EXPECTATION OF ORDERED RANDOM VARIATES
AND RELATED RESULTS

ABSTRACT
OF THE
THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF
Doctor of Philosophy
IN
STATISTICS

BY
SABA KHALID KHWAJA

UNDER THE SUPERVISION OF
Dr. HASEEB ATHAR

DEPARTMENT OF STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202 002 (INDIA)

2014
EXPECTATION OF ORDERED RANDOM VARIATES AND RELATED RESULTS

ABSTRACT
Order statistics and related general models of ordered random variables are important in statistical theory and its applications. The independent and identically random variables, that can be interpreted as results of an experiment measuring values of a certain random variable arranged in order of magnitude, are called order statistics. In statistical model of many experiments, for instance in reliability analysis, life time studies, in testing of strength of materials, etc., the realizations arise in non decreasing order, therefore the use of order statistics is necessary. Furthermore, order statistics are extensively used in statistical inferences; in the estimation theory and hypothesis testing. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics.

Order statistics has immense role in characterization problems. Characterization results are located on the borderline between probability theory and mathematical statistics, and utilize numerous classical tools of mathematical analysis. The useful characterizations results are those which shed light on modelling consequences of certain distributional assumptions and those which have potential for development of hypothesis tests for model assumptions. There are many results on characterizations of probability distributions through properties of order statistics. For detailed survey, one may refer to Ali and Khan (1987), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998), Khan and Abouammmoh (2000), Khan and Athar (2004) and references therein.
The distribution theory of order statistics has been widely described in several monographs written by outstanding statisticians and there are numerous papers devoted to the theory of order statistics and its applications as well as asymptotic results and inferences based on order statistics. Contributions of Sarhan and Greenberg (1958), Arnold et al. (1992), David and Nagaraja (2003), Ahsanullah et al. (2013) are significant.

Order statistics and their moments have received attention from the beginning of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics. The moment of order statistics did, subsequently, assume considerable importance in the statistics literature and have been numerically tabulated extensively for several distributions. For example one can refer to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold et al. (1989, 1992) for details.


The concept of generalized order statistics have been introduced and extensively studied by Kamps (1995 a, b). The use of such concept has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been
separately treated in statistical literature. Some types of ordered random variables such as order statistics, upper record values, sequential order statistics, progressive type II censored order statistics can be discussed as special case of generalized order statistics. Contributions of Keseling (1999), Bieniek and Szynal (2003), Cramer et al. (2004), Khan and Alzaid (2004), Athar and Islam (2004), Ahsanullah (2005), Khan et al. (2006) and Samuel (2008) are significant. [Bairamov, 2007].

Generalized order statistics can be easily applicable in practice problems except that when $F(.)$ is so called inverse distribution function. So the concept of lower generalized order statistics is needed. Pawlas and Szynal (2001a) introduced the concept of lower generalized order statistics (Igos) to enable a common approach to descending ordered random variables like reversed order statistics and lower record values. Further, the concept of Igos was extensively studied by Burkschat et al. (2003) with the name dual generalized order statistics (dgos). For more detailed survey on dgos one may refer to Ahsanullah (2005), Mbah and Ahsanullah (2007), Athar et al. (2008), Athar et al. (2010) and references therein.

The thesis entitled “Expectation of Ordered Random Variates and Related Results” is based on six chapters, which are described as below:

**Chapter I** is introductory in nature and deals with the basic concepts and results concerning order statistics, record values, generalized order statistics and lower generalized order statistics. Some well known continuous distributions are discussed as well.

**Chapter II** is based on recurrence relations for single and product moments of generalized order statistics from Pareto distribution. Special cases of generalized order statistics such as order statistics and record
Abstract

values are also discussed and at the end a characterization theorem is given.

**Chapter III** contains explicit expression for moments of order statistics for extended type-I generalized logistic distribution and some computational work is carried out. Further, recurrence relations for marginal and joint moment generating functions of order statistics are derived.

**Chapter IV** extends the results of Chapter III for dual generalized order statistics. Results are deduced for order statistics and lower record values.

**Chapter V** consists of recurrence relations for moments of generalized order statistics, from doubly truncated Makeham distribution; a characterizing result based on conditional expectation of generalized order statistics is also presented.

**Chapter VI** is related to the characterization of some generalized probability distributions through conditional expectation of function of generalized order statistics conditioned on non-adjacent generalized order statistics. Further, some deductions for order statistics and records are also discussed.

In the end, a comprehensive bibliography is given which has been referred in these chapters.

**References**


Galton, F. (1902): The most suitable proportion between the values of first and second prized. *Biometrika*, 1, 385-390.


Khan, A.H. and Abouammoh, A.M. (2000): Characterization of
distributions by conditional expectation of order statistics. *J. Appl.

through linear regression of non-adjacent generalized order statistics,

through linear regression of non-adjacent generalized order statistics.


continuous distributions through record statistics. *Commun. Korean

continuous distributions through conditional expectation of

of distributions based on linear regression of order statistics and

distribution by conditional expectation of record values.
*Sankhyā, Ser. A*, 58, 135-141.


EXPECTATION OF ORDERED RANDOM VARIATES AND RELATED RESULTS

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

SABA KHALID KHWAJA

UNDER THE SUPERVISION OF

Dr. HASEEB ATHAR

DEPARTMENT OF STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202 002 (INDIA)

2014
Candidate’s Declaration

I, Saba Khalid Khwaja, a Research Student of Department of Statistics & Operations Research certify that the work embodied in this Ph.D. thesis is my own bonafide work carried out by me under the supervision of Dr. Haseeb Athar at Aligarh Muslim University, Aligarh. The matter embodied in this Ph.D. thesis has not been submitted for the award of any other degree.

I declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and the body of the thesis. I further certify that I have not willfully lifted up some other’s work, para, text, data, result, etc. reported in the journals, books, magazines, reports, dissertations, theses, etc., or available at websites and included them in this Ph.D. thesis and cited as my own work.

Dated 22-05-2014

(Saba Khalid Khwaja)

Certificate from the Supervisor

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Dr. Haseeb Athar)
Assistant Professor

(Signature of the Chairman of the Department with seal)

Phone: +91-571-2701251 (Office), e-mail: haseebathar@hotmail.com
Mohammad Masood Khalid
Ph.D. (Agt)
Professor & Chairman

Course/Comprehensive Examination/Pre-submission Seminar Completion Certificate

This is to certify that Ms. Saba Khalid Khwaja, Department of Statistics & Operations Research has satisfactorily completed the course work/comprehensive examination and pre-submission seminar requirement which is part of her Ph.D. (Statistics) programme.

Dated:

(Prof. M.M. Khalid)
Chairman
Chairman
Dept. of Statistics & O.R.
A.M.U., Aligarh

Phone: +91 571 2701251 (Office), e-mail: chairman.stats@gmail.com
Copyright Transfer Certificate

Title of the Thesis : Expectation of Ordered Random Variates and Related Results

Candidate’s Name : Saba Khalid Khwaja

Copyright Transfer

The undersigned hereby assigns to the Aligarh Muslim University, Aligarh copyright that may exist in and for the above thesis submitted for the award of the Ph.D. degree.

(Saba Khalid Khwaja)

Note: However, the author may reproduce or authorize others to reproduce material extracted verbatim from the thesis or derivative of the thesis for author’s personal use provide that the source and the University’s copyright notice are indicated.
Papers Published


Homage to my Parents
and
Dedicated to my husband
# CONTENTS

**Acknowledgements**  

**Preface**  

<table>
<thead>
<tr>
<th>Chapter I</th>
<th>Preliminaries and Basic Concepts</th>
<th>1 - 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Order Statistics</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>Overview</td>
<td>1</td>
</tr>
<tr>
<td>2.2</td>
<td>Definition and distribution</td>
<td>2</td>
</tr>
<tr>
<td>2.3</td>
<td>Truncated and conditional distribution</td>
<td>6</td>
</tr>
<tr>
<td>2.4</td>
<td>Some important results</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Record Values and Record Times</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Overview</td>
<td>9</td>
</tr>
<tr>
<td>3.2</td>
<td>Definition</td>
<td>10.</td>
</tr>
<tr>
<td>3.3</td>
<td>Distribution of record values</td>
<td>10</td>
</tr>
<tr>
<td>3.4</td>
<td>$k$-records</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>Generalized Order Statistics</td>
<td></td>
</tr>
<tr>
<td>4.1</td>
<td>Overview</td>
<td>13</td>
</tr>
<tr>
<td>4.2</td>
<td>Definition</td>
<td>13</td>
</tr>
<tr>
<td>4.3</td>
<td>Distribution of generalized order statistics</td>
<td>14</td>
</tr>
<tr>
<td>4.4</td>
<td>Some important results</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>Lower Generalized Order Statistics</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>Overview</td>
<td>18</td>
</tr>
<tr>
<td>5.2</td>
<td>Definition</td>
<td>19</td>
</tr>
<tr>
<td>5.3</td>
<td>Distribution of lower generalized order statistics</td>
<td>19</td>
</tr>
<tr>
<td>5.4</td>
<td>Some important results</td>
<td>21</td>
</tr>
<tr>
<td>6</td>
<td>Expectation Properties of Ordered Random Variables</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>Some Continuous Distributions</td>
<td>27</td>
</tr>
</tbody>
</table>

**Chapter II**  

**Moment Properties of Pareto Distribution Based on Generalized Order Statistics and its Characterization**  

<p>| 1 | Introduction | 33 |
| 2 | Single Moments | 34 |
| 3 | Product Moments | 43 |
| 4 | Characterization | 53 |</p>
<table>
<thead>
<tr>
<th>Chapter III</th>
<th>Moments of Order Statistics from Extended Type-I Generalized Logistic Distribution</th>
<th>57 - 66</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td></td>
<td>57</td>
</tr>
<tr>
<td>2 Exact Moments</td>
<td></td>
<td>58</td>
</tr>
<tr>
<td>3 Recurrence Relations for Moment Generating Functions</td>
<td></td>
<td>62</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter IV</th>
<th>Moment Generating Functions of Lower Generalized Order Statistics from Extended Type-I Generalized Logistic Distribution</th>
<th>67 - 77</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td></td>
<td>67</td>
</tr>
<tr>
<td>2 Marginal Moment Generating Function</td>
<td></td>
<td>68</td>
</tr>
<tr>
<td>3 Joint Moment Generating Function</td>
<td></td>
<td>74</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter V</th>
<th>Generalized Order Statistics from Doubly Truncated Makeham Distribution</th>
<th>79 - 90</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td></td>
<td>79</td>
</tr>
<tr>
<td>2 Single Moments</td>
<td></td>
<td>80</td>
</tr>
<tr>
<td>3 Product Moments</td>
<td></td>
<td>85</td>
</tr>
<tr>
<td>4 Characterization</td>
<td></td>
<td>87</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter VI</th>
<th>Characterization of Probability Distributions through Generalized Order Statistics</th>
<th>91 - 105</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td></td>
<td>91</td>
</tr>
<tr>
<td>2 Characterization of Distributions</td>
<td></td>
<td>92</td>
</tr>
<tr>
<td>3 Examples</td>
<td></td>
<td>103</td>
</tr>
</tbody>
</table>

References  
107 - 117
ACKNOWLEDGEMENTS

This thesis is the end of my journey in obtaining my research degree. I have not traveled in a vacuum in this journey. This thesis has been kept on track and seen through to completion with the support and encouragement of numerous people. I would like to thank all those people who made this thesis possible and an unforgettable experience for me.

Firstly, I bow my head to Almighty 'Allah' "the one universal Being", who blessed me with strength to overcome all the obstacles in the way of this toilsome journey.

At this moment of accomplishment, I am extremely grateful to my research guide, Dr. Haseeb Athar, for his valuable guidance, scholarly inputs and consistent encouragement I received throughout the research work. I have been amazingly fortunate to have an advisor who gave me the freedom to explore on my own, and at the same time the guidance to recover when my steps faltered. This feat was possible only because of the unconditional support provided by him.

I owe a great deal of appreciation and gratitude to Prof. A. H. Khan. I am grateful to him for holding me to a high research standard and enforcing strict validations for each result, and thus teaching me how to do research. I will remain ever so thankful to him for his kindness as a teacher in the real sense of the term.

My thanks to Prof. M.M. Khalid, Chairman, Prof. H. M. Islam and Dr. R. U. Khan for their moral support and wholehearted cooperation.

I am also thankful to other faculty members of the Department, especially Prof. Q.M. Ali, Prof. A.A. Khan, Dr. M. Faizan and Dr. Zaki Anwar for their help and support.

Collectively and individually, my colleagues added value to this thesis. Most importantly, my thanks are due to Dr. Ziaul Haque (J.S.S.), who was always there when I really needed. Thanks doesn't seem sufficient but it is said with appreciation.
and respect to him for his support, understanding and precious friendship. Dr. M. I. Khan, Mr. Nayabuddin, Ms. Zubdah-e-Noor, Mr. Zuber Akhter, Mr. Mohammad Azam Khan deserve special mention here for their constant support and help in the completion of this work. I am also thankful to those whom I failed to mention. Thanks are also due to all non-teaching staff members of the department and all the seminar library staff for their cooperation.

I pay homage to my parents, Late Dr. K. K. Ahmad and Late Mrs. Rukhsana Khan, this work would not have been possible without their blessings. Just because of their faith in me I successfully overcame many difficulties. I can’t forget them. Their unflinching courage and conviction will always inspire me, and I hope to continue to work with their noble thoughts. I wish their soul roots in peace and solace in the heaven. It is to them that I dedicate this work.

Words fail me to express my appreciation to my husband, Mr. K. M. Khan for his support, generous care and encouragement. He was always beside me during the happy and hard moments to push me and motivate me. A journey is easier when you travel together. Interdependence is certainly more valuable than independence. I would like to thank him for always believing in me. Without him I could not have made it here. I am also extremely indebted to my beloved daughters Almas and Sehar for their love, care, support and creating a pleasant atmosphere for me at home. I convey special thanks to my in-laws Mr. Iqbal Ali Khan and Mrs. Shamim Begum, Dr. Mohd. Asif, Dr. Mohd. Yusuf and Mr. Mohd. Imran. My sisters Samreen and Sana deserve my sincere expression of thanks for their encouragement and inspiration throughout my research work and lifting me uphill this phase of life.

For any errors or inadequacies that may remain in this work the responsibility is entirely my own.

Dated: 24-05-2014

Saba K. Khuraja
Saba Khalid Khuraja
Order statistics and related general models of ordered random variables are important in statistical theory and its applications. The independent and identically random variables, that can be interpreted as results of an experiment measuring values of a certain random variable arranged in order of magnitude, are called order statistics. In statistical model of many experiments, for instance in reliability analysis, life time studies, in testing of strength of materials, etc., the realizations arise in non decreasing order, therefore the use of order statistics is necessary. Furthermore, order statistics are extensively used in statistical inferences; in the estimation theory and hypothesis testing. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics.

Order statistics has immense role in characterization problems. Characterization results are located on the borderline between probability theory and mathematical statistics, and utilize numerous classical tools of mathematical analysis. The useful characterizations results are those which shed light on modelling consequences of certain distributional assumptions and those which have potential for development of hypothesis tests for model assumptions. There are many results on characterizations of probability distributions through properties of order statistics. For detailed survey, one may refer to Ali and Khan (1987), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998), Khan and Abouammoh (2000), Khan and Athar (2004) and references there in.

The distribution theory of order statistics has been widely described in several monographs written by outstanding statisticians and there are numerous papers devoted to the theory of order statistics and its applications as well as asymptotic results and inferences based on order statistics. Contributions of Sarhan and Greenberg (1958), Arnold et al. (1992), David and Nagaraja (2003) are significant.
Order statistics and their moments have received attention from the beginning of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics. The moment of order statistics did, subsequently, assume considerable importance in the statistics literature and have been numerically tabulated extensively for several distributions. For example one can refer to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold et al. (1989, 1992) for details.

The theory of records is very closely connected with the theory of extreme order statistics. Numerous papers and several monographs appeared since Chandler (1952) presented the basic definitions and the first theoretical results on records.

Kamps (1995a, b) introduced the generalized order statistics and shown that all known models of ordered random variables are contained in the model of generalized order statistics in the distributional and theoretical sense. [Bairamov, 2007].

Generalized order statistics can be easily applicable in practice problems except that when \( F() \) is so called inverse distribution function. So the concept of lower generalized order statistics is needed. Pawlas and Szynal (2001a) introduced the concept of lower generalized order statistics \( (lgos) \) to enable a common approach to descending ordered random variables like reversed order statistics and lower record values. Further, the concept of \( lgos \) was extensively studied by Burkschat et al. (2003) with the name dual generalized order statistics \( (dgos) \).

The thesis entitled “Expectation of Ordered Random Variates and Related Results” is based on six chapters, which are described as below:

**Chapter I** is introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

**Chapter II** is based on recurrence relations for single and product moments of generalized order statistics from Pareto distribution. Special cases of
generalized order statistics such as order statistics and record values are also discussed and at the end a characterization theorem is given.

Chapter III contains explicit expression for moments of order statistics for extended type-I generalized logistic distribution and some computational work is carried out. Further, recurrence relations for marginal and joint moment generating functions of order statistics are derived.

Chapter IV extends the results of Chapter III for dual generalized order statistics. Results are deduced for order statistics and lower record values.

Chapter V consists of recurrence relations for moments of generalized order statistics, from doubly truncated Makeham distribution; a characterizing result based on conditional expectation of generalized order statistics is also presented.

Chapter VI is related to the characterization of some generalized probability distributions through conditional expectation of function of generalized order statistics conditioned on non-adjacent generalized order statistics. Further, some deductions for order statistics and records are also discussed.

In the end, a comprehensive bibliography is given which has been referred in these chapters.
1. INTRODUCTION

In this chapter we have introduced those concepts/results which are needed to grasp the idea in the subsequent chapters. Section 2 deals with some basic definition and applications of order statistics. In this section distribution theory of order statistics and some well known results are also presented. In Section 3, basic concept and applications of record values are given. Section 4 is devoted to definition and basic distribution theory of generalized order statistics, whereas Section 5, deals with concept of dual (lower) generalized order statistics. In Section 6, literature based on expectation properties of ordered random variables is given while in Section 7, some basic continuous distributions are discussed.

2. ORDER STATISTICS

2.1 Overview

Order statistics is that body of knowledge which utilizes the rank or order of an observation as well as its magnitude. It combines the techniques of conventional statistics (which consider the magnitude of the observation) with those of rank order statistics (which consider only the relative rank whether or not the original observations were measured on an ordinal scale). Order statistics can be applied in instances in which short-cut or time-saving devices are appropriate for problems of estimation and/or tests of significance. For example, use of the median to estimate central tendency, use of the range to estimate dispersion, or an entire short-cut analysis of variance as suggested by Hartley (1942). Applications in which no other known technique provides a suitable answer with as high power efficiency. Such instances occur with censored data and in deciding how to form groups or intervals for a frequency distribution. For example, the response to a drug may be so rapid that the reaction occurs before
the experimenter can measure it, or so feeble that it is below the smallest level of measurement.

Order statistics and functions of order statistics play a very important role in statistical theory and methodology. In many cases, methods based on order statistics are the most efficient. In other cases, they are used because of their simplicity or their robustness, even at the cost of some loss of efficiency. For example, the sample mean and standard deviations provide efficient estimators of the corresponding population parameters under the assumption of normality, but the sample range is simpler to use than the sample median in statistical quality control, and the sample median furnish more robust estimators when the population may have longer tails than the normal. Extreme (largest and smallest) values are important in hydrology (floods and droughts), aeronautics (gust loads), oceanography (waves and tides), material strength ("weakest link" theory) and meteorology, duration of life on humans, other organisms and devices of various kinds. Order statistics occur naturally in life testing (Wilks, 1948; Sarhan and Greenberg, 1958).

David and Nagaraja (2003) is the basic book on order statistics dealing in detail with its different aspects. Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). Also, references may be made to Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991), Arnold et al. (1992) and the references therein.

2.2 Definition and distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a continuous population having probability density function (PDF) $f(x)$ and distribution function (DF) $F(x)$. Let they be arranged in ascending order of magnitude as $X_{1n} \leq X_{2n} \leq \ldots \leq X_{rn} \leq \ldots \leq X_{nn}$, then $X_{1n}, X_{2n}, \ldots, X_{nn}$ are collectively called the order statistics of the sample and $X_{rn}$ ($r = 1, 2, \ldots, n$) is called the $r$-th order statistic of the sample. $X_{1n} = \min(X_1, X_2, \ldots, X_n)$ and
$X_{nn} = \max(X_1, X_2, \ldots, X_n)$ are called extreme order statistics or the smallest and the largest order statistics.

The pdf of \(X_{r,n}\), the \(r-th\) order statistic is given by (David and Nagaraja, 2003)

$$f_{rn}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x), \ -\infty < x < \infty. \quad (2.1)$$

The pdfs of smallest and largest order statistics are,

$$f_{1n}(x) = n [1 - F(x)]^{n-1} f(x) ; \ -\infty < x < \infty, \quad (2.2)$$

$$f_{nn}(x) = n [F(x)]^{n-1} f(x) ; \ -\infty < x < \infty. \quad (2.3)$$

The df of \(X_{rn}\) is given by

$$F_{rn}(x) = P(X_{rn} \leq x)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} {n \choose i} [F(x)]^i [1 - F(x)]^{n-i} ; \ -\infty < x < \infty \quad (2.4)$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{0}^{F(x)} u^{r-1} (1-u)^{n-r} du \quad (2.5)$$

$$= I_{F(x)}(r, n-r+1). \quad (2.6)$$

RHS is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991).

$$F_{rn}(x) = \sum_{i=0}^{n-r} {n-i \choose r-1} [F(x)]^i [1 - F(x)]^{n-r-i}, \ -\infty < x < \infty. \quad (2.7)$$

For continuous case the pdf of \(X_{rn}\) may also be obtained by differentiating (2.5) w.r.t. \(x\).
From the density function given in (2.1), we may obtain the $k$-th moment of $X_{\psi_n}$ as below:

$$\mu_{\psi_n}^{(k)} = E[X_{\psi_n}^k] = \int_{-\infty}^{\infty} x^k f_{\psi_n}(x) \, dx . \quad (2.8)$$

The joint pdf of $X_{\phi,n}$, $X_{\psi,n}$, $1 \leq r < s \leq n$ is given by

$$f_{r,s,n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{s-r-1} \times [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x)f(y) ; -\infty < x < y < \infty . \quad (2.9)$$

The joint df of $X_{\phi,n}$ and $X_{\psi,n}$, $(1 \leq r < s \leq n)$ can be obtained as follows:

$$F_{r,s,n}(x,y) = P(X_{\phi,n} \leq x, X_{\psi,n} \leq y)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x$$

and at least $s$ of $X_1, X_2, \ldots, X_n$ are at most $y)$$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^i [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j} . \quad (2.10)$$

We can write the joint df of $X_{\phi,n}$ and $X_{\psi,n}$ in (2.10) equivalently as

$$F_{r,s,n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^{F(x)} \int_0^{F(y)} u^{r-1} (v - u)^{s-r-1}$$

$$\times (1 - v)^{n-s} \, du \, dv$$

$$= I_{F(x),F(y)}(r, s-r, n-s+1) ; -\infty < x < y < \infty , \quad (2.11)$$

which is incomplete bivariate beta function.

It may be noted that for $x \geq y$

$$F_{r,s,n}(x,y) = F_{s,n}(y) . \quad (2.12)$$
The product moments of the $j$-th and $k$-th order of $X_{rn}$ and $X_{xn}$ respectively, $(1 \leq r < s \leq n)$ is given by

$$\mu_{r,s,n}^{(j,k)} = E[X_{r,n}^j X_{s,n}^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{r,s,n}(x,y) \, dx \, dy.$$  

(2.13)

In general, the joint pdf of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n}, x_{i_2:n}, \ldots, x_{i_k:n}) = n! \left\{ \prod_{j=1}^{k} f(x_{i_j}) \right\} \prod_{j=0}^{k} \left[ \frac{[F(x_{i_{j+1}}) - F(x_{i_j})]^{i_{j+1} - i_j - 1}}{(i_{j+1} - i_j - 1)!} \right]$$

$$-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty,$$  

(2.14)

where $x_0 = -\infty, x_{k+1} = +\infty, i_0 = 0, i_{k+1} = n + 1$.

**Remarks:**

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.

2. Regarding the probability integral transformation, if $X_{rn}, 1 \leq r \leq n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{rn} = F(X_{rn})$ produces a random variable which is the $r$-th order statistic from a uniform distribution on $U(0,1)$.

3. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

4. Let $X_1, X_2, \ldots, X_n$ be iid random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

5. Let $X$ be a continuous random variable with $E[X_{rn}] = \mu_{rn}$.
a) If $\mu = E(X)$ exists then $\mu_{r,n}$ exists, but converse is not necessarily true. That is, $\mu_{r,n}$ may exist for certain (but not all) values of $r$, even though $\mu$ does not exist.

b) $\mu_{r,n}$ for all $n$ determine the distribution completely.

2.3 Truncated and conditional distribution

Let $X$ be a continuous random variable having pdf $f(x)$ and df $F(x)$ in the interval $[-\infty, \infty]$.

Let $\int_{-\infty}^{Q_1} f(x) \, dx = Q$ and $\int_{-\infty}^{P_1} f(x) \, dx = P$, \hspace{2cm} (2.15)

where $Q_1$ and $P_1$ are known constants. Then doubly truncated pdf of $X$ is given by

$$\frac{f(x)}{P-Q} ; \ x \in (Q_1, P_1)$$ \hspace{2cm} (2.16)

and the corresponding df is given by

$$\frac{F(x) - Q}{P-Q} ; \ x \in (Q_1, P_1).$$ \hspace{2cm} (2.17)

The lower and upper truncation points are $Q_1, P_1$ respectively; the degrees of truncation are $Q$ (from below) and $1-P$ (from above). If we put $Q = 0$, the distribution will be truncated to the right. Similarly, for $P = 1$, the distribution will be truncated to the left. Whereas for $Q = 0, P = 1$, we get the non truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

2.4 Some important results

Result 1 (David and Nagaraja, 2003): Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with the df $F(x)$ and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics obtained from this sample.
Then the conditional distribution of \( X_{r:n} \), given that \( X_{s:n} = y \) for \( s > r \), is the same as the distribution of the \( r \)-th order statistic obtained from a sample of size \((s-1)\) from a population whose distribution is truncated on the right at \( y \).

**Result 2 (David and Nagaraja, 2003):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with the df \( F(x) \) and pdf \( f(x) \), and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order statistics obtained from this sample. Then the conditional distribution of \( X_{s:n} \), given that \( X_{r:n} = x \) for \( r < s \), is the same as the distribution of the \((s-r)-th\) order statistic obtained from a sample of size \((n-r)\) from a population whose distribution is truncated on the left at \( x \).

**Result 3 (David and Nagaraja, 2003):** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with df \( F(x) \) and pdf \( f(x) \), and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order statistics obtained from this sample. Then the conditional distribution of \( X_{s:n} \) given that \( X_{r:n} = x \) and \( X_{k:n} = z \) for \( 1 \leq r < s < k \leq n \), is the same as the distribution of the \((s-r)-th\) order statistic obtained from a sample of size \((k-r-1)\) from a population whose distribution is truncated on the left at \( x \) and on the right at \( z \).

**Result 4:** Order statistics in a sample from a continuous distribution form a Markov chain, that is

\[
f(X_{k:n} | X_{1:n} = x_1, \ldots, X_{r:n} = x_r, \ldots, X_{s:n} = x_s, \ldots, X_{n:n} = x_n) = f(X_{k:n} | X_{r:n} = x_r, X_{s:n} = x_s), \quad r < k < s.
\]

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

**Result 5 (Ali and Khan, 1997):** Let \( g(x) \) be a Borel measurable function of \( x \) in the interval \([\alpha, \beta]\) then, for \( 1 \leq r \leq n \), \( n = 1, 2, \ldots \)
(i) \( E[g(X_{r:n})] - E[g(X_{r-1:n-1})] \)
\[
= \left( \frac{n-1}{r-1} \right) \int_{Q_1} g'(x) [F(x)]^{r-1} [1-F(x)]^{n-r+1} \, dx. \tag{2.18}
\]

(ii) \( E[g(X_{r:n})] - E[g(X_{r-1:n})] \)
\[
= \left( \frac{n}{r-1} \right) \int_{Q_1} g'(x) [F(x)]^{r-1} [1-F(x)]^{n-r+1} \, dx. \tag{2.19}
\]

(iii) \( E[g(X_{r-1:n-1})] - E[g(X_{r-1:n})] \)
\[
= \left( \frac{n-1}{r-2} \right) \int_{Q_1} g'(x) [F(x)]^{r-1} [1-F(x)]^{n-r+1} \, dx. \tag{2.20}
\]

In view of (2.18), (2.19) and (2.20), we have
\[
(n-r+1) E[g(X_{r-1:n})] + (r-1) E[g(X_{r:n})] = n E[g(X_{r-1:n-1})]. \tag{2.21}
\]

At \( g(x) = x \) in (2.21), we get the well known relation established by (David and Nagaraja, 2003).

**Result 6 (Ali and Khan, 1998):** If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)
\[
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})] \]
\[
= \frac{C_{r,s:n}}{(n-s+1)} \int_{Q_1} \int_{x<y \leq R_1} \frac{\partial}{\partial y} g(x,y) [F(x)]^{r-1}
\times [F(y) - F(x)]^{s-r-1} [1-F(y)]^{n-s+1} f(x) \, dx \, dy. \tag{2.22}
\]

**Result 7 (Khan et al., 2001):** If \( g() \) is a Borel measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), then for \( 1 \leq r < s \leq n, \ n = 1, 2, \ldots \)
\[
E[g(X_{r:n}, X_{s:n})] - E[g(X_{r-1:n}, X_{s:n})] \]
\[
= \frac{C_{r,s:n}}{(s-r)} \int_{Q_1} \int_{x \leq R_1} \frac{\partial}{\partial x} g(x,y) [F(x)]^{r-1}
\times [F(y) - F(x)]^{s-r} [1-F(y)]^{n-s} f(y) \, dy \, dx. \tag{2.23}
\]
3. RECORD VALUES AND RECORD TIMES

3.1 Overview

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recoding them, e.g. Olympic records or world records in sports.

Records are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test. The easiest way to explain how to statistically define the theory of records is by examples.

Suppose we have the following ten observations from a given experiment

\[
10, 12, 6, 15, 20, 18, 17, 5, 22, 3
\]

The lower Record values are

\[
10, 6, 5, 3
\]

The upper record values are

\[
10, 12, 15, 20, 22
\]

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.

To be precise, record values are defined by means of record times. That is, those times have to be described at which successively largest values appear.

Chandler (1952) shows several properties of record values and notes their Markovian structure. Two recent books on records by Ahsanullah (1995) and Arnold et al. (1998) are worth mentioning.
3.2 Definition

Suppose that \( X_1, X_2, \ldots, X_n \) is a sequence of independent and identically distributed random variables with df \( F(x) \). Let \( Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\} \) for \( n \geq 1 \). We say \( X_j \) is an upper (lower) record values of \( \{X_n, n \geq 1\} \), if \( Y_j > (<) Y_{j-1}, j > 1 \). By definition \( X_1 \) is an upper as well as lower record values. One can transform the upper record by replacing the original sequence of \( \{X_j\} \) by \( \{-X_j, j \geq 1\} \) or if \( P(X_i > 0) = 1 \) for all \( i \) by \( \left\{ \frac{1}{X_i}, i \geq 1 \right\} \), the lower record value of this sequence will correspond to the upper record values of the original sequence (Ahsanullah, 1995).

The indices at which upper record values occur are given by the record times \( \{U(n)\}, n > 0 \). That is \( X_{U(n)} \) is the \( n \)-th upper record, where \( U(n) = \min\{j | j > U(n-1), X_j > X_{U(n-1)} \}, n > 1 \) and \( U(n) = 1 \). The distribution of \( U(n), n \geq 1 \) does not depend on \( F \). Further, we will denote \( L(n) \) as the indices where the lower record values occur. By assumption \( U(1) = L(1) = 1 \). The distribution of \( L(n) \) also does not depend on \( F \).

3.3 Distribution of record values

Let \( R(x) \) be a continuous function of \( x \) with \( R(x) = -\ln F(x) \) and \( 0 < \bar{F}(x) = 1 - F(x) \), where 'ln' is the natural logarithm.

If we define \( F_n(x) \) as the df of \( X_{U(n)} \) for \( n \geq 1 \), then we have (Ahsanullah, 1995)

\[
F_n(x) = P(X_{U(n)} \leq x)
= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty \tag{3.1}
\]

and the pdf \( f_n(x) \) of \( X_{U(n)} \) is
The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$
 f_{i,j}(x_i, x_j) = \frac{(R(x_j))^{j-1}}{(i-1)!} \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j),
 -\infty < x_i < x_j < \infty.
$$ (3.3)

The joint pdf of the $n$ record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$
 f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \cdots r(x_{n-1}) f(x_n),
 -\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty,
$$ (3.4)

where

$$
 r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{1 - F(x)},
 0 < F(x) < 1
$$

is known as hazard rate.

In particular at $i = 1, j = n$, we have

$$
 f_{1,n}(x_1, x_n) = r(x_1) \frac{(R(x_n) - R(x_1))^{n-2}}{(n-2)!} f(x_n),
 -\infty < x_1 < x_2 < \infty.
$$

The conditional distribution of $X_{U(j)} \mid X_{U(i)} = x_i$ is

$$
 f(X_{U(j)} \mid X_{U(i)} = x_i) = \frac{f_{i,j}(x_i, x_j)}{f_i(x_i)}
 = \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)},
 -\infty < x_i < x_j < \infty
$$ (3.5)

and for $X_{U(i)} \mid X_{U(j)} = x_j$ is

$$
 f(X_{U(i)} \mid X_{U(j)} = x_j)
 = \frac{(j-1)!}{(i-1)!(j-i-1)!} \left[ \frac{R(x_i)}{R(x_j)} \right]^{j-i-1} \left[ 1 - \frac{R(x_i)}{R(x_j)} \right]^{j-i-1} \frac{r(x_i)}{R(x_j)},
 -\infty < x_i < x_{i+1} < \infty.
$$ (3.6)
3.4 $k$-Records

In some situations record values themselves are viewed as ‘outlier’ and hence second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example.

Let $X_1, X_2, \cdots, X_n$ be an identically and independent sequence of random variables with a continuous distribution function $F(x)$ and let $k$ be a positive integer.

Then the random variables $L^{(k)}(n)$ is given by (Kamps, 1995a)

$$L^{(k)}(n) = 1,$$

$$L^{(k)}(n + 1) = \min\{j \in N; X_{j_1, j+k-1} > X_{L^{(k)}(n), L^{(k)}(n)+k-1}\}, n \in N$$

are called $k$-th record times and the quantities $X_{L^{(k)}(n)}, n \in N$ are called $k$-th record values or $k$-records.

We can obtain ordinary record values at $k = 1$.

The joint density of the $k$-records $X_{L^{(k)}(1)}, \cdots, X_{L^{(k)}(r)}$ is given as

$$f_{X_{L^{(k)}(1)}, \cdots, X_{L^{(k)}(r)}}(x_1, \cdots, x_r)$$

$$= k^r \left( \prod_{i=1}^{r-1} \frac{f(x_i)}{1-F(x_i)} \right) \left[1-F(x_r)\right]^{k-1} f(x_r)$$

and the marginal densities and marginal distribution functions are given by

$$f_{X_{L^{(k)}(r)}}(x) = \frac{k^r}{(r-1)!} [R(x)]^{r-1} [1-F(x)]^{k-1} f(x)$$

(3.8)

and

$$F_{X_{L^{(k)}(r)}}(x) = 1-[1-F(x)]^k \sum_{j=0}^{r-1} \frac{1}{j!} [k R(x)]^j.$$  

(3.9)
4. GENERALIZED ORDER STATISTICS

4.1 Overview

The concept of generalized order statistics (gos) have been introduced and extensively studied by Kamps (1995 a, b). The use of such concept has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been separately treated in statistical literature. Some types of ordered random variables such as order statistics, upper record values, sequential order statistics, progressive type II censored order statistics can be discussed as special case of generalized order statistics.

4.2 Definition

Let \( n \geq 2 \) be a given integer and \( \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}, \ k \geq 1 \) be the parameters such that

\[
\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n - 1.
\]

Then \( X(1,n, \tilde{m}, k), X(2,n, \tilde{m}, k), \ldots, X(n,n, \tilde{m}, k) \) are called generalized order statistics if their joint pdf has the form

\[
k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n)
\]

(4.1)
on the cone \( F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1) \) of \( \mathbb{R}^n \)

with absolutely continuous distribution function (df) \( F() \) and probability density function (pdf) \( f() \). The model of generalized order statistics contains as special cases such as ordinary order statistics \( (\gamma_i = n - i + 1; i = 1, 2, \ldots, n \text{ i.e. } m_1 = m_2 = \ldots = m_{n-1} = 0, k = 1) \), \( k \)-th record values \( (\gamma_i = k, \text{ i.e. } m_1 = m_2 = \ldots = m_{n-1} = -1, k \in N) \), sequential order statistics \( (\gamma_i = (n - i + 1)\alpha_i; \alpha_1, \alpha_2, \ldots, \alpha_n > 0) \), order statistics with non-integral sample
size \((\gamma_i = \alpha - i + 1; \alpha > 0)\), Pfeifer’s record values \((\gamma_i = \beta_i; \beta_1, \beta_2, \ldots, \beta_n > 0)\) and progressive type II censored order statistics \((m_i \in N_0, k \in N)\) are obtained [Kamps (1995a,b), Kamps and Cramer (2001)].

4.3 Distribution of generalized order statistics

**Case I: ** \(m_i = m; i = 1, 2, \ldots, n - 1\)

The marginal density of the \(r\)-th generalized order statistic \((gos)\) is given by [Kamps, 1995 a, b]

\[
f_X(r, n, m, k)(x) = \frac{C_{r-1}}{(r-1)!} [1 - F(x)]^{r-1} f(x) g_m^{-1}(F(x))
\]

and the joint pdf of \(X(r, n, m, k)\) and \(X(s, n, m, k), 1 \leq r < s \leq n\) is

\[
f_X(r, n, m, k), X(s, n, m, k)(x, y) = \frac{C_{s-1}}{(s-r-1)!} [1 - F(x)]^m g_m^{-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{r-1} f(x) f(y)
\]

where \(C_{r-1} = \prod_{i=1}^{r} y_i, \ y_i = k + (n-i)(m+1)\)

\[
h_m(x) = \begin{cases} 
- \frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\
- \log(1-x), & m = -1
\end{cases}
\]

\[
g_m(x) = \int_0^x (1-t)^m dt = h_m(x) - h_m(0), \ x \in [0,1].
\]

The conditional pdf of \(X(s, n, m, k)\) given \(X(r, n, m, k) = x\), \(1 \leq r < s \leq n\) is given by

\[
f_{X(s, n, m, k)|X(r, n, m, k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1} [1 - F(y)]^{r-1} f(y)}{[1 - F(x)]^{r-1}}, \ x < y
\]

\[(4.4)\]
and the conditional pdf of $X(r, n, m, k)$ given $X(s, n, m, k) = y$, $1 \leq r < s \leq n$ is

$$f_{X(r, n, m, k)|X(s, n, m, k)}(x | y) = \frac{(s-1)! (m+1)}{(r-1)! (s-r-1)!} \times \frac{[\overline{F}(x)]^m [1 - (\overline{F}(x))^{m+1}]^{r-1} [(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}]^{s-r-1}}{[1 - (\overline{F}(y))^{m+1}]^{s-1}} f(x), \quad x < y. \quad (4.5)$$

**Case II:** $\gamma_i \neq \gamma_j$ ; $i, j = 1, 2, ..., n-1$

The pdf of $X(r, n, m, k)$ is [Kamps and Cramer, 2001]

$$f_{X(r, n, m, k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i - 1} \quad (4.6)$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$ is

$$f_{X(r, n, m, k), X(s, n, m, k)}(x, y) = C_{s-1} \sum_{i=r+1}^{s} a_i(r) (1 - F(x))^{\gamma_i} \times \left[ \sum_{i=1}^{r} a_i(r) (1 - F(x))^{\gamma_i} \right] \frac{f(x) f(y)}{(1 - F(x))(1 - F(y))} \quad (4.7)$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + \sum_{j=m_j}^{n-1} m_j$$

$$a_i(r) = \prod_{j=i}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n$$

and

$$a_i^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r + 1 \leq i \leq s \leq n.$$ 

Thus, the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \leq r < s \leq n$ is given by
\[
f_{X(s,n,\tilde{m},k)\mid X(r,n,\tilde{m},k)}(y \mid x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{1-F(y)}{1-F(x)} \right]^{\gamma_i} \frac{f(y)}{[1-F(y)]}, \quad x \leq y
\]

and the conditional pdf of \( X(r,n,\tilde{m},k) \) given \( X(s,n,\tilde{m},k) = y \), \( 1 \leq r < s \leq n \) is given by

\[
f_{X(r,n,\tilde{m},k)\mid X(s,n,\tilde{m},k)}(x \mid y) = \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{F(y)}{F(x)} \right]^{\gamma_i} \left\{ \sum_{j=1}^{r} a_i^{(r)}(s)[\frac{F(y)}{F(x)}]^{\gamma_j} \right\}^{\gamma_i} f(x)}{\left\{ \sum_{i=1}^{s} a_i(s)[\frac{F(y)}{F(x)}]^{\gamma_i} \right\}^{\gamma_i}}.
\]

It may be noted that for \( \gamma_i \neq \gamma_j \) but \( m_i = m_j = m \neq -1; i \neq j = 1,2,\ldots \)

\[
a_i^{(r)} = \frac{(-1)^{r-i}}{(m+1)^{r-i}(r-i)!} \binom{r-1}{r-i}
\]

\[
a_i^{(s)} = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \binom{s-r-1}{s-i}.
\]

Therefore pdf of \( X(r,n,\tilde{m},k) \) given in (4.6) reduces to pdf of \( X(r,n,m,k) \) as given in (4.2). Similarly the joint pdf of \( X(r,n,\tilde{m},k) \) and \( X(s,n,\tilde{m},k) \) given in (4.7) reduces to the joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k) \) as given in (4.3).

4.4 Some important results

**Result 1 (Athar and Islam, 2004):**

Let \( \xi(x) \) is a measurable function of \( x \) which is differentiable, then for any arbitrary distribution function \( F \) and \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1,2,\ldots, \) following relations hold:
Case I: \( m_i = m; i = 1, 2, \ldots, n-1 \)

\[
E[\xi\{X(r, n, m, k)\}] - E[\xi\{X(r-1, n, m, k)\}] = \frac{C_{r-2}}{(r-1)!} \left[ \int \xi'(x)[1-F(x)]^{r} g_m^{r-1}(F(x)) dx \right].
\] (4.10)

\[
E[\xi\{X(r-1, n, m, k)\}] - E[\xi\{X(r-1, n-1, m, k)\}] = -\frac{(m+1)}{\gamma_1} \int \xi'(x)[1-F(x)]^{r} g_m^{r-1}(F(x)) dx.
\] (4.11)

\[
E[\xi\{X(r, n, m, k)\}] - E[\xi\{X(r-1, n-1, m, k)\}] = \int \xi'(x)[1-F(x)]^{r} g_m^{r-1}(F(x)) dx.
\] (4.12)

Case II: \( \gamma_i \neq \gamma_j; i \neq j = 1, 2, \ldots, n-1 \)

\[
E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r-1, n, \tilde{m}, k)\}] = C_{r-2} \int \xi'(x) \sum_{i=1}^{r} a_i(r)[1-F(x)]^{\gamma_i} dx.
\] (4.13)

\[
E[\xi\{X(r-1, n, \tilde{m}, k)\}] - E[\xi\{X(r-1, n-1, \tilde{m}^*, k)\}] = -\sum_{j=1}^{r-1} m_j \int \xi'(x) \sum_{i=1}^{r} a_i(r)[1-F(x)]^{\gamma_i} dx.
\] (4.14)

\[
E[\xi\{X(r, n, \tilde{m}, k)\}] - E[\xi\{X(r-1, n-1, \tilde{m}^*, k)\}] = \int \xi'(x) \sum_{i=1}^{r} a_i(r)[1-F(x)]^{\gamma_i} dx.
\] (4.15)

where \( \tilde{m}^* = (m_2, m_3, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \).
Result 2 (Athar and Islam, 2004):

For $1 \leq r < s \leq n - 1$, $n \geq 2$ and $k = 1, 2, \ldots$

Case I: $m_i = m_j = m; i \neq j = 1, 2, \ldots, n - 1$

$$E[\xi\{X(r, n, m, k) \cdot X(s, n, m, k)\}] - E[\xi\{X(r, n, m, k) \cdot X(s - 1, n, m, k)\}]$$

$$= \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int_0^1 \int_0^1 \frac{\partial}{\partial x} \xi(x, y)\left[1 - F(x)\right]^m f(x) g_m r^{r-1} (F(x))$$

$$\times \left[h_m (F(y)) - h_m (F(x))\right]^{s-r-1} \left[1 - F(y)\right]^y dy dx. \quad (4.16)$$

Case II: $\gamma_i \neq \gamma_j; i \neq j = 1, 2, \ldots, n - 1$

$$E[\xi\{X(r, n, \tilde{m}, k) \cdot X(s, n, \tilde{m}, k)\}] - E[\xi\{X(r, n, \tilde{m}, k) \cdot X(s - 1, n, \tilde{m}, k)\}]$$

$$= C_{s-2} \int_0^1 \int_0^1 \frac{\partial}{\partial y} \xi(x, y) \left[\sum_{i=r+1}^s a_i(r) s \left(\frac{1 - F(y)}{1 - F(x)}\right)^{\gamma_i}\right]$$

$$\times \left[\sum_{i=1}^r a_i(r) (1 - F(x))^{\gamma_i}\right] \frac{f(x)}{1 - F(x)} dy dx \quad (4.17)$$

where, $\xi(x, y) = \xi_1(x) \xi_2(y)$.

5. LOWER GENERALIZED ORDER STATISTICS

5.1 Overview

Generalized order statistics can be easily applicable in practice problems except that when $F()$ is so called inverse distribution function. So the concept of lower generalized order statistics is needed. Pawlas and Szynal (2001a) introduced the concept of lower generalized order statistics (lgos) to enable a common approach to descending ordered rv's like reversed order statistics and lower record values. Further, the concept of lgos was extensively studied by Bukschat et al. (2003) with the name dual generalized order statistics (dgos). For more detailed survey on dgos one may refer to Ahsanullah (2005), Mbah
and Ahsanullah (2007), Athar et al. (2008), Athar et al. (2010), and references therein.

5.2 Definition

Let \( n \geq 2 \) be a given integer and \( \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1} \), \( k \geq 1 \) be the parameters such that

\[
\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n - 1.
\]

The random variables \( X'(n, n, m, k), \ldots, X'(1, n, \tilde{m}, k) \) are said to be lower generalized order statistics from an absolutely continuous distribution function \( F() \) with the probability density function \( f() \), if their joint density function is of the form

\[
f_{r}(x^*) = k^{n-1} \prod_{j=1}^{n-1} \left( F(x_i) \right)^{m_i} f(x_i) \left[ F(x_n) \right]^{k-1} f(x_n)
\]

\begin{equation}
(5.1)
\end{equation}

for \( F^{-1}(l) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0) \).

If \( m_i = m; i = 1, 2, \ldots, n-1 \) and \( k = 1 \), we obtain the joint pdf of the reverse order statistics and for \( m_i = -1, k \in N \), we get joint pdf of \( k-\text{th} \) lower record values.

Here it may be noted that the joint density (5.1) is obtained by replacing \( 1 - F(x) \) with \( F(x) \) in (4.1).

5.3 Distribution of lower generalized order statistics

Case I: \( m_i = m; i = 1, 2, \ldots, n-1 \).

The density function of \( r - \text{th} \) lower generalized order statistic is given by

\[
f_{X'(r, n, m, k)}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{r-1} f(x) g_m^{-1}(F(x)) \cdot
\]

\begin{equation}
(5.2)
\end{equation}
The joint density function of $r$-th and $s$-th lower generalized order statistics is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_{s-1}} f(y), \alpha \leq y < x \leq \beta,$$

(5.3)

where,

$$h_m(x) = \begin{cases} \frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\log x, & m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1)$.

**Case II:** $\gamma_i \neq \gamma_j, \quad i, j = 1, 2, \cdots, n-1$.

The pdf of $r$-th lower generalized order statistic is given by

$$f_{X'(r,n,m,k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_{i-1}}$$

(5.4)

and the joint pdf of $r$-th and $s$-th lower generalized order statistics is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{F(y)}{F(x)} \right]^{\gamma_i}$$

$$\times \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \frac{f(x)f(y)}{F(x)F(y)}, \alpha \leq y < x \leq \beta,$$

(5.5)

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j$$

$$a_i(r) = \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_i)^{\gamma_i}} , \quad 1 \leq i \leq r \leq n$$
and 
\[ a_i^{(r)}(s) = \prod_{j=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r + 1 \leq i \leq s \leq n. \]

### 5.4 Some important results (Athar et al., 2008)

**Result 1:** Let \( \xi(x) \) is a measurable function of \( x \) which is differentiable, then for any arbitrary distribution function \( F \) and \( 2 \leq r \leq n, \ n \geq 2 \) and \( i \neq j = 1, 2, \ldots, n - 1 \), following relations hold:

\[
E[\xi(X'(r,n,m,k))] - E[\xi(X'(r-1,n,m,k))] \\
= -C_{r-2} \frac{\beta}{\alpha} \int \xi'(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \, dx, \ \gamma_i \neq \gamma_j .
\]

\[
= -C_{r-2} \frac{\beta}{(r-1)!} \int \xi'(x) [F(x)]^{\gamma_r} \ g_m^{r-1}(F(x)) \, dx, \ m_i = m.
\]

\[
E[\xi(X(r-1,n,m,k))] - E[\xi(X(r-1,n-1,m^*,k))] \\
= \frac{(m+1)}{\gamma_1} C_{r-2}^{(n)} \frac{\beta}{\alpha} \int \xi'(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \, dx, \ \gamma_i \neq \gamma_j ,
\]

\[
= \frac{(m+1)}{\gamma_1} C_{r-2}^{(n)} \frac{\beta}{(r-2)!} \int \xi'(x) [F(x)]^{\gamma_r} \ g_m^{r-1}(F(x)) \, dx, \ m_i = m,
\]

\[
E[\xi(X(r,n,m,k))] - E[\xi(X(r-1,n-1,m^*,k))] \\
= -\frac{\gamma_r}{\gamma_1} C_{r-2} \frac{\beta}{\alpha} \int \xi'(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \, dx, \ \gamma_i \neq \gamma_j ,
\]

\[
= -\frac{C_{r-2}^{(n-1)} \beta}{(r-1)!} \int \xi'(x) [F(x)]^{\gamma_r} \ g_m^{r-1}(F(x)) \, dx, \ m_i = m,
\]

where \( m^* = (m_2,m_3,\ldots,m_{n-1}) \in \mathcal{R}^{n-1} \).
Result 2: For \(1 \leq r < s \leq n-1\), \(n \geq 2\) and \(i \neq j = 1, 2, \ldots, n-1\)

\[
E[\xi(X(r,n,m,k), X(s,n,m,k))] - E[\xi(X(r,n,m,k), X(s-1,n,m,k))]
\]
\[
= -C_{s-2} \int_{\alpha \leq x \leq y \leq \beta} \xi(x,y) \left[ \sum_{i=r+1}^{s} a_i(r) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \times \left[ \sum_{i=1}^{r} a_i(r) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \frac{f(x)}{F(x)} \, dy \, dx, \quad \gamma_i \neq \gamma_j,
\]
\]
\[
= -\frac{C_{s-2}}{(r-1)!(s-r-1)!} \int_{\alpha \leq x \leq y \leq \beta} \xi(x,y)[F(x)]^m f(x) g_{m-r-1}(F(x))
\]
\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s} \, dy \, dx, \quad m_i = m,
\]
\]

where, \(\xi(x,y) = \xi_1(x) \xi_2(y)\).

6. EXPECTATION PROPERTIES OF ORDERED RANDOM VARIABLES

Order statistics and their expectation have received attention from the beginning of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics. The moment of order statistics did, subsequently, assume considerable importance in the statistics literature and have been numerically tabulated extensively for several distributions. For example one can refer to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold and Balakrishnan (1989), Arnold et al. (1992) for details.

There are mainly three reasons due to which recurrence relations and identities have great importance.

(i) Reduce the amount of direct computations and hence reduce the time and labour.

(ii) They express the higher order moments in terms of the lower order moments and hence make the evaluation of higher order moments easy.
(iii) Provide some simple checks to test the accuracy of computation of moments of order statistics.


Joshi (1978) obtained recurrence relation between the moment of order statistics from the exponential and right truncated exponential distribution.

Joshi (1979 a, b) obtained similar recurrence relation for the moments of order statistics from doubly truncated exponential and gamma distribution respectively.

Joshi (1982) also obtained some recurrence relations for mixed moments of order statistics for exponential and truncated exponential distribution.


Khan et al. (1983a) developed general results for finding the $k-th$ moment of order statistics without considering any particular distribution. Further, these results were utilized to obtain recurrence relation for doubly truncated and non-truncated distributions, thus unifying all the known results on recurrence relations for moments of order statistics. Khan et al. (1983 b) also extended the results for product moments of order statistics for doubly truncated and non-truncated distributions.

and product moments of order statistics from doubly truncated Weibull distribution.


Balakrishnan et al. (1988) obtained recurrence relations and identities for moments of order statistics for some specific continuous distributions whereas Balakrishnan et al. (1992) established general relations and identities for order statistics from non-independent non-identical variables.

Ali and Khan (1995) have obtained ratio and product moment of two order statistics of different order from Burr distribution. Further, they deduced the moments and inverse moments of single order statistics from the product moments.


Balakrishnan and Ahsanullah (1994a, 1994b) established recurrence relations for single and product moments of record values from generalized Pareto
distribution and Lomax distribution respectively. Further, Balakrishnan and Ahsanullah (1995) obtained recurrence relations for single and product moments of record values from exponential distribution.

Pawlas and Szynal (1998) developed relations for single and product moments of $k$–th record values from exponential and Gumbel distributions, whereas Pawlas and Szynal (1999) established relations for single and product moments of $k$–th record values from Pareto, generalized Pareto and Burr distributions.

Kamps (1995a) investigated recurrence relations for moments of generalized order statistics based on non-identically distributed random variables, which contains order statistics and record values as special cases.

Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases.

Pawlas and Szynal (2001a) derived recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Pawlas and Szynal (2001b) defined the concept of lower generalized order statistics and obtained the recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution.

Athar and Islam (2004) established some recurrence relations between expectation of function of single and joint generalized order statistics from a general class of distribution. Further Athar et al. (2008) generalized the result of Athar and Islam (2004) and established the relations for the expectation of function of gos for truncated distributions.

Athar et al. (2007) obtained the ratio and inverse moments of generalized order statistics from Weibull distribution whereas Athar et al. (2008) obtained some recurrence relations for lower generalized order statistics.

Athar et al. (2011, 2013) obtained the expressions for order statistics and lower generalized order statistics for moments of extended type-I generalized logistic distribution.

The problem of characterization of distributions has always been the topic of interest among researchers. Various approaches are available in literature. Conditional expectation of ordered random variables are extensively used in characterizing the probability distributions. Khan and Abu-Salih (1989) have characterized some general form of distributions through conditional expectation of function of order statistics fixing adjacent order statistics. Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the distributions when the conditioning is not adjacent. Further, Samuel (2008) characterized the distributions considered by Khan and Abu-Salih (1989) for gos. Keseling (1999) has generalized the result of Franco and Ruiz (1995) in terms of generalized order statistics and characterized some general form of distributions. Khan et al. (2006) established characterizing relationship for the distributions through generalized order statistics and characterized several distributions through conditional expectation of function of generalized order statistics. Khan et al. (2009) characterized a family of continuous probability distributions through the difference of two conditional expectations; where as Khan et al. (2010a) characterized the same family of distributions through record statistics. Further, Athar and Noor (2013) characterized two general classes of distributions through conditional expectation of difference of pair of order statistics.

For more detailed survey on characterization one may refer to Franco and Ruiz (1995, 1997), López-Blázquez and Moreno-Rebollo (1997), Dembińska and

7. SOME CONTINUOUS DISTRIBUTIONS

7.1 Pareto distribution

A random variable $X$ is said to have the Pareto distribution if its probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$ are of the form given below:

$$f(x) = \frac{\lambda^p x^{-(p+1)}}{\Lambda}; \quad \lambda \leq x < \infty; \quad \lambda, p > 0.$$

$$F(x) = 1 - \frac{\lambda^p x^{-p}}{\Lambda}; \quad \lambda \leq x < \infty; \quad \lambda, p > 0.$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

7.2 Power function distribution

A random variable $X$ is said to have a power function distribution if its pdf and df are of the form given below:

$$f(x) = \frac{p \lambda^p x^{p-1}}{\Lambda}; \quad 0 < x < \lambda; \quad \lambda, p > 0.$$

$$F(x) = \frac{\lambda^p x^p}{\Lambda}; \quad 0 < x < \lambda; \quad \lambda, p > 0.$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if $X$ has a power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

7.3 Beta distribution

i) Beta distribution of first kind

A random variable $X$ is said to have the beta distribution of first kind if its pdf is of the form
Chapter I

Ax) = \frac{1}{B(p, q)} x^{p-1} (1-x)^{q-1}; \quad 0 \leq x \leq 1, \; p, q > 0.

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r,n}$ is an ordered sample from $U(0,1)$, then $X_{r,n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents $p$ and $q$ equal to 1. If $q=1$, the distribution reduces to power function distribution.

ii) Beta distribution of second kind

The continuous random variable $X$ which is distributed according to probability law:

$$f(x) = \frac{1}{B(p, q)} \frac{x^{p-1}}{(1+x)^{p+q}} \quad (p, q) > 0, \quad 0 < x < \infty.$$ 

is known as a beta variate of the second kind with parameters $p$ and $q$.

Remark: Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $\frac{1}{y}$.

Usage: The Beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with “uncertain” transition probabilities.

7.4 Weibull distribution

A random variable $X$ is said to have a Weibull distribution if its pdf is given by:

$$f(x) = \theta x^{p-1} e^{-\theta x^p}; \quad 0 \leq x < \infty; \; \theta > 0, \; p > 0$$

and the df is given by

$$F(x) = 1 - e^{-\theta x^p}; \quad 0 \leq x < \infty; \; \theta > 0, \; p > 0.$$
Remark 7.4.1: If we put \( p = 1 \) in Weibull distribution, we get the pdf of exponential distribution.

Remark 7.4.2: If we put \( p = 2 \), it gives pdf of Rayleigh distribution.

Remark 7.4.3: If \( X \) has a Weibull distribution, then the pdf of \( Y = -p \log \left( \frac{X}{\alpha} \right) \) is

\[
f(y) = e^{-y} e^{-e^{-y}},
\]

which is a form of an Extreme Value distribution.

Remark 7.4.4: The pdf and the cdf of inverse Weibull distribution is given by

\[
f(x) = \theta p x^{-(p+1)} e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0.
\]

\[
F(x) = e^{-\theta x^{-p}}, \quad 0 \leq x < \infty; \quad \theta > 0, \quad p > 0.
\]

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

7.5 Exponential distribution

A random variable \( X \) is said to have an exponential distribution if its pdf is given by

\[
f(x) = \theta e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0
\]

and the cdf is given by

\[
F(x) = 1 - e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0.
\]

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a
continuous random variable $X$ assuming non-negative values satisfies the assumption,
\[ P(X > s + t | X > s) = P(X > t) \] for all $s$ and $t$.
then $X$ will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

7.6 Rectangular distribution
A random variable $X$ is said to have a rectangular distribution if its pdf is given by
\[ f(x) = \frac{1}{\lambda - \beta}; \beta \leq x \leq \lambda \]
and the df is given by
\[ F(x) = \frac{x - \beta}{\lambda - \beta}; \beta \leq x \leq \lambda. \]
The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(x)$ follows rectangular distribution $R(0,1)$. This distribution is used in “rounding off” errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

7.7 Burr distribution
Let $X$ be a continuous random variable, then different forms of cumulative distribution function of $X$ are listed below (Johnson and Kotz, 1970):

i) \[ F(x) = x, \quad 0 < x < 1 \]

ii) \[ F(x) = (1 + e^{-x})^{-k}, \quad -\infty < x < \infty \]

iii) \[ F(x) = (1 + x^{-c})^{-k}, \quad 0 \leq x < \infty \]
iv) \[ F(x) = \left[1 + \left(\frac{c - x}{x}\right)^{1/c}\right]^{-k}, \quad 0 \leq k \leq c \]

v) \[ F(x) = [1 + ce^{-\tan x}]^{-k}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \]

vi) \[ F(x) = [1 + ce^{-k \sinh x}]^{-k}, \quad -\infty < x < \infty \]

vii) \[ F(x) = 2^{-k}(1 + \tanh x)^{k}, \quad -\infty < x < \infty \]

viii) \[ F(x) = \left(\frac{2}{\pi} \tan^{-1} e^{x}\right), \quad -\infty < x < \infty \]

ix) \[ F(x) = 1 - \frac{2}{c[(1 + e^{x})^{k} - 1] + 2}, \quad -\infty < x < \infty \]

x) \[ F(x) = (1 + e^{-x^2})^{k}, \quad 0 \leq x < \infty \]

xi) \[ F(x) = \left(x - \frac{1}{2\pi} \sin 2\pi x\right)^{k}, \quad 0 \leq x \leq 1 \]

xii) \[ F(x) = 1 - (1 + x^c)^{-k}, \quad 0 \leq x < \infty, \]

where \( k \) and \( c \) are positive parameters.

Special attention is given to type XII, whose pdf is given as:

\[ f(x) = kc x^{c-1}(1 + x^c)^{-(k+1)}; \quad 0 \leq x < \infty; \quad k, c > 0. \]

This distribution is frequently used for the purpose of graduation and in reliability theory. At \( c = 1 \), it is called Lomax distribution whereas at \( k = 1 \), it is known as Log-logistic distribution.

7.8 Cauchy distribution

The special form of the Pearson type VII distribution, with pdf

\[ f(x) = \frac{1}{\pi\lambda} \frac{1}{\left[1 + \{(x - \theta)/\lambda\}^2\right]^{2}}, \quad -\infty < x < \infty; \quad \lambda > 0; \quad -\infty < \theta < \infty \]

is called the Cauchy distribution.
The **cdf** is given by

\[
F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\theta}{\lambda}\right) - \infty < x < \infty; \lambda > 0; -\infty < \theta < \infty.
\]

The distribution is symmetrical about \(x = \theta\). The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, \(\theta\) and \(\lambda\) are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting \(\theta = 0, \lambda = 1\). The standard probability density function is given by

\[
f(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \quad -\infty < x < \infty
\]

and the standard cumulative distribution function is

\[
F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x \quad -\infty < x < \infty
\]
CHAPTER II

MOMENT PROPERTIES OF PARETO DISTRIBUTION BASED ON GENERALIZED ORDER STATISTICS AND ITS CHARACTERIZATION

1. INTRODUCTION

The Pareto Distribution was first proposed as a model for the distribution of incomes. It is also used as a model for the distribution of city populations within a given area. The Pareto distribution provides reasonably good fit to the distribution of income property values (Malik, 1970). It has been observed that Pareto curve gives good fit at the extremities of the income range. In this chapter recurrence relations for single and product moments of generalized order statistics from Pareto distribution have been derived and subsequently explicit expressions for single and product moments are obtained.

Special cases of generalized order statistics such as order statistics and record values are also discussed and at the end of chapter a characterization theorem is given.

A random variable $X$ is said to have the Pareto distribution if its pdf is of the form

$$f(x) = \theta \alpha^\theta x^{-\alpha-1}, \quad x \geq \alpha, \theta > 0 \tag{1.1}$$

and the corresponding df

$$F(x) = \alpha^\theta x^{-\alpha}, \quad \alpha \leq x < \infty \tag{1.2}$$

where, $F(x) = 1 - F(x)$. 

Now, in view of (1.1) and (1.2), we have

$$F(x) = \frac{x}{\theta} f(x). \tag{1.3}$$

The relation (1.3) will be exploited to obtain relations between moments of gos from Pareto distribution.
2. SINGLE MOMENTS

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \cdots, n - 1$.

**Theorem 2.1.** For Pareto distribution as given in (1.1) and $n \in \mathbb{N}, \bar{m} \in \mathbb{R}$, $k > 0$, $(\theta \gamma_r - j) > 0$, $1 \leq r \leq n$, $j = 1, 2, \cdots$

\[
E[X^j(r, n, \bar{m}, k)] = \frac{\theta \gamma_r - j}{\theta \gamma_r - j} E[X^j(r - 1, n, \bar{m}, k)] \quad (2.1)
\]

**Proof.** In view of (1.4.6), we have

\[
E[X^j(r, n, \bar{m}, k)] - E[X^j(r - 1, n, \bar{m}, k)]
\]

\[
= C_{r-1} \int_{x=r}^{\infty} x^j f(x) \sum_{i=1}^{r} a_i(r) [\overline{F}(x)]^{\gamma_i - 1} \, dx
\]

\[-C_{r-2} \int_{x=r}^{\infty} x^j f(x) \sum_{i=1}^{r-1} a_i(r-1) [\overline{F}(x)]^{\gamma_i - 1} \, dx
\]

\[= C_{r-1} \int_{x=r}^{\infty} x^j f(x) \sum_{i=1}^{r-1} a_i(r) [\overline{F}(x)]^{\gamma_i - 1} \, dx
\]

\[+ C_{r-1} \int_{x=r}^{\infty} x^j f(x) a_r(r) [\overline{F}(x)]^{\gamma_r - 1} \, dx
\]

\[-C_{r-2} \int_{x=r}^{\infty} x^j f(x) \sum_{i=1}^{r-1} a_i(r-1) [\overline{F}(x)]^{\gamma_i - 1} \, dx
\]

\[= C_{r-2} \int_{x=r}^{\infty} x^j \left[ f(x) \sum_{i=1}^{r-1} a_i(r) [\overline{F}(x)]^{\gamma_i - 1} \{\gamma_r - (\gamma_r - \gamma_i)\} \right.
\]

\[+ f(x) a_r(r) \gamma_r [\overline{F}(x)]^{\gamma_r - 1} \] \, dx
\]

\[= C_{r-2} \sum_{i=1}^{r} a_i(r) \int_{x=r}^{\infty} x^j \gamma_i [\overline{F}(x)]^{\gamma_i - 1} f(x) \, dx,
\]

as $C_{r-1} = \gamma_r C_{r-2}$ and $a_i(r - 1) = (\gamma_r - \gamma_i) a_i(r)$. 

Let
\[ v_i(x) = -[F(x)]^\gamma_i \]  \hspace{1cm} (2.2)\]
\[ v'_i(x) = \gamma_i [F(x)]^{\gamma_i-1} f(x). \]

Thus,
\[ E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = C_{r-2} \sum_{i=1}^{r} a_i(r) \int_{\alpha}^{\beta} x^j v'_i(x) \, dx. \hspace{1cm} (2.3)\]

Integrating (2.3) by parts, and using the value of \( v'_i(x) \) from (2.2), we get
\[ E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] \]
\[ = C_{r-2} \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \, dx, \hspace{1cm} (2.4)\]

as \( \sum_{i=1}^{r} a_i(r) = 0, \ r \geq 2. \)

Now on application of (1.3) in (2.4), we get
\[ E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] \]
\[ = \frac{j}{\theta \gamma_r} \sum_{i=1}^{\infty} \int_{\alpha}^{\beta} x^j \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} g_m^{r-1} f(x) \, dx \]
\[ = \frac{j}{\theta \gamma_r} E[X^j(r,n,\tilde{m},k)], \]

which after re-arranging yields (2.1).

**Case II:** \( m_i = m, i = 1,2,\ldots,n-1. \)

**Theorem 2.2.** For Pareto distribution as given in (1.1) and \( n \in \mathbb{N}, m \in \mathbb{R}, k > 0, (\theta \gamma_i - j) > 0, 1 \leq r \leq n, \ j = 1,2,\ldots \)

\[ E[X^j(r,n,m,k)] = \frac{\theta \gamma_r}{\theta \gamma_r - j} E[X^j(r-1,n,m,k)] \hspace{1cm} (2.5)\]

and subsequently,
\[ E[X^j (r,n,m,k)] = \alpha^j \theta^r \prod_{i=1}^{r} \frac{\gamma_i}{\theta \gamma_i - j}. \]  

(2.6)

**Proof.** From (1.4.10), we have

\[
E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}]
\]

\[
= \frac{C_{r-2}}{(r-1)!} \int \xi'(x)[1 - F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx.
\]

Let \( \xi(x) = x^j \), then

\[
E[X^j (r,n,m,k)] - E[X^j (r-1,n,m,k)]
\]

\[
= \frac{jC_{r-2}}{(r-1)!} \int x^{j-1}[1 - F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx. \tag{2.7}
\]

The relation (2.5) can be established on the lines of Theorem 2.1 and on application of (1.3).

Since \( X_{0,n,m,k} = \alpha \), the minimum of \( X \) in Pareto distribution, then we have

\[
E[X^j_{1,n,m,k}] = \left( \frac{\theta \gamma_1}{\theta \gamma_1 - j} \right) \alpha^j. \tag{2.8}
\]

Expression (2.6) can be obtained by writing (2.1) recursively and using (2.8) as initial value.

**Remark 2.1:** At \( m = 0, k = 1 \), we get the result for order statistics

\[
E(X_{r:n}^j) = \frac{\theta(n-r+1)}{\theta(n-r+1) - j} E(X^j_{r-1:n})\tag{2.9}
\]

\[
= \alpha^j \theta^r \prod_{i=1}^{r} \frac{(n-i+1)}{\theta(n-i+1) - j}. \tag{2.10}
\]

Expression (2.10) may also be expressed as
\[ E(X_{r:n}^j) = \alpha^j \frac{\Gamma(n+1) \Gamma(n-r-j+1)}{\Gamma(n-r+1) \Gamma(n-j+1)} \]

as obtained by Malik (1966).

Further, since
\[(n-r)\mu_{r:n}^{(k)} + r\mu_{r+1:n}^{(k)} = n\mu_{r:n-1}^{(k)} \]
(David and Nagaraja, 2003)

Therefore (2.9) can also be written as
\[ E(X_{r:n}^j) = \frac{n\theta}{n\theta - j} E(X_{r-1:n-1}^j) \]

as obtained by Khan et al. (1983 a).

**Remark 2.2:** Setting \( m = -1 \) in Theorem 2.1, we get the result for \( k \)-th upper record values from Pareto distribution.

\[ E(X_{U(r)}^{(k)})^j = \frac{\theta k}{\theta k - j} E(X_{U(r-1)}^{(k)})^j \]

\[ = \alpha^j \left( \frac{\theta k}{\theta k - j} \right)^r. \]
First four moments of order statistics:

Table 2.1

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.2500</td>
<td>1.6667</td>
<td>2.500</td>
<td>5.0000</td>
<td>1.2000</td>
<td>1.5000</td>
<td>2.0000</td>
<td>3.0000</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.1111</td>
<td>1.2500</td>
<td>1.4285</td>
<td>1.6667</td>
<td>1.0909</td>
<td>1.2000</td>
<td>1.3333</td>
<td>1.5000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.3889</td>
<td>2.0833</td>
<td>3.5714</td>
<td>8.3333</td>
<td>1.3090</td>
<td>1.8000</td>
<td>2.6667</td>
<td>4.5000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.0714</td>
<td>1.1538</td>
<td>1.2500</td>
<td>1.3636</td>
<td>1.0588</td>
<td>1.1250</td>
<td>1.2000</td>
<td>1.2857</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.1905</td>
<td>1.4423</td>
<td>1.7857</td>
<td>2.2727</td>
<td>1.1550</td>
<td>1.3500</td>
<td>1.6000</td>
<td>1.9285</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.4880</td>
<td>2.4038</td>
<td>4.4642</td>
<td>11.3636</td>
<td>1.3861</td>
<td>2.0250</td>
<td>3.2000</td>
<td>5.7856</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.0526</td>
<td>1.1111</td>
<td>1.1764</td>
<td>1.2500</td>
<td>1.0435</td>
<td>1.0909</td>
<td>1.1429</td>
<td>1.2000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.1278</td>
<td>1.2820</td>
<td>1.4706</td>
<td>1.7045</td>
<td>1.1049</td>
<td>1.2272</td>
<td>1.3714</td>
<td>1.5429</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.2537</td>
<td>1.6025</td>
<td>2.1008</td>
<td>2.8409</td>
<td>1.2053</td>
<td>1.4727</td>
<td>1.8286</td>
<td>2.3143</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.5664</td>
<td>2.6709</td>
<td>5.2521</td>
<td>14.2045</td>
<td>1.4463</td>
<td>2.2090</td>
<td>3.6571</td>
<td>6.99429</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.0416</td>
<td>1.0869</td>
<td>1.1363</td>
<td>1.1905</td>
<td>1.0344</td>
<td>1.0714</td>
<td>1.1111</td>
<td>1.1538</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.0964</td>
<td>1.2077</td>
<td>1.3368</td>
<td>1.4880</td>
<td>1.0794</td>
<td>1.1688</td>
<td>1.2698</td>
<td>1.3846</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.1748</td>
<td>1.3935</td>
<td>1.6711</td>
<td>2.0929</td>
<td>1.1430</td>
<td>1.3150</td>
<td>1.5238</td>
<td>1.7802</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.3053</td>
<td>1.7419</td>
<td>2.3873</td>
<td>3.8203</td>
<td>1.2468</td>
<td>1.5779</td>
<td>2.0317</td>
<td>2.6703</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.6317</td>
<td>2.9031</td>
<td>5.9682</td>
<td>16.9102</td>
<td>1.4962</td>
<td>2.3668</td>
<td>4.0634</td>
<td>8.0100</td>
</tr>
</tbody>
</table>
Table 2.2

Moment Properties of Pareto Distribution

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>j = 1</th>
<th>j = 2</th>
<th>j = 3</th>
<th>j = 4</th>
<th>j = 1</th>
<th>j = 2</th>
<th>j = 3</th>
<th>j = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2.5000</td>
<td>6.6667</td>
<td>20.0000</td>
<td>80.0000</td>
<td>2.4000</td>
<td>6.0000</td>
<td>16.0000</td>
<td>48.0000</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.2222</td>
<td>5.0000</td>
<td>11.4286</td>
<td>26.6667</td>
<td>2.1818</td>
<td>4.8000</td>
<td>10.6667</td>
<td>24.0000</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.7778</td>
<td>8.3333</td>
<td>28.5714</td>
<td>133.3333</td>
<td>2.6181</td>
<td>7.2000</td>
<td>21.3333</td>
<td>72.0000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2.1428</td>
<td>4.6154</td>
<td>10.0000</td>
<td>21.8181</td>
<td>2.1176</td>
<td>4.5000</td>
<td>9.6000</td>
<td>20.5714</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.3809</td>
<td>5.7692</td>
<td>14.2860</td>
<td>36.3636</td>
<td>2.3101</td>
<td>5.4000</td>
<td>12.8000</td>
<td>30.8571</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.9762</td>
<td>9.6154</td>
<td>35.4173</td>
<td>181.8181</td>
<td>2.7721</td>
<td>8.1000</td>
<td>25.6000</td>
<td>92.5714</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.2556</td>
<td>5.1282</td>
<td>11.7650</td>
<td>27.2727</td>
<td>2.2097</td>
<td>4.9090</td>
<td>10.9714</td>
<td>24.6857</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.5062</td>
<td>6.4102</td>
<td>16.8067</td>
<td>45.4545</td>
<td>2.4106</td>
<td>5.8909</td>
<td>14.6286</td>
<td>37.0826</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3.1328</td>
<td>10.6837</td>
<td>42.0168</td>
<td>227.2727</td>
<td>2.8927</td>
<td>8.8363</td>
<td>29.2571</td>
<td>111.086</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.1930</td>
<td>4.8309</td>
<td>10.6952</td>
<td>23.8095</td>
<td>2.1589</td>
<td>4.6753</td>
<td>10.1587</td>
<td>22.1538</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.3496</td>
<td>5.5741</td>
<td>13.3690</td>
<td>32.4675</td>
<td>2.2859</td>
<td>5.2597</td>
<td>12.1905</td>
<td>28.4835</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2.6107</td>
<td>6.9676</td>
<td>19.0985</td>
<td>54.1125</td>
<td>2.4937</td>
<td>6.3116</td>
<td>16.2540</td>
<td>42.7252</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3.2633</td>
<td>11.6127</td>
<td>47.7463</td>
<td>47.7463</td>
<td>2.9924</td>
<td>9.4675</td>
<td>32.5080</td>
<td>128.176</td>
</tr>
</tbody>
</table>
Here in the above tables, it may be seen that the condition

$$\sum_{i=1}^{n} E(X_{i,n}) = n \ E(X)$$

is satisfied. (David and Nagaraja, 2003)
First four moments of upper record values:

### Table 2.4

<table>
<thead>
<tr>
<th>$r$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2500</td>
<td>1.6667</td>
<td>2.5000</td>
<td>5.0000</td>
<td>1.2000</td>
<td>1.5000</td>
<td>2.0000</td>
<td>3.0000</td>
</tr>
<tr>
<td>2</td>
<td>1.5625</td>
<td>2.7778</td>
<td>6.2500</td>
<td>25.0000</td>
<td>1.4400</td>
<td>2.2500</td>
<td>4.0000</td>
<td>9.0000</td>
</tr>
<tr>
<td>3</td>
<td>1.9531</td>
<td>4.6296</td>
<td>15.6250</td>
<td>125.0000</td>
<td>1.7280</td>
<td>3.3750</td>
<td>8.0000</td>
<td>27.0000</td>
</tr>
<tr>
<td>4</td>
<td>2.4414</td>
<td>7.7160</td>
<td>39.0625</td>
<td>625.0000</td>
<td>2.0736</td>
<td>5.0625</td>
<td>16.0000</td>
<td>81.0000</td>
</tr>
<tr>
<td>5</td>
<td>3.0518</td>
<td>12.8600</td>
<td>97.6563</td>
<td>3125.0000</td>
<td>2.4883</td>
<td>7.5934</td>
<td>32.0000</td>
<td>243.0000</td>
</tr>
</tbody>
</table>

### Table 2.5

<table>
<thead>
<tr>
<th>$r$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5000</td>
<td>6.6667</td>
<td>20.0000</td>
<td>80.0000</td>
<td>2.4000</td>
<td>6.0000</td>
<td>16.0000</td>
<td>48.0000</td>
</tr>
<tr>
<td>2</td>
<td>3.1250</td>
<td>11.1111</td>
<td>50.0000</td>
<td>400.0000</td>
<td>2.8800</td>
<td>9.0000</td>
<td>32.0000</td>
<td>144.0000</td>
</tr>
<tr>
<td>3</td>
<td>3.9062</td>
<td>18.5185</td>
<td>125.0000</td>
<td>2000.00</td>
<td>3.4560</td>
<td>13.5000</td>
<td>64.0000</td>
<td>432.0000</td>
</tr>
<tr>
<td>4</td>
<td>4.8828</td>
<td>30.8642</td>
<td>312.5000</td>
<td>10000.0</td>
<td>4.1472</td>
<td>20.2500</td>
<td>128.0000</td>
<td>1296.00</td>
</tr>
<tr>
<td>5</td>
<td>6.1035</td>
<td>51.4403</td>
<td>781.2500</td>
<td>50000.0</td>
<td>4.9766</td>
<td>30.3750</td>
<td>256.0000</td>
<td>3888.00</td>
</tr>
</tbody>
</table>
### Table 2.6

<table>
<thead>
<tr>
<th>( r )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
<th>( \theta = 5 )</th>
<th>( \theta = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.7500</td>
<td>15.0000</td>
<td>67.5000</td>
<td>405.000</td>
<td>3.6000</td>
<td>13.5000</td>
</tr>
<tr>
<td>2</td>
<td>4.6870</td>
<td>25.0000</td>
<td>168.750</td>
<td>2025.000</td>
<td>4.3200</td>
<td>20.2500</td>
</tr>
<tr>
<td>3</td>
<td>5.8594</td>
<td>41.6670</td>
<td>421.875</td>
<td>10125.0</td>
<td>5.1840</td>
<td>30.3750</td>
</tr>
<tr>
<td>4</td>
<td>7.3242</td>
<td>69.4440</td>
<td>1054.68</td>
<td>50625.0</td>
<td>6.2208</td>
<td>45.5625</td>
</tr>
<tr>
<td>5</td>
<td>9.1552</td>
<td>115.7407</td>
<td>2636.72</td>
<td>253125</td>
<td>7.4650</td>
<td>68.3434</td>
</tr>
</tbody>
</table>
3. PRODUCT MOMENTS

Case I. \( \gamma_i \neq \gamma_j, i, j = 1, 2, \ldots, n-1 \).

**Theorem 3.1.** For Pareto distribution as given in (1.1). Fix a positive integer \( k \) and for \( n \in N, \tilde{m} \in \mathbb{R}, (\theta \gamma_s - j) > 0, 1 \leq r < s \leq n, \ i, j = 1, 2, \ldots \)

\[
E[X^i(r,n,\tilde{m},k).X^j(s,n,\tilde{m},k)] = \frac{\theta \gamma_s}{\theta \gamma_s - j} E[X^i(r,n,\tilde{m},k).X^j(s-1,n,\tilde{m},k)]
\]

\[ (3.1) \]

**Proof.** We have,

\[
E[X^i(r,n,\tilde{m},k).X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k).X^j(s-1,n,\tilde{m},k)]
\]

\[
= C_{s-1}^{\beta} \int \int x^i y^j \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\tilde{F}(y)}{F(x)} \right]^{\gamma_i} \times \left( \sum_{i=1}^{r} a_i^{(r)}(\tilde{F}(x))^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \, dy \, dx
\]

\[
- C_{s-2}^{\beta} \int \int x^i y^j \sum_{i=r+1}^{s-1} a_i^{(r)}(s-1) \left[ \frac{\tilde{F}(y)}{F(x)} \right]^{\gamma_i} \times \left( \sum_{i=1}^{r} a_i^{(r)}(\tilde{F}(x))^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \, dy \, dx.
\]

\[
= \gamma_s C_{s-2}^{\beta} \int \int x^i y^j \left\{ \sum_{i=r+1}^{s-1} a_i^{(r)}(s) \left[ \frac{\tilde{F}(y)}{F(x)} \right]^{\gamma_i} + a_s^{(r)}(s) \left[ \frac{\tilde{F}(y)}{F(x)} \right]^{\gamma_s} \right\} \times \left( \sum_{i=1}^{r} a_i^{(r)}(\tilde{F}(x))^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \, dy \, dx
\]

\[
- C_{s-2}^{\beta} \int \int x^i y^j \left\{ \sum_{i=r+1}^{s-1} a_i^{(r)}(s)(\gamma_s - \gamma_i) \left[ \frac{\tilde{F}(y)}{F(x)} \right]^{\gamma_i} \right\} \times \left( \sum_{i=1}^{r} a_i^{(r)}(\tilde{F}(x))^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \, dy \, dx.
\]
\begin{align*}
= C_{s-2} \int_{\alpha}^{\beta} x^j \sum_{i=r+1}^{s} a_i^{(r)}(s) y_i \left[ \frac{\bar{F}(y)}{F(x)} \right]^r \\
\quad \times \left( \sum_{i=1}^{r} a_i(r) \frac{f(x)}{F(x)} f(y) \right) dy dx. \tag{3.2}
\end{align*}

Let \( v(x, y) = -\sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{F(x)} \right]^r \), then

\begin{align*}
\frac{\partial}{\partial y} v(x, y) &= \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{F(x)} \right]^r \frac{f(y)}{F(y)}.
\end{align*}

Putting the value of (3.3) into (3.2), we get

\begin{align*}
E[X^i(r, n, \bar{m}, k), X^j(s, n, \bar{m}, k)] - E[X^i(r, n, \bar{m}, k), X^j(s-1, n, \bar{m}, k)] \\
= C_{s-2} \int_{\alpha}^{\beta} x^j \frac{\partial}{\partial y} v(x, y) \sum_{i=1}^{r} a_i(r) \frac{f(x)}{F(x)} dy dx.
\end{align*}

Now consider,

\begin{align*}
\int_{x}^{y} y \frac{\partial}{\partial y} v(x, y) dy &= \int_{x}^{y} y^{-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{F(x)} \right]^r dy.
\end{align*}

Therefore, we have,

\begin{align*}
E[X^i(r, n, \bar{m}, k), X^j(s, n, \bar{m}, k)] - E[X^i(r, n, \bar{m}, k), X^j(s-1, n, \bar{m}, k)] \\
= jC_{s-2} \int_{\alpha}^{\beta} x^j \left( \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{F(x)} \right]^r \right) \\
\quad \times \left\{ \int_{x}^{y} y^{-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{F(x)} \right]^r \right\} dy dx. \tag{3.4}
\end{align*}
Then in view of (1.3), we get

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

$$= \frac{j}{\theta y_s} C_{s-1} \int_0^\infty \int_a^x x^i y^j [\sum_{i=r+1}^s a_i(r)(F(y)^{\gamma_i})^j]$$

$$\times [\sum_{i=1}^s a_i(r)F(x)^{\gamma_i}] \frac{f(x) f(y) y^{\gamma_i}}{F(x) F(y)} dy dx.$$

$$= \frac{j}{\theta y_s} E[X^i(r,n,m,k)X^j(s,n,m,k)]$$

and hence the result.

**Case II:** \(m_i = m, i = 1, 2, \ldots, n-1\).

**Theorem 3.2.** For Pareto distribution as given in (1.2). Fix a positive integer \(k\) and for \(n \in N, m \in \Re, \theta y_u > (i + j), (\theta y_v - j) > 0, 1 \leq r < s \leq n, i, j = 1, 2, \ldots\)

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{\theta y_u}{\theta y_v - j} E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

and subsequently,

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \theta^s \alpha^{i+j} \left( \prod_{u=1}^r \frac{\gamma_u}{\theta y_u - i - j} \right) \left( \prod_{v=r+1}^s \frac{\gamma_v}{\theta y_v - j} \right).$$

**Proof.** From (1.4.16), we have

$$E[\xi(X(r,n,m,k),X(s,n,m,k))] - E[\xi(X(r,n,m,k),X(s-1,n,m,k))]$$

$$= \frac{C_{s-2}}{(r-1)!(s-r-1)!} \int_a^x \frac{\partial^j}{\partial y^j} \xi(x,y)[1 - F(x)]^m f(x) \xi_{m-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[1 - F(y)]^r dy dx.$$
Let \( \xi(x, y) = x^i y^j \), then

\[
E[X^i(r, n, m, k), X(s, n, m, k)] - E[X^i(r, n, m, k), X(s-1, n, m, k)]
= \frac{jC_{s-2}}{(r-1)!(s-r-1)!} \int \int x^i y^{i-1} [1 - F(x)]^m f(x) g_{m}^{r-1} (F(x))
\times [h_{m}(F(y)) - h_{m}(F(x))]^{s-r-1} [1 - F(y)]^r \, dy \, dx.
\] (3.8)

Using (1.3) in (3.8) and proceeding on the lines of Theorem 3.1, relation (3.6) can be established.

To obtain (3.7), we write (3.6) recursively.

**Remark 3.1:** Recurrence relation for product moments of order statistics (at \( m = 0, k = 1 \)) is

\[
E[X_r^i, X_s^j] = \theta(n-s+1) \frac{\theta(n-s+1)}{\theta(n-s+1) - j} E[X_{r-1}^i, X_{s-1}^j] \quad \text{(Khan et al., 1983b)}
\]

\[
= \theta^s \alpha^{i+j} \left( \prod_{u=1}^{r} \frac{(n-u+1)}{\theta(n-u+1) - i - j} \right) \left( \prod_{v=r+1}^{s} \frac{(n-v+1)}{\theta(n-v+1) - j} \right).
\] (3.9)

Expression (3.9) may also be expressed as

\[
E[X_r^i, X_s^j] = \alpha^{i+j} \frac{\Gamma(n+1) \Gamma\left(n-r+1-\frac{i+j}{\theta}\right) \Gamma\left(n-s+1-\frac{j}{\theta}\right)}{\Gamma(n-s+1) \Gamma\left(n-r+1-\frac{j}{\theta}\right) \Gamma\left(n+1-\frac{i+j}{\theta}\right)}.
\]

**Remark 3.2:** Recurrence relation for product moments of \( k \)-th record values will be

\[
E[(X_{U(r)}^{(k)})^i.(X_{U(s)}^{(k)})^j] = \frac{\theta k}{\theta k - j} E[(X_{U(r)}^{(k)})^i.(X_{U(s-1)}^{(k)})^j].
\]
\[ = \alpha^{i+j} \left( \frac{k\theta}{k\theta - i - j} \right)^r \left( \frac{k\theta}{k\theta - j} \right)^{s-r}. \]  

(3.10)

**Remark 3.3:** At \( i = 0 \), result is reduced for single moment as obtained in Theorem 2.1.
# Variance - covariance of order statistics from Pareto distribution:

## Table 3.1

<table>
<thead>
<tr>
<th>(n)</th>
<th>(s)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.1042</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.0154</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.0059</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.0031</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.0019</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.0195</td>
<td>0.1542</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.0195</td>
<td>0.1542</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.0065</td>
<td>0.0250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.0065</td>
<td>0.0250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0082</td>
<td>0.0313</td>
<td>0.1886</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.0082</td>
<td>0.0313</td>
<td>0.1886</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.0082</td>
<td>0.0313</td>
<td>0.1886</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.0037</td>
<td>0.0113</td>
<td>0.0322</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.0037</td>
<td>0.0113</td>
<td>0.0322</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.0046</td>
<td>0.0140</td>
<td>0.0403</td>
<td>0.2172</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.0046</td>
<td>0.0140</td>
<td>0.0403</td>
<td>0.2172</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>0.0046</td>
<td>0.0140</td>
<td>0.0403</td>
<td>0.2172</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.0021</td>
<td>0.0058</td>
<td>0.0133</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.0021</td>
<td>0.0058</td>
<td>0.0133</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0.0025</td>
<td>0.0065</td>
<td>0.0149</td>
<td>0.0381</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.0025</td>
<td>0.0065</td>
<td>0.0149</td>
<td>0.0381</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.0031</td>
<td>0.0082</td>
<td>0.0185</td>
<td>0.0474</td>
<td>0.2406</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.0031</td>
<td>0.0082</td>
<td>0.0185</td>
<td>0.0474</td>
<td>0.2406</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3.2

<table>
<thead>
<tr>
<th>n</th>
<th>s</th>
<th>( r )</th>
<th>( \alpha = 2, \theta = 5 )</th>
<th>( \alpha = 2, \theta = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.1470</td>
<td>0.2400</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.0618</td>
<td>0.0397</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.0771 0.6171</td>
<td>0.0478 0.3455</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.0238</td>
<td>0.0157</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.2640 0.1005</td>
<td>0.0171 0.0634</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0328 0.1254 0.7580</td>
<td>0.0207 0.0761 0.4154</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.0125</td>
<td>0.0080</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.0134 0.0405</td>
<td>0.0014 0.0262</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0150 0.0450 0.1291</td>
<td>0.0095 0.0287 0.0799</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.0182 0.0562 0.1616 0.8195</td>
<td>0.0113 0.0344 0.0958 0.4686</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.0098</td>
<td>0.0054</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.0113 0.0207</td>
<td>0.0055 0.0145</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0090 0.0233 0.0534</td>
<td>0.0060 0.0153 0.1343</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.0095 0.0257 0.0593 0.1519</td>
<td>0.0064 0.0168 0.0375 0.0930</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.0120 0.0323 0.0744 0.1901 0.9636</td>
<td>0.0077 0.0202 0.0452 0.1119 0.5130</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.3

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.9375</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.1391</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0894</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.1736</td>
<td>1.3890</td>
<td></td>
<td></td>
<td></td>
<td>0.1075</td>
<td>0.7771</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.0528</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0354</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0589</td>
<td>0.2258</td>
<td></td>
<td></td>
<td></td>
<td>0.0386</td>
<td>0.1424</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0734</td>
<td>0.2822</td>
<td>1.7050</td>
<td></td>
<td></td>
<td>0.0489</td>
<td>0.1710</td>
<td>0.9344</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.0270</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0187</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0292</td>
<td>0.0904</td>
<td></td>
<td></td>
<td></td>
<td>0.0187</td>
<td>0.0560</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0259</td>
<td>0.0985</td>
<td>0.2854</td>
<td></td>
<td></td>
<td>0.0212</td>
<td>0.0630</td>
<td>0.1795</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0406</td>
<td>0.1255</td>
<td>0.3595</td>
<td>1.9505</td>
<td></td>
<td>0.0262</td>
<td>0.8572</td>
<td>0.2151</td>
<td>1.0548</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.0170</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0118</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0181</td>
<td>0.0498</td>
<td></td>
<td></td>
<td></td>
<td>0.0123</td>
<td>0.0334</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0192</td>
<td>0.0568</td>
<td>0.1204</td>
<td></td>
<td></td>
<td>0.0099</td>
<td>0.0365</td>
<td>0.0777</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0215</td>
<td>0.0587</td>
<td>0.1337</td>
<td>0.3421</td>
<td></td>
<td>0.0144</td>
<td>0.0376</td>
<td>0.0845</td>
<td>0.2092</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0267</td>
<td>0.0733</td>
<td>0.1672</td>
<td>0.4278</td>
<td>2.1679</td>
<td>0.0167</td>
<td>0.0452</td>
<td>0.1014</td>
<td>0.2511</td>
<td>1.1536</td>
</tr>
</tbody>
</table>

Here in the above tables, it may be seen that the condition

$$\sum_{r=1}^{n} \sum_{s=1}^{n} \sigma_{rsn} = n \sigma^2$$

is satisfied. (David and Nagaraja, 2003)

where $\sigma_{rsn} = \text{cov}(X_{rn}, X_{sn})$, $\sigma^2 = \text{V}(X)$. 
**Variance - Covariance between upper record values:**

Table 3.4

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 1, \theta = 5$</th>
<th></th>
<th>$\alpha = 1, \theta = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r$</td>
<td></td>
<td>$r$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.1042</td>
<td></td>
<td>0.0060</td>
</tr>
<tr>
<td>2</td>
<td>0.1301</td>
<td>0.3363</td>
<td>0.0720</td>
</tr>
<tr>
<td>3</td>
<td>0.1627</td>
<td>0.4204</td>
<td>0.8150</td>
</tr>
<tr>
<td>4</td>
<td>0.2035</td>
<td>0.5255</td>
<td>1.0187</td>
</tr>
<tr>
<td>5</td>
<td>0.2543</td>
<td>0.6570</td>
<td>1.2632</td>
</tr>
</tbody>
</table>

Table 3.5

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 2, \theta = 5$</th>
<th></th>
<th>$\alpha = 2, \theta = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r$</td>
<td></td>
<td>$r$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.4167</td>
<td></td>
<td>0.2400</td>
</tr>
<tr>
<td>2</td>
<td>0.5208</td>
<td>1.3454</td>
<td>0.2880</td>
</tr>
<tr>
<td>3</td>
<td>0.6512</td>
<td>1.6820</td>
<td>3.2601</td>
</tr>
<tr>
<td>4</td>
<td>0.8138</td>
<td>2.1024</td>
<td>4.0749</td>
</tr>
<tr>
<td>5</td>
<td>1.0172</td>
<td>2.6280</td>
<td>5.0937</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 3, \theta = 5$</td>
<td></td>
<td>$\alpha = 3, \theta = 6$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$r$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0.9375</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.1740</td>
<td>3.0320</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.4648</td>
<td>3.7870</td>
<td>7.3341</td>
</tr>
<tr>
<td>4</td>
<td>1.8310</td>
<td>4.7340</td>
<td>9.1680</td>
</tr>
</tbody>
</table>
4. CHARACTERIZATION

Theorem 4.1. Let \( X(r,n,m,k), \ r=1,2,\ldots,n \) be gos based on continuous distribution function \( F() \). Then for two consecutive values \( r \) and \( r+1, 2 \leq r+1 \leq s \leq n \), the conditional expectation of \( X(s,n,m,k) \) given \( X(r,n,m,k) = x \), is given by

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = g_{s|r}(x) = a_{s|r}x, \ l = r, r+1
\] (4.1)

if and only if \( X \) has the df

\[
\overline{F}(x) = \left( \frac{\alpha}{x} \right)^{\theta}, \ \alpha \leq x < \infty, \ \theta > 0
\] (4.2)

where,

\[
a_{s|r} = \prod_{i=r+1}^{s} \frac{\theta y_i}{\theta y_i - 1},
\]

Proof. We have for \( s \geq r+1, \)

\[
g_{s|r}(x) = E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_{x}^{\infty} \left( \frac{\overline{F}(y)}{\overline{F}(x)} \right)^{y-1} \left[ 1 - \left( \frac{\overline{F}(y))}{\overline{F}(x)} \right)^{m+1} \right]^{s-r-1} \frac{f(y)}{\overline{F}(x)} \ dy
\] (4.3)

Let \( u = \frac{\overline{F}(y)}{\overline{F}(x)} = \left( \frac{x}{y} \right)^{\theta} \), then \( y = xu \frac{1}{\theta} \).

Thus (4.3) becomes

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_{0}^{1} x u \frac{1}{\theta} u^{r-1} \left[ 1 - u^{m+1} \right]^{s-r-1} du.
\]
Set \( u^{m+1} = t \), to get

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{c_{s-1}}{c_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_0^1 x t \left( \frac{\frac{1}{\theta(m+1)} + \frac{1}{m+1}}{1-t} \right)^{s-r-1} dt
\]

\[
= \frac{c_{s-1} x}{c_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_0^1 \frac{\gamma_{s-1}}{t^\frac{\gamma_s}{\theta(m+1)} + s-r} \left( 1-t \right)^{s-r-1} dt
\]

\[
= \frac{c_{s-1} x}{c_{r-1}(s-r-1)! (m+1)^{s-r-1}} \Gamma \left( \frac{\gamma_s}{\theta(m+1)} + s-r \right)
\]

\[
= x \frac{c_{s-1}}{c_{r-1}(s-r-1)! (m+1)^{s-r-1}} \prod_{i=r+1}^s \frac{\Gamma \left( \frac{\gamma_i}{m+1} + s-r \right)}{\Gamma \left( \frac{\gamma_i}{m+1} \right)}
\]

\[
= x \frac{\prod_{i=r+1}^s \gamma_i}{(m+1)^{s-r-1}} \frac{\theta[k + (n-s-1)(m+1) - 1]}{m+1} \prod_{i=r+1}^s \frac{\theta[k + (n-r-1)(m+1) - 1]}{m+1}
\]

\[
= x \prod_{i=r+1}^s \frac{\theta \gamma_i}{\theta \gamma_i - 1}
\]

\[= a_{s|r} x.\]
where,

\[ a_{slr} = \prod_{i=r+1}^{s} \frac{\theta \gamma_i}{\theta \gamma_i - 1} \]  

[Khan and Alzaid, 2004].

To show that (4.1) implies (4.2), we have

\[ E[X(s,n,m,k) | X(r,n,m,k) = x] = g_{slr}(x) \]

That is,

\[ \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_{x}^{\infty} y \left[ (F(x))^{m+1} - (F(y))^{m+1} \right]^{s-r-1} \]

\[ \times [\bar{F}(y)]^{r+1} f(y)dy = g_{slr}(x)[\bar{F}(x)]^{r+1}. \]  

(4.4)

Differentiating (4.4) both sides w.r.t. \( x \), we get

\[ \frac{C_{s-1}[F(x)]^m f(x)}{C_{r-1}(s-r-2)! (m+1)^{s-r-2}} \]

\[ \times \int_{x}^{\infty} y [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2} [\bar{F}(y)]^{r+1} f(y)dy \]

\[ = g'_{s[slr]}(x)[\bar{F}(x)]^{r+1} - \gamma_{r+1} g_{s[slr]}(x)[\bar{F}(x)]^{r+1-1} f(x) \]

\[ - [\bar{F}(x)]^{r+2} [\bar{F}(x)]^m \gamma_{r+1} f(x) g_{s[slr]}(x) \]

\[ = g'_{s[slr]}(x)[\bar{F}(x)]^{r+1} - \gamma_{r+1} g_{s[slr]}(x)[\bar{F}(x)]^{r+1-1} f(x), \]

or,

\[ - \gamma_{r+1} g_{s[slr]}(x) = \frac{g'_{s[slr]}(x)[\bar{F}(x)]}{f(x)} - \gamma_{r+1} g_{s[slr]}(x) \]

\[ \frac{f(x)}{\bar{F}(x)} = \frac{-1}{\gamma_{r+1}} \frac{g'_{s[slr]}(x)}{[g_{s[slr]}(x) - g_{s[slr]}(x)]} \]  

(4.5)
Now, consider

$$g_{s[r+1]}(x) - g_{s[r]}(x) = a_{s[r+1]} - a_{s[r]} x$$

$$= (a_{s[r+1]} - a_{s[r]}) x$$

$$= -\frac{a_{s[r]}}{\gamma r+1} x$$

and

$$g'_{s[r]}(x) = a_{s[r]} x.$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{\theta}{x}.$$

Implying that

$$F(x) = \left(\frac{\alpha}{x}\right)^\theta, \quad \alpha \leq x < \infty.$$ 

Hence the theorem.
CHAPTER III

MOMENTS OF ORDER STATISTICS FROM EXTENDED TYPEI GENERALIZED LOGISTIC DISTRIBUTION

1. INTRODUCTION

The concept of extended type-I generalized logistic distribution was introduced by Olapade (2004), which is the particular case of the general class considered by Wu et al. (2000). The importance of the logistic distribution has already been felt in many areas of human endeavor. Verhulst (1845) used it in economic and demographic studies. Berkson (1944, 1951 and 1953) used the distribution extensively in analyzing the response data. The simplicity of the logistic distribution and its importance as a growth curve has made it one of the many important statistical distributions. The shape of the logistic distribution that is similar to that of the normal distribution makes it simpler and also profitable on suitable occasions to replace the normal distribution by the logistic distribution with negligible errors in the respective theories. Balakrishnan and Leung (1988) shown the probability density function of a random variable that has type I generalized logistic distribution. Wu et al. (2000) proposed an extended form of the generalized logistic distribution which is referred to as the five-parameter generalized logistic distribution.

A random variable \( X \) is said to have extended type-I generalized logistic distribution if its pdf is given by

\[
f(x) = \frac{ae^{-x}}{(1 + ae^{-x})^{p+1}}, \quad -\infty < x < \infty, \quad a, p > 0,
\]

(1.1)

and the corresponding df is

\[
F(x) = \frac{1}{(1 + ae^{-x})^p}.
\]

(1.2)

When \( p = a = 1 \), we have the ordinary logistic distribution and when \( a = 1 \), we have the type - I generalized logistic distribution (Balakrishnan and Leung, 1988).
In this chapter explicit expression for moments of order statistics for the given distribution is obtained and some computational work is also carried out. Further, recurrence relations for marginal and joint moment generating functions of order statistics are derived.

2. EXACT MOMENTS

For extended type I generalized logistic distribution, the pdf of $r$-th order statistics, $X_{r,n}$, $1 \leq r \leq n$, may be written as

$$f_{r,n}(x) = [B(r, n-r + 1)]^{-1}(1 + a e^{-x})^{-p(r+1)}$$

$$\times [1 - (1 + a e^{-x})^{-p}]^{n-r} a p e^{-x} (1 + a e^{-x})^{-p-1}$$

$$= a p [B(r, n-r + 1)]^{-1}\sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} e^{-x} [1 + a e^{-x}]^{-p(r+i+1)}$$

(2.1)

**Theorem 2.1.** For the distribution as given in (1.1) and $1 \leq r \leq n$, $k = 1, 2, \ldots$

$$\mu_{r,n}^{(k)} = \frac{a p}{B(r, n-r + 1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \frac{\partial^k}{\partial[-p(r+i)]^k} [a^{-1} B(p(r+i), 1)].$$

(2.2)

**Proof.** We have

$$\mu_{r,n}^{(k)} = \frac{a p}{B(r, n-r + 1)} \int_{-\infty}^{\infty} x^k \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} e^{-x} [1 + a e^{-x}]^{-p(r+i+1)} dx$$

$$= \frac{a p}{B(r, n-r + 1)} \int_{-\infty}^{\infty} x^k \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} e^{-x} e^{p(r+i)x+x} (e^x + a)^{p(r+i)+1} dx$$

$$= \frac{a p}{B(r, n-r + 1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} x^k \frac{e^{-(-p(r+i))x}}{[e^x + a]^{p(r+i)+1}} dx.$$  

(2.3)

Now, on using the result from Prudnikov *et al.* (1986)

$$\int_{-\infty}^{\infty} x^n e^{-px} \frac{dx}{(e^x + 2)^\rho} = (-1)^n \frac{\partial^n}{\partial \rho^n} [z^{-\rho-\rho} B(-\rho, (p+\rho))],$$

$[\arg z < \pi; -\Re \rho < \Re \rho < 0; n = 1, 2, 3, \ldots]$.
we have,

\[
\mu_{r,n}^{(k)} = \frac{a}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i+k} \binom{n-r}{i} \\
\times \frac{\partial^k}{\partial[-p(r+i)]^k} [a^{(-p(r+i))} p(r+i-1) B(p(r+i),1)]
\]

\[
= \frac{a}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i+k} \binom{n-r}{i} \frac{\partial^k}{\partial[-p(r+i)]^k} [a^{-1} B(p(r-i),1)]
\]

and hence the required result.

**Remark 2.1:** When \( p=a=1 \), the expression (2.2) reduces to ordinary logistic distribution and

\[
\mu_{r,n}^{(k)} = \frac{1}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i+k} \binom{n-r}{i} \frac{\partial^k}{\partial[-(r+i)]^k} B(r+i,1).
\]

**Remark 2.2:** When \( a=1 \), the expression (2.2) reduces to type-I generalized logistic distribution and

\[
\mu_{r,n}^{(k)} = \frac{p}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i+k} \binom{n-r}{i} \frac{\partial^k}{\partial[-p(r+i)]^k} B(p(r+i),1).
\]

When \( k=1 \) and \( k=2 \), we have

\[
\mu_{r,n} = \frac{a}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i} \binom{n-r}{i} \int_{-\infty}^{\infty} \frac{e^{-(p(r+i))x}}{[e^x + a]^{(p(r+i)+1)}} dx.
\]

Now, using the result from Prudnikov et al. (1986)

\[
\int_{-\infty}^{\infty} \frac{x^n e^{-px}}{(e^x + z)^\rho} dx = (-1)^n \frac{\partial^n}{\partial z^n} [z^{-\rho} B(-p,(p + \rho))]
\]

\[
= (-1)^n \frac{\partial^n}{\partial z^n} \{z^{-\rho} B(-p,(p + \rho))[\ln z + \psi(-p) - \psi(p+\rho)]\}.
\]

\[
\mu_{r,n} = \frac{p}{B(r,n-r+1)} \frac{p}{i=0} (-1)^{i} \binom{n-r}{i} \\
\times [B(p(r+i),1) \{(\log(a) + \psi(p(r+i)) - \psi(1)) \}]
\]

(2.4)
and

\[ \mu_{rn}^{(2)} = \frac{p}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} 
\times B(p(r+i), 1) \{ \psi(p(r+i)) - \psi(1) + [\log(a) + \psi(p(r+i)) - \psi(1)]^2 \} \]

(2.5)

where \( \psi(x) \) is a digamma function defined by

\[ \psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x \neq 0, -1, -2, \ldots \]

as obtained in Balakrishnan and Leung (1988).

Thus making use of the explicit expression in (2.4) and (2.5), we can obtain the mean and variance of \( r-th \) order statistic.

### Means of order statistics of type-I generalized logistic distribution

<table>
<thead>
<tr>
<th>( n )</th>
<th>( r )</th>
<th>( p = 1 )</th>
<th>( p = 1.5 )</th>
<th>( p = 2 )</th>
<th>( p = 2.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0000</td>
<td>0.6137</td>
<td>1.0000</td>
<td>1.2804</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1.0000</td>
<td>-0.2726</td>
<td>0.1667</td>
<td>0.4774</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.0000</td>
<td>1.5000</td>
<td>1.8333</td>
<td>2.0833</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1.5000</td>
<td>-0.6928</td>
<td>-0.2167</td>
<td>0.1151</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0000</td>
<td>0.5678</td>
<td>0.9333</td>
<td>1.2021</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.5000</td>
<td>1.9661</td>
<td>2.2833</td>
<td>2.5239</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-1.8333</td>
<td>-0.9642</td>
<td>-0.4595</td>
<td>-0.1116</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.5000</td>
<td>0.1213</td>
<td>0.5119</td>
<td>0.7951</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.5000</td>
<td>1.0143</td>
<td>1.3547</td>
<td>1.6089</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.8333</td>
<td>2.2833</td>
<td>2.5928</td>
<td>2.8289</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-2.0833</td>
<td>-1.1632</td>
<td>-0.6353</td>
<td>-0.2742</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.8333</td>
<td>-0.1676</td>
<td>0.2436</td>
<td>0.5387</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0000</td>
<td>0.5547</td>
<td>0.9143</td>
<td>1.1797</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.8333</td>
<td>1.3208</td>
<td>1.6484</td>
<td>1.8952</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2.0833</td>
<td>2.5239</td>
<td>2.8289</td>
<td>3.0624</td>
</tr>
</tbody>
</table>
Here, it may be noted that the well known property of order statistics

\[ \sum_{i=1}^{n} E(X_{(i:n)}) = nE(X) \]  

(David and Nagaraja, 2003)

is satisfied.

Variances of order statistics of type-I generalized logistic distribution

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>( p = 1 )</th>
<th>( p = 1.5 )</th>
<th>( p = 2 )</th>
<th>( p = 2.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3.2898</td>
<td>2.5797</td>
<td>2.2898</td>
<td>2.1352</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.2898</td>
<td>1.5485</td>
<td>1.2621</td>
<td>1.1148</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.2898</td>
<td>2.0398</td>
<td>1.9287</td>
<td>1.8662</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2.0398</td>
<td>1.2786</td>
<td>0.9929</td>
<td>0.8490</td>
</tr>
<tr>
<td>2</td>
<td>1.2898</td>
<td>1.0288</td>
<td>0.9187</td>
<td>0.8586</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.0398</td>
<td>1.8936</td>
<td>1.8262</td>
<td>1.7875</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.9287</td>
<td>1.1534</td>
<td>0.8676</td>
<td>0.7257</td>
</tr>
<tr>
<td>2</td>
<td>1.0398</td>
<td>0.7708</td>
<td>0.6611</td>
<td>0.6025</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0398</td>
<td>0.8881</td>
<td>0.8211</td>
<td>0.7836</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.9287</td>
<td>1.8262</td>
<td>1.7780</td>
<td>1.7501</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.8662</td>
<td>1.0801</td>
<td>0.7939</td>
<td>0.6533</td>
</tr>
<tr>
<td>2</td>
<td>0.9287</td>
<td>0.6535</td>
<td>0.5439</td>
<td>0.4863</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.7898</td>
<td>0.6337</td>
<td>0.5670</td>
<td>0.5303</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9287</td>
<td>0.8229</td>
<td>0.7749</td>
<td>0.7477</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.8662</td>
<td>1.7875</td>
<td>1.7501</td>
<td>1.7282</td>
<td></td>
</tr>
</tbody>
</table>
3. RECURRENCE RELATIONS FOR MOMENT GENERATING FUNCTIONS

For extended type-I generalized logistic distribution defined in (1.1), we have the relation

\[ F(x) = \frac{(a + e^x)}{ap} f(x). \]  \hspace{1cm} (3.1)

Using the relation (3.1), we shall derive the recurrence relations for moment generating function \((mgf)\) of order statistics from extended type-I generalized logistic distribution.

Let us denote the \(mgf\) of \(X_{rn}\) by \(M_{rn}(t)\).

Thus,

\[ M_{rn}(t) = C_{rn} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) dx \]

and the joint \(mgf\) of \(X_{rn}\) and \(X_{sn}\) \((1 \leq r < s \leq n)\) by \(M_{r,s;n}(t_1, t_2)\)

\[ M_{r,s;n}(t_1, t_2) = C_{r,s;n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} \]
\[ \times [1 - F(y)]^{n-r} f(x) f(y) dx dy, \]

where

\[ C_{rn} = \frac{n!}{(r-1)!(n-r)!} \]

and

\[ C_{r,s;n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}. \]

**Lemma 3.1.** For the distribution as given in (1.1) and \(1 \leq r \leq n\), \(M_{rn}(t)\) exists.

*Proof.* We have,

\[ M_{rn}(t) = C_{rn} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x) dx \]
\[
\frac{ap}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} e^{tx} e^{-x} \left[1 + ae^{-x}\right]^{-p(r+i)+1} dx
\]

\[
\frac{ap}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} e^{-x(1-r)} \left[1 + ae^{-x}\right]^{-p(r+i)-1} dx
\]

Let \( ae^{-x} = u \), implying \( x = \log \left( \frac{a}{u} \right) \).

Hence,

\[
M_{r,n}(t) = \frac{p}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{0}^{\infty} e^{t \log(a/u)} \left[1 + u\right]^{-p(r+i)+1} du.
\]

\[
M_{r,n}(t) = \frac{a^t p}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{0}^{\infty} \frac{u^{-t}}{(1+u)^{p(r+i)+1}} du.
\]

Since,

\[
\int_{0}^{\infty} \frac{x^{\alpha-1}}{(x+z)^\beta} dx = Z^{\alpha-\beta} B(\alpha, \beta - \alpha); \quad [\arg z < \pi; \ 0 < \Re \alpha < \Re \beta].
\]

(Prudnikov et al., 1986)

Thus, we get

\[
M_{r,n}(t) = \frac{a^t p}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} B[1-t, p(r+i)+t], \quad t < 1.
\]

Hence the lemma.

**Lemma 3.2.** For any arbitrary distribution and \( 2 \leq r \leq n, \ n \geq 2 \).

(i) \( M_{r,n}(t) - M_{r-1,n}(t) = \binom{n}{r-1} t \int_{-\infty}^{\infty} e^{tx}[F(x)]^{-1} [1 - F(x)]^{n-r+1} dx \). \hspace{1cm} (3.2)

(ii) \( M_{r,n}(t) - M_{r-1,n-1}(t) = \binom{n-1}{r-1} t \int_{-\infty}^{\infty} e^{tx}[F(x)]^{-1} [1 - F(x)]^{n-r+1} dx \). \hspace{1cm} (3.3)
(iii) \( M_{r-ln}(t) - M_{r-ln}(t) = \left( \begin{array}{c} n-1 \\ r-2 \end{array} \right) t \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} [1 - F(x)]^{n-r+1} dx. \) \( (3.4) \)

Proof. We have from Ali and Khan (1997), that

\[
E[g(X_{r,n})] - E[g(X_{r-1,n})] = \left( \begin{array}{c} n \\ r-1 \end{array} \right) \int_{\alpha}^{\beta} g'(x) [F(x)]^{r-1} [1 - F(x)]^{n-r+1} dx,
\] \( (3.5) \)

where, \( g(x) \) is a Borel measurable function of \( x \) and \( x \in (\alpha, \beta) \).

Let \( g(x) = e^{tx}, -\infty < x < \infty \), then \( (3.2) \) can be proved in view of \( (3.5) \). The equations \( (3.3) \) and \( (3.4) \) can be seen on the lines of \( (3.2) \).

**Theorem 3.1.** For extended type-I generalized logistic distribution as given in
\( (1.1) \) and \( 2 \leq r \leq n, \ n \geq 2, \ j = 1,2,\cdots \)

(i) \( M_{r/l}^{(j)}(t) = \left( \frac{t}{p(r-1)} \right) M_{r-1/l}^{(j)}(t) \)
\[+ \frac{1}{p(r-1)} \left( \frac{t}{\alpha} M_{r-2/l}^{(j-1)}(t) + \frac{t}{\beta} M_{r-1/l}^{(j)}(t+1) + \frac{1}{\alpha} M_{r-1/l}^{(j-1)}(t+1) \right), \] \( (3.6) \)

(ii) \( M_{r/l}^{(j)}(t) = M_{r-1/l}^{(j)}(t) \)
\[+ \frac{n-r+1}{np(r-1)} \left( t M_{r-1/l}^{(j)}(t) + t M_{r-2/l}^{(j-1)}(t) + \frac{t}{\alpha} M_{r-1/l}^{(j)}(t+1) + \frac{1}{\alpha} M_{r-1/l}^{(j-1)}(t+1) \right), \] \( (3.7) \)

(iii) \( M_{r-1/l}^{(j)}(t) = \left( \frac{t}{np} \right) M_{r-1/l}^{(j)}(t) \)
\[+ \frac{j}{np} M_{r-1/l}^{(j)}(t) + \frac{j}{np} M_{r-1/l}^{(j)}(t+1) + \frac{j}{np} M_{r-1/l}^{(j)}(t+1), \] \( (3.8) \)

where \( M_{r/l}^{(j)}(t) \) is the \( j \)-th derivative of \( M_{r/l}(t) \).

Proof. In view of equations \( (3.1) \) and \( (3.2) \), we have

\[
M_{r/n}(t) - M_{r-1/n}(t) = \frac{t}{a} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-2} [1 - F(x)]^{n-r+1} (a + e^x) f(x) dx
\]
Moments of Order Statistics...

After rearranging the terms, we get

$$M_{r,n}(t) = \left(1 + \frac{t}{p(r-1)} \right) M_{r-1,n}(t) + \frac{t}{ap(r-1)} M_{r-1,n}(t+1). \quad (3.9)$$

Differentiating (3.9) w.r.t. \( t \), \( j \) times, we get (3.6).

The equations (3.7) and (3.8) can be established on the lines of (3.6).

**Lemma 3.3** For any arbitrary distribution and \( 2 \leq r \leq n, \ n \geq 2 \),

$$M_{r,s,n}(t_1,t_2) - M_{r-1,s,n}(t_1,t_2) = \frac{n!t_1}{(r-1)!(s-r)!(n-s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x+t_2y} \left[ F(y) - F(x) \right]^{s-r} \left[ 1 - F(y) \right]^{n-s} f(y) dy dx \quad (3.10)$$

**Proof.** We have from Khan et al. (2001),

$$E[g(X_{r,n},X_{s,n})] - E[g(X_{r-1,n},X_{s,n})]$$

$$= \frac{C_{r,s,n}}{(s-r)} \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial x} g(x,y) \left[ F(x) \right]^{r-1} \left[ F(y) - F(x) \right]^{s-r} \left[ 1 - F(y) \right]^{n-s} f(y) dy dx \quad (3.11)$$

where \( g(x,y) \) is a measurable function of \( (x,y) \).

The lemma can be proved by letting \( g(x,y) = e^{t_1x+t_2y} \) in (3.11).
Theorem 3.2. For the distribution as given in (1.1) and
\[ 2 \leq r \leq n, \quad j, k = 0, 1, 2, \ldots \]
\[
M_{r,s}^{(j,k)}(t_1, t_2) = \left(1 + \frac{t_1}{p(r-1)}\right) M_{r-1,s}^{(j,k)}(t_1, t_2) + \left(\frac{j}{p(r-1)}\right) M_{r-1,s}^{(j-1,k)}(t_1, t_2)
+ \left(\frac{t_1}{ap(r-1)}\right) M_{r-1,s}^{(j,k)}(t_1 + 1, t_2) + \left(\frac{j}{ap(r-1)}\right) M_{r-1,s}^{(j-1,k)}(t_1 + 1, t_2)
\]
(3.12)
where, \( M_{r,s}^{(j,k)}(t_1, t_2) \) is the \( j, k \)-th partial derivative of \( M_{r,s}^{(j,k)}(t_1, t_2) \).

Proof. In view of (3.1) and (3.10), we get
\[
M_{r,s}^{(j,k)}(t_1, t_2) = \left(1 + \frac{t_1}{p(r-1)}\right) M_{r-1,s}^{(j,k)}(t_1, t_2) + \frac{t_1}{ap(r-1)} M_{r-1,s}^{(j,k)}(t_1 + 1, t_2)
\]
(3.13)
Differentiating (3.13) \( j \) times w.r.t. \( t_1 \) and \( k \) times w.r.t. \( t_2 \), we get the required result.
CHAPTER IV

MOMENT GENERATING FUNCTIONS OF LOWER GENERALIZED ORDER STATISTICS FROM EXTENDED TYPE-I GENERALIZED LOGISTIC DISTRIBUTION

1. INTRODUCTION

The use of concept of gos has been steadily growing along the years. This is due to the fact that such concept describes random variables arranged in ascending order of magnitude and includes important well known concept that have been separately treated in statistical literature.

To enable a common approach to descending ordered random variables like reverse order statistics and lower record values, Pawlas and Szynal (2001a) introduced the concept of lower gos which was further extensively studied by Burkschat et al. (2003) with the name dual order statistics (dgos).

Relations for marginal and joint moment generating functions of record values and gos for some specific distributions are investigated by several authors in literature. For more detailed survey one may refer to Ahsanullah and Raqab (1999), Raqab and Ahsanullah (2000, 2003), Saran and Pandey (2003), Al-Hussaini et al. (2005, 2007), Khan et al. (2010), Maswadah and Faheem (2012), Nayabuddin and Athar (2014), Athar and Nayabuddin (2014) and references therein.

A random variable $X$ is said to have extended type-I generalized logistic distribution (Olapade, 2004) if its pdf is of the form

$$f(x) = \frac{a p e^{-x}}{(1 + a e^{-x})^{p+1}}, \quad -\infty < x < \infty, \quad a, p > 0,$$

and the corresponding df is given as

$$F(x) = \frac{1}{(1 + a e^{-x})^p}$$
At \( p = a = 1 \), we have the ordinary logistic distribution and when \( a = 1 \), we have the type-I generalized logistic distribution (Balakrishnan and Leung, 1988).

In this chapter we have established some recurrence relations for marginal and joint moment generating function of lower generalized order statistics from extended type-I generalized logistic distribution. Further, results are deduced for order statistics and lower record values.

Note that in view of (1.1) and (1.2)

\[
F(x) = \frac{(a + e^x)}{ap} f(x). \tag{1.3}
\]

Using the relation (1.3), we shall derive the recurrence relations for marginal and joint moment generating function of the extended type-I generalized logistic distribution.

Let us denote the mgf of \( X'(r,n,m,k) \) by \( M_{X'(r,n,m,k)}(t) \) as below

\[
M_{X'(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} F(x)^{r-1} f(x) g^{r-1}_m(F(x)) \, dx \tag{1.4}
\]

and the joint mgf of \( X'(r,n,m,k) \) and \( X'(s,n,m,k), 1 \leq r < s \leq n \) by \( M_{X'(r,s,n,m,k)}(t_1,t_2) \)

\[
M_{X'(r,s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)! (s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} [F(x)]^m f(x)
\]

\[
\times g^{r-1}_m [F(x) [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1}] \, dy \, dx \tag{1.5}
\]

2. MARGINAL MOMENT GENERATING FUNCTION

Before coming to the main result, first we shall show the existence of \( M_{X'(r,n,m,k)}(t) \).
Lemma 2.1. For the distribution as given in (1.1) and \(1 \leq r \leq n\),

\[ M_{X'(r, n, m, k)}(t) \] exists.

Proof. Expanding \( g_{m}^{b}[F(x)] = \left[ \frac{1}{m+1}(1 -(F(x))^{m+1}) \right]^{b} \) binomially in (1.4)

and then using (1.1) and (1.2), we get

\[
M_{X'(r, n, m, k)}(t) = \frac{C_{r-1}}{(r-1)!} \frac{p^{\gamma_{r}+(m+1)i}}{(m+1)^{r-1}} \times \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} -p(\gamma_{r}+(m+1)i-1) \\
\times \left(1 - \text{Pr}(\text{X'}(r, n, m, k) = t)\right) \times \int_{-\infty}^{\infty} e^{-x(1-t)} (1 + ae^{-x})^{-p(\gamma_{r}+(m+1)i-1)} dx . \tag{2.1}
\]

Since,

\[
\int_{0}^{\infty} x^{\alpha-1} \frac{e^{-\alpha x}}{(x+z)^{\beta}} dx = Z^{\alpha-\beta} B(\alpha, \beta - \alpha) \quad \text{[Prudnikov et al. (1986)]}.
\]

Therefore,

\[
M_{X'(r, n, m, k)}(t) = a^{r-1} p \frac{C_{r-1}}{(r-1)!} \frac{(m+1)^{r-1}}{\sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i}} \times B\left((1-t), p(\gamma_{r}+(m+1)i) + t\right) , \quad t < 1.
\]

Hence the lemma.

Lemma 2.2. For any arbitrary distribution and \(2 \leq r \leq n, n \geq 2\) and \(k = 1, 2 \cdots\)

(i) \( M_{X'(r, n, m, k)}(t) - M_{X'(r-1, n, m, k)}(t) = -\frac{C_{r-2}}{(r-1)!} \int_{-\infty}^{\infty} e^{\alpha x} [F(x)]^{\gamma_{r}} g_{m}^{-1}[F(x)] dx . \tag{2.2} \)

(ii) \( M_{X'(r-1, n, m, k)}(t) - M_{X'(r-1, n-1, m, k)}(t) = \frac{(m+1)C_{r-2}}{\gamma_{1}(r-2)!} \times \int_{-\infty}^{\infty} e^{\alpha x} [F(x)]^{\gamma_{r}} g_{m}^{-1}[F(x)] dx . \tag{2.3} \)
(iii) \[ M_{X'(r,n,m,k)}(t) - M_{X'(r-1,n-1,m,k)}(t) = -\frac{C_{r-1}}{\gamma_1(r-1)!} \times t \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} x^r g_m^{-1}(F(x)) dx. \] (2.4)

Proof. We have by Athar et al. (2008),

\[ E[X'(r,n,m,k)] - E[X'(r-1,n,m,k)] \]

\[ = -\frac{C_{r-2}}{(r-1)!} \int_{\alpha}^{\beta} \xi'(x) [F(x)]^{r-1} g_m^{-1}(F(x)) dx. \] (2.5)

where, \( \xi(x) \) is a Borel measurable function of \( x \) and \( x \in (\alpha, \beta) \).

Let \( \xi(x) = e^{tx} \), then (2.2) can be proved in view of (2.5). Relations (2.3) and (2.4) can be established on the similar lines of (2.2).

**Theorem 2.1** For the extended type-I generalized logistic distribution as given in (1.1) and for \( n \in \mathbb{N} \), \( m \in \mathbb{R} \), \( 2 \leq r \leq n \), \( n \geq 2 \), \( k = 1, 2, \ldots \)

\[ M^{(j)}_{X'(r,n,m,k)}(t) = \left( \frac{p \gamma_r}{p \gamma_r + t} \right) M^{(j)}_{X'(r-1,n,m,k)}(t) - \left( \frac{t}{a(p \gamma_r + t)} \right) M^{(j)}_{X'(r,n,m,k)}(t+1) \]

\[ - \frac{j}{p \gamma_r + t} \left( M^{(j-1)}_{X'(r,n,m,k)}(t) + \frac{1}{a} M^{(j-1)}_{X'(r,n,m,k)}(t+1) \right). \] (2.6)

where,

\( M^{(j)}_{X'(r,n,m,k)}(t) \) is the \( j \) -th derivative of \( M_{X'(r,n,m,k)}(t) \).

Proof. In view of equations (1.3) and (2.2), we have

\[ M_{X'(r,n,m,k)}(t) - M_{X'(r-1,n,m,k)}(t) \]

\[ = -\frac{C_{r-1}}{\gamma_r(r-1)!} \int_{-\infty}^{\infty} e^{tx} (F(x))^{r-1} \left\{ \frac{(a + e^x)}{ap} f(x) \right\} g_m^{-1}(F(x)) dx. \]
\[
= -\frac{t}{p} \frac{C_{r-1}}{\gamma_r (r-1)!} \left[ \int_{-\infty}^{\infty} e^{tx} (F(x))^{r_{r-1}} g_{m-1}^{(r-1)} (F(x)) f(x) \, dx \\
+ \frac{1}{a} \int_{-\infty}^{\infty} e^{(t+1)x} (F(x))^{r_{r-1}} g_{m-1}^{(r-1)} (F(x)) f(x) \, dx \right]
\]

\[
= -\frac{t}{p \gamma_r} \left[ M_{X'(r,n,m,k)}(t) + \frac{1}{a} M_{X'(r,n,m,k)}(t+1) \right].
\]

After rearranging the terms, we get

\[
M_{X'(r,n,m,k)}(t) = \left( \frac{p \gamma_r}{p \gamma_r + t} \right) M_{X'(r-1,n,m,k)}(t)
\]

\[
- \left( \frac{t}{a(p \gamma_r + t)} \right) M_{X'(r,n,m,k)}(t+1).
\] (2.7)

Differentiating (2.7) \( j \) times w.r.t. \( t \), we get

\[
M_{X'(r,n,m,k)}^{(j)}(t) - M_{X'(r-1,n,m,k)}^{(j)}(t)
= -\frac{t}{p \gamma_r} \left[ M_{X'(r,n,m,k)}^{(j)}(t) + \frac{1}{a} M_{X'(r,n,m,k)}^{(j)}(t+1) \right]
\]

\[
- \frac{j}{p \gamma_r} \left[ M_{X'(r,n,m,k)}^{(j-1)}(t) + \frac{1}{a} M_{X'(r,n,m,k)}^{(j-1)}(t+1) \right]
\]

\[
M_{X'(r,n,m,k)}^{(j)}(t) \left( 1 + \frac{t}{p \gamma_r} \right)
= \left[ M_{X'(r-1,n,m,k)}^{(j)}(t) - \frac{t}{ap \gamma_r} M_{X'(r,n,m,k)}^{(j)}(t+1) \right]
\]

\[
- \frac{j}{p \gamma_r} \left[ M_{X'(r,n,m,k)}^{(j-1)}(t) + \frac{1}{a} M_{X'(r,n,m,k)}^{(j-1)}(t+1) \right].
\]

Hence the result.
Remark 2.2. At \( m = 0, k = 1 \) in (2.6), we get the result for the order statistics

\[
M_{X_{n-r+1,n}}^{(j)}(t) = \left( \frac{p(n-r+1)}{p(n-r+1)+t} \right) M_{X_{n-r+2,n}}^{(j)}(t)
\]

\[
- \left( \frac{t}{a(p(n-r+1)+t)} \right) M_{X_{n-r+1,n}}^{(j)}(t+1)
\]

\[
- \frac{j}{t} \left( M_{X_{n-r+1,n}}^{(j-1)}(t) + \frac{1}{a} M_{X_{n-r+1,n}}^{(j-1)}(t+1) \right)
\]

Replacing \( n-r+1 \) by \( r-1 \) in (2.8), we get

\[
M_{X_{r,n}}^{(j)}(t) = \left( \frac{1}{r-1} \right) M_{X_{r-1,n}}^{(j)}(t) + \frac{j}{r-1} M_{X_{r-1,n}}^{(j)}(t+1)
\]

as obtained in Chapter III.

Remark 2.3. Setting \( m = -1 \) in (2.6), we get the recurrence relation for marginal moment generating function of the \( k-th \) lower record values from extended type-1 generalized logistic distribution

\[
M_{X_{L_k(r)}}^{(j)}(t) = \left( \frac{p k}{p k+t} \right) M_{X_{L_k(r-1)}}^{(j)}(t) - \left( \frac{t}{a(p k+t)} \right) M_{X_{L_k(r)}}^{(j)}(t+1)
\]

\[
- \frac{j}{p k+t} \left( M_{X_{L_k(r)}}^{(j-1)}(t) + \frac{1}{a} M_{X_{L_k(r)}}^{(j-1)}(t+1) \right)
\]

Theorem 2.2 For the extended type-I generalized logistic distribution as given in (1.1) and the condition as stated in Theorem 2.1,

\[ (i) \quad M_{X(r,n,m,k)}^{(j)}(t) = \left( \frac{P y_1}{P y_1+t} \right) M_{X(r-1,n-1,m,k)}^{(j)}(t) - \left( \frac{t}{a(P y_1+t)} \right) M_{X(r,n,m,k)}^{(j)}(t+1) \]

\[
- \frac{j}{P y_1+t} \left( M_{X(r,n,m,k)}^{(j-1)}(t) + \frac{1}{a} M_{X(r,n,m,k)}^{(j-1)}(t+1) \right). \]
(ii) \( M_{X'_{(r-1,n,m,k)}}^{(j)}(t) = M_{X'_{(r-1,n-1,m,k)}}^{(j)}(t) \)

\[
+ \left( \frac{\gamma_1 - \gamma_r}{p \gamma_1 \gamma_r} \right) \left( t M_{X'_{(r,n,m,k)}}^{(j)}(t) + j M_{X'_{(r,n,m,k)}}^{(j)}(t) \right)
\]

\[
+ \left( \frac{\gamma_1 - \gamma_r}{ap \gamma_1 \gamma_r} \right) \left( t M_{X'_{(r,n,m,k)}}^{(j-1)}(t+1) + j M_{X'_{(r,n,m,k)}}^{(j-1)}(t+1) \right) .
\] (2.11)

**Proof.** In view of equations (1.3) and (2.3), we have

\[
M_{X'_{(r,n,m,k)}}(t) - M_{X'_{(r-1,n-1,m,k)}}(t)
\]

\[
= - \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_{-\infty}^{\infty} e^{t \cdot x} (F(x))^{\gamma_1-1} \left\{ \frac{a + e^x}{ap} f(x) \right\} g_{m-1}^{-1} (F(x)) \, dx
\]

\[
= - \frac{t}{p \gamma_1} \left[ \int_{-\infty}^{\infty} e^{t \cdot x} (F(x))^{\gamma_1-1} g_{m-1}^{-1} (F(x)) f(x) \, dx \right]
\]

\[
+ \frac{1}{a} \int_{-\infty}^{\infty} e^{t \cdot x} (F(x))^{\gamma_1-1} g_{m}^{-1} (F(x)) f(x) \, dx
\]

\[
= - \frac{t}{p \gamma_1} \left[ M_{X'_{(r,n,m,k)}}(t) + \frac{1}{a} M_{X'_{(r,n,m,k)}}(t+1) \right].
\]

\[
M_{X'_{(r,n,m,k)}}(t) = \left( \frac{p \gamma_1}{p \gamma_1 + t} \right) M_{X'_{(r-1,n-1,m,k)}}(t)
\]

\[
- \left( \frac{t}{a(p \gamma_1 + t)} \right) M_{X'_{(r,n,m,k)}}(t+1).
\] (2.12)

Differentiating (2.12) on both sides \( j \) times w.r.t. \( t \), we obtain

\[
M_{X'_{(r,n,m,k)}}^{(j)}(t) - M_{X'_{(r-1,n-1,m,k)}}^{(j)}(t)
\]

\[
= - \frac{t}{p \gamma_1} \left[ M_{X'_{(r,n,m,k)}}^{(j)}(t) + \frac{1}{a} M_{X'_{(r-1,n,m,k)}}^{(j)}(t+1) \right]
\]

\[
- \frac{j}{p \gamma_1} \left[ M_{X'_{(r,n,m,k)}}^{(j-1)}(t) + \frac{1}{a} M_{X'_{(r,n,m,k)}}^{(j-1)}(t+1) \right]
\]
\[
\left( \frac{p \gamma_1 + t}{p \gamma_1} \right) M_{X'(r,n,m,k)}^{(j)}(t) = M_{X'(r-1,n-1,m,k)}^{(j-1)}(t) - \frac{t}{a \gamma_1} M_{X'(r,n,m,k)}^{(j)}(t+1)
\]

\[
- \frac{j}{p \gamma_1} \left[ M_{X'(r,n,m,k)}^{(j-1)}(t) + \frac{1}{a} M_{X'(r,n,m,k)}^{(j-1)}(t+1) \right].
\]

Hence the result.

Similarly (2.11) can be proved on the same lines in view of (2.4).

3. JOINT MOMENT GENERATING FUNCTION

Lemma 3.1 For \(1 \leq r < s \leq n-1, n \geq 2\) and \(k = 1,2, \ldots\)

\[
M_{X'(r,s,n,m,k)}(t_1,t_2) - M_{X'(r,s-1,n,m,k)}(t_1,t_2)
= -\frac{t_2}{\gamma_s} \frac{C_{s-1}}{(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} [F(x)]^m f(x) g_{m-1}^r(F(x))

\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{r+1} \, dy \, dx,
\quad -\infty < y < \infty,
\]

(3.1)

Proof. Lemma can be proved by letting \(\xi(x,y) = e^{t_1 x + t_2 y}\) in 1.5.13.

Theorem 3.1 For the distribution given in (1.1) and \(n \in \mathbb{N}, m \in \mathbb{R}, \)
1 \(\leq r < s - 1 \leq n, \quad k \geq 1, \quad\)

\[
M_{X'(r,s,n,m,k)}^{(i,j+1)}(t_1,t_2) = \left( \frac{p \gamma_s}{p \gamma_s + t_2} \right) M_{X'(r,s-1,n,m,k)}^{(i,j+1)}(t_1,t_2)

- \frac{t_2}{a(p \gamma_s + t_2)} M_{X'(r,s,n,m,k)}^{(i,j+1)}(t_1,t_2 + 1)

- \frac{j+1}{p \gamma_s + t_2} \left[ M_{X'(r,s,n,m,k)}^{(i,j)}(t_1,t_2) + \frac{1}{a} M_{X'(r,s,n,m,k)}^{(i,j)}(t_1,t_2 + 1) \right]
\]

(3.2)

where,

\(M_{X'(r,s,n,m,k)}^{(i,j)}(t_1,t_2)\) is the \(i,j\)-th derivative of \(M_{X'(r,s,n,m,k)}(t_1,t_2)\).
Proof. Using relations (1.3) in (3.1), we get

\[
M_{X'(r,s,n,m,k)}(t_1, t_2) - M_{X'(r,s-1,n,m,k)}(t_1, t_2)
\]

\[
= \frac{t_2}{ap \gamma_s (r-1)(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tx+t_2y} [F(x)]^m f(x) g_m^{r-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1} (a + e^y) f(y) dy \, dx
\]

(3.3)

\[
= \frac{t_2}{p \gamma_s (r-1)(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tx+(1+t_2)y} [F(x)]^m f(x) g_m^{r-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1} f(y) dy \, dx
\]

\[
= \frac{-t_2}{ap \gamma_s (r-1)(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tx+(1+t_2)yz} [F(x)]^m f(x) g_m^{r-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{r-1} f(y) dy \, dx
\]

\[
M_{X'(r,s,n,m,k)}(t_1, t_2) - M_{X'(r,s-1,n,m,k)}(t_1, t_2)
\]

\[
= -\left( \frac{t_2}{p \gamma_s} \right) \left( M_{X'(r,s,n,m,k)}(t_1, t_2) + \frac{1}{a} M_{X'(r,s,n,m,k)}(t_1, t_2 + 1) \right). \tag{3.4}
\]

Differentiating (3.4) \(i\) times w.r.t. \(t_1\) and \(j+1\) times w.r.t. \(t_2\), we get

\[
M_{X'(r,s,n,m,k)}^{(i,j+1)}(t_1, t_2) - M_{X'(r,s-1,n,m,k)}^{(i,j+1)}(t_1, t_2)
\]

\[
= -\left( \frac{t_2}{p \gamma_s} \right) \left( M_{X'(r,s,n,m,k)}^{(i,j+1)}(t_1, t_2) + \frac{1}{a} M_{X'(r,s,n,m,k)}^{(i,j+1)}(t_1, t_2 + 1) \right)
\]

\[
- \left( \frac{j+1}{p \gamma_s} \right) \left( M_{X'(r,s,n,m,k)}^{(i,j)}(t_1, t_2) + \frac{1}{a} M_{X'(r,s,n,m,k)}^{(i,j)}(t_1, t_2 + 1) \right)
\]
or,

\[
\left( \frac{p \gamma_1 + t_2}{p \gamma_1} \right) M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2) = M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2)
\]

\[
- \left( \frac{t_2}{ap \gamma_1} \right) \left( M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2 + 1) \right)
\]

\[
- \left( \frac{j + 1}{p \gamma_1} \right) \left( M_{X^{(i,j)}}^{(i,j)}(t_1, t_2) + \frac{1}{a} M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2 + 1) \right).
\]

Hence the required result.

**Remark 3.1.** Putting \( m = 0, k = 1 \) in Theorem 3.1, we get the recurrence relations for joint moment generating function of order statistics

\[
M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2) = \left( 1 + \frac{t_2}{p(n-s+1)} \right) M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2)
\]

\[
+ \left( \frac{j + 1}{p(n-s+1)} \right) M_{X^{(i,j+1)}}^{(i,j+1)}(t_1, t_2 + 1)
\]

Replacing \((n-r+1)\) by \(s\) and \((n-s+1)\) by \((r-1)\), we get the result as obtained in Chapter III.

**Remark 3.2.** Substituting \( m = -1 \) in Theorem 3.1, we get the result for \(k\)-th lower record values from extended type I generalized logistic distribution in the form
\[
M^{(i,j+1)}_{X_{L(r)}^{(k)} X_{L(s)}^{(k)}} = \left( \frac{pk}{pk + t_2} \right) M^{(i,j+1)}_{X_{L(r)}^{(k)} X_{L(s)}^{(k)}}(t_1, t_2) \\
- \left( \frac{t_2}{a(pk + t_2)} \right) M^{(i,j+1)}_{X_{L(r)}^{(k)} X_{L(s)}^{(k)}}(t_1, t_2 + 1) \\
- \frac{j + 1}{pk + t_2} \left[ M^{(i,j)}_{X_{L(r)}^{(k)} X_{L(s)}^{(k)}}(t_1, t_2) + \frac{1}{a} M^{(i,j)}_{X_{L(r)}^{(k)} X_{L(s)}^{(k)}}(t_1, t_2 + 1) \right].
\]

**Remark 3.3.** Theorem 2.1 can be deduced from Theorem 3.1 by letting \( t_2 \) tends to zero.
CHAPTER V

GENERALIZED ORDER STATISTICS FROM DOUBLY TRUNCATED MAKEHAM DISTRIBUTION

1. INTRODUCTION

Makeham distribution is an important life distribution and has been widely used to fit actuarial data (Marshall and Olkin, 2007). This distribution is used to model adult lifetimes by actuaries. For a description on the genesis and applications of Makeham distribution one may refer to Makeham (1860). Aboutahoun and Al-Otaibi (2009) have derived some recurrence relations for the moments of order statistics arising from doubly truncated Makeham distribution. Pandit and Math (2009) have dealt with some inference problems of Makeham distribution. In this chapter, we have obtained some recurrence relations based on single and product moments of generalized order statistics. A characterizing result based on conditional expectation of gos is also presented.

A random variable \( X \) is said to have Makeham distribution, if the pdf is of the form.

\[
f_1(x) = [1 + \theta(1 - e^{-x})]e^{-x - \theta(x + e^{-x} - 1)}, \quad x \geq 0, \quad \theta \geq 0
\]  

(1.1)

with the corresponding df

\[
F_1(x) = 1 - e^{-x - \theta(x + e^{-x} - 1)}, \quad x \geq 0, \quad \theta \geq 0.
\]  

(1.2)

Now, if for the given \( P_1 \) and \( Q_1 \)

\[
\int_0^{Q_1} f_1(x)dx = Q_1 \quad \text{and} \quad \int_0^{P_1} f_1(x)dx = P_1
\]

Then, the truncated pdf is given by

\[
f(x) = \frac{1}{P - Q_1}[1 + \theta(1 - e^{-x})]e^{-x - \theta(x + e^{-x} - 1)}, \quad Q_1 \leq x \leq P_1
\]  

(1.3)
and the corresponding truncated survival function

\[ \bar{F}(x) = \frac{f(x)}{1 + \theta(1 - e^{-\lambda x})} - P_2, \]

where,

\[ 1 - P = e^{-R - \theta(R + e^{-\lambda x})} \]
\[ 1 - Q = e^{-Q - \theta(Q + e^{-\lambda x})} \]
\[ Q_2 = \frac{1 - Q}{P - Q} \quad \text{and} \quad P_2 = \frac{1 - P}{P - Q}. \]

2. SINGLE MOMENTS

**Lemma 2.1.** For \( 2 \leq r \leq n, \ n \geq 2 \) and \( k = 1, 2, \ldots \)

(i) \( E[X^r (r, n, m, k)] - E[X^r (r - 1, n, m, k)] \)

\[ = \frac{C_{r-2}}{(r - 1)!} \int_{Q_1}^{R} x^{j-1} [\bar{F}(x)]^r g_{m-1}^r (F(x)) \, dx. \quad (2.1) \]

(ii) \( E[X^r (r - 1, n, m, k)] - E[X^r (r - 1, n - 1, m, k)] \)

\[ = -\frac{(m + 1)C_{r-2}}{\gamma_1 (r - 2)!} \int_{Q_1}^{R} x^{j-1} [\bar{F}(x)]^r g_{m-1}^r (F(x)) \, dx. \quad (2.2) \]

(iii) \( E[X^r (r, n, m, k)] - E[X^r (r - 1, n - 1, m, k)] \)

\[ = \frac{1}{\gamma_1 (r - 1)!} \int_{Q_1}^{R} x^{j-1} [\bar{F}(x)]^r g_{m-1}^r (F(x)) \, dx. \quad (2.3) \]

**Proof.** The relation (2.1) can be established by letting \( \xi(x) = x^j \) in the (1.4.10). Similarly, relations (2.2) and (2.3) can be seen in view of (1.4.11) and (1.4.12).
Theorem 2.1. For the given Makeham distribution and \( n \in \mathbb{N} \), \( m \in \mathbb{R} \), \( k > 1 \),
\[ 2 \leq r \leq n, \quad j = 1, 2, \ldots \]

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] = -P_2 K \left\{ E[X^j(r,n-1,m,k+m)] - E[X^j(r-1,n-1,m,k+m)] \right\} \\
+ \frac{j}{\gamma_1} E[\psi \{X(r,n,m,k)\}], \quad (2.4)
\]

where,
\[
K = \frac{C_{r-2}^{(n-1)}}{C_{r-2}^{(n-1,k+m)}} = \left( \prod_{i=1}^{r-1} \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1,k+m)}} \right), \quad \gamma_i^{(n-1)} = k + (n-1-i)(m+1)
\]

and \( \psi(x) = \frac{x^{j-1}}{1 + \theta(1 - e^{-x})} \).

Proof. In view of (1.4) and (2.3), we have

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] = \frac{C_{r-1}}{\gamma_1(r-1)!} \int_0^{r_1} x^{j-1} \left[ \bar{F}(x) \right]^{(n-1)} \left\{ -P_2 + \frac{f(x)}{1 + \theta(1 - e^{-x})} \right\} g_m^{r-1} (F(x)) \, dx \\
+ \frac{C_{r-1}}{\gamma_1(r-1)!} \int_0^{r_1} x^{j-1} \left[ \bar{F}(x) \right]^{(n-1,k+m)} g_m^{r-1} (F(x)) \, f(x) \, dx \\
= -P_2 \frac{C_{r-2}}{(r-1)!} \int_0^{r_1} x^{j-1} \left[ \bar{F}(x) \right]^{(n-1,k+m)} g_m^{r-1} (F(x)) \, f(x) \, dx \\
+ \frac{j C_{r-1}}{\gamma_1(r-1)!} \int_0^{r_1} \psi(x) \left[ \bar{F}(x) \right]^{(n-1,k+m)} g_m^{r-1} (F(x)) \, f(x) \, dx
\]
\[ \begin{align*}
&= -P_2 \frac{C_{r-2}^{(n-1)}}{(r-1)!} \int_0^1 x^{j-1} \left[ \hat{F}(x) \right]^{(n-1,k+m)} g_r^{r-1}(F(x)) \, dx \\
&\quad + \frac{j}{\gamma_1} E[\psi \{ X(r,n,m,k) \}] \\
as \quad \gamma_j - 1 = \gamma_{j(r-1,k+m)} = (k + m) + (n - 1 - r)(m + 1), \quad \gamma_j^{(n,k)} = \gamma_j
\end{align*} \]

and \( C_{r-1}^{(n)} = C_{r-1} = \gamma_1 C_{r-2}^{(n-1)} \).

Hence the required result.

**Remark 2.1:** At \( P = 1, Q = 0 \) we get the relation for non-truncated case

\[ E[X^j(r,n,m,k)] - E[X^j(r-1,n-1,m,k)] = -\frac{j}{\gamma_1} E[\psi \{ X(r,n,m,k) \}] \]

**Remark 2.2:** Recurrence relation for single moments of order statistics \((m = 0, k = 1)\) is

\[ E(X_{r,0}^j) - E(X_{r-1,0}^j) = -P_2 \left\{ E(X_{r,n}^j) - E(X_{r-1,n}^j) \right\} + \frac{j}{n} E[\psi(X_{r,n})] \]

or

\[ E(X_{r,n}^j) = -P_2 E(X_{r-1,n}^j) + Q_2 E(X_{r-1,n-1}^j) + \frac{j}{n} E[\psi(X_{r,n})] \]

At \( j = 1 \), result is obtained by Aboutahoun and Al-Otaibi (2009).

By convention, we use \( X_{n+1} = P_1 \) and \( X_{0n-1} = Q_1 \).

**Remark 2.3:** For \( k-th \) record statistics \((m = -1)\), recurrence relation for single moments reduces as

\[ E\left( X_{U(r)}^{(k)} \right)^j - E\left( X_{U(r-1)}^{(k)} \right)^j = -P_2 \left( \frac{k}{k-1} \right)^{j-1} \left\{ E\left( X_{U(r)}^{(k-1)} \right)^j - E\left( X_{U(r-1)}^{(k-1)} \right)^j \right\} \\
+ \frac{j}{k} E[\psi(X_{U(r)}^{(k)})] \]
Similarly, the recurrence relation for single moments of order statistics with non-integral sample size is for \( m = 0 \), and \( k = \alpha - n + 1 \), \( \alpha \in \mathbb{R}_+ \), and for sequential order statistics for \( m = \alpha - 1 \), \( k = \alpha \) may be obtained.

**Theorem 2.2.** For the given Makeham distribution and \( n \in \mathbb{N} \), \( m \in \mathbb{R} \), \( 2 \leq r \leq n \), \( j = 1, 2, \ldots \)

\[
E[X^j (r, n, m, k)] - E[X^j (r - 1, n - 1, m, k)]
= \frac{K^* (P - Q)}{\gamma_1} j E[\phi \{X (r, n, m, k + 1)\}]
\]
\[
= \frac{j}{\gamma_1} \left[ - (1 - P) E[\psi \{X (r, n, m, k)\}] + E[\phi \{X (r, n, m, k)\}] \right].
\]

where,

\[
\phi(x) = \frac{x^{j-1} e^{\{x+\theta(x+e^{-x})\}}}{1 + \theta(1 - e^{-x})},
\]

\[
K^* = \frac{C_{r-1}}{C_{r-1}^{(k+1)}} = \prod_{i=1}^{r} \left( \frac{\gamma_i}{\gamma_i + 1} \right)
\]
and \( C_{r-1}^{(k+1)} = \prod_{i=1}^{r} [(k + 1) + (n - i)(m + 1)] = \prod_{i=1}^{r} \gamma_i^{(k+1)} \).

**Proof.** On using (1.3) in (2.3), we get

\[
E[X^j (r, n, m, k)] - E[X^j (r - 1, n - 1, m, k)]
= \frac{C_{r-1}}{\gamma_1 (r-1)!} \int_0^1 x^{j-1} \left[ \overline{F} (x) \right]^{r-1} \left( \frac{(P - Q) e^{\{x+\theta(x+e^{-x})\}}}{1 + \theta(1 - e^{-x})} \right) f(x) g_m^{-1} (F(x)) dx
\]
\[
= \frac{(P - Q)}{\gamma_1} \frac{C_{r-1}}{C_{r-1}^{(k+1)}} j \left[ \frac{C_{r-1}^{(k+1)}}{(r-1)!} \left[ \phi (x) \right]^{(k+1)-1} F (x) \right] g_m^{-1} (F(x)) dx,
\]
and hence the Theorem.
To prove (2.6), note that

\[ \frac{\bar{F}(x)}{f(x)} = \frac{1}{1+\theta(1-e^{-x})} - (1-P)\frac{e^{\{x+\theta(x+e^{-x})\}}}{1+\theta(1-e^{-x})}. \]

Then, from (2.3) we have,

\[ E[X^r(n,m,k)] - E[X^{r-1}(n-1,m,k)] = \int_0^\infty \frac{1}{1+\theta(1-e^{-x})} \frac{e^{\{x+\theta(x+e^{-x})\}}}{1+\theta(1-e^{-x})} \, dx \]

\[ = \frac{1}{\gamma_1} \frac{C_{r-1}}{(r-1)!} \int_0^\infty \frac{x^{r-1}}{1+\theta(1-e^{-x})} \, dx \]

\[ = \frac{1}{\gamma_1} \frac{C_{r-1}}{(r-1)!} \int_0^\infty \frac{x^{r-1}e^{\{x+\theta(x+e^{-x})\}}}{1+\theta(1-e^{-x})} \, dx \]

\[ \times \left[ \frac{\bar{F}(x)^{r-1}}{1+\theta(1-e^{-x})} \right] g_{r-1}^m (F(x)) f(x) \, dx. \]

Hence the result.

Remark 2.4: At Q = 0 and P = 1, relation (2.6) reduces to

\[ E[X^r(n,m,k)] - E[X^{r-1}(n-1,m,k)] = -\frac{j}{\gamma_1} \int_{X(r,n,m,k)} \phi(X) \, dX. \]

Theorem 2.3. For the given Makeham distribution and n \in \mathbb{N}, m \in \mathbb{R},

\[ 2 \leq r \leq n, \ k > 1, \ j = 1,2,\ldots \]

\[ E[X^r(n,m,k)] - E[X^{r-1}(n-1,m,k)] \]

\[ = -P_2 K^{*2} E[X^r(n-1,m,k+m)] - E[X^r(n-1,n-1,m,k+m)] \]

\[ + \frac{j}{\gamma_r} E[\psi(X(n,m,k))] \quad (2.7) \]
where,

\[ K^{**} = \frac{C_{r-2}}{C_{r-2}^{n-1,k+m}} = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i^{(n-1)} + m} \right) = \prod_{i=1}^{r-1} \left( \frac{\gamma_i}{\gamma_i - 1} \right). \]

**Proof.** From (2.1), we have

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{C_{r-2}}{(r-1)!} \int_{0}^{1} x^j-1 \left( \bar{F}(x) \right)^{r-1} g_m^{r-1} (F(x)) \, dx.
\] (2.8)

Now in view of equation (1.4), we get

\[
= -P_2 \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} \times \frac{C_{r-2}^{(n-1,k+m)}}{(r-1)!} \int_{0}^{1} x^j-1 \left( \bar{F}(x) \right)^{r-1} g_m^{r-1} (F(x)) \, dx
\]

\[ + \frac{C_{r-1}}{\gamma_r} \int_{0}^{1} \left( \bar{F}(x) \right)^{r-1} f(x) g_m^{r-1} (F(x)) \, dx. \]

Hence the result (2.7) is proved on the lines of Theorem 2.1.

**3. PRODUCT MOMENTS**

**Theorem 3.1.** For the Makeham distribution as given in (1.1) and \( n \geq 2 \), \( m \in \mathbb{R} \), \( 1 \leq r < s \leq n - 1 \), \( k > 1 \), \( i, j = 1, 2, ... \)

\[
E[X^i(r,n,m,k).X^j(s,n,m,k)] = E[X^i(r,n,m,k).X^j(s-1,n,m,k)]
\]

\[
= -P_2 K_1 \left\{ E[X^i(r,n-1,m,k+m).X^j(s-1,n-1,m,k+m)]
\]

\[ + \frac{j}{\gamma_s} E[X(r,n,m,k).X(s,n,m,k)] \right\}
\] (3.1)

where \( K_1 = \frac{C_{s-2}^{(n)}}{C_{s-2}^{(n-1,k+m)}} = \prod_{i=1}^{s-1} \left( \frac{\gamma_i}{\gamma_i^{(n-1)} + m} \right) = \prod_{i=1}^{s-1} \left( \frac{\gamma_i}{\gamma_i - 1} \right) \)

and \( \psi(x,y) = \frac{x^i.y^{j-1}}{1 + \theta(1 - e^{-y})} \).
Proof. From (1.4.16), we have

\[ E[X^i (r,n,m,k) X^j (s,n,m,k)] - E[X^i (r,n,m,k) X^j (s-1,n,m,k)] \]

\[ = \frac{C_{s-2}}{(r-1)! (s-r-1)!} \int_{Q_r}^{P_r} x^i y^{j-1} \left[ \tilde{F}(x) \right]^m f(x) g_m^{-1} (F(x)) \]

\[ \times [h_m (F(y)) - h_m (F(x))]^{r-r-1} [\tilde{F}(y)]^{r_s-1} dy \, dx \]  
(3.2)

On using (1.4) in (3.2), we get

\[ = \frac{C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} \int_{Q_r}^{P_r} x^i y^{j-1} \left[ \tilde{F}(x) \right]^m g_m^{-1} (F(x)) f(x) \]

\[ \times [h_m (F(y)) - h_m (F(x))]^{r-r-1} [\tilde{F}(y)]^{r_s-1} dy \, dx \]

\[ + \frac{P_2 C_{s-1}}{\gamma_s (r-1)! (s-r-1)!} \int_{Q_r}^{P_r} \psi(x,y) \left[ \tilde{F}(x) \right]^m g_m^{-1} (F(x)) \]

\[ \times [h_m (F(y)) - h_m (F(x))]^{r-r-1} [\tilde{F}(y)]^{r_s-1} f(x) f(y) dy \, dx \, . \]

Hence (3.1) can be established after noting that

\[ \gamma_s - 1 = \gamma_s^{(n-1,k+m)}, \quad \gamma_s^{(n)} = \gamma_s, \]

\[ C_{s-1} = \gamma_s C_{s-2} . \]

Remark 3.1: Recurrence relation for product moments of order statistics

\[(m = 0, k = 1) \]

\[ E(X_{r,n}^i \cdot X_{s,n}^j) = E(X_{r,n}^i \cdot X_{s-1,n}^j) \]

\[ - P_2 \frac{n}{n-s+1} \left\{ E(X_{r,n-1}^i \cdot X_{s,n-1}^j) - E(X_{r,n-1}^i \cdot X_{s-1,n-1}^j) \right\} \]

\[ + \frac{j}{n-s+1} E[\psi(X_{r:n}, X_{s:n})] . \]
At \( i = j = 1 \), result is obtained by Aboutahoun and Al-Otaibi (2009).

**Remark 3.2:** For \( k \)-th record statistics \((m = -1)\), recurrence relation for product moments reduces as

\[
E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j] = E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j]\]

\[
= -P_2 \left( \frac{k}{k-1} \right)^{s-1} \left\{ E[(X_{U(r)}^{(k-1)})^i(X_{U(s)}^{(k-1)})^j] - E[(X_{U(r)}^{(k-1)})^i(X_{U(s)}^{(k-1)})^j] \right\}
\]

\[
+ \frac{j}{k} E \left\{ \psi[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j] \right\}.
\]

**Remark 3.3:** At \( i = 0 \), we obtain recurrence relations for single moments.

### 4. CHARACTERIZATION

**Theorem 4.1.** Let \( X \) be an absolutely continuous random variable with distribution function \( F(x) \) and probability density function \( f(x) \) with \( F(x) < 1 \), for all \( x \in (0, \infty) \). Then for two consecutive values \( r \) and \( r - 1 \), \( 2 \leq r + 1 \leq s \leq n \),

\[
E \left[ X(s, n, m, k) + \frac{\theta}{\theta + 1} e^{-X(s,n,m,k)} \middle| X(l, n, m, k) = x \right] = g_{s|r}(x)
\]

\[
= \left( x + \frac{\theta}{\theta + 1} e^{-x} \right) + \frac{1}{\theta + 1} \sum_{j=r+1}^{s} \frac{1}{j} j_{j}, \quad l = r, r + 1
\]

(4.1)

if and only if

\[
F(x) = 1 - e^{-x - \theta(x + e^{x-1})}, \quad x \geq 0, \quad \theta \geq 0.
\]

(4.2)

**Proof.** We have for \( s \geq r + 1 \),

\[
g_{s|r}(x) = E \left[ X(s, n, m, k) + \frac{\theta}{\theta + 1} e^{-X(s,n,m,k)} \middle| X(r, n, m, k) = x \right]
\]
Let

$$u = \frac{\bar{F}(y)}{F(x)} = e^{-((y-x)(1+\theta)+\theta(e^{-y}-e^{-x}))},$$

then

$$y + \frac{\theta}{\theta + 1} e^{-y} = x + \frac{\theta}{\theta + 1} e^{-x} - \frac{1}{\theta + 1} \log u.$$

Thus (4.3) becomes

$$E \left[ X(s,n,m,k) + \frac{\theta}{\theta + 1} e^{-X(s,n,m,k)} \mid X(r,n,m,k) = x \right] = \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_0^1 \left( x + \frac{\theta}{\theta + 1} e^{-x} - \frac{\log u}{\theta + 1} \right) u^{\gamma_{s-1}} [1-u^{m+1}]^{s-r-1} du.$$

Set $$u^{m+1} = t$$, to get

$$E \left[ X(s,n,m,k) + \frac{\theta}{\theta + 1} e^{-X(s,n,m,k)} \mid X(r,n,m,k) = x \right] = \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r-1}} \int_0^1 \left( x + \frac{\theta}{\theta + 1} e^{-x} \right)^{\gamma_{s-1}} t^{m+1} [1-t]^{s-r-1} dt$$

$$- \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r+1}(\theta + 1)} \int_0^1 \log t \ t^{m+1} [1-t]^{s-r-1} dt.$$
\begin{align*}
&= \left( x + \frac{\theta}{\theta + 1} e^{-x} \right) \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r}} \frac{1}{\beta\left(\frac{\gamma_s}{m+1}, s-r \right)} \\
&= \frac{C_{s-1}}{C_{r-1}(s-r-1)! (m+1)^{s-r+1}(\theta + 1)} \times \beta\left(\frac{\gamma_s}{m+1}, s-r \right) \left[ \psi\left(\frac{\gamma_s}{m+1}\right) - \psi\left(\frac{\gamma_r}{m+1}\right) \right],
\end{align*}

where \( \psi(x) = \frac{d}{dx} \ln \Gamma(x) \) [Gradshteyn and Ryzhik (2007), pp. 540].

Since \( \psi(x-n) - \psi(x) = -\sum_{k=1}^{n} \frac{1}{x-k} \) [Gradshteyn and Ryzhik (2007), pp. 905],

Thus, we have

\begin{align*}
\prod_{j=r+1}^{s} \gamma_j &= \frac{1}{(s-r-1)! (m+1)^{s-r}} \frac{\Gamma\left(\frac{k+(n-s)(m+1)}{m+1}\right)}{\Gamma\left(\frac{k+(n-s)(m+1)+(m+1)(s-r)}{m+1}\right)} \\
&\times \left( x + \frac{\theta}{\theta + 1} e^{-x} + \frac{1}{(m+1)(\theta + 1)} \sum_{j=1}^{s-r} \frac{1}{k+(n-r)} \right) \\
= \frac{\prod_{j=r+1}^{s} \gamma_j}{(m+1)^{s-r}} \left( \frac{k+(n-s-1)(m+1)}{m+1} \right)^{s-r} \left[ x + \frac{\theta}{\theta + 1} e^{-x} + \frac{1}{\theta + 1} \sum_{j=1}^{s-r} \frac{1}{\gamma_r + j} \right] \\
= \left( x + \frac{\theta}{\theta + 1} e^{-x} \right) + \frac{1}{\theta + 1} \sum_{j=r+1}^{s} \frac{1}{\gamma_j}.
\end{align*}

To show that (4.1) implies (4.2), we have
\[ g_{slr+1}(x) - g_{slr}(x) = \frac{1}{(\theta + 1)\gamma_{r+1}} \]

and

\[ g'_{slr}(x) = 1 - \frac{\theta}{(\theta + 1)} e^{-x}. \]

Therefore,

\[ -\frac{1}{\gamma_{r+1}} \frac{g'_{slr}(x)}{g_{slr+1}(x) - g_{slr}(x)} = [1 + \theta(1 - e^{-x})], \quad \text{[Khan et al., 2006]} \]

and hence

\[ \frac{f(x)}{F(x)} = [1 + \theta(1 - e^{-x})]. \]

Implying that

\[ F(x) = e^{-x - \theta(x + e^{-x} - 1)}, \quad x \geq 0, \quad \theta \geq 0. \]
CHAPTER VI

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH GENERALIZED ORDER STATISTICS

1. INTRODUCTION


In this chapter, characterizations of general form of continuous probability distributions through conditional expectation of function of generalized order statistics are presented. Here an attempt is made to characterize many well known continuous probability distributions through

\[ E[g\{X(s,n,m,k)\} | X(r,n,m,k) = x]; \ 1 \leq r < s \leq n \]
and

\[ E[g(X(r,n,m,k)) \mid X(s,n,m,k) = y] \mid 1 \leq r < s \leq n \]

where \( g(x) = e^{h(x)} \).

Further, some deductions for order statistics and records are also discussed.

2. CHARACTERIZATION OF DISTRIBUTIONS

**Theorem 2.1.** Let \( X \) be an absolutely continuous random variable (rv) with df \( F(x) \) and pdf \( f(x) \) over the support \((\alpha, \beta)\), and \( g(x) = e^{h(x)} \) be a monotonic and differentiable function of \( x \), then for two values \( r \) and \( s \), \( 1 \leq r < s \leq n \),

\[ E[g(X(s,n,m,k)) \mid X(r,n,m,k) = x] = g_{s|r} (x) \]

\[ = e^{-b/a} \sum_{i=0}^{\infty} \frac{[a \cdot h(x) + b]^i}{a^i i!} \prod_{j=r+1}^{s} \left( \frac{c \gamma_j}{c \gamma_j + i} \right); a \neq 0, c \gamma_j + i \neq 0 \quad (2.1) \]

if and only if

\[ \overline{F}(x) = [ah(x) + b]^c. \quad (2.2) \]

where \( a, b, \) and \( c \) are so chosen that \( F(x) \) be a df.

**Proof.** To prove necessary part, for \( s \geq r + 1 \),

\[ E[g(X(s,n,m,k)) \mid X(r,n,m,k) = x] \]

\[ = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_{0}^{\beta} e^{h(y)} \left[ 1 - \left( \frac{\overline{F}(y)}{\overline{F}(x)} \right)^{m+1} \right]^{s-r-1} \]

\[ \times \left[ \frac{\overline{F}(y)}{\overline{F}(x)} \right]^{\gamma_y - 1} \frac{f(y)}{\overline{F}(x)} dy. \quad (2.3) \]

Set \( u = \frac{\overline{F}(y)}{\overline{F}(x)} = \left[ \frac{ah(y) + b}{ah(x) + b} \right]^c \).
which implies
\[ h(y) = \frac{1}{a} u^{1/c} \{ ah(x) + b \} - b \].

Then the RHS of (2.3) reduces to
\[ \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \int_0^{u^{m+1}} e^a \frac{u^{1/c} \{ ah(x) + b \} - b}{(1-u^{m+1})^{s-r-1}} u^{y-1} dt. \]

Let \( u^{m+1} = t \), then we get
\[ E[g\{X(s,n,m,k)\} | X(r,n,m,k) = x] \]
\[ = \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \int_0^{\gamma} e^{b/a} \frac{1}{c} \int_0^{u^{m+1}} t^{c(m+1)-1} (1-t)^{y-1} dt \]
\[ = \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)! (m+1)^{s-r}} \sum_{i=0}^{\infty} \frac{[ah(x) + b]^i}{a^i!} \Gamma \left( \frac{c \gamma_j + i}{c(m+1)}, s-r \right) \]
\[ = \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)! (m+1)^{s-r}} \sum_{i=0}^{\infty} \frac{[ah(x) + b]^i}{a^i!} \frac{\Gamma \left( \frac{c \gamma_j + i}{c(m+1)} \right)}{\Gamma \left( \frac{c \gamma_j + i}{c(m+1)} + (s-r) \right)} \]

which after simplification, yields
\[ E[g\{X(s,n,m,k)\} | X(r,n,m,k) = x] \]
\[ = e^{-b/a} \sum_{i=0}^{\infty} \frac{[ah(x) + b]^i}{a^i!} \prod_{j=r+1}^s \left( \frac{c \gamma_j}{c \gamma_j + i} \right) \]
hence the ‘if’ part.
To prove sufficiency part, we have

\[ g_{s|r}(x) = e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+1}^{s} \frac{c \gamma_j}{c \gamma_j + i}. \]

Therefore,

\[ g_{s|r+1}(x) = e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+2}^{s} \frac{c \gamma_j}{c \gamma_j + i}. \]

Then in view of Khan et al. (2006)

\[ \frac{f(x)}{F(x)} = \frac{g_{s|r}(x)}{\gamma_{r+1} [g_{s|r+1}(x) - g_{s|r}(x)]} \]

\[ = -\frac{a h'(x) e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+1}^{s} \frac{c \gamma_j}{c \gamma_j + i}}{\gamma_{r+1} \left[ e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+2}^{s} \frac{c \gamma_j}{c \gamma_j + i} - e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+1}^{s} \frac{c \gamma_j}{c \gamma_j + i} \right]} \]

\[ = -\frac{a c h'(x)}{(ah(x) + b)^c} \]

implying that

\[ \overline{F}(x) = [ah(x) + b]^c. \]

Hence (2.2).

**Remark 2.1:** At \( m = 0, k = 1 \) in (2.1), we get the characterization result for order statistics, that is

\[ E[g(X_{x:n})|X_{r:n} = x] = e^{-b/a} \sum_{i=0}^{\infty} \frac{(ah(x) + b)^i}{a^i i!} \prod_{j=r+1}^{s} \frac{c (n-j+1)}{c (n-j+1) + i}, \]

if and only if \( \overline{F}(x) = [ah(x) + b]^c. \)
Remark 2.2: If we put \( m = -1, \ k = 1 \) in (2.1), we get the characterization result for record values as

\[
E[g(X_{U(r)}^{(k)}) | X_{U(r)}^{(k)} = x] = e^{-\frac{b}{a}} \sum_{i=0}^{\infty} \frac{[a h(x) + b]^{i}}{a^{i} i!} \left( \frac{c}{c + i} \right)^{i-r}, \ a \neq 0, c + i \neq 0.
\]

if and only if \( \overline{F}(x) = [ah(x) + b]^{c} \).

Theorem 2.2. Let \( X \) be an absolutely continuous rv with df \( F(x) \) and pdf \( f(x) \) over the support \( (\alpha, \beta) \) and \( g(x) = e^{h(x)} \) be a monotonic and differentiable function of \( x \), then for \( 1 \leq r < s \leq n, \)

\[
E[g\{X(r, n, m, k)\} | X(s, n, m, k) = y] = g_{r|s}(y) = e^{-\frac{b}{a}} \sum_{i=0}^{\infty} \frac{[a h(y) + b]^{i}}{a^{i} i!} \left( \frac{c}{c + i} \right)^{i-r}, \ a \neq 0, c + i \neq 0 \quad (2.4)
\]

if and only if

\[
1 - [\overline{F}(x)]^{m+1} = [ah(x) + b]^{c}, \ m \neq -1 \quad (2.5)
\]

and

\[
- \log \overline{F}(x) = [ah(x) + b]^{c}, \ m = -1. \quad (2.6)
\]

where \( a, b, \) and \( c \) are so chosen that \( F(x) \) is a df.

Proof. To prove (2.5) implies (2.4), we have

\[
g_{r|s}(y) = \frac{(s-1)! (m+1)}{(r-1)! (s-r-1)!} \int_{\alpha}^{\beta} e^{h(x)} \left[ 1 - \frac{(\overline{F}(x))^{m+1}}{1 - (\overline{F}(y))^{m+1}} \right]^{r-1} \left[ 1 - \frac{1 - (\overline{F}(x))^{m+1}}{1 - (\overline{F}(y))^{m+1}} \right]^{s-r-1} \frac{(\overline{F}(x))^{m}}{1 - (\overline{F}(y))^{m+1}} f(x) dx.
\]

Let

\[
u = \frac{1 - (\overline{F}(x))^{m+1}}{1 - (\overline{F}(y))^{m+1}} = \left[ \frac{ah(x) + b}{ah(y) + b} \right]^{c}, \text{ then}
\]
\[ g_{rsl}(y) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \int_0^1 e^{\frac{u}{c}(ah(y)+b)} u^{r-1} (1-u)^{s-r-1} du \]

\[ = \frac{(s-1)!}{(r-1)!(s-r-1)!} e^{-b/a} \int_0^1 e^{a\left(\frac{u}{c}(ah(y)+b)\right)} u^{r-1} (1-u)^{s-r-1} du \]

\[ = \frac{(s-1)!}{(r-1)!(s-r-1)!} e^{-b/a} \sum_{i=0}^\infty \frac{(ah(y)+b)^i}{a^i i!} \int_0^1 u^{r+i-1} (1-u)^{s-r-1} du \]

\[ = \frac{(s-1)!}{(r-1)!(s-r-1)!} e^{-b/a} \sum_{i=0}^\infty \frac{(ah(y)+b)^i}{a^i i!} B\left(r+i, s-r\right), \]

which after simplification leads to (2.4).

To prove (2.4) implies (2.5), we have

\[ g_{rsl}(y) = e^{-b/a} \sum_{i=0}^\infty \frac{(ah(y)+b)^i}{a^i i!} \prod_{j=r}^{s-1} \frac{c^j}{c^j+i} \]

and

\[ g_{rsl-1}(y) = e^{-b/a} \sum_{i=0}^\infty \frac{(ah(y)+b)^i}{a^i i!} \prod_{j=r}^{s-2} \frac{c^j}{c^j+i}. \]

Therefore in view of Khan et al. (2006), we have

\[ \frac{(m+1)f(y)(\bar{F}(y))^m}{1-[\bar{F}(y)]^{m+1}} = \frac{1}{(s-1)} \frac{g'_{rsl}(y)}{[g_{rsl-1}(y) - g_{rsl}(y)]}. \]

Consider,

\[ g'_{rsl}(y) = ah'(y)e^{-b/a} \sum_{i=0}^\infty \frac{i(a h(y)+b)^{i-1}}{a^i i!} \prod_{j=r}^{s-1} \frac{c^j}{c^j+i} \]
and

\[
g_{rj-1}(y) - g_{rj}(y) = e^{-b/a} \sum_{i=0}^{\infty} \frac{[a h(y) + b]^i}{a^i i!} \prod_{j=r}^{s-2} \left( \frac{c j}{c j + i} \right) - e^{-b/a} \sum_{i=0}^{\infty} \frac{[a h(y) + b]^i}{a^i i!} \prod_{j=r}^{s-1} \left( \frac{c j}{c j + i} \right).
\]

Thus, we have

\[
\frac{(m+1)f(y)(\overline{F}(y))^m}{1-[\overline{F}(y)]^{m+1}} = \frac{a c h'(y)}{[a h(y) + b]}
\]

implying that

\[
1-[\overline{F}(x)]^{m+1} = [ah(x) + b]^c, \; m \neq -1
\]

and hence the (2.5).

Now to prove (2.6), taking the limit as \( m \to -1 \) in the LHS of (2.7), we get.

\[
-\frac{f(y)}{\overline{F}(y)} \frac{1}{\log \overline{F}(y)} = \frac{a c h'(y)}{[a h(y) + b]} = A(y).
\]

Note that \(-\log \overline{F}(x), \alpha < x < \beta\) is a non-decreasing function in \((0, \infty)\), therefore there exists a \( q \) such that

\[
-\log \overline{F}(q) = 1.
\]

Therefore,

\[
-\int_{x}^{q} \frac{f(y)}{F(y) \log \overline{F}(y)} dy = -\log[-\log \overline{F}(x)]
\]

\[
= \int_{x}^{q} A(y) dy,
\]

which gives,

\[
-\log \overline{F}(x) = [ah(x) + b]^c, \; m = -1.
\]

Hence (2.6).
**Theorem 2.3.** Under the condition as stated in the Theorem 2.1 and for two values \( r \) and \( s \), \( 1 \leq r < s \leq n \),

\[
E[g\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = g_{sr}(x)
\]

\[
= e^{h(x)} \prod_{j=r+1}^{s} \left( \frac{a \gamma_j}{a \gamma_j - 1} \right), \quad a \gamma_j \neq 1
\]  

(2.8)

if and only if

\[
\bar{F}(x) = e^{-ah(x)}.
\]

(2.9)

**Proof.** First we shall prove (2.9) implies (2.8).

For any \( s \geq r + 1 \), we have

\[
E[g\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_{x}^{\bar{F}(y)} e^{h(y)} \left[ 1 - \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right]^{s-r-1} \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{y_{r-1}} \frac{f(y)}{\bar{F}(x)} dy.
\]

(2.10)

Set \( u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{-ah(y)}}{e^{-ah(x)}} \),

which implies \( du = \frac{f(y)}{\bar{F}(x)} dy \)

and \( h(y) = h(x) - \frac{1}{a} \ln u \).

Then the RHS of (2.10), becomes

\[
= \frac{e^{h(x)} C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_{0}^{1} e^{-\ln u / a} (1 - u^{m+1})^{s-r-1} u^{[k+(n-s)(m+1)]-1} du.
\]

Now, let \( u^{m+1} = t \).
Then we get,

\[
E[g\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1} (m+1)^{s-r}}
\]

\[
\times e^{h(x)} \int_0^1 e^{-\frac{1}{a(m+1)}} \log (t) (1-t)^{s-r-1} t^{\frac{1}{a(m+1)}} dt
\]

\[
\times e^{h(x)} \int_0^1 e^{\log (t)} \frac{1}{a(m+1)} (1-t)^{s-r-1} t^{\frac{1}{a(m+1)}} dt
\]

\[
\times e^{h(x)} \int_0^1 e^{\log (t)} \frac{1}{a(m+1)} (1-t)^{s-r-1} t^{\frac{1}{a(m+1)}} dt
\]

\[
\times e^{h(x)} \int_0^1 e^{\log (t)} \frac{1}{a(m+1)} (1-t)^{s-r-1} t^{\frac{1}{a(m+1)}} dt
\]

\[
= \frac{C_{r-1}}{(s-r-1)!C_{r-1} (m+1)^{s-r}}
\]

\[
\times e^{h(x)} \int_0^1 t^{\frac{a[k+(n-s)(m+1)]}{a(m+1)}-1} (1-t)^{s-r-1} dt
\]

\[
= \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)! (m+1)^{s-r}}
\]

\[
\times e^{h(x)} B\left(\frac{a[k+(n-s)(m+1)]}{a(m+1)}, s-r\right)
\]

\[
= \frac{\prod_{j=r+1}^s \gamma_j}{(s-r-1)! (m+1)^{s-r}}
\]

\[
\times e^{h(x)} \Gamma\left(\frac{a[k+(n-s)(m+1)]}{a(m+1)}-1\right) \frac{\Gamma(s-r)}{\Gamma\left(\frac{a[k+(n-s)(m+1)]}{a(m+1)}-1\right) + (s-r)}
\]
\[
\prod_{j=r+1}^{s} \gamma_j \cdot e^{h(x)} \cdot \frac{a[k + (n-s-1)(m+1)]-1}{a(m+1)} \cdot \frac{1}{\left(\frac{a[k + (n-r-1)(m+1)]-1}{a(m+1)}\right)}
\]

which after simplification, gives

\[
= \frac{e^{h(x)}}{(m+1)^{s-r}} \prod_{j=r+1}^{s} \gamma_j \cdot \frac{1}{\left(\frac{a(m+1) \gamma_j}{a(\gamma_j - 1)}\right)}
\]

\[
= e^{h(x)} \cdot \prod_{j=r+1}^{s} \frac{a \gamma_j}{a(\gamma_j - 1)}
\]

Hence (2.8).

Now to prove sufficiency part, we have

\[
g_{sfr}(x) = E[\{g \{X(s,n,m,k)\} | X(r,n,m,k) = x\}]
\]

Therefore,

\[
g_{sfr}(x) [\bar{F}(x)]^{\gamma_{r+1}} = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \\
\times \int_{x}^{\theta} e^{h(y)} [(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1} [\bar{F}(y)]^{\gamma_{s-1}} f(y) dy.
\]

Differentiating both sides w.r.t. \(x\) and adjusting the terms, we get

\[
\frac{f(x)}{\bar{F}(x)} = -\frac{1}{\gamma_{r+1}} \frac{g'_{sfr}(x)}{g_{sfr+1}(x) - g_{sfr}(x)} \quad [\text{Khan et al. 2006}]
\]
Consider,
\[ g_{s|r}(x) = e^{h(x)} h'(x) \prod_{i=r+1}^{s} \left( \frac{a \gamma_i}{a \gamma_i - 1} \right) \]
and \[ g_{s|r+1}(x) - g_{s|r}(x) = -\frac{e^{h(x)}}{a \gamma_{r+1} - 1} \prod_{j=r+2}^{s} \left( \frac{a \gamma_j}{a \gamma_j - 1} \right). \]
Thus, we have
\[ \frac{f(x)}{F(x)} = ah'(x), \]
implying that
\[ F(x) = e^{-ah(x)}. \]

**Remark 2.3:** If we put \( m = 0, n = 1 \) in (2.8), we get the characterization result
for order statistics
\[ E[g(X_{s,n}) | X_{r,n} = x] = e^{h(x)} \prod_{j=r+1}^{s} \left( \frac{a(n-j+1)}{a(n-j+1)-1} \right), a(n-j+1) \neq 1. \]
if and only if \( F(x) = e^{-ah(x)}. \)

**Remark 2.4:** If we put \( m = -1, n = 1 \) in (2.8), we get the characterization result
for record values as
\[ E[g(X_{U(s)}) | X_{U(r)} = x] = e^{h(x)} \left( \frac{a}{a-1} \right)^{s-r}, a \neq 1. \]
if and only if \( F(x) = e^{-ah(x)}. \)

**Theorem 2.4.** Under the condition as stated in Theorem 2.2,
\[ E[g\{X(r,n,m,k)\} | X(s,n,m,k) = y] = g_{rs}(y) \]
\[ = e^{h(y)} \prod_{j=r}^{s} \left( \frac{a_j}{a_j - 1} \right), aj \neq 1 \quad (2.11) \]
if and only if
\[
1 - [\overline{F}(x)]^{m+1} = e^{-ah(x)}, \ m \neq -1
\] (2.12)
and
\[
\overline{F}(x) = \exp \left[ - \exp \left( \frac{h(q) - h(x)}{\delta} \right) \right], \ m = -1, \ \delta = \frac{1}{a} \neq 0.
\] (2.13)

Proof. Necessary part can be proved on the lines of Theorem 2.3.

To prove (2.11) implies (2.12), we have
\[
E[g\{X(r,n,m,k)\} | X(s,n,m,k) = y] = g_{ris}(y)
\]
Now, in view of Khan et al., 2006, we get
\[
\frac{(m + 1)f(y)\overline{F}(y)^m}{1 - [\overline{F}(y)]^{m+1}} = -ah'(y),
\] (2.14)
implying that
\[
1 - [\overline{F}(x)]^{m+1} = e^{-ah(x)}, \ m \neq -1
\]
and hence the (2.12).

Now to prove (2.13), taking the limit as \( m \to -1 \) in the LHS of (2.14), we get
\[
- \frac{f(y)}{F(y) \log F(y)} \frac{1}{F(y) \log F(y)} = -ah'(y),
\]
or
\[
\log[-\log \overline{F}(x)] = -\int q ah'(y)dy, \ q \in (\alpha, \beta),
\]
which gives,
\[
\overline{F}(x) = \exp \left[ - \exp \left( \frac{h(q) - h(x)}{\delta} \right) \right], \ m = -1, \ \delta = \frac{1}{a} \neq 0.
\]
Hence the theorem.
### 3. EXAMPLES

Table 3.1: Examples based on $F(x) = 1 - [ah(x) + b]^c$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$F(x)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power function</td>
<td>$a^{-p} x^p, 0 \leq x \leq a$</td>
<td>$-a^{-p}$</td>
<td>1</td>
<td>1</td>
<td>$x^p$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$1 - a^p x^{-p}, a \leq x &lt; \infty$</td>
<td>$a^{-q}$</td>
<td>0</td>
<td>$p/q$</td>
<td>$x^q, q \neq 0$</td>
</tr>
<tr>
<td>Beta of the first kind</td>
<td>$1 - (1-x)^p, 0 \leq x \leq 1$</td>
<td>1</td>
<td>0</td>
<td>$p/q$</td>
<td>$(1-x)^q, q \neq 0$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$1 - e^{-\theta x^p}, 0 \leq x \leq \infty$</td>
<td>1</td>
<td>$-\theta/c$</td>
<td>1</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Inverse Weibull</td>
<td>$e^{-\theta x^p}, 0 \leq x &lt; \infty$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$e^{-\theta x^p}$</td>
</tr>
<tr>
<td>Burr type II</td>
<td>$[1 + e^{-x}]^{-k}, -\infty &lt; x &lt; \infty$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$[1 + e^{-x}]^{-k}$</td>
</tr>
<tr>
<td>Burr type III</td>
<td>$[1 + x^{-c}]^{-k}, 0 \leq x &lt; \infty$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$[1 + x^{-c}]^{-k}$</td>
</tr>
<tr>
<td>Burr type IV</td>
<td>$\left[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right]^{-k}, 0 \leq x &lt; c$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$\left[ 1 + \left( \frac{c-x}{x} \right)^{1/c} \right]^{-k}$</td>
</tr>
<tr>
<td>Burr type V</td>
<td>$[1 + ce^{-\tan x}]^{-k}, -\pi/2 \leq x \leq \pi/2$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$[1 + ce^{-\tan x}]^{-k}$</td>
</tr>
<tr>
<td>Burr type VI</td>
<td>$[1 + ce^{-k \sinh x}]^{-k}, -\infty &lt; x &lt; \infty$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$[1 + ce^{-k \sinh x}]^{-k}$</td>
</tr>
<tr>
<td>Burr type VII</td>
<td>$2^{-k} [1 + \tanh x]^k, -\infty &lt; x &lt; \infty$</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>$[1 + \tanh x]^k$</td>
</tr>
<tr>
<td>Burr type VIII</td>
<td>$\left( \frac{2}{\pi} \tan^{-1} e^x \right)^k, -\infty &lt; x &lt; \infty$</td>
<td>$\left( \frac{2}{\pi} \right)^k$</td>
<td>1</td>
<td>1</td>
<td>$[\tan^{-1} e^x]^k$</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Burr type IX</th>
<th>$\frac{2}{c[(1 + e^x)^k - 1] + 2}$, $-\infty &lt; x &lt; \infty$</th>
<th>$\frac{c}{2}$</th>
<th>$1 - \frac{c}{2}$</th>
<th>$-1$</th>
<th>$[1 + e^x]^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Burr type X</td>
<td>$[1 - e^{-x^2}]^k$, $0 &lt; x &lt; \infty$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$[1 - e^{-x^2}]^k$</td>
</tr>
<tr>
<td>Burr type XI</td>
<td>$\left(x - \frac{1}{2\pi} \sin 2\pi x\right)^k$, $0 \leq x \leq 1$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$(x - \frac{1}{2\pi} \sin 2\pi x)^k$</td>
</tr>
<tr>
<td>Burr type XII</td>
<td>$1 - (1 + \theta x^p)^{-m}$, $0 \leq x &lt; \infty$</td>
<td>$\theta$</td>
<td>$1$</td>
<td>$-m$</td>
<td>$x^p$</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$\frac{1}{2} + \frac{1}{\pi} \tan^{-1} x$, $-\infty &lt; x &lt; \infty$</td>
<td>$-\frac{1}{\pi}$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
<td>$\tan^{-1} x$</td>
</tr>
</tbody>
</table>
### Table 3.2: Examples based on the $F(x) = 1 - e^{-ah(x)}$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$F(x)$</th>
<th>$a$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>$1 - e^{-\theta x}$, $0 &lt; x &lt; \infty$</td>
<td>$\theta$</td>
<td>$x$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$1 - e^{-\theta x^p}$, $0 &lt; x &lt; \infty$</td>
<td>$\theta$</td>
<td>$x^p$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$1 - \left(\frac{x}{a}\right)^{-\theta}$, $0 &lt; x &lt; \infty$</td>
<td>$\theta$</td>
<td>$\log\left(\frac{x}{a}\right)$</td>
</tr>
<tr>
<td>Lomax</td>
<td>$1 - \left(1 + \left(\frac{x}{a}\right)^p\right)^{\frac{1}{p}}$, $0 &lt; x &lt; \infty$</td>
<td>$p$</td>
<td>$\log\left[1 + \left(\frac{x}{a}\right)^p\right]$</td>
</tr>
<tr>
<td>Gompertz</td>
<td>$1 - \exp\left[-\frac{\lambda}{\mu}(e^{\mu x} - 1)\right]$, $0 &lt; x &lt; \infty$</td>
<td>$\frac{\lambda}{\mu}$</td>
<td>$e^{\mu x} - 1$</td>
</tr>
<tr>
<td>Beta of the I kind</td>
<td>$1 - (1 - x)^{-\theta}$, $0 &lt; x &lt; 1$</td>
<td>$-\theta$</td>
<td>$-\log(1 - x)$</td>
</tr>
<tr>
<td>Beta of the II kind</td>
<td>$1 - (1 + x)^{-1}$, $0 &lt; x &lt; 1$</td>
<td>$1$</td>
<td>$\log(1 + x)$</td>
</tr>
<tr>
<td>Extreme value I</td>
<td>$1 - \exp(-e^x)$, $0 &lt; x &lt; \infty$</td>
<td>$1$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>Log logistic</td>
<td>$1 - \left[1 + \theta x^p\right]^{-1}$, $0 &lt; x &lt; \infty$</td>
<td>$1$</td>
<td>$\log(1 - \theta x^p)$</td>
</tr>
<tr>
<td>Burr type IX</td>
<td>$1 - \left[\frac{c(1-e^{x^p})-1}{2} + 1\right]^{-1}$, $-\infty &lt; x &lt; \infty$</td>
<td>$1$</td>
<td>$\log\left[\frac{c(1-e^{x^p})-1}{2} + 1\right]$</td>
</tr>
<tr>
<td>Burr type XII</td>
<td>$1 - (1 + \theta x^p)^{-m}$, $0 &lt; x &lt; \infty$</td>
<td>$m$</td>
<td>$\log(1 + \theta x^p)$</td>
</tr>
</tbody>
</table>
REFERENCES


Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992): *A First Course in

New York.

Athar, H. and Islam, H.M. (2004): Recurrence relations between single and
product moments of generalized order statistics from a general class of

Athar, H. and Nayabuddin (2012): On moment generating function of
generalized from extended generalized half logistic distribution and its

Athar, H. and Nayabuddin (2014): Expectation identities of generalized order
statistics from Marshall-Olkin extended uniform distribution and its

through conditional expectation of function of pair of order statistics.
*Aligarh J. Statist.*, 23, 97-105.


Galton, F. (1902): The most suitable proportion between the values of first and second prized. *Biometrika*, 1, 385-390.


