof in two cases.

Case I: If \(Z \cap I = \{0\}\), we obtain that \(d(x)y + yd(x) = 0\) i.e.;

\[
d(x)y = -yd(x)
\] (5.2.5)
for all $x, y \in I$. Substituting $ry$, where $r \in R$ for $y$ in the relation (5.2.5) and using it again we arrive at $d(x)ry = rd(x)y$ i.e.; $d(x), R y = \{0\}$. This implies that $d(x), R I_s = \{0\} = [d(x), R] I^*$, where $0 \neq s \in R$. Now by Lemma 5.2.1, we infer that $d(x), R I_s = \{0\}$ for all $x \in I$ i.e.; $d(I) \subseteq Z$. Finally, Lemma 5.2.4 assures that $R$ is commutative.

Case II: If $Z \cap I \neq \{0\}$, there exists $0 \neq z \in Z \cap I$. By hypothesis we have $d(x)y + yd(x) \in Z$ for all $x, y \in I$. In particular we conclude that $d(x)z + zd(x) \in Z$ i.e.; $d(x), R I_z = \{0\}$ for all $x \in I$. Now we have $2d(x)zr = 2rd(x)z$ for all $x \in I$ and $r \in R$. This yields that $[d(x), r] z = 0$ for all $x \in I$ and $r \in R$ i.e.; $[d(x), R] I_z = \{0\}$ for all $x \in I$. We already know that $0 \neq z \in Z \cap I$. Since $I = I^*$ and $Z = Z^*$, the latter relation implies that $0 \neq z^* \in Z \cap I$. Now using $z^*$ in place of $z$ and arguing in the similar way as in just above lines we arrive at $[d(x), R] I_{z^*} = \{0\}$ for all $x \in I$. Finally we conclude that $[d(x), R] I_z = \{0\} = [d(x), R] I_{z^*}$ for all $x \in I$, where $0 \neq z$. Using Lemma 5.2.1 & Lemma 5.2.4, we get the required result for this case.

The following example demonstrates that the $*$-primeness in the hypothesis of the above theorem can not be omitted.

**Example 5.2.2.** Let $R = \mathbb{R}[x] \times Q$, where $\mathbb{R}[x]$ is the polynomial ring over the ring $\mathbb{R}$ of real numbers and $Q$ is the ring of real quaternions. $R$ is clearly a ring of characteristic different from 2. Define $D : R \rightarrow R$ as $D(f(x), q) = (d(f(x)), 0)$, where $d$ is the usual differentiation of the polynomial ring $\mathbb{R}[x]$. Also define $*: R \rightarrow R$ as $(f(x), q) = (f(-x), \bar{q})$, where $f(x) \in \mathbb{R}[x]$ and $\bar{q} = \alpha - \beta i - \gamma j - \delta k$, where $q = \alpha + \beta i + \gamma j + \delta k \in Q$.

It can be easily shown that $D$ and $*$ are a nonzero derivation and an involution of $R$ respectively. Suppose that $I = \mathbb{R}[x] \times \{0\}$. It is obvious that $I$ is a $*$-ideal of $R$. Let $0 \neq u(x) \in \mathbb{R}[x]$ and $0 \neq v \in Q$. Then we have $\langle u(x), 0 \rangle R(0, v) = \{(0, 0)\} = \langle u(x), 0 \rangle R(0, v)^*$, where $(0, 0) \neq (u(x), 0), (0, 0) \neq (0, v) \in R$. This implies that $R$ is not a $*$-prime ring but it is a semiprime ring. It can be easily seen that $D(m)n + nD(m) \in Z(R)$ for all $m, n \in I$, where $Z(R)$ stands for the center of the ring $R$, but $R$ is noncommutative.

**Theorem 5.2.5.** Let $R$ be a $*$-prime ring of characteristic different from 2, $I$ a nonzero $*$-ideal of $R$. If $R$ admits a nonzero derivation $d$ which commutes with $*$ such that $d[x, y] \pm (xoy) \subseteq Z$ for all $x, y \in I$. Then $R$ is commutative.

**Proof.** It is clear that $d[x, y] \pm (xoy) \subseteq I$ for all $x, y \in I$ also. Now in view of our hypothesis we conclude that $d[x, y] \pm (xoy) \subseteq Z \cap I$ for all $x, y \in I$.

Case I: If $Z \cap I = \{0\}$, then $d[x, y] \pm (xoy) = 0$ for all $x, y \in I$, using Theorem 5.2.3 we get our required result.
Case II: If \( Z \cap I \neq \{0\} \), then suppose \( 0 \neq z \in Z \cap I \). Replacing \( y \) by \( z \), we arrive at \( d[x, z] \pm (xoz) \in Z \cap I \) for all \( x \in I \) i.e., \( 2xz \in Z \) for all \( x \in I \) and hence \( xz \in Z \) for all \( x \in I \) i.e., \( xzr = rxz \) for all \( r \in R \). This implies that \( [x, R]Rz = \{0\} \) for all \( x \in I \).

Since \( 0 \neq z^* \in Z \cap I \), arguing in the similar lines as above we also obtain that \( [x, R]Rz^* = \{0\} \) for all \( x \in I \). By \(*\)-primeness of \( R \) we conclude that \( I \subseteq Z \). Finally by Lemma 5.2.5, the result follows.

**Theorem 5.2.6.** Let \( R \) be a \(*\)-prime ring of characteristic different from 2 and \( I \) a nonzero \(*\)-ideal of \( R \). If \( R \) admits a nonzero derivation \( d \) which commutes with \(*\) such that \( d(xoy) \pm [x, y] \in Z \) for all \( x, y \in I \), then \( R \) is commutative.

**Proof.** It is clear that \( d(xoy) \pm [x, y] \in I \) for all \( x, y \in I \) also. Now including the hypothesis we conclude that \( d(xoy) \pm [x, y] \in Z \cap I \) for all \( x, y \in I \).

Case I: If \( Z \cap I = \{0\} \), we find that \( d(xoy) \pm [x, y] = 0 \) for all \( x, y \in I \) and hence using Theorem 5.2.3 we get our required result.

Case II: Suppose \( Z \cap I \neq \{0\} \). Let \( 0 \neq z \in Z \cap I \). Replacing \( y \) by \( z \), we arrive at \( d(xoz) \in Z \cap I \) for all \( x \in I \) i.e., \( 2d(xz) \in Z \) for all \( x \in I \) and hence \( d(xz) \in Z \) for all \( x \in I \) i.e., \( d(xz) = xzd(z) = xzd(x)z + xzd(z) \) for all \( x \in I \). Using the fact that \( d(z) \subseteq Z \), we conclude that \( [d(x), x] \in Z \) for all \( x \in I \). Since \( Z^* = Z \) and \( I^* = I \), we obtain that \( 0 \neq z^* \in Z \cap I \). Now arguing in the similar lines as above we also obtain that \( [d(x), x] \in Z \) for all \( x \in I \). By \(*\)-primeness of \( R \) we conclude that \( [d(x), x] = 0 \) for all \( x \in I \). Finally by Lemma 5.2.3, the result follows.

**Corollary 5.2.2.** Let \( R \) be a prime ring of characteristic not 2, \( I \) a nonzero ideal and \( d \) a nonzero derivation of \( R \) satisfying either of the following differential identities (i) \( d[x, y] \pm (xoy) \in Z \) for all \( x, y \in I \) or (ii) \( d(xoy) \pm [x, y] \in Z \) for all \( x, y \in I \). Then \( R \) is commutative.

**Proof.** Assume that \( d \) is a nonzero derivation of \( R \) such that (i) \( d[x, y] \pm (xoy) \in Z \) for all \( x, y \in I \) or (ii) \( d(xoy) \pm [x, y] \in Z \) for all \( x, y \in I \). Since \( R \) is a prime ring of characteristic not 2, consider \( \mathcal{R} = R \times R^* \), which is clearly a \(*_{ex}\)-prime ring of characteristic different from 2. Set \( I = I \times I \) a nonzero \(*_{ex}\)-ideal of \( \mathcal{R} \). Now define \( D : \mathcal{R} \longrightarrow \mathcal{R} \) by \( D(x, y) = (d(x), d(y)) \). Using hypothesis it can be easily proved that \( D \) is a nonzero derivation of \( \mathcal{R} \). Moreover \( D^*_{ex} = *_{ex}D \) and (i) \( D[x, y] \pm (xoy) \in Z(\mathcal{R}) \) for all \( x, y \in I \) or (ii) \( D(xoy) \pm [x, y] \in Z(\mathcal{R}) \) for all \( x, y \in I \), where \( Z(\mathcal{R}) \) denotes the center of the ring \( \mathcal{R} \). In view of Theorems 5.2.5 & 5.2.6 we deduce that \( \mathcal{R} \) is commutative and in turn we obtain that \( R \) is also commutative.

\[\square\]
The following example shows that the **-primeness in hypotheses of Theorems 5.2.5 & 5.2.6 can not be omitted.

**Example 5.2.3.** Let $R = \mathbb{R}[x] \times Q$, where $\mathbb{R}[x]$ is the polynomial ring over ring $\mathbb{R}$ of real numbers and $Q$ is the ring of real quaternions. Clearly, $R$ is a ring of characteristic different from 2. Define $D : R \rightarrow R$ as $D(f(x), q) = (0, d_i(q))$, where $d_i$ is the inner derivation of $Q$, determined by $i \in Q$, i.e.; $d_i(q) = [i, q]$ for all $q \in Q$. Also define $* : R \rightarrow R$ as $*(f(x), q) = (f(x), \bar{q})$, where $f(x) \in \mathbb{R}[x]$ and $\bar{q} = \alpha - \beta i - \gamma j - \delta k$. It can be easily shown that $D$ and $*$ are a nonzero derivation and an involution of $R$ respectively such that $D* = *D$. Suppose that $I = \mathbb{R}[x] \times \{0\}$.

It is obvious that $I$ is a **-ideal of $R$. Let $0 \neq u(x) \in \mathbb{R}[x]$ and $0 \neq v \in Q$. Then we have $(u(x), 0)R(0, v) = \{(0, 0)\} = (u(x), 0)R(0, v)^*$, where $(0, 0) \neq (u(x), 0), (0, 0) \neq (0, v) \in R$. This implies that $R$ is not a **-prime ring but it is a semiprime ring. Here it is obvious to observe that (i) $D[m, n] \pm (mon) \in Z(R)$ for all $m, n \in I$ and (ii) $D(mon) \pm [m, n] \in Z(R)$ for all $m, n \in I$, where $Z(R)$ stands for the center of the ring $R$. However $R$ is noncommutative.

We now consider differential identities involving anticommutators in the next two results and show that there does not exist nonzero derivation satisfying these differential identities.

**Theorem 5.2.7.** Let $R$ be a **-prime ring of characteristic different from 2 and $I$ a nonzero **-ideal of $R$ such that $Z \cap I \neq \{0\}$. Then there exists no nonzero derivation $d$ such that $d(x)oy = xod(y)$ for all $x, y \in I$.

**Proof.** By hypothesis we have $d(x)y + yd(x) - xd(y) - d(y)x = 0$ for all $x, y \in I$. Let $z \in Z \cap I$. Replacing $y$ by $z$ in the hypothesis, we arrive at $d(x)z + zd(x) - xd(z) - d(z)x = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Now since $R$ has characteristic different from 2, and $d(Z) \subseteq Z$, we find that $d(x)z - zd(x) = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Substituting $xy$, where $y \in I$ for $x$ in the last relation and using the same again we conclude that $d(x)yz = 0$ for all $x, y \in I$ and for all $z \in Z \cap I$. But since $Z^* = Z$ and $I^* = I$, we also have $Z^* \cap I^* = Z \cap I$. These arguments show that $d(x)yz^* = 0$ for all $x, y \in I$ and for all $z \in Z \cap I$. Finally we infer that $d(x)Iz = \{0\} = d(x)Iz^*$ for all $x \in I$ and for all $z \in Z \cap I$. Lemma 5.2.1 and the fact that $Z \cap I \neq \{0\}$ insure that $d(x) = 0$ for all $x \in I$. Replacing $x$ by $xr$, where $r \in R$ in the last relation and using the same again we arrive at $Id(r) = \{0\}$. This implies that $sId(r) = \{0\} = s*Id(r)$, where $0 \neq s \in R$. Finally by Lemma 5.2.1, we obtain that $d = 0$. \[\square\]
Theorem 5.2.8. Let $R$ be a $*$-prime ring of characteristic different from 2 and $I$ a nonzero $*$-ideal of $R$ such that $Z \cap I \neq \{0\}$. Then there exists no nonzero derivation $d$ which commutes with $*$ and satisfies either (i) $d(x)oy = d(x)oD(y)$ for all $x, y \in I$ or (ii) $xod(y) = d(x)od(y)$ for all $x, y \in I$.

Proof. (i) By hypothesis we have $d(x)y + yd(x) - d(x)d(y) - d(y)d(x) = 0$ for all $x, y \in I$. Let $z \in Z \cap I$. Replacing $x$ by $z$ in the hypothesis, we arrive at $d(z)y + yd(z) - d(z)d(y) - d(y)d(z) = 0$ for all $y \in I$ and for all $z \in Z \cap I$. But since $R$ has characteristic different from 2 and $d(Z) \subseteq Z$ we arrive at $d(z)y - d(z)d(y) = 0$ for all $y \in I$ and for all $z \in Z \cap I$. Now we infer that $d(z)I(d(y) - y) = \{0\}$ for all $y \in I$ and for all $z \in Z \cap I$. But it is obvious to see that $Z^* \cap I^* = Z \cap I$. Since $d^* = d^*$, we also observe that $\{d(z)\}^*I(d(y) - y) = \{0\}$ for all $y \in I$ and for all $z \in Z \cap I$. Using Lemma 5.2.1 we obtain that $d(z) = 0$ for all $z \in Z \cap I$ or $d(y) = y$ for all $y \in I$. If first case holds, then hypothesis gives us $d(x)oz = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Since $R$ has characteristic different from 2, Lemma 4.3.1 provides us $d(x)z = 0$ for all $x \in I$ and for all $z \in Z \cap I$. This implies that $d(x)iz = \{0\} = d(x)iz^*$ for all $x \in I$ and for all $z \in Z \cap I$. Lemma 5.2.1 and the fact that $Z \cap I \neq \{0\}$ insure that $d(x) = 0$ for all $x \in I$. Now arguing in the similar way as in the above Theorem 5.2.7, we conclude that $d = 0$.

(ii) Using similar arguments as above, one can obtain the proof. \hfill \Box

The following example justifies that "characteristic different from 2" in the hypothesis of the above Theorems 5.2.7 and 5.2.8 is not superfluous.

Example 5.2.4. Consider $R, D, d, *_{ex}, I$ and $\mathcal{I}$ as discussed in the Example 5.2.1. It is obvious to observe that $Z(R) \cap \mathcal{I} = \mathcal{I} \neq \{0\}$, where $Z(R)$ denotes the center of the ring $R$. It is easy to check that (i) $D(x)oy = xoD(y)$ for all $x, y \in \mathcal{I}$, (ii) $D(x)oy = D(x)oD(y)$ for all $x, y \in \mathcal{I}$ and (iii) $xoD(y) = D(x)oD(y)$ for all $x, y \in \mathcal{I}$. However $D \neq 0$.

Theorem 5.2.9. Let $R$ be a $*$-prime ring of characteristic different from 2 and $I$ a nonzero $*$-ideal of $R$ such that $Z \cap I \neq \{0\}$. If $R$ admits a nonzero derivation $d$ which commutes with $*$ and satisfies $d(x)od(y) = xoy$ for all $x, y \in I$, then $R$ is commutative.
Proof. Given that \( d(x)od(y) = xoy \) for all \( x, y \in I \). Choose \( z \in Z \cap I \). Replacing \( y \) by \( yz \) in the hypothesis we obtain that \( d(x)od(yz) = xoyz \) for all \( x \in I \) and for all \( z \in Z \cap I \).

Now we have

\[
\begin{align*}
    d(x)od(yz) &= d(x)(d(y)z + yd(z)) + (d(y)z + yd(z))d(x) \\
               &= d(x)d(y)z + d(x)yd(z) + d(y)zd(x) + yd(z)d(x)
\end{align*}
\]

and on the other hand using the hypothesis we obtain that

\[
\begin{align*}
xoyz &= (xoy)z \\
      &= (d(x)od(y))z \\
      &= d(x)d(y)z + d(y)d(x)z.
\end{align*}
\]

Equating the above two expressions and using the fact that \( d(Z) \subseteq Z \), we conclude that 

\[
(d(x)y + yd(x))d(z) = 0 \quad \text{i.e.;} \quad (d(x)y + yd(x))Id(z) = \{0\} \quad \text{for all } x, y \in I, z \in Z \cap I. \]

It is obvious that \( Z^* \cap I^* = Z \cap I \). Since \( d^* = *d \), we also infer that \( (d(x)y + yd(x))Id(z)^* = \{0\} \) for all \( x, y \in I, z \in Z \cap I \). By Lemma 5.2.1 we arrive at \( (d(x)y + yd(x)) = 0 \) for all \( x, y \in I \) or \( d(z) = 0 \) for all \( z \in Z \cap I \). We claim that \( d(z) \neq 0 \) for all \( z \in Z \cap I \). For otherwise hypothesis provides us \( xoz = 2xz = d(x)od(z) = 0 \) for all \( x \in I \) and \( z \in Z \cap I \). Since \( R \) has characteristic different from 2, Lemma 4.3.1 insures that \( xz = 0 \) i.e.; \( xIz = \{0\} \) for all \( x \in I, z \in Z \cap I \). Since \( Z^* \cap I^* = Z \cap I \). This fact shows that \( xIz^* = \{0\} \) for all \( x \in I, z \in Z \cap I \). By Lemma 5.2.1, we deduce that either \( I = \{0\} \) or \( Z \cap I = \{0\} \). This leads to a contradiction. Finally we conclude that \( (d(x)y + yd(x)) = 0 \) for all \( x, y \in I \). Replacing \( y \) by \( yr \), where \( r \in I \) in the last relation and using the same again we obtain that \( y[d(x), r] = 0 \) for all \( x, y \in I, r \in R \). This implies that \( Il[d(x), R] = \{0\} = I^*[d(z), R], \) for all \( x \in I \), where \( 0 \neq l \in R \). By Lemma 5.2.1, we find that \( d(I) \subseteq Z \). Finally using Lemma 5.2.4, we get that \( R \) is commutative.

\[ \square \]

Theorem 5.2.10. Let \( R \) be a \(*\)-prime ring of characteristic different from 2 and \( I \) a nonzero \(*\)-ideal of \( R \) such that \( Z \cap I \neq \{0\} \). If \( R \) admits a nonzero derivation \( d \) which commutes with \( * \) and satisfies \( (d(x)y) - (xod(y)) \in Z \) for all \( x, y \in I \), then \( R \) is commutative.

Proof. Replacing \( y \) by \( z \), where \( z \in Z \cap I \) in the hypothesis and using the fact that \( d(Z) \subseteq Z \), we arrive at \( 2(d(x)z - xzd(z)) \in Z \) for all \( x \in I \). This implies that \( (d(x)z - xzd(z)) \in Z \) for all \( x \in I \) and for all \( z \in Z \cap I \) and hence \( (d(x)z - xzd(z))x = x(d(x)z - xzd(z)) \) for all \( x \in I \) and for all \( z \in Z \cap I \). In turn we conclude that \( (d(x)x - xod(x))z = 0 \) i.e.;
\[(d(x)x - xd(x))Iz = \{0\}\] for all \(x \in I\) and for all \(z \in Z \cap I\). Since \(Z^* \cap I^* = Z \cap I\),
we obtain that \((d(x)x - xd(x))Iz^* = \{0\}\) for all \(x \in I\) and for all \(z \in Z \cap I\). Now
hypothesis and Lemma 5.2.1 provide us that \([d(x), x] = 0\) for all \(x \in I\). Finally Lemma 5.2.3,
completes the proof.

The following example demonstrates that the \(*\)-primeness in the hypothesis in the above
theorem is necessary.

**Example 5.2.5.** Let \(S = M_{2 \times 2}(\mathbb{R}[x])\), the ring of all \(2 \times 2\) matrices over ring \(\mathbb{R}[x]\),
where \(\mathbb{R}[x]\) is the polynomial ring over ring of real numbers. Suppose that \(R = S \times S\),
which is clearly a ring of characteristic different from 2. Define \(D : R \rightarrow R\) as
\[D(A, B) = (0, D'(B))\],

\[B = \begin{pmatrix} f(x) & g(x) \\ h(x) & u(x) \end{pmatrix}\]
\[d\] is the usual differentiation of the polynomial ring \(\mathbb{R}[x]\).

Also define \(\ast : R \rightarrow R\) as \(\ast(A, B) = (A^t, B^t)\), where \(A^t\) and \(B^t\) are the transpose
of the matrices \(A\) and \(B\) respectively. It can be easily shown that \(D\) and \(\ast\) are a
nonzero derivation and an involution of \(R\) respectively such that \(D\ast = \ast D\). Suppose
that \(I = M_{2 \times 2}(\mathbb{R}[x]) \times \{0\}\). It is obvious that \(I\) is a \(*\)-ideal of \(R\) and \(Z(R) \cap I \neq \{0\}\),
where \(Z(R)\) is the center of \(R\). Let \(0 \neq U, 0 \neq V \in M_{2 \times 2}(\mathbb{R}[x])\). Then we have
\[(U, 0)R(0, V) = \{(0, 0)\} = (U, 0)R(0, V)^*, \] where \((0, 0) \neq (U, 0), (0, 0) \neq (0, V) \in R\).
This implies that \(R\) is not a \(*\)-prime ring but it is a semiprime ring. Here it is obvious
to see that \((D(p)q) - (pdD(q)) \in Z(R)\) for all \(p, q \in I\), but \(R\) is noncommutative.

### 5.3 Some extension theorems on the ring of quotients of
\(*\)-prime rings

There has been a great deal of work on extension problems of a semiprime ring \(R\) to
its different types of quotient rings i.e.; \(Q_{mr}\) and \(Q_s\) etc. For examples we know the
following: (i) an automorphism ( resp. antiautomorphism) of a semiprime ring \(R\) can
be uniquely extended to \(Q_{mr}\) and \(Q_s\) (resp. \(Q_s\)). (ii) derivation \(d\) of a semiprime ring \(R\)
can be uniquely extended to \(Q_{mr}\) and \(Q_s\). (iii) if \(R\) is a prime ring with involution \(\ast\),
then \(\ast\) can be uniquely extended to an involution of its right symmetric Martindale
quotient ring. (iv) let \(R\) is a prime (resp. semiprime) ring, then so are its quotient rings

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Motivated by the above nice extensions, we have obtained some possible analogues for \( \ast \)-prime rings as follows: (i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. (ii) if \( R \) is a \( \ast \)-prime ring, then so is its right symmetric Martindale quotient ring. (iii) every \( \ast \)-derivation of a \( \ast \)-prime ring can be uniquely extended to its right symmetric Martindale quotient ring.

It is well known that \( Q_s = \{ q \in Q_{mr} \mid qJ \subseteq R \text{ for some } J \in \mathcal{I} \} \). Here \( \mathcal{I} = \mathcal{I}(R) = \{ I \mid I \text{ is an ideal of } R \text{ and } l(I) = \{0\} \} \), where \( l(I) \) denotes the left annihilator of the ideal \( I \) in the ring \( R \). It is obvious to see that \( \mathcal{I} \) consists of precisely the dense ideals of \( R \).

Next suppose that \( q_1, q_2, \ldots, q_n \in Q_{mr} \). Then the set \( T = \{ q_1, q_2, \ldots, q_n \} \) is called \( C \)-dependent if there exist \( c_1, c_2, \ldots, c_n \in C \) not all zero such that \( c_1q_1 + c_2q_2 + \cdots + c_nq_n = 0 \).

On the other hand if \( T \) is not \( C \)-dependent, then it is called \( C \)-independent. It is well known that \( C \) is field if \( R \) is a prime ring and it is also to be noticed that if \( R \) is a semiprime ring and \( C \) a field, then \( R \) must be a prime ring. Further it is to be noted that if \( R \) is a prime ring then two nonzero elements \( q_1, q_2 \in Q_{mr} \) will be \( C \)-dependent if and only if \( q_1 = \lambda q_2 \) for some \( \lambda \in C \).

In the end of this section we have obtained a sufficient condition under which two nonzero elements of \( Q_s \) become \( C \)-dependent if \( Q_s \) is the right symmetric Martindale quotient ring of a \( \ast \)-prime ring \( R \).

We facilitate our discussion with the following lemmas which are essential for developing the proof of our main results of this section. The proof of Lemma 5.3.1 can be found in [20, Theorem 2.3.3].

**Lemma 5.3.1.** Let \( R \) be a semiprime ring, \( Q = Q_{mr}(R) \), \( C = Z(Q) \), where \( Z(Q) \) stands for the center of the ring \( Q \) and \( q_1, q_2, \ldots, q_n \in Q \). Suppose that \( q_1 \not\in \sum_{i=2}^{n} Cq_i \).

Then there exists an element \( p = \sum_{i=1}^{m} l_i r_i \in R_{(l)}R_{(r)} \) such that \( q_1 p = \sum_{i=1}^{m} a_i q_i r_i \not\leq 0 \) and \( q_j p = 0 \) for \( j \geq 2 \). Here \( R_{(l)} \) (resp. \( R_{(r)} \)) denotes the subring of \( \text{End}_C(Q) \) generated by all left (resp. right) multiplications by elements of \( R \), where \( \text{End}_C(Q) \) denotes the ring of all homomorphisms of \( Q \) as left-\( C \) modules.

In the year 1989 Brešar and Vukman [32, Proposition 1] proved that if a prime \( \ast \)-ring \( R \) admits a nonzero \( \ast \)-derivation, then \( R \) is commutative. We have shown that this result holds even for \( \ast \)-prime rings. In fact, we have obtained the following.

**Lemma 5.3.2.** Let \( R \) be a \( \ast \)-prime ring. If it admits a nonzero \( \ast \)-derivation \( d \), then \( R \) is commutative.
Proof. By hypothesis we have, for all \( x, y, z \in R \)

\[
d((xy)z) = d((xy)z^*) + xyd(z) = \{d(x)y^* + xd(y)\}z^* + xyd(z) = d(x)y^*z^* + xd(y)z^* + xyd(z).
\]

Also

\[
d(x(yz)) = d(x)(yz)^* + xd(yz) = d(x)z^*y^* + x\{d(y)z^* + yd(z)\} = d(x)z^*y^* + xd(y)z^* + xyd(z).
\]

Combining the above two relations, we get
\[
d(x)y^*z^* = d(x)z^*y^* \text{ for all } x, y, z \in R.
\]

Putting \( y^* \) and \( z^* \) in the places of \( y \) and \( z \) respectively, we find that
\[
d(x)yz = d(x)zy, \text{ for all } x, y, z \in R. \tag{5.3.1}
\]

Now replacing \( y \) by \( yr \) where \( r \in R \), in the relation (5.3.1) and using it again we arrive at
\[
d(x)yrz = d(x)yzr \text{ i.e.};
\]

\[
d(x)R[r, z] = \{0\}, \text{ for all } x, z, r \in R. \tag{5.3.2}
\]

Replacing \( r \) and \( z \) by \( r^* \) and \( z^* \) respectively in the relation (5.3.2) we also obtain that
\[
d(x)R[r, z]^* = \{0\}, \text{ for all } x, z, r \in R. \tag{5.3.3}
\]

Since \( d \neq 0 \) and \( R \) is a \( * \)-prime ring, using the relations (5.3.2) and (5.3.3), we conclude that \( rz = zr \) for all \( z, r \in R \) and hence \( R \) is commutative. \( \square \)

Following example demonstrates that the \( * \)-primeness in the hypothesis of Lemma 5.3.2 can not be omitted.

Example 5.3.1. Let \( Q \) and \( C \) be the ring of real quaternions and complex numbers respectively. Assume \( R = Q \times C \) is the ring of cartesian product of \( Q \) and \( C \) with regard to componentwise addition and multiplication. Let \( *_1, *_2 \) and \( * \) denote the involutions of rings \( Q, C \) and \( R \) respectively, defined by \( q^*_1 = \alpha - \beta i - \gamma j - \delta k \), where \( q = \alpha + \beta i + \gamma j + \delta k \in Q; z^*_2 = x - iy \), where \( z = x + iy \in C \) and \( (q, z)^* = (q^*_1, z^*_2) \) for all \( (q, z) \in R \). Define \( d : R \rightarrow R \) such that \( d(q, z) = (0, \eta(z - z^2)) \) where \( \eta \) is any fixed complex number. It can be easily verified that \( R \) is a semiprime ring but not a \( * \)-prime ring and \( d \) is a nonzero \( * \)-derivation of \( R \). However, \( R \) is not commutative. \( \text{108} \)
If $R$ is a prime ring with involution $*$, then we know that $R$ is a $*$-prime ring. Using this fact, we get the following:

**Corollary 5.3.1** ([32, Proposition 1]). *If a prime $*$-ring $R$ admits a nonzero $*$-derivation, then $R$ is commutative.*

Now we prove the main results of this section:

**Theorem 5.3.1.** *Let $R$ be a semiprime ring with involution $*$'. Then $*$ can be uniquely extended to an involution of its right symmetric Martindale quotient ring.

*Proof.* Since $R$ is a semiprime ring, $Q_{mr}$ and $Q_s$ will exist. We will also denote the extension of $*$, the involution of $R$ to $Q_s = Q$ by the same $*$. Let $q \in Q$. This implies that $q \in Q_{mr}$ and there exists $I \in \mathcal{I}$ such that $qI \cup Iq \subseteq R$. It is easy to see that $I^*$ is also a dense ideal, and therefore $I^* \subseteq I$. Now we define a relation $f : I^* \rightarrow R$ such that $f(i^*) = (iq)^*$. It is easy to check that $f$ is a well defined map and in addition it is a homomorphism of right $R$-modules. Therefore $[f; I^*] \subseteq Q_{mr}$. Let us say $q^* = [f; I^*]$. Consider $q^*i^* = [f; I^*][I^*; R] = [f1^*; l_{i^*}^{-1}(I^*)] = [l_{(q_i)}; R] = (iq^*)$ for all $i \in I$. Also consider $i^*q^* = [I^*; R][f; I^*] = [l_{*}f; f^{-1}(R)] = [l_{*}f; I^*] = [l_{(q_0)}; R] = (qi)^*$ for all $i \in I$.

Now we obtain the following two relations

$$q^*i^* = (iq)^*, \text{ for all } i \in I. \quad (5.3.4)$$

and

$$i^*q^* = (qi)^*, \text{ for all } i \in I. \quad (5.3.5)$$

From the above two relations it is clear that $q^*I^* \cup I^*q^* \subseteq R$. Therefore $q^* \in Q$.

Next we define a mapping $q \mapsto q^*$ of $Q$ into itself, where $q^* = [f; I^*]$. We will prove that this is our required unique extension of involution $*$ of $R$. Let $q_1, q_2 \in Q$. This implies that $q_1 + q_2 \in Q$. There exists a dense ideal $J$ of $R$ i.e.; $J \in \mathcal{I}$ such that $q_1J \cup Jq_1$, $q_2J \cup Jq_2$, $(q_1 + q_2)J \cup J(q_1 + q_2)$ are all contained in $R$. It is obvious that relations (5.3.4) and (5.3.5) will be true if we replace $q$ by $q_1, q_2$ or $(q_1 + q_2)$ and $I$ by $J$. Therefore for all $j \in J$, we have $(q_1 + q_2)^*j^* = (j(q_1 + q_2))^* = (j_1q_1 + j_2q_2)^* = (j_1q_1)^* + (j_2q_2)^* = (q_1^* + q_2^*)j^*$. Finally we arrive at $\{(q_1 + q_2)^* - (q_1^* + q_2^*)\}J^* = \{0\}$. Since $J^* \in \mathcal{I}$, by characterization of $Q_s$ we conclude that $(q_1 + q_2)^* = q_1^* + q_2^*$ showing that $*$ is an additive map. Let $q_1, q_2 \in Q$. This implies that $q_1q_2 \in Q$. There exists a dense ideal $K$ of $R$ i.e.; $K \in \mathcal{I}$ such that $q_1K \cup Kq_1$, $q_2K \cup Kq_2$, $(q_1q_2)K \cup K(q_1q_2)$ are all contained in $R$ and let $L = K^2$. Then $q_1L, Lq_1, q_2L, Lq_2 \subseteq K$. It is obvious that $L \in \mathcal{I}$. Like above for all $l \in L$ we have
(q_1 q_2)^* l^* = (l q_1 q_2)^* = q_2^* (l q_1)^* = q_2^* q_1^* l^*$. This implies that $(q_1 q_2)^* = q_2^* q_1^*$. Since $'*$ is an involution on $R$, operating $'*$ on both sides of relation (5.3.4) we obtain that $i(q^*)^* = i q$
for all $i \in I$. Now we arrive at $I \{ (q^*)^* - q \} = \{ 0 \}$. Since $I \subseteq I$ and $\{ (q^*)^* - q \} \subseteq Q$, we conclude that $(q^*)^* = q$. Including all the above arguments we obtain that $'*$ is an involution of $Q$.

Finally we have to prove that this extension is unique. Let us suppose that $'\phi'$ and $'*'$ be two extensions of the involution of $R$. From above arguments it is clear that for any $q \in Q$, there exists $I \subseteq I$ such that $q I \cup I q \subseteq R$. It is obvious that for all $i \in I$, $q_i \in R$.

Using the fact that $r^\phi = r^*$ for all $r \in R$, we obtain that $(q_i)^\phi = (q_i)^*$ for all $i \in I$. This implies that $i^{\phi} q^{\phi} = i^{\phi} q^*$ for all $i \in I$. Now we conclude that $I^* (q^\phi - q^*) = \{ 0 \}$. But $I^* \subseteq I$, therefore using the characterization of $Q$, we arrive at $q^* = q^\phi$ for all $q \in Q$ and hence this is a unique extension.

**Theorem 5.3.2.** Right symmetric Martindale quotient ring of a $*$-prime ring is also a $*$-prime ring.

**Proof.** Since $R$ is a $*$-prime ring, it must be a semiprime ring also. Therefore its right symmetric Martindale quotient ring $Q_s$ will exist. By the above theorem it is clear that involution $'\phi'$ of $R$ can be uniquely lifted to an involution of $Q_s$. Therefore we can assume that $'*'$ is defined on whole of $Q_s$. Finally we conclude that $Q_s$ is a $'*'$-ring.

Now we have to prove that $Q = Q_s$ is also a $*$-prime ring. Suppose that $q_1, q_2 \in Q$ such that $q_1 Q q_2 = \{ 0 \}$ and $q_1 Q q_2^* = \{ 0 \}$, then we have to prove that either $q_1 = 0$ or $q_2 = 0$. Suppose on contrary that $q_1 \neq 0$ and $q_2 \neq 0$. There exist dense ideals $J_1, J_2 \subseteq I$ such that $q_1 J_1 \cup J_1 q_1 \subseteq R$ and $q_2 J_2 \cup J_2 q_2 \subseteq R$. By characterization of $Q_s$, we have $x \in J_1$ and $y \in J_2$ such that $0 \neq q_1 x \in R$ and $0 \neq q_2 y \in R$. But now by using hypothesis we have $(q_1 x) R (q_2 y) = \{ 0 \}$ and $(q_1 x) R (q_2 y)^* = \{ 0 \}$. Contradicting the fact that $R$ is a $*$-prime ring. Finally we conclude that $Q$ is a $*$-prime ring.

**Theorem 5.3.3.** Let $R$ be a commutative semiprime ring with involution $'\ast'$ admitting a $\ast$-derivation $d$. Then $d$ can be uniquely extended to a $\ast$-derivation of its right symmetric Martindale quotient ring.

**Proof.** Since $R$ is a commutative semiprime ring, its right symmetric Martindale quotient ring $Q = Q_s$ will exist and will also be commutative. For this case we will also have $Q_{mr} = Q_s$. By Theorem 5.3.1, involution $'\ast'$ of $R$ can be uniquely extended to an involution of $Q$. Therefore we can assume that $'\ast'$ is defined on whole $Q$. We shall let $d$ also denote its extension to $Q$. $d(q)$, where $q \in Q$ will be denoted by $q^d$.

Given any $q \in Q$. This implies that $q \in Q_{mr}$ and there exists $J \subseteq I$ such that
It is also obvious that $J$ is a dense right ideal of $R$. Now we set $J_d = \sum_{x \in J} x \{(x^d : J)_R\}^*$. Since $x^d \in R$, $(x^d : J)_R$ is a dense right ideal. Here '*' is an automorphism of $R$, therefore $\{(x^d : J)_R\}^*$ is also a dense right ideal of $R$. Next we claim that $J_d$ is a dense right ideal of $R$. It is obvious to observe that $J_d$ is a right ideal of $R$. Let $0 \neq r_1, r_2 \in R$. Since $J$ is a dense right ideal of $R$, $0 \neq s_1 r_1$ and $s_2 r_2 \in J$ for some $s \in R$. As we already know that $\{(\langle r_2 s \rangle^d : J)_R\}^*$ is a dense right ideal of $R$. Therefore $0 \neq r_1 s t$ for some $t \in \{(\langle r_2 s \rangle^d : J)_R\}^*$, it is due to the fact that the left annihilator of any dense right ideal in a semiprime ring vanishes. Clearly $r_2 s t \in J_d$ and so our claim stands proved. Also we observe that $J_d \subseteq J$ and $(J_d)^d \subseteq J$. Since $J_d$ is a dense right ideal of $R$, $(J_d)^* \subseteq J$ is also a dense right ideal of $R$. We define $f : (J_d)^* \to R$ by the rule $f(x^* d) = (qx)^d - qx^d$ for all $x^* \in (J_d)^*$. It is easy to see that $f$ is a well defined map and additive also. For all $x^* \in (J_d)^*$ and $r \in R$ we have $f(x^* r) = f(x^* r) = (qx)^d - qx^d = (qx)^d x^* + qx^d x^* - qx x^* = (qx)^d x^* - qx x^* = \{q x^* d\} x^* = f(x^*) r$; where $r = l^*$ for some $l \in R$. Arguments given above show that $f$ is a homomorphism of right $R$-modules. Therefore $[f; (J_d)^*] \in Q_m$. Now we put $q^d = [f; (J_d)^*]$. Due to commutativity of $Q$, it is trivial to see that $(J_d)^* \in I$ and $q^d (J_d)^* \cup (J_d)^* q^d \subseteq R$. Finally we arrive at $q^d \in Q = Q_s$. Let us define a map $q \mapsto q^d$ of $Q$ into itself, where $q^d = [f; (J_d)^*]$. We will prove that this is our required unique extension of $*$-derivation $d$ of $R$. First we compute the following:

$$q^d x^* = (qx)^d - qx^d \quad (5.3.6)$$

for all $x \in J_d$. Let $q_1, q_2 \in Q$. This implies that $q_1 + q_2 \in Q$. There exists a $K \in I$ such that $q_1 K \cup K q_1, K q_2 \cup K q_2, (q_1 + q_2) K \cup K (q_1 + q_2)$ are all contained in $R$. It is obvious that relation (5.3.6) will be true if we replace $q$ by $q_1$, $q_2$ or $q_1 + q_2$ and $J_d$ by $K_d$ where $K_d = \sum_{x \in K} x \{(x^d : K)_R\}^*$. Therefore for all $k \in K_d$, we have $(q_1 + q_2) k^d = (q_1 + q_2) k^d - (q_1 + q_2) k^d = (q_1 k^d + q_2 k^d - q_1 k^d - q_2 k^d = (q_1 + q_2) k^d = (q_1 + q_2) k^d \in K_d \in I$, using characterization of $Q$, we conclude that $(q_1 + q_2)^d = q_1^d + q_2^d$ showing that $d$ is an additive map. Let $q_1, q_2 \in Q$. This implies that $q_1 q_2 \in Q$. By above arguments it is clear that there exists $T_d \in I$ such that $q_1 T_d \cup T_d q_1, q_2 T_d \cup T_d q_2, q_1 q_2 T_d \cup T_d q_1 q_2$ are all contained in $R$. It is obvious that relation (5.3.6) will be true if we replace $q$ by $q_1, q_2$ or $q_1 q_2$ and $J_d$ by $T_d$. Let $I = (T_d)^2$.

Then $q_1 I, I q_1, q_2 I, I q_2 \subseteq T_d$. For all $i \in I$, we have $(q_1 q_2)^d i^* = (q_1 q_2)^d - q_1 q_2 i^d = \ldots$
Finally we arrive at \((q_1q_2)^d - q_1q_2^d - q_1q_2^d\) and \(q_1q_2^d + q_1q_2^d\). Finally we conclude that \((q_1q_2)^d = q_1^d q_2^d + q_1^d q_2^d\). Therefore \(d\) is a \(\ast\)-derivation of \(Q\).

Finally we have to prove that this extension is unique. Let us suppose that \(\delta\) and \(d\) be two extensions of the \(\ast\)-derivation of \(R\). From above arguments it is clear that for any \(q \in Q\), there exists \(J \subseteq I\) such that \(qJ \cup Jq \subseteq R\). It is obvious that for all \(j \in J\), \(qj \in R\).

Using the fact that \(r^\delta = r^d\) for all \(r \in R\), we obtain that \((qj)^\delta = (qj)^d\) for all \(j \in J\). This implies that \(q^\delta j^\delta + qj^\delta = q^d j^\delta + qj^d\) for all \(j \in J\). Now we infer that \((q^\delta - q^d)J^\ast = \{0\}\).

But \(J^\ast \in I\), therefore by characterization of \(Q\), we find that \(q^\delta = q^d\) for all \(q \in Q\), thus this extension is unique. \(\square\)

In the light of Lemma 5.3.2 and Proposition 1 of [32], we obtained the following:

**Corollary 5.3.2.** Let \(R\) be a \(\ast\)-prime ring (resp. prime ring with involution \(\ast\)) admitting a \(\ast\)-derivation \(d\). Then \(d\) can be uniquely extended to a \(\ast\)-derivation of its right symmetric Martindale quotient ring.

It has been proved in [20, Theorem 2.3.4] that if \(R\) is a prime ring, \(Q = Q_{mr}\) and \(a, b \in Q\). Suppose that \(ab = ba\) for all \(x \in R\). Then \(a\) and \(b\) are \(C\)-dependent.

We have extended this result in the setting of \(\ast\)-prime rings as follows:

**Theorem 5.3.4.** Let \(R\) be a \(\ast\)-prime ring, \(Q = Q_s\) and \(0 \neq a, 0 \neq b \in Q\). Suppose that \(a \ast b^\ast = b \ast a\) and \(a \ast x b^\ast = b^\ast x a\) for all \(x \in R\). Then \(a \in Cb^\ast\) and hence \(a\) and \(b^\ast\) are \(C\)-dependent.

**Proof.** Since \(R\) is a \(\ast\)-prime ring, it will be a semiprime also and \(Q = Q_s\) exists. By Theorem 5.3.1, \(\ast\)' can be assumed to be defined on whole of \(Q\). We have to prove that \(a \in Cb^\ast\). Suppose on contrary i.e.; \(a \notin Cb^\ast\), then by Lemma 5.3.1 there exists an element \(p = \sum_{i=1}^{n} l_i r_i y_i \in R(l) R(r)\) such that \(d = ap \neq 0\) and \(b^\ast p = 0\). Using the condition \(a \ast b^\ast = b \ast a\) for all \(x \in R\), we have \(0 = ap = \sum_{i=1}^{n} x_i b^\ast y_i = \sum_{i=1}^{n} (ax_i b^\ast) + \sum_{i=1}^{n} (br_i a) y_i = br \sum_{i=1}^{n} (x_i a y_i) = brd\) for all \(r \in R\) and hence we obtain that

\[
bRd = \{0\}. \quad (5.3.7)
\]

If we use the condition \(a^\ast b^\ast = b^\ast a\) for all \(x \in R\), on the other hand we also obtain that \(0 = a^\ast r \sum_{i=1}^{n} x_i b^\ast y_i = \sum_{i=1}^{n} (a^\ast r x_i b^\ast) y_i = \sum_{i=1}^{n} (b^\ast r x_i a) y_i = b^\ast r \sum_{i=1}^{n} (x_i a y_i) = b^\ast rd\) for all \(r \in R\) and therefore

\[
b^\ast Rd = \{0\}. \quad (5.3.8)
\]

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It is given that $0 \neq b$ and $0 \neq d \in Q_s$. By characterization of $Q_s$, we conclude that there exist $J, U \in \mathcal{I}$ such that $0 \neq bj \in R$ and $0 \neq du \in R$ for some $j \in J$ and $u \in U$. Using the relations (5.3.7) and (5.3.8), we also conclude that $(bj)R(du) = \{0\}$ and $(bj)^*R(du) = \{0\}$, leading to a contradiction due to the fact that $R$ is a $*$-prime ring. \qed
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Research Publications
ON PERMUTING $n$-DERIVATIONS IN NEAR-RINGS

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ABSTRACT. In this paper, we introduce the notion of permuting $n$-derivations in near-ring $N$ and investigate commutativity of addition and multiplication of $N$. Further, under certain constraints on a $n!$-torsion free prime near-ring $N$, it is shown that a permuting $n$-additive mapping $D$ on $N$ is zero if the trace $d$ of $D$ is zero. Finally, some more related results are also obtained.

1. Introduction

Throughout this paper $N$ will denote a zero-symmetric left near ring. A near ring $N$ is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that in a left near ring $x0 = 0$ for all $x \in N$). $N$ is called prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semi prime if $xNx = \{0\}$ implies $x = 0$. Near-ring $N$ is called $n$-torsion free if $nxyz = 0$ implies $x = 0$. The symbol $Z$ will represent the multiplicative center of $N$, that is, $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. As usual, for $x, y \in N, [x, y]$ will denote the commutator $xy - yx$, while $(x, y)$ will indicate the additive group commutator $x + y - x - y$. The symbol $C$ will represent the set of all additive commutators of near ring $N$. For terminologies concerning near-rings we refer to G. Pilz [10].

An additive map $f : N \rightarrow N$ is called a derivation if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in N$. The concepts of symmetric bi-derivation, permuting tri-derivation and permuting $n$-derivation have already been introduced in rings by G. Maksa, M. A. Öztürk and K. H. Park in [4, 5, 6], and [8], respectively. These concepts of symmetric bi-derivations and permuting tri-derivations have been studied in near-rings by M. A. Öztürk and K. H. Park in [7] and [9], respectively. In the present paper, motivated by these concepts, we define permuting $n$-derivations in near-rings and study some properties involved there. Some relations between permuting $n$-derivations and $C$, the set of all additive commutators in near-ring $N$ have also been studied.
ON \((\sigma, \tau)\)-\(n\)-DERIVATIONS IN NEAR-RINGS

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In the present paper, we introduce the notion of \((\sigma, \tau)\)-\(n\)-derivation in near-ring \(N\) and investigate some properties involving \((\sigma, \tau)\)-\(n\)-derivations of a prime near-ring \(N\) which force \(N\) to be a commutative ring. Additive commutativity of near-ring \(N\) satisfying certain identities involving \((\sigma, \tau)\)-\(n\)-derivations of a prime near-ring \(N\) has also been obtained. Related examples to justify the hypotheses in various theorems have also been provided.

Keywords: Prime near-ring; derivation; \((\sigma, \tau)\)-derivation; \(n\)-derivation; \((\sigma, \tau)\)-\(n\)-derivation and commutativity.

AMS Subject Classification: 16W25, 16Y30

1. Introduction

A nonempty set \(N\) equipped with two binary operations \(+\) and \(\cdot\) is called a left near-ring provided that \((N, +)\) is a group (not necessarily abelian), \((N, \cdot)\) is a semigroup and \(x \cdot (y + z) = x \cdot y + x \cdot z\) for all \(x, y, z \in N\). For the sake of convenience the product \(x \cdot y\) between two elements of \(N\) will be denoted by \(xy\). A left near-ring \(N\) is called zero symmetric if \(0x = 0\) holds for all \(x \in N\) (recall that in a left near-ring \(x0 = 0\) for all \(x \in N\)). Throughout this paper, unless otherwise specified, we will use the word near-ring denoted by \(N\) to mean zero symmetric left near-ring. Further, \(N\) is called a prime near-ring if \(xy = 0\) implies \(x = 0\) or \(y = 0\). It is called semiprime if \(xN = 0\) implies \(x = 0\). For a given integer \(n > 1\), near-ring \(N\) is said to be \(n\)-torsion free, if for \(x \in N\), \(nx = 0\) implies \(x = 0\). The symbol \(Z\) will denote the multiplicative center of \(N\), that is, \(Z = \{x \in N \mid xy = yx\ \text{for all} \ y \in N\}\). For any \(x, y \in N\) the symbols \([x, y] = xy - yx\) and \((x, y) = x + y - x - y\) stand for multiplicative commutator and additive commutator of \(x\) and \(y\) respectively,
On *-derivations in near-rings with involution

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Abstract. The purpose of this paper is to introduce the notions of involution and *-derivation in near-rings. Let \( N \) be a left near-ring. An additive mapping \( \phi : N \rightarrow N \) is said to be an involution on \( N \) if (i) \( (x^*)^* = x \) and (ii) \( (xy)^* = y^*x^* \) hold for all \( x, y \in N \). A near-ring equipped with an involution \( \ast \) is called a *-near-ring. An additive map \( D \) on a *-near ring \( N \) is called a *-derivation on \( N \) if \( D(xy) = D(x)y^* + xD(y) \) holds for all \( x, y \in N \). Analogues of some ring theoretic results have been obtained in the setting of *-near-rings. In fact, if a prime *-near ring \( N \) possesses a nonzero *-derivation (resp. reverse *-derivation) \( D \), then it is shown that \( N \) is a commutative ring. Further, some related properties of *-derivation in semiprime *-near-rings have been studied. Finally, some results concerning composition of *-derivations of prime *-near-rings, have also been obtained.

Keywords: Left near-ring; Zerosymmetric; Involution; Derivation; *-derivation; Reverse *-derivation; Prime *-near-ring; Semiprime *-near-ring; Quotient near-ring.

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1 Introduction

Throughout the discussion, unless otherwise mentioned, \( N \) will denote a zero symmetric left near-ring. \( N \) is called zero symmetric if \( 0x = 0 \) holds for all \( x \in N \) (Recall that in a left near-ring \( x0 = 0 \) holds for all \( x \in N \)). \( N \) is called a prime near-ring if \( xNy = \{0\} \) implies \( x = 0 \) or \( y = 0 \). It is called semiprime if \( xNx = \{0\} \) implies \( x = 0 \). Given an integer \( n > 1 \), near-ring \( N \) is said to be \( n \)-torsion free, if for \( x \in N \), \( nx = 0 \) implies \( x = 0 \). A nonempty subset \( U \) of \( N \) is called a semigroup left ideal (resp. a semigroup right ideal) if \( NU \subseteq U \) (resp. \( UN \subseteq U \)) and if \( U \) is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. If \( K \) is a nonempty subset of \( N \), then a normal subgroup \( (K,+) \) of \( (N,+) \) is called a right ideal (resp. a left ideal) of \( N \) if

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GENERALIZED DERIVATIONS ON SEMIGROUP IDEALS
AND COMMUTATIVITY OF PRIME NEAR-RINGS

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Abstract: A non empty subset \( U \) of a near-ring \( N \) is said to be a semigroup left (resp. right) ideal of \( N \) if \( NU \subseteq U \) (resp. \( UN \subseteq U \)). and if \( U \) is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal. In the present paper, we investigate the commutativity of prime near-rings satisfying certain identities involving generalized derivations on semigroup ideals or ideals. Furthermore, we give examples to show that the restrictions imposed on the hypothesis of the various theorems are not superfluous.

1. INTRODUCTION

Throughout the paper, \( N \) will denote a zero symmetric left near-ring. \( N \) is called a prime near-ring if \( xy = \{0\} \) implies \( x = 0 \) or \( y = 0 \). Given an integer \( n > 1 \), near-ring \( N \) is said to be \( n \)-torsion free, if for \( x \in N \), \( nx = 0 \) implies \( x = 0 \). A nonempty subset \( U \) of \( N \) is called semigroup left ideal (resp. semigroup right ideal)if \( NU \subseteq U \) (resp. \( UN \subseteq U \)). and if \( U \) is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. The symbol \( Z \) will denote the multiplicative center of \( N \), that is, \( Z = \{ x \in N \mid xy = yx \text{ for all } y \in N \} \). For any \( x, y \in N \) the symbol \( [x, y] = xy - yx \) stands for multiplicative commutator of \( x \) and \( y \), while the symbol \( xoy \) will denote \( xy + yx \). Finally the notation \( \pm(xoy) \) represents either \( + (xoy) \) i.e.; \( xy + yx \) or \( -(xoy) \) i.e.; \( -(yx) - (xy) \).

An additive mapping \( d \) from \( N \) to \( N \) is called a derivation of \( N \) if \( d(xy) = d(x)y + xd(y) \)

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\(^{2}\)Keywords and Phrases: Ideals, semigroup ideals, generalized derivation, prime near-rings, commutativity.